3.1 Newton-Leibniz Formula

Motivation. Consider the following problem: for a given function F find a function f such that $dF(x) = f(x)\,dx$, over $[a,b]$, that is, $\int\limits_c^d f(t)\,dt = F(d) - F(c)$ for any subinterval $[c,d]$ of $[a,b]$.

Suppose that such an f exists. Since the value of f at a single point does not affects the integrals, we cannot say anything about the value of f at any given point. But if f is continuous at a point x_0, its value is uniquely defined by F.

To be precise, the difference quotient $\frac{F(x)-F(x_0)}{x-x_0}$ tends to $f(x_0)$ as x tends to x_0. Indeed, $F(x) = F(x_0) + \int\limits_{x_0}^{x} f(t)\,dt$. Furthermore, $\int\limits_{x_0}^{x} f(t)\,dt = f(x_0)(x-x_0) + \int\limits_{x_0}^{x} (f(t) - f(x_0))\,dt$. Also, $|\int\limits_{x_0}^{x} (f(t) - f(x_0)\,dt| \le \operatorname{var}_f[x_0,x]|x-x_0|$. Consequently

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| \le \operatorname{var}_f[x,x_0] \tag{3.1.1}$$

However, $\operatorname{var}_f[x,x_0]$ can be made arbitrarily small by choosing x sufficiently close to x_0, since $\operatorname{var}_f x_0 = 0$.

Infinitesimally small functions. A set is called a *neighborhood* of a point x if it contains all points *sufficiently close* to x, that is all points y such that $|y-x|$ is less then a positive number ε.

We will say that a function f is *locally bounded* (above) by a constant C at a point x, if $f(x) \le C$ for all y sufficiently close to x.

A function $o(x)$ is called *infinitesimally small* at x_0, if $|o(x)|$ is locally bounded at x_0 by any $\varepsilon > 0$.

Lemma 1. *If the functions o and ω are infinitesimally small at x_0 then $o \pm \omega$ are infinitesimally small at x_0.*

Proof. Given a $\varepsilon > 0$. Let O_1 be a neighborhood of x_0 where $|o(x)| < \varepsilon/2$, and O_2 be a neighborhood of x_0 where $|\omega(x)| < \varepsilon/2$, then $O_1 \cap O_2$ is a neighborhood where both inequalities hold. Hence for all $x \in O_1 \cap O_2$ one has $|o(x) \pm \omega(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Lemma 2. *If $o(x)$ is infinitesimally small at x_0 and $f(x)$ is locally bounded at x_0, then $f(x)o(x)$ is infinitesimally small.*

Proof. The neighborhood where $|f(x)o(x)|$ is bounded by a given $\varepsilon > 0$ can be constructed as the intersection of a neighborhood U, where $|f(x)|$ is bounded by a constant C, and a neighborhood V, where $|o(x)|$ is bounded by ε/C. □

Definition. *One says that a function $f(x)$ tends to A as x tends to x_0 and writes $\lim\limits_{x\to x_0} f(x) = A$, if $f(x) = A + o(x)$ on the complement of x_0, where $o(x)$ is infinitesimally small at x_0.*

Corrolary 3. *If limits $\lim\limits_{x\to x_0} f(x)$ and $\lim\limits_{x\to x_0} g(x)$ exist, then $\lim\limits_{x\to x_0} (f(x)+g(x))$ also exists and $\lim\limits_{x\to x_0} (f(x)+g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$.*

Proof. This follows immediately from Lemma **1**. $\square$

Lemma 4. *If the limits* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then also* $\lim_{x\to x_0} f(x)g(x)$ *and* $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x)$ *exist.*

Proof. If $f(x) = A + o(x)$ and $g(x) = B + \omega(x)$, then $f(x)g(x) = AB + A\omega(x) + Bo(x) + \omega(x)o(x)$, where $A\omega(x)$, $Bo(x)$ and $\omega(x)o(x)$ all are infinitesimally small at x_0 by Lemma **2**, and their sum is infinitesimally small by Lemma **1**. $\square$

Definition. *A function f is called* continuous *at x_0, if* $\lim_{x\to x_0} f(x) = f(x_0)$.

A function is said to be continuous (without mentioning a point), if it is continuous at all points under consideration.

The following lemma gives a lot of examples of continuous functions.

Lemma 5. *If f is a monotone function on $[a,b]$ such that $f[a,b] = [f(a), f(b)]$ then f is continuous.*

Proof. Given a positive ε. Suppose f is nondecreasing. For a given point x denote by $x^\varepsilon = f^{-1}(f(x)+\varepsilon)$ and $x_\varepsilon = f^{-1}(f(x)-\varepsilon)$. Then $[x_\varepsilon, x^\varepsilon]$ contains a neighborhood of x, and for any $y \in [x_\varepsilon, x^\varepsilon]$ one has $f(x)+\varepsilon = f(x_\varepsilon) \leq f(y) \leq f(x^\varepsilon) = f(x)+\varepsilon$. Hence the inequality $|f(y) - f(x)| < \varepsilon$ holds locally at x for any ε. $\square$

The following theorem immediately follows from the Corollary **3** and Lemma **4**.

Theorem 6. *If the functions f and g are continuous at x_0, then $f+g$ and fg are continuous at x_0.*

The following property of continuous function is very important.

Theorem 7. *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g(f(x))$ is continuous at x_0.*

Proof. Given $\varepsilon > 0$, we have to find a neighborhood U of x_0, such that $|g(f(x)) - g(f(x_0))| < \varepsilon$ for $x \in U$. As $\lim_{y\to f(x_0)} g(y) = g(f(x_0))$, there exists a neighborhood V of $f(x_0)$ such that $|g(y) - g(y_0)| < \varepsilon$ for $y \in V$. Thus it is sufficient to find a U such that $f(U) \subset V$. And we can do this. Indeed, by definition of neighborhood there is $\delta > 0$, such that V contains $V_\delta = \{y \mid |y - f(x_0)| < \delta\}$. Since $\lim_{x\to x_0} f(x) = f(x_0)$, there is a neighborhood U of x_0 such that $|f(x) - f(x_0)| < \delta$ for all $x \in U$. Then $f(U) \subset V_\delta \subset V$. $\square$

Definition. *A function f is called* differentiable *at a point x_0 if the* difference quotient $\frac{f(x)-(f_0)}{x-x_0}$ *has a limit as x tends to x_0. This limit is called the* derivative *of the function F at the point x_0, and denoted* $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

Immediately from the definition one evaluates the derivative of linear function.

$$(ax+b)' = a \tag{3.1.2}$$

The following lemma is a direct consequence of Lemma **3**.

Lemma 8. *If f and g are differentiable at x_0, then $f+g$ is differentiable at x_0 and* $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

Linearization. Let f be differentiable at x_0. Denote by $o(x)$ the difference $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x)(x - x_0), \tag{3.1.3}$$

where $o(x)$ is infinitesimally small at x_0. We will call such a representation a *linearization* of $f(x)$.

Lemma 9. *If f is differentiable at x_0, then it is continuous at x_0.*

Proof. All summands but $f(x_0)$ on the right-hand side of (3.1.3) are infinitesimally small at x_0; hence $\lim_{x\to x_0} f(x) = f(x_0)$. □

Lemma 10 (on uniqueness of linearization). *If $f(x) = a + b(x - x_0) + o(x)(x - x_0)$, where $\lim_{x\to x_0} o(x) = 0$, then f is differentiable and at x_0, $a = f(x_0)$ and $b = f'(x_0)$.*

Proof. The difference $f(x) - f(x_0)$ is infinitesimally small at x_0, because f is continuous at x_0, and the difference $f(x) - a = b(x - x_0) + o(x)(x - x_0)$ is infinitesimally small by the definition of linearization. Hence $f(x_0) - a$ is infinitesimally small. But it is constant, hence $f(x_0) - a = 0$.

Thus we established $a = f(x_0)$. The difference $\frac{f(x)-a}{x-x_0} - b = o(x)$ is infinitesimally small as well as $\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$. But $\frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x)-a}{x-x_0}$. Therefore $b - f'(x_0)$ is infinitesimally small. That is $b = f'(x_0)$. □

Lemma 11. *If f and g are differentiable at x_0, then fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$*

Proof. Consider leneariations $f(x_0) + f'(x_0)(x - x_0) + o(x)(x - x_0)$ and $g(x_0) + g'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Their product is $f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))(x - x_0) + (f(x)\omega(x) + f(x_0)o(x))(x - x_0)$. This is the linearization of $f(x)g(x)$ at x_0, because $f\omega$ and go are infinitesimally small at x_0. □

Theorem 12. *If f is differentiable at x_0, and g is differentiable at $f(x_0)$ then $g(f(x))$ is differentiable at x_0 and $(g(f(x_0)))' = g'(f(x_0))f'(x_0)$*

Proof. Denote $f(x_0)$ by y_0 and substitute into the linearization $g(y) = g(y_0) + g'(y_0)(y - y_0) + o(y)(y - y_0)$ another linearization $y = f(x_0) + f'(x_0)(x - x_0) + \omega(x)(x - x_0)$. Since $y - y_0 = f'(x_0)(x - x_0) + \omega(x)(x - x_0)$, we get $g(y) = g(y_0) + g'(y_0)f'(x_0)(x - x_0) + g'(y_0)(x - x_0)\omega(x) + o(f(x))(x - x_0)$. Due to the Lemma on linearization, it is sufficient to prove that $g'(y_0)\omega(x) + o(f(x))$ is infinitesimally small at x_0. The first summand is obviously infinitesimally small. To prove that the second one also is infinitesimally small, we remark that $o(f(x_0)) = 0$ and $o(y)$ is continuous at $f(x_0)$ and that $f(x)$ is continuous at x_0 due to Lemma **9**. Hence by Theorem 3.1 the composition is continuous at x_0 and infinitesimally small. □

Theorem 13. *Let f be a virtually monotone function on $[a, b]$. Then $F(x) = \int_a^x f(t)\,dt$ is virtually monotone and continuous on $[a, b]$. It is differentiable at any point x_0 where f is continuous, and $F'(x_0) = f(x_0)$.*

Proof. If f has a constant sign, then F is monotone. So, if $f = f_1 + f_2$ is a monotonization of f, then $\int_a^x f_1(x)\,dx + \int_a^x f_1(x)\,dx$ is a monotonization of $F(x)$. This proves that $F(x)$ is virtually monotone.

To prove continuity of $F(x)$ at x_0, fix a constant C which bounds f in some neighborhood U of x_0. Then for $x \in U$ one proves that $|F(x) - F(x_0)|$ is infinitesimally small via inequalities $|F(x) - F(x_0)| = |\int_{x_0}^x f(x)\,dx| \leq |\int_{x_0}^x C\,dx| = C|x - x_0|$.

Now suppose f is continuous at x_0. Then $o(x) = f(x_0) - f(x)$ is infinitesimally small at x_0. Therefore $\lim_{x \to x_0} \frac{1}{x - x_0} \int_{x_0}^x o(x)\,dx = 0$. Indeed for any $\varepsilon > 0$ the inequality $|o(x)| \leq \varepsilon$ holds over $[x_\varepsilon, x_0]$ for some x_ε. Hence $|\int_{x_0}^x o(x)\,dx| \leq |\int_{x_0}^x \varepsilon\,dx| = \varepsilon|x - x_0|$ for any $x \in [x_0, x_\varepsilon]$.

Then $F(x) = F(x_0) + f(x_0)(x - x_0) + (\frac{1}{x - x_0} \int_{x_0}^x o(t)\,dt)(x - x_0)$ is a linearization of $F(x)$ at x_0. □

Corrolary 14. *The functions* ln, sin, cos *are differentiable and* $\ln'(x) = \frac{1}{x}$, $\sin' = \cos$, $\cos' = -\sin$.

Proof. Since $d \sin x = \cos x\,dx$, $d \cos x = -\sin x\,dx$, due to Theorem **13** both $\sin x$ and $\cos x$ are continuous, and, as they are continuous, the result follows from Theorem **13**. And $\ln' x = \frac{1}{x}$, by Theorem **13**, follows from continuity of $\frac{1}{x}$. The continuity follows from Lemma **5**. □

Since $\sin'(0) = \cos 0 = 1$ and $\sin 0 = 0$, the linearization of $\sin x$ at 0 is $x + xo(x)$. This implies the following very important equality

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad (3.1.4)$$

Lemma 15. *If* $f'(x) > 0$ *for all* $x \in [a, b]$*, then* $f(b) > f(a)|$

Proof. Suppose $f(a) \geq f(b)$. We construct a sequence of intervals $[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$. such that their lengths tend to 0 and $f(a_k) \geq f(b_k)$. All steps of construction are the same. The general step is: let m be the middle point of $[a_k, b_k]$. If $f(m) \leq f(a_k)$ we set $[a_{k+1}, b_{k+1}] = [a_k, m]$, otherwise $f(m) > f(a_k) \geq f(b_k)$ and we set $[a_{k+1}, b_{k+1}] = [m, b_k]$.

Now consider a point x belonging to all $[a_k, b_k]$. Let $f(y) = f(x) + (f'(x) + o(x))(y - x)$ be the linearization of f at x. Let U be neighborhood where $|o(x)| < f'(x)$. Then $\operatorname{sgn}(f(y) - f(x)) = \operatorname{sgn}(y - x)$ for all $y \in U$. However for some n we get $[a_n, b_n] \subset U$. If $a_n \leq x < b_n$ we get $f(a_n) \leq f(x) < f(b_n)$ else $a_n < x$ and $f(a_n) < f(x) \leq f(b_n)$. In the both cases we get $f(a_n) < f(b_n)$. This is contradiction with our construction of the sequence of intervals. □

Theorem 16. *If* $f'(x) = 0$ *for all* $x \in [a, b]$*, then* $f(x)$ *is constant.*

Proof. Set $k = \frac{f(b)-f(a)}{b-a}$. If $k < 0$ then $g(x) = f(x) - kx/2$ has derivative $g'(x) = f'(x) - k/2 > 0$ for all x. Hence by Lemma **15** $g(b) > g(a)$ further $f(b) - f(a) > k(b-a)/2$. This contradict to definition of k. If $k > 0$ then the same contradiction one gets considering $g(x) = -f(x) + kx/2$. □

Theorem 17 (Newton-Leibniz). *If $f'(x)$ is a continuous virtually monotone function on an interval $[a, b]$, then* $\int\limits_a^b f'(x)\,dx = f(b) - f(a)$

Proof. Due to Theorem **13**, the derivative of the difference $\int\limits_a^x f'(t)\,dt - f(x)$ is zero. Hence the difference is constant by Theorem **16**. Substituting $x = a$ we find the constant which is $f(a)$. Consequently, $\int\limits_a^x f'(t)\,dt - f(x) = f(a)$ for all x. In particular, for $x = b$ we get the Newton-Leibniz formula.

□

Problems.

1. $(1/x)'$, $\sqrt{x}'$, $(\sqrt{\sin x^2})'$
2. $\exp' x$
3. $\operatorname{arctg}' x$, $\tan' x$
4. $|x|'$, $\operatorname{Re} z'$
5. $f'(x) \equiv 1 \iff f(x) = x + const$
6. $\left(\int\limits_x^{x^2} \frac{\sin t}{t}\,dt\right)'_x$
7. $\sqrt{1-x^2}'$
8. $\left(\int\limits_0^1 \frac{\sin kt}{t}\,dt\right)'_k$
9. If f is continuous at a and $\lim\limits_{n\to\infty} x_n = a$ then $\lim\limits_{n\to\infty} f(x_n) = f(a)$.
10. $\left(\int\limits_0^y [x]\,dx\right)'_y$.
11. $arcsin' x$
12. $\int \frac{dx}{2+3x^2}$
13. If $f'(x) < 0$ for all $x < m$ and $f'(x) > 0$ for all $x > m$ then $f'(m) = 0$
14. If $f'(x)$ is bounded on $[a, b]$ then f is virtually monotone