3.3 Euler Formula

Complex Newton-Leibniz. For a function of a complex variable $f(z)$ the derivative is defined by the same formula $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. We will denote it also by $\frac{df(z)}{dz}$, to distinguish from derivative of pathes: complex valued functions of real variable. For a path $p(t)$ its derivative will be denoted either $p'(t)$ or $\frac{dp(t)}{dt}$. The Newton-Leibniz formula for real functions can be expressed The equality $\frac{df(t)}{dt}dt = df(t)$. Now we extend this formula to complex functions.

The linearization of a complex function $f(z)$ at z_0 has the same form $f(z_0)+f'(z_0)(z-z_0)+o(z)(z-z_0)$, where $o(z)$ is an infinitesimally small function of complex variable. The same arguments as for real numbers prove the basic rules of differentiation: the derivative of sum, product and composition.

Theorem 1. $\frac{dz^n}{dz} = nz^{n-1}$

Proof. $\frac{dz}{dz} = 1$ one gets immediately from the definition of the derivative. Suppose the equality $\frac{dz^n}{dz} = nz^{n-1}$ is proved for n. Then $\frac{dz^{n+1}}{dz} = \frac{dzz^n}{dz} = z\frac{dz^n}{dz} + z^n\frac{dz}{dz} = znz^{n-1}+z^n = (n+1)z^n dz$. And the theorem is proved by induction. □

A smooth path is a differentiable mapping $p\colon [a,b] \to \mathbb{C}$ with continuous bounded derivative. A function $f(z)$ of complex variable is called *virtually monotone* if for any smooth path $p(t)$ functions $\operatorname{Re} f(p(t))$ and $\operatorname{Im} f(p(t))$ are virtually monotone.

Lemma 2. *If $f'(z)$ is bounded, then $f(z)$ is virtually monotone.*

Proof. Consider a smooth path p. Then $\frac{df(p(t))}{dt} = f'(p(t))p'(t)$ is bounded by some K. Due to Lemma **15** one has $|f(p(t)) - f(p(t_0))| \leq K|t-t_0|$. Hence any partial variation of $f(p(t))$ does not exceed $K(b-a)$. Therefore $\operatorname{var}_{f(p(t))}[a,b] \leq K$. □

Theorem 3. *If a complex function $f(z)$ has a bounded virtually monotone continuous complex derivative over the image of a smooth path $p\colon [a,b] \to \mathbb{C}$, then $\int_p f'(z)\,dz = f(p(b)) - f(p(a))$.*

Proof. $\frac{df(p(t))}{dt} = f'(p(t))p'(t) = \frac{d\operatorname{Re} f(p(t))}{dt} + i\frac{d\operatorname{Im} f(p(t))}{dt}$. All functions here are continuous and virtually monotone by hypothesis. Passing to differential forms one gets $\frac{df(p(t))}{dt}\,dt = \frac{d\operatorname{Re} f(p(t))}{dt}\,dt + i\,dt\operatorname{Im} f(p(t))\,dt = d(\operatorname{Re} f(p(t))) + i\,d(\operatorname{Im} f(p(t))) = d(\operatorname{Re} f(p(t)) + i\operatorname{Im} f(p(t))) = d(f(p(t)))$. Hence $\int_p f'(z)\,dz = \int_p df(z)$.

□

Corrolary 4. *If $f'(z) = 0$ then $f(z)$ is constant*

Proof. Consider $p(t) = z_0 + (z-z_0)t$, then $f(z) - f(z_0) = \int_p f'(\zeta)\,d\zeta = 0$. □

Differentiation of series. Let us say that a series of complex numbers $\sum\limits_{k=1}^{\infty} a_k$ majorizes (eventually) another such series $\sum\limits_{k=1}^{\infty} b_k$ if $|b_k| \leq |a_k|$ for all k (resp. for k onward some n).

The series $\sum\limits_{k=1}^{\infty} kc_k(z-z_0)^{k-1}$ is called a *formal derivative* of $\sum\limits_{k=0}^{\infty} c_k(z-z_0)^k$.

Lemma 5. *Any power series* $\sum\limits_{k=0}^{\infty} c_k(z-z_0)^k$ *eventually majorizes its formal derivative* $\sum\limits_{k=0}^{\infty} kc_k(z_1-z_0)^{k-1}$ *if* $|z_1-z_0| < |z-z_0|$.

Proof. The ratio of n-th term of the derivative to the n-th term of the series tends to 0 as n tends to infinity. Indeed, this ratio is $\frac{k(z_1-z_0)^k}{(z-z_0)^k} = kq^k$, where $|q| < 1$ since $|z_1 - z_0| < |z - z_0$. The fact that $\lim\limits_{n\to\infty} nq^n = 0$ follows from convergence of $\sum\limits_{k=1}^{\infty} kq^k$ which we already have proved before. This series is eventually majorized by any geometric series $\sum\limits_{k=0}^{\infty} AQ^k$ with $Q > q$. □

A path $p(t)$ is called *monotone* if both $\operatorname{Re} p(t)$ and $\operatorname{Im} p(t)$ are monotone.

Lemma 6. *Let* $p\colon [a,b] \to \mathbb{C}$ *be a smooth monotone path, and let* $f(z)$ *be virtually monotone. If* $|f(p(t))| \leq c$ *for* $t \in [a,b]$ *then* $\left|\int\limits_p f(z)\,dz\right| \leq 4c|p(b) - p(a)|$

Proof. Integration of the inequalities $-c \leq \operatorname{Re} f(p(t)) \leq c$ against $d\operatorname{Re} z$ along the path gives $\left|\int\limits_p \operatorname{Re} f(z)\,d\operatorname{Re} z\right| \leq c|\operatorname{Re} p(b) - \operatorname{Re} p(a)| \leq c|p(b)-p(a)|$. The same arguments prove $\left|\int\limits_p \operatorname{Im} f(z)\,d\operatorname{Im} z\right| \leq c|\operatorname{Im} p(b) - \operatorname{Im} p(a)| \leq c|p(b) - p(a)|$. The sum of these inequalities gives $\left|\operatorname{Re}\int\limits_p f(z)\,dz\right| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. The same arguments yields $\left|\operatorname{Im}\int\limits_p f(z)\,dz\right| \leq 2c|\operatorname{Re} p(b) - \operatorname{Re} p(a)|$. And the addition of the two last inequalities allow us to accomplish the proof of the Lemma because $\left|\int\limits_p f(z)\,dz\right| \leq \left|\operatorname{Re}\int\limits_p f(z)\,dz\right| + \left|\int\limits_p f(z)\,dz\right|$. □

Lemma 7. $|z^n - \zeta^n| \leq n|z-\zeta| \max\{|z^{n-1}|, |\zeta^{n-1}|\}$

Proof. $(z^n - \zeta^n) = (z-\zeta)\sum\limits_{k=0}^{n-1} z^k\zeta^{n-k-1}$ and $|z^k\zeta^{n-k-1}| \leq \max\{|z^{n-1}|, |\zeta^{n-1}|\}$. □

A *linear path* from z_0 to z_1 is defined as a linear mapping $p\colon [a,b] \to \mathbb{C}$, such that $p(a) = z_0$ and $p(b) = z_1$, that is $p(t) = z_0(t-a) + (z_1 - z_0)(t-a)/(b-a)$.

We denote by $\int\limits_a^b f(z)\,dz$ the integral along the linear path from a to b.

Lemma 8. *For any complex z, ζ and natural $n > 0$ one has*

$$|z^n - z_0^n - nz_0^{n-1}(z - z_0)| \le 2n(n-1)|z - z_0|^2 \max\{|z|^{n-2}, |z_0|^{n-2}\} \tag{3.3.1}$$

Proof. By the Newton-Leibniz formula, $z^n - z_0^n = \int\limits_{z_0}^{z} n\zeta^{n-1}\,d\zeta$. Further,

$$\int\limits_{z_0}^{z} n\zeta^{n-1}\,d\zeta = \int\limits_{z_0}^{z} nz_0^{n-1}\,d\zeta + \int\limits_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\,d\zeta =$$

$$nz_0^{n-1} + \int\limits_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\,d\zeta.$$

Consequently, the left-hand side of (3.3.1) is equal to $\left|\int\limits_{z_0}^{z} n(\zeta^{n-1} - z_0^{n-1})\,d\zeta\right|$. Due to Lemma **7** the absolute value of the integrand along the linear path does not exceed $(n-1)|z-z_0|\max\{|z^{n-2}|, |z_0^{n-2}|\}$. Now the estimation of the integral by Lemma **6** gives just the inequality (3.3.1). □

Theorem 9. *If $\sum\limits_{k=0}^{\infty} c_k(z_1 - z_0)^k$ converges absolutely, then $\sum\limits_{k=0}^{\infty} c_k(z - z_0)^k$ and $\sum\limits_{k=1}^{\infty} kc_k(z - z_0)^{k-1}$ absolutely converge provided by $|z - z_0| < |z_1 - z_0|$, and the function $\sum\limits_{k=1}^{\infty} kc_k(z - z_0)^{k-1}$ is the complex derivative of $\sum\limits_{k=0}^{\infty} c_k(z - z_0)^k$.*

Proof. Series $\sum\limits_{k=0}^{\infty} c_k(z - z_0)^k$ and its formal derivative are eventually majorized by $\sum\limits_{k=0}^{\infty} c_k(z_1 - z_0)^k$ if $|z - z_0| \le |z_1 - z_0|$ by the Lemma **5**. Hence they absolutely converge in the circle $|z - z_0| \le |z_1 - z_0|$. Consider

$$R(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k - \sum_{k=0}^{\infty} c_k(\zeta - z_0)^k - (z - \zeta)\sum_{k=1}^{\infty} kc_k(\zeta - z_0)^{k-1}. \tag{3.3.2}$$

To prove that the formal derivative is the derivative of $\sum\limits_{k=0}^{\infty} c_k(z - z_0)^k$ at ζ it is sufficient to prove that $R(z) = o(z)(z - \zeta)$, where $o(z)$ is infinitesimally small at ζ. One has $R(z) = \sum\limits_{k=1}^{\infty} c_k\left((z - z_0)^k - (\zeta - z_0)^k - k(\zeta - z_0)^{k-1}\right)$. By Lemma **8** one gets the following estimate: $|R(z)| \le \sum\limits_{k=1}^{\infty} 2|c_k|k(k-1)|z - \zeta|^2|z_2 - z_0|^{n-2}$, where $|z_2 - z_0| = \max\{|z - z_0|, |\zeta - z_0|\}$. Hence all we need now is to prove that $\sum\limits_{k=1}^{\infty} 2k(k-1)|c_k||z_2 - z_0|^{k-2}|z - \zeta|$ is infinitesimally small at ζ. And this in its turn follows from the convergence of $\sum\limits_{k=1}^{\infty} 2k(k-1)|c_k||z_2 - z_0|^{k-2}$. The later may be deduced from the Lemma **5**. Indeed, consider z_3, such that $|z_2 - z_0| < |z_3 - z_0| < |z_1 - z_0|$. The convergence of $\sum\limits_{k=1}^{\infty} k|c_k||z_3 - z_0|^{k-1}$ follows from the convergence of $\sum\limits_{k=0}^{\infty} |c_k||z_1 - z_0|^k$ by the Lemma **5**. And convergence of $\sum\limits_{k=2}^{\infty} k(k-1)|c_k||z_2 - z_0|^{k-2}$ follows from convergence of $\sum\limits_{k=1}^{\infty} k|c_k||z_3 - z_0|^{k-1}$ by the same Lemma □

Corrolary 10. *Let* $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ *absolutely converges for* $|z| < r$*, and* a, b *have absolute values less then* r*. Then* $\int\limits_a^b f(z)\,dz = \sum\limits_{k=0}^{\infty} \frac{c_k}{k+1}(b^{k+1} - a^{k+1})$

Proof. Consider $F(z) = \sum\limits_{k=0}^{\infty} \frac{c_k z^{k+1}}{k+1}$. This series is term-wise majorized by the series of $f(z)$, hence it converges absolutely for $|z| < r$. By the Theorem **9** $f(z)$ is its derivative for $|z| < r$. In our case $f(z)$ is differentiable and its derivative is bounded by $\sum\limits_{k=0}^{\infty} k|c_k|r_0^k$, where $r_0 = \max\{|a|, |b|\}$. Hence $f(z)$ is continuous and virtually monotone and our result now follows from the Theorem **3**. □

Exponenta in $\mathbb{C}$. The exponenta for any complex number z is defined as $\exp z = \sum\limits_{k=0}^{\infty} \frac{z^k}{k!}$. The definition works because the series $\sum\limits_{k=0}^{\infty} \frac{z^k}{k!}$ absolutely converges for any $z \in \mathbb{C}$.

Theorem 11. . *The exponenta is a differentiable function of complex variable with derivative* $\exp' z = \exp z$*, such that for all complex* z*,* ζ *holds the following* addition formula $\exp(z + \zeta) = \exp z \exp \zeta$.

Proof. The derivative of the exponenta can be evaluated termwise by the Theorem **9**. And this evaluation gives $\exp' z = \exp z$. To prove the addition formula consider the following function $r(z) = \frac{\exp(z+\zeta)}{\exp z}$. Differentiation of the equality $r(z)\exp z = \exp(z + \zeta)$ gives $r'(z)\exp z + r(z)\exp z = \exp(z + \zeta)$. Division by $\exp z$ gives $r'(z) + r(z) = r(z)$. Hence $r(z)$ is constant. This constant is determined by substitution $z = 0$ as $r(z) = \exp \zeta$. This proves the addition formula. □

Lemma 12. *Let* $p\colon [a, b] \to \mathbb{C}$ *be a smooth path contained in the complement of a neighborhood of* 0*. Then* $\exp \int\limits_p \frac{1}{\zeta}\, d\zeta = \frac{p(b)}{p(a)}$.

Proof. First consider the case when p is contained in a circle $|z - z_0| < |z_0|$ with center $z_0 \neq 0$. In this circle, $\frac{1}{z}$ expands in a power series:

$$\frac{1}{\zeta} = \frac{1}{z_0 - (z_0 - \zeta)} = \frac{1}{z_0} \frac{1}{1 - \frac{z_0 - \zeta}{z_0}} = \sum_{k=0}^{\infty} \frac{(z_0 - \zeta)^k}{z_0^{k+1}} \tag{3.3.3}$$

The integration of this series is possible to do termwise due to Corollary **10**. Hence the result of the integration does not depends on path. And the Theorem **9** provides differentiability of the termwise integral and possibility of its termwise differentiation. Such differentiation simply gives the original series, which represents $\frac{1}{z}$ in this circle.

Consider the function $l(z) = \int\limits_{z_0}^{z} \frac{1}{\zeta}\, d\zeta$. Then $l'(z) = \frac{1}{z}$. The derivative of the composition $\exp l(z)$ is $\frac{\exp l(z)}{z}$. Hence the composition satisfies the differential equation $y'z = y$. We search for a solution of this equation in the form $y = zw$. Then $y' = w + w'z$ and our equation turns into $wz + w'z^2 = wz$. Therefore $w' = 0$ and w is constant. To find this constant substitute $z = z_0$ and get $1 = \exp 0 = \exp l(z_0) = wz_0$. Hence $w = \frac{1}{z_0}$ and $\exp l(z) = \frac{z}{z_0}$.

To prove the general case consider a partition $\{x_k\}_{k=0}^{n}$ of $[a, b]$. Denote by p_k the restriction of p over $[x_k, x_{k+1}]$. Choose the partition so small that $|p(x) - p(x_k)| < |p(x_k)|$ for all $x \in [x_k, x_{k+1}]$. Then any p_k satisfies the requirement of the above considered case. Hence $\exp \int\limits_{p_k} \frac{1}{\zeta}\, d\zeta = \frac{p(x_{k+1})}{p(x_k)}$. Further $\exp \int\limits_{p} \frac{1}{\zeta}\, d\zeta = \exp \sum\limits_{k=0}^{n-1} \int\limits_{p_k} \frac{1}{\zeta}\, d\zeta = \prod_{k=0}^{n-1} \frac{p(x_{k+1})}{p(x_k)} = p(b)/p(a)$. ☐

Theorem 13 (Euler Formula). *For any real ϕ one has*

$$\boxed{\exp i\phi = \cos\phi + i \sin\phi} \tag{3.3.4}$$

Proof. In Lecture 2.5 we have evaluated $\int\limits_{p} \frac{1}{z}\, dz = i\phi$ for $p(t) = \cos t + i \sin t$, $t \in [0, \phi]$. Hence Lemma **12** applied to $p(t)$ immediately gives the Euler formula. ☐

Trigonometric functions in $\mathbb{C}$. The Euler formula gives power series expansions for $\sin x$ and $\cos x$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \tag{3.3.5}$$

These expansions are used to define the trigonometric functions for complex variable. On the other hand the Euler formula allows us to express trigonometric functions via the exponenta:

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i} \qquad \cos z = \frac{\exp(iz) + \exp(-iz)}{2} \tag{3.3.6}$$

The other trigonometric functions tan, cot, sec, cosec are defined for complex variable by usual formulas via sin and cos.

Problems.

1. Evaluate $\sum\limits_{k=1}^{\infty} \frac{\sin k}{k!}$

2. Prove the formula of Joh. Bernoulli $\int\limits_0^1 x^x\, dx = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k}$

3. Find $\ln(-1)$.

4. Solve the equation $\exp z = i$.
5. Evaluate i^i.
6. Prove $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$
7. Prove the identity $\sin^2 z + \cos^2 z = 1$
8. Solve the equation $\sin z = 5/3$
9. Evaluate $\sum\limits_{k=1}^{\infty} \frac{\sin n}{n!}$
10. Solve the equation $\cos z = 2$
11. Evaluate $\sum\limits_{k=0}^{\infty} \frac{\cos n}{n!}$
12. Evaluate $\oint\limits_{|z|=1} \dfrac{dz}{z^2}$
13. Evaluate $\sum\limits_{k=1}^{\infty} q^k \dfrac{\sin kx}{k}$
14. Expand into a power series $e^x \cos x$