3.6 Analytic Functions

On the contents of the lecture. This lecture introduces the reader into phantastically beautiful world of analytic functions. Integral Cauchy formula, Taylor series, Fundamental Theorem of Algebra. The reader will see all of these treasures in a single lecture.

Theorem 1 (Integral Cauchy Formula). *If function f is complex differentiable in the domain D, then for any interior point $z \in D$ one has*

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(\zeta)\,dz}{\zeta - z}} \tag{3.6.1}$$

Proof. The function $\frac{f(z)}{z-z_0}$ has its only singular point inside the circle. This is z_0, which is a simple pole. The residue of $\frac{f(z)}{z-z_0}$ by Lemma **7** is equal to $\lim\limits_{z\to z_0}(z-z_0)\frac{f(z)}{z-z_0} = \lim\limits_{z\to z_0} f(z) = f(z_0)$. And by the formula (3.5.8) the integral is equal to $2\pi i f(z_0)$. □

Lemma 2. *Let $\sum\limits_{k=1}^{\infty} f_k$ be a series of virtually monotone complex functions, which is termwise majorized by a convergent positive series $\sum\limits_{k=1}^{\infty} c_k$ on a monotone curve Γ (that is $|f_k(z)| \leq c_k$ for natural k and $z \in \Gamma$) and such that $F(z) = \sum\limits_{k=1}^{\infty} f_k(z)$ is virtually monotone. Then*

$$\sum_{k=1}^{\infty}\int_{\Gamma} f_k(z)\,dz = \int_{\Gamma}\sum_{k=1}^{\infty} f_k(z)\,dz \tag{3.6.2}$$

Proof. By the Lemma on Estimate one has the following inequalities:

$$\left|\int_{\Gamma} f_k(z)\,dz\right| \leq 4c_k \operatorname{diam}\Gamma \qquad \left|\int_{\Gamma}\sum_{k=n}^{\infty} f_k(z)\,dz\right| \leq 4\operatorname{diam}\Gamma\sum_{k=n}^{\infty} c_k \tag{3.6.3}$$

Set $F_n(z) = \sum\limits_{k=1}^{n-1} f_k(z)$. By the left inequality of (3.6.3), the module of difference between $\int_{\Gamma} F_n(z)\,dz = \sum\limits_{k=1}^{n-1}\int_{\Gamma} f_k(z)\,dz$ and the left-hand side of (3.6.2) does not exceed $4\operatorname{diam}\Gamma\sum\limits_{k=n}^{\infty} c_k$. Hence this module is infinitesimally small as n tends to infinity. On the other hand, by the right inequality of (3.6.3) one gets $\left|\int_{\Gamma} F_n(z)\,dz - \int_{\Gamma} F(z)\,dz\right| \leq 4\operatorname{diam}\Gamma\sum\limits_{k=n}^{\infty} c_k$. This implies that the difference between the left-hand and right-hand sides of (3.6.2) is infinitesimally small as n tends to infinity. But this difference does not depend on n. Hence it is zero. □

Lemma 3. *If a real function f defined over interval $[a, b]$ is locally bounded, then it is bounded.*

Proof. The proof is by contradiction. Suppose that f is unbounded. Divide the interval $[a, b]$ in half. Then the function has to be unbounded at least on one of the halves. Consider this half and divide it in half. Choose the half where the function is unbounded. So we construct a nested infinite sequence of intervals converging to a point, which coincides with the intersection of all the intervals. And f is obviously not locally bounded at this point. □

Corrolary 4. *A complex function $f(z)$ continuous on the boundary of a domain D is bounded on ∂D.*

Proof. Consider a path $p\colon [a, b] \to \partial D$. Then $|f(p(t))|$ is continuous on $[a, b]$, hence it is locally bounded, hence it is bounded. Since ∂D can be covered by images of finitely many pathes this implies boundedness of f over ∂D. □

Theorem 5. *If a function $f(z)$ is complex differentiable in the disk $|z - z_0| \leq R$, then for $|z - z_0| < R$*

$$f(z) = \sum_{k=0}^{\infty} (z - z_0)^k \oint_{z_0}^{R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta, \tag{3.6.4}$$

where the series on the right-hand side absolutely converges for $|z - z_0| < R$.

Proof. Fix a point z such that $|z - z_0| < R$ and consider ζ as a variable. For $|\zeta - z_0| > |z - z_0|$ one has

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} \tag{3.6.5}$$

On the circle $|\zeta - z_0| = R$ the series on the right-hand side is majorized by a convergent series $\sum_{k=0}^{\infty} \frac{|z - z_0|^k}{R^{k+1}}$ for $r > |z - z_0|$. The function $f(\zeta)$ is bounded on $|\zeta - z_0| = R$ by the Corollary **4**. Therefore after multiplication of (3.6.5) by $f(\zeta)$ all conditions of the Lemma **2** are satisfied. The termwise integration gives:

$$f(z) = \oint_{z_0}^{R} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{k=0}^{\infty} (z - z_0)^k \oint_{z_0}^{R} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{k+1}} \tag{3.6.6}$$

□

Analytic functions. A function $f(z)$ of complex variable is called an *analytic function* in a point z_0 if there is a positive ε such that $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for all z from a disk $|z - z_0| \leq \varepsilon$ and the series absolutely converges. Since one can differentiate power series termwise (Theorem **9**), any function which is analytic at z is also complex differentiable at z. The theorem **5** gives a converse. Thus, we get the following:

Corrolary 6. *A function $f(z)$ is analytic at z, iff it is complex differentiable in some neighborhood of z.*

Theorem 7. *If f is analytic at z then f' is analytic at z. If f and g are analytic at z then $f+g$, $f-g$, fg are analytic at z. If f is analytic at z and g is analytic at $f(z)$ then $g(f(z))$ is analytic at z.*

Proof. The termwise differentiation of a power series representing f in a neighborhood of z gives the power series for its derivative. Hence f' is analytic. The differentiability of $f \pm g$, fg and $g(f(z))$ follows from corresponding differentiation rules. $\square$

Lemma 8 (on isolated zeros). *If $f(z)$ is analytic and is not identically equal to 0 in some neighborhood of z_0, then $f(z) \neq 0$ for all $z \neq z_0$ sufficiently close to z_0.*

Proof. Let $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ in a neighborhood U of z_0. Let c_m be the first nonzero coefficient. Then $\sum_{k=m}^{\infty} c_k(z-z_0)^{k-m}$ converges in U to a differentiable function $g(z)$ by Theorem **9**. Since $g(z_0) = c_m \neq 0$ and $g(z)$ is continuous at z_0, the inequality $g(z) \neq 0$ holds for all z sufficiently close to z_0. As $f(z) = g(z)(z-z_0)^m$, the same is true for $f(z)$. $\square$

Theorem 9 (Uniqueness Theorem). *If two power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_0)^k$ converge in a neighborhood of z_0 and their sums coincide for some infinite sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \neq z_0$ for all k and $\lim_{k\to\infty} z_k = z_0$, then $a_k = b_k$ for all k.*

Proof. Set $c_k = a_k - b_k$. Then $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ has a non-isolated zero at z_0. Hence $f(z) = 0$ in a neighborhood of z_0. We get a contradiction by considering function $g(z) = \sum_{k=m}^{\infty} c_k(z-z_0)^{k-m}$, which is nonzero for all z sufficiently close to z_0 (cf. the proof of Lemma on isolated zero), and satisfies equation $f(z) = g(z)(z-z_0)^m$. $\square$

Taylor series. Set $f^{(0)} = f$ and by induction define the $(k+1)$-th derivative $f^{(k+1)}$ of f as the derivative of its k-th derivative $f^{(k)}$. For the first and the second derivatives one prefers notation f' and f''. For example, the k-th derivative of z^n is $n^{\underline{k}} z^{n-k}$. (Recall that $n^{\underline{k}} = n(n-1)\dots(n-k+1)$)

The following series is called the *Taylor series* of a function f at point z_0:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \tag{3.6.7}$$

The Taylor series is defined for any analytic function, because an analytic function has derivative of any order due to Theorem **7**.

Theorem 10. *If a function f is analytic in the disk $|z-z_0| < r$ then $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ for any z from the disk.*

Proof. By the Theorem **5**, $f(z)$ is presented in the disk by a convergent power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$. To prove our theorem we prove that

$$a_k = \oint_{z_0}^{R} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta = \frac{f^{(k)}(z_0)}{k!}. \tag{3.6.8}$$

Indeed, $a_0 = f(z_0)$ and termwise differentiating of $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ applied n times gives $f^{(n)}(z) = \sum_{k=n}^{\infty} k^{\underline{n}} a_k (z-z_0)^k$. Putting $z = z_0$, one gets $f^{(n)}(z_0) = n^{\underline{n}} a_n = a_n n!$. □

Theorem 11 (Liouville). *If a function f is analytic and bounded on the whole complex plane, then f is constant.*

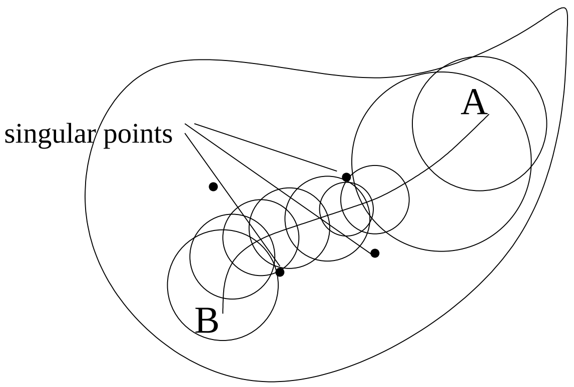
Proof. If f is analytic on the whole plane then $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where a_k is defined by (3.6.8). If $|f(z)| \leq B$ by Lemma on Estimate one gets

$$|a_k| = \left| \oint_{0}^{R} \frac{f(\zeta)}{z^{k+1}}\, d\zeta \right| \leq 4 \cdot 4 \frac{B}{R^{k+1}} \frac{R}{\sqrt{2}} = \frac{C}{R^k} \tag{3.6.9}$$

Consequently a_k for $k > 0$ is infinitesimally small as R tends to infinity. But a_k does not depend on R, hence it is 0. Therefore $f(z) = a_0$. □

Theorem 12 (Fundamental Theorem of Algebra). *Any nonconstant polynomial $P(z)$ has a complex root.*

Proof. If $P(z)$ has no roots the function $f(z) = \frac{1}{P(z)}$ is analytic on the whole plane. Since $\lim_{z\to\infty} f(z) = 0$ the inequality $|f(z)| < 1$ holds for $|z| = R$ if R is sufficiently large. Therefore the estimation (3.6.9) for k-th coefficient of f holds with $B = 1$ for sufficiently large R. Hence the same arguments as in proof of Liouville Theorem show that $f(z)$ is constant. This is a contradiction. □



Analytic continuation.

Lemma 13. *If an analytic function $f(z)$ has finitely many singular points in a domain D and a non isolated zero at a point $z_0 \in D$ then $f(z) = 0$ for all regular $z \in D$*

Proof. For any nonsingular point $z \in D$, we construct a sequence of sufficiently small disks $D_0, D_1, D_2, \dots D_n$ without singular points with the following properties: 1) $z_0 \in D_0 \subset U$, 2) $z \in D_n$, 3) z_k, the center of D_k, belongs to D_{k-1} for all $k > 0$. Then by induction we prove that $f(D_k) = 0$. First step: if z_0 is a non-isolated zero of f, then the Taylor series of f vanishes at z_0 by Uniqueness theorem. But this series represents $f(z)$ on D_0 due to Theorem **10**, since D_0 does not contain singular points. Hence, $f(D_0) = 0$. Suppose we have proved already that $f(D_k) = 0$. Then z_{k+1} is a non-isolated zero of f by the third property of the sequence $\{D_k\}_{k=0}^n$. Consequently, the same arguments as above for $k = 0$ prove that $f(D_{k+1}) = 0$. And finally we get $f(z) = 0$. $\square$

Consider any formula which you know from school about trigonometric functions. For example, $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$. The above Lemma implies that this formula remains true for complex x and y. Indeed, consider the function $T(x,y) = \tan(x+y) - \frac{\tan x + \tan y}{1 - \tan x \tan y}$. For a fixed x function $T(x,y)$ is analytic and has finitely many singular points in any disk. This function has non-isolated zeros in all real points, hence this function is zero in any disk intersecting the real line. This implies that $T(x,y)$ is zero for all y. The same arguments applied to $T(x,y)$ with fixed y and variable x prove that $T(x,y)$ is zero for all complex x, y.

The same arguments prove the following theorem

Theorem 14. *If some analytic relation between analytic functions holds on a curve Γ, it holds for any $z \in \mathbb{C}$, which can be connected with Γ by a paths avoiding singular points of the functions.*

Lemma 15. $\sin t \geq \frac{2t}{\pi}$ *for* $t \in [0, \pi/2]$

Proof. Let $f(t) = \sin t - \frac{2t}{\pi}$. Then $f'(x) = \cos t - \frac{2}{\pi}$. Set $y = \arccos \frac{2}{\pi}$. Then $f'(x) \geq 0$ for $x \in [0, y]$. Therefore f is nondecreasing on $[0, y]$, and nonnegative, because $f(0) = 0$. On the interval $[y, \pi/2]$ the derivative of f is negative. Hence $f(x)$ is non-increasing and nonnegative, because its value on the end of the interval is 0. $\square$

Lemma 16 (Jordan). *Let $f(z)$ be an analytic function in the upper half-plane such that $\lim\limits_{z\to\infty} f(z) = 0$. Denote by Γ_R the upper half of the circle $|z| = R$. Then for any natural m*

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z) \exp(miz)\, dz = 0 \tag{3.6.10}$$

Proof. Consider the parameterization $z(t) = R\cos t + Ri \sin t$, $t \in [0, \pi]$ of Γ_R. Then the integral (3.6.10) turns into

$$\begin{aligned} &\int_0^\pi f(z) \exp(iRm\cos t - Rm\sin t)\, d(R\cos t + Ri\sin t) = \\ &\quad = \int_0^\pi Rf(z) \exp(iRm\cos t) \exp(-Rm\sin t)(-\sin t + i\cos t)\, dt. \end{aligned} \tag{3.6.11}$$

If $|f(z)| \le B$ on Γ_R, then $|f(z)\exp(iRm\cos t)(-\sin t + i\cos t)| \le B$ on Γ_R. And the module of the integral (3.6.11) can be estimated from above by

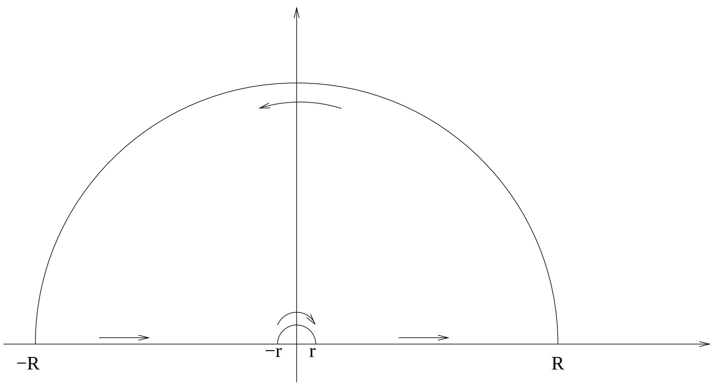
$$BR\int_0^\pi \exp(-Rm\sin t)\,dt.$$

Since $\sin(\pi - t) = \sin t$, the latter integral is equal to $2BR\int_0^{\pi/2}\exp(-Rm\sin t)\,dt$. Since $\sin t \ge \frac{2t}{\pi}$, the latter integral does not exceed

$$2BR\int_0^{\pi/2}\exp(-2Rmt/\pi)\,dt = 2BR\frac{1-\exp(-Rm)}{2Rm} \le \frac{B}{m} \tag{3.6.12}$$

Since B can be chosen arbitrarily small for sufficiently large R, this proves Lemma. □

Evaluation of $\int_{-\infty}^{+\infty}\frac{\sin x}{x}\,dx = \lim_{N\to\infty}\int_{-N}^{N}\frac{\sin x}{x}\,dx$. Since $\sin x = \operatorname{Im} e^{ix}$ our integral is equal to $\operatorname{Im}\int_{-\infty}^{+\infty}\frac{e^{iz}}{z}\,dz$. Set $\Gamma(r) = \{z \mid |z| = r, \operatorname{Im} z \ge 0\}$. This is a semicircle. Let us orient it counter-clockwise, so that its initial point is r.



Consider the domain $D(R)$ bounded by semicircles $-\Gamma(r)$, $\Gamma(R)$ and intervals $[-R,-r]$ and $[r,R]$, where $r = \frac{1}{R}$ and $R > 1$. The function $\frac{e^{iz}}{z}$ has not singular points inside $D(R)$. Hence $\oint_{\partial D(R)}\frac{e^{iz}}{z}dz = 0$. Hence for any R

$$\int_{-r}^{-R}\frac{e^{iz}}{z}\,dz + \int_{r}^{R}\frac{e^{iz}}{z}\,dz = \int_{\Gamma(r)}\frac{e^{iz}}{z}\,dz - \int_{\Gamma(R)}\frac{e^{iz}}{z}\,dz \tag{3.6.13}$$

The second integral on the right-hand side tends to 0 as R tends to infinity due to Jordan's Lemma. The function $\frac{e^{iz}}{z}$ has a simple pole at 0, hence the first integral on the right-hand side of (3.6.13) tends to $\pi i \operatorname{res}\frac{e^{iz}}{z} = \pi i$ due to Remark

1. As a result, the right-hand side of (3.6.13) tends to πi as R tends to infinity. Consequently the left-hand side of (3.6.13) also tends to πi as $R \to \infty$. The imaginary part of left-hand side of (3.6.13) is equal to $\int\limits_{-R}^{R} \frac{\sin x}{x}\,dx - \int\limits_{-r}^{r} \frac{\sin x}{x}\,dx$. The last integral tends to 0 as $r \to 0$, because $|\frac{\sin x}{x}| \leq 1$. Hence $\int\limits_{-R}^{R} \frac{\sin x}{x}\,dx$ tends to π as $R \to \infty$. Finally $\int\limits_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \pi$.

Problems.

1. Prove that an even analytic function f, i.eȧ function that $f(z) = f(-z)$ has Taylor series at 0 consisting only of even powers
2. Prove that analytic function which has Taylor series only with even powers is an even function.
3. If an analytic function $f(z)$ takes real values on $[0,1]$, then $f(x)$ is real for any real x.
4. $\int\limits_{-\infty}^{+\infty} \dfrac{1}{1+x^4}$
5. $\int\limits_{-\pi}^{+\pi} \dfrac{d\phi}{5+3\cos\phi}$
6. Evaluate $\int\limits_{0}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx \ (a > 0)$
7. $\int\limits_{-\infty}^{+\infty} \frac{x\sin x}{x^2+4x+20}\,dx$
8. $\int\limits_{0}^{\infty} \frac{\cos ax}{x^2+b^2}\,dx \ (a, b > 0)$