4.2 Bernoulli numbers

On the contents of the lecture. In this lecture we give explicit formulas for the telescoping of powers. These formulas involve a remarkable numbers, which were discovered by Jacob Bernoulli. They will appear in formulas for sums of series of reciprocal powers. In particular, we will see that $\frac{\pi^2}{6}$, the sum of Euler series, contains the second Bernoulli number $\frac{1}{6}$.

Summation Polynomials. Jacob Bernoulli found a general formula for the sum $\sum_{k=1}^{n} k^q$. To be precise he discovered that there is a sequence of numbers $B_0, B_1, B_2, \dots, B_n \dots$ such that

$$\sum_{k=1}^{n} k^q = \sum_{k=0}^{q+1} B_k \frac{q^{\underline{k-1}} n^{q+1-k}}{k!} \tag{4.2.1}$$

The first 11 of the *Bernoulli numbers* are $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}$. The right-hand side of (4.2.1) is a polynomial of degree $q+1$ in n. Let us denote this polynomial by $\psi_{q+1}(n)$. It has the following remarkable property: $\Delta\psi_{q+1}(x) = (1+x)^q$. Indeed the latter equality holds for any natural value n of the variable, hence it holds for all x, because two polynomials coinciding in infinitely many points coincide. Replacing in (4.2.1) $q+1$ by m, n by x and reversing the order of summation, one gets the following:

$$\psi_m(x) = \sum_{k=0}^{m} B_{m-k} \frac{(m-1)^{\underline{m-k-1}}}{(m-k)!} x^k = \sum_{k=0}^{m} B_{m-k} \frac{(m-1)!}{k!(m-k)!} x^k = \\ = \sum_{k=0}^{m} B_{m-k} \frac{(m-1)^{\underline{k-1}}}{k!} x^k \tag{4.2.2}$$

Today's lecture is devoted to the proof of this Bernoulli theorem.

Telescoping powers. Newton's Formula represents x^m as a factorial polynomial $\sum_{k=0}^{n} \frac{\Delta^k 0^m}{k!} x^{\underline{k}}$, where $\Delta^k 0^m$ denotes the value of $\Delta^k x^m$ at $x = 0$. Since $\Delta x^{\underline{k}} = k x^{\underline{k-1}}$, one immediately gets a formula for a polynomial $\phi_{m+1}(x)$ which telescopes x^m in the form

$$\phi_{m+1}(x) = \sum_{k=0}^{\infty} \frac{\Delta^k 0^m}{(k+1)!} x^{\underline{k+1}} \tag{4.2.3}$$

This polynomial has the property $\phi_{m+1}(n) = \sum_{k=0}^{n-1} k^m$ for all n.

Polynomials $\phi_m(x)$, as follows from Lemma **2**, are characterized by two conditions:

$$\Delta\phi_m(x) = x^{m-1}, \qquad \phi_m(1) = 0 \tag{4.2.4}$$

Lemma 1 (on differentiation). $\phi'_{m+1}(x) = \phi'_{m+1}(0) + m\phi_m(x)$

Proof. Differentiation of $\Delta\phi_{m+1}(x) = x^m$ gives $\Delta\phi'_{m+1}(x) = mx^{m-1}$. The polynomial $m\phi_m$ has the same differences, hence $\Delta(\phi'_{m+1}(x) - m\phi_m(x)) = 0$. By the Lemma **2** this implies that $\phi'_{m+1}(x) - m\phi_m(x)$ is constant. Therefore, $\phi'_{m+1}(x) - m\phi_m(x) = \phi'_{m+1}(0) - m\phi_m(0)$. But $\phi_m(1) = 0$ and $\phi_m(0) = \phi_m(1) - \Delta\phi_m(0) = 0 - 0^{m-1} = 0$. □

Bernoulli polynomials. Let us introduce the m-th *Bernoulli number* B_m as $\phi'_{m+1}(0)$, and define the *Bernoulli polynomial* of degree $m > 0$ as $B_m(x) = m\phi_m(x) + B_m$. Bernoulli polynomial $B_0(x)$ of degree 0 is defined as identically equal to 1. Consequently $B_m(0) = B_m$ and $B'_{m+1}(0) = (m+1)B_m$.

The Bernoulli polynomials satisfy the following condition:

$$\Delta B_m(x) = mx^{m-1} \quad (m > 0) \tag{4.2.5}$$

In particular, $\Delta B_m(0) = 0$ for $m > 1$, and therefore we get the following *boundary conditions* for Bernoulli polynomials:

$$B_m(0) = B_m(1) = B_m \quad \text{for } (m > 1) \text{ and } \quad B_1(0) = -B_1(1) = B_1 \tag{4.2.6}$$

The Bernoulli polynomials, in contrast to $\phi_m(x)$, are *normed*: its leading coefficient is equal to 1 and they have a simpler rule for differentiation:

$$B'_m(x) = mB_{m-1}(x) \tag{4.2.7}$$

Indeed, $B'_m(x) = m\phi'_m(x) = m((m-1)\phi_{m-1}(x) + \phi'_m(0)) = mB_{m-1}(x)$, by Lemma **1**.

Differentiating $B_m(x)$ at 0, k times, we get $B_m^{(k)}(0) = m^{\underline{k-1}}B'_{m-k+1}(0) = m^{\underline{k-1}}(m-k+1)B_{m-k} = m^{\underline{k}}B_{m-k}$. Hence Taylor formula gives the following representation of the Bernoulli polynomial:

$$B_m(x) = \sum_{k=0}^{m} \frac{m^{\underline{k}}B_{m-k}}{k!}x^k \tag{4.2.8}$$

Characterization theorem. The following important property of Bernoulli polynomials will be called the *Balance property*:

$$\int_0^1 B_m(x)\,dx = 0 \quad (m > 0) \tag{4.2.9}$$

Indeed, $\int_0^1 B_m(x)\,dx = \int_0^1 (m+1)B'_{m+1}(x)\,dx = \Delta B_{m+1}(0) = 0$

The Balance property and the Differentiation rule allows us to evaluate Bernoulli polynomials recursively. Thus, $B_1(x)$ has 1 as leading coefficient and zero integral on $[0,1]$, this allows us identify $B_1(x)$ with $x - 1/2$. Integration of $B_1(x)$ gives $B_2(x) = x^2 - x + C$, where C is defined by (4.2.9) as $-\int_0^1 x^{\underline{2}}\,dx = \frac{1}{6}$. Integrating $B_2(x)$ we get $B_3(x)$ modulo a constant which we find by (4.2.9) and so on. Thus we obtain the following theorem:

Theorem 2 (characterization). *If a sequence of polynomials $\{P_n(x)\}$ satisfies the following conditions:*

- $P_0(x) = 1$,
- $\int\limits_0^1 P_n(x)\,dx = 0$ *for* $n > 0$,
- $P_n'(x) = nP_{n-1}(x)$ *for* $n > 0$,

then $P_n(x) = B_n(x)$ *for all* n.

Analytic properties.

Lemma 3 (on reflection). $B_n(x) = (-1)^n B_n(1-x)$ *for any* n.

Proof. We prove that the sequence $T_n(x) = (-1)^n B_n(1-x)$ satisfies all conditions of the Theorem **2**. Indeed, $T_0 = B_0 = 1$, $\int\limits_0^1 T_n(x)\,dx = (-1)^n \int\limits_1^0 B_n(x)\,dx = 0$ and

$$T_n(x)' = (-1)^n B_n'(1-x) = (-1)^n nB_{n-1}(1-x)(1-x)' =$$
$$(-1)^{n+1} nB_{n-1}(x) = nT_{n-1}(x)$$

□

Lemma 4 (on roots). *For any odd* $n > 1$ *the polynomial* $B_n(x)$ *has on* $[0,1]$ *just three roots:* $0, \frac{1}{2}, 1$.

Proof. For odd n, the reflection Lemma **3** implies that $B_n(\frac{1}{2}) = -B_n(\frac{1}{2})$, that is $B_n(\frac{1}{2}) = 0$. Furthermore, for $n > 1$ one has $B_n(1) - B_n(0) = n0^{n-1} = 0$. Hence $B_n(1) = B_n(0)$ for any Bernoulli polynomial of degree $n > 1$. By reflection formula for an odd n one obtains $B_n(0) = -B_n(1)$. Thus any Bernoulli polynomial of odd degree greater than 1 has roots $0, \frac{1}{2}, 1$.

The proof that there are no more roots is by contradiction. In the opposite case consider $B_n(x)$, of the least odd degree > 1 which has a root α different from the above mentioned numbers. Say $\alpha < \frac{1}{2}$. By Rolle's Theorem $B_n'(x)$ has at least three roots $\beta_1 < \beta_2 < \beta_3$ in $(0,1)$. To be precise, $\beta_1 \in (0,\alpha)$, $\beta_2 \in (\alpha, \frac{1}{2})$, $\beta_3 \in (\frac{1}{2}, 1)$. Then $B_{n-1}(x)$ has the same roots. By Rolle's Theorem $B_{n-1}'(x)$ has at least two roots in $(0,1)$. Then at least one of them differs from $\frac{1}{2}$ and is a root of $B_{n-2}(x)$. By the minimality of n one concludes $n-2 = 1$. However, $B_1(x)$ has the only root $\frac{1}{2}$. This is a contradiction. □

Theorem 5. $B_n = 0$ *for any odd* $n > 1$. *For* $n = 2k$, *the sign of* B_n *is* $(-1)^{k+1}$. *For any even n one has either* $B_n = \max_{x\in[0,1]} B_n(x)$ *or* $B_n = \min_{x\in[0,1]} B_n(x)$. *The first occurs for positive* B_n, *the second for negative.*

Proof. $B_{2k+1} = B_{2k+1}(0) = 0$ for $k > 0$ by the Lemma **4**. Above we have found that $B_2 = \frac{1}{6}$. Suppose we have established that $B_{2k} > 0$ and that this is the maximal value for $B_{2k}(x)$ on $[0,1]$. Let us prove that $B_{2k+2} < 0$ and is the minimal value for $B_{2k+2}(x)$ on $[0,1]$. The derivative of B_{2k+1} in this case is positive at the ends of $[0,1]$, hence $B_{2k+1}(x)$ is positive for $0 < x < \frac{1}{2}$ and negative for $\frac{1}{2} < x < 1$, by Lemma **4** on roots and Theorem on Intermediate values. Hence, $B_{2k+2}'(x) > 0$ for $x < \frac{1}{2}$ and $B_{2k+2}'(x) < 0$ for $x > \frac{1}{2}$. Therefore, $B_{2k+2}(x)$ takes the maximal value in the middle of $[0,1]$ and takes the minimal values at the ends of $[0, 1]$. Since the integral of the polynomial along $[0, 1]$ is zero 0 and the polynomial is not constant, its minimal value has to be negative. The same arguments prove that if B_{2k} is negative and minimal, then B_{2k+2} is positive and maximal. □

Lemma 6 (Lagrange Formula). *If f is a differentiable function on $[a, b]$, then there is a $\xi \in (a, b)$, such that*

$$f(b) = f(a) + f'(\xi)\frac{f(b) - f(a)}{b - a} \tag{4.2.10}$$

Proof. The function $g(x) = f(x) - (x - a)\frac{f(b)-f(a)}{b-a}$ is differentiable on $[a, b]$ and $g(b) = g(a) = 0$. By Rolle's Theorem $g'(\xi) = 0$ for some $\xi \in [a, b]$. Hence $f'(\xi) = \frac{f(b)-f(a)}{b-a}$. Substitution of this value of $f'(\xi)$ in (4.2.10) gives the equality. □

Generating function. The following function of two variables is called the *generating function of Bernoulli polynomials*

$$B(x, t) = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} \tag{4.2.11}$$

Since $B_k \leq \frac{k!}{2^k}$, the series on the right-hand side converges for $t < 2$ for any x. Let us differentiate it termwise as a function of x, for a fixed t. We get $\sum\limits_{k=0}^{\infty} kB_{k-1}(x)\frac{t^k}{k!} = tB(x, t)$. Consequently $(\log B(x, t))'_x = \frac{B'_x(x,t)}{B(x,t)} = t$ and $\log B(x, t) = xt + c(t)$, where the constant $c(t)$ depends on t. It follows that $B(x, t) = \exp(xt)k(t)$, where $k(t) = \exp(c(t))$. For $x = 0$ we get $B(0, t) = k(t) = \sum\limits_{k=0}^{\infty} B_k\frac{t^k}{k!}$. To find $k(t)$ consider the difference $B(x + 1, t) - B(x, t)$. It is equal to $\exp(xt + t)k(t) - \exp(xt)$. On the other hand the difference is $\sum\limits_{k=0}^{\infty} \Delta B_k(x)\frac{t^k}{k!} = \sum\limits_{k=0}^{\infty} kB_{k-1}(x)\frac{t^k}{k!} = tB(x, t)$. Comparing these expressions we get explicit formulas for the generating functions of Bernoulli numbers:

$$k(t) = \frac{t}{\exp t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}t^k\ , \tag{4.2.12}$$

and Bernoulli polynomials:

$$B(x, t) = \sum_{k=+}^{0-1} B_k(x)\frac{t^k}{k!} = \frac{t\exp(tx)}{\exp t - 1} \tag{4.2.13}$$

From (4.2.11) one gets $t = (\exp t - 1)\sum\limits_{k=0}^{\infty} B_k\frac{t^k}{k!}$. Substituting $\exp t - 1 = \sum\limits_{k=1}^{\infty} \frac{t^k}{k!}$ in this equality, by the Uniqueness Theorem, one gets the equalities for the coefficients of the power series

$$\sum_{k=1}^{n} \frac{B_{n-k}}{(n-k)!k!} = 0 \quad \text{for } n > 1 \tag{4.2.14}$$

Add $\frac{B_n}{n!}$ to both sides of this equality and multiply both sides by $n!$ to get

$$B_n = \sum_{k=0}^{n} \frac{B_k n^{\underline{k}}}{k!} \quad \text{for } n > 1 \tag{4.2.15}$$

The latter equality one memorizes via the formula $B^n = (B+1)^n$, where after expansion of the right hand side, one should move down all the exponents at B turning the powers of B into Bernoulli numbers.

Now we are ready to prove that

$$\phi_m(1+x) = \frac{B_m(x+1)}{m} - \frac{B_m}{m} = \sum_{k=0}^{m} B_{m-k} \frac{(m-1)^{\underline{k-1}}}{k!} x^k = \psi_m(x) \tag{4.2.16}$$

Putting $x = 0$ in the right hand side one gets $\psi_m(0) = B_m(m-1)^{\underline{-1}} = \frac{B_m}{m}$. The left-hand side takes the same value at $x = 0$, because $B_m(1) = B_m(0) = B_m$. It remains to prove the equality of the coefficients in (4.2.16) for positive degrees.

$$\frac{B_m(x+1)}{m} = \frac{1}{m} \sum_{k=0}^{m} \frac{m^{\underline{k}} B_{m-k}}{k!} (1+x)^k = \frac{1}{m} \sum_{k=0}^{m} \frac{m^{\underline{k}} B_{m-k}}{k!} \sum_{j=0}^{k} \frac{k^{\underline{j}} x^j}{j!} =$$

Now let us change the summation order and change the summation index of the interior sum by $i = m - k$:

$$= \frac{1}{m} \sum_{j=0}^{m} \frac{x^j}{j!} \sum_{k=j}^{m} \frac{m^{\underline{k}} B_{m-k}}{k!} k^{\underline{j}} = \frac{1}{m} \sum_{j=0}^{m} \frac{x^j}{j!} \sum_{i=0}^{m-j} \frac{m^{\underline{m-i}} B_i}{(m-i)!} (m-i)^{\underline{j}} =$$

Now we change $\frac{m^{\underline{m-i}}(m-i)^{\underline{j}}}{(m-i)!}$ by $\frac{(m-j)^{\underline{i}} m^{\underline{j}}}{i!}$ and apply the identity (4.2.15)

$$= \sum_{j=0}^{m} \frac{x^j m^{\underline{j}}}{mj!} \sum_{i=0}^{m-j} \frac{B_i (m-j)^{\underline{i}}}{i!} = \sum_{j=0}^{m} \frac{(m-1)^{\underline{j-1}} x^j}{j!} B_{m-j}$$

Problems.

1. Evaluate $\int_0^1 B_n(x) \sin 2\pi x$.
2. Expand $x^4 - 3x^2 + 2x - 1$ as a polynomial in $(x-2)$.
3. Calculate the first 20 Bernoulli numbers.
4. Prove the inequality $|B_n(x)| \le |B_n|$ for even n.
5. Prove the inequality $|B_n(x)| \le \frac{n}{4}|B_{n-1}|$ for odd n.
6. Prove that $\frac{f(0)+f(1)}{2} = \int_0^1 f(x)\,dx + \int_0^1 f'(x) B_1(x)\,dx$
7. Prove that $\frac{f(0)+f(1)}{2} = \int_0^1 f(x)\,dx + \frac{\Delta f'(0)}{2} - \int_0^1 f''(x) B_2(x)\,dx$

8. Deduce $\Delta B_n(x) = nx^{n-1}$ from the balance property and the differentiation rule.

9. Prove that $B_n(x) = B_n(1-x)$, using the generating function.

10. Prove that $B_{2n+1} = 0$, using the generating function.

11. Prove that $B_m(nx) = n^{m-1} \sum_{k=0}^{n-1} B_m\left(x + \frac{k}{n}\right)$

12. Evaluate $B_n(\frac{1}{2})$.

13. Prove that $B_{2k}(x) = P(B_2(x))$, where $P(x)$ is a polynomial with positive coefficient (Jacobi Theorem).

14. Prove that $B_n = \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k 0^n}{k+1}$

15. * Prove that $B_m + \sum \frac{1}{k+1}[k+1 \text{ is prime and } k \text{ is divisor of } m]$ is an integer. (Staudt theorem).