## 4.3 Euler-Maclaurin Formula

**On the contents of the lecture.** From this lecture we will learn how Euler managed to calculate 18 digit places of the sum $\sum_{k=0}^{\infty} \frac{1}{k^2}$.

**Symbolic derivation.** Taylor expansion of a function $f$ at point $x$ gives

$$f(x+1) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}. \tag{4.3.1}$$

Hence $\Delta f(x) = \sum_{k=1}^{\infty} \frac{\mathbf{D}^k f(x)}{k!}$, where $\mathbf{D}$ is the operation of differentiation. One expresses this equality symbolically as

$$\Delta = \exp \mathbf{D} - \mathbf{1}. \tag{4.3.2}$$

We are searching for $F$ such that $F(n) = \sum_{k=1}^{n-1} f(k)$ for all $n$. Then $\Delta F(x) = f(x)$, or symbolically $F = \Delta^{-1} f$. So we have to invert the operation of the difference. From (4.3.2), the inversion is given formally by the formula $(\exp \mathbf{D} - \mathbf{1})^{-1}$. This function has a singularity at 0 and cannot be expanded into a power series in $\mathbf{D}$. However we know the expansion $\frac{t}{\exp t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$. This allows us to give a symbolic solution of our problem in the form

$$\Delta^{-1} = \mathbf{D}^{-1} \frac{\mathbf{D}}{\exp \mathbf{D} - \mathbf{1}} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \mathbf{D}^{k-1} = \mathbf{D}^{-1} - \frac{1}{2}\mathbf{1} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k!} \mathbf{D}^{2k-1} \tag{4.3.3}$$

Here we take into account that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$ for $k > 0$. Since $\sum_{k=1}^{n-1} f(k) = F(n) - F(1)$, the latter symbolic formula gives the following summation formula:

$$\sum_{k=1}^{n-1} f(k) = \int_1^n f(x)\,dx - \frac{f(n) - f(1)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(1)) \tag{4.3.4}$$

For $f(x) = x^m$ this formula gives the Bernoulli polynomial $\phi_{m+1}$.

**Euler's estimate.** Euler applied this formula to $f(x) = \frac{1}{(x+9)^2}$ and estimated the sum $\sum_{k=10}^{\infty} \frac{1}{k^2}$. In this case the $k$-th derivative of $\frac{1}{(x+9)^2}$ at 1 has the absolute value $\frac{k!}{10^{k+2}}$. Hence the module of $k$-th term of the summation formula does not exceed $\frac{B_k}{k10^{k+2}}$. For an accuracy of 18 digit places it is sufficient to sum up the first 14 terms of the series, only 8 of them do not vanish. Euler conjectured, and we will prove, that the value of error does not exceed of the value of the first rejected term, which is $\frac{B_{16}}{16 \cdot 10^{16}}$. Since $B_{16} = -\frac{3617}{510}$ this gives the promised accuracy.

| $B_1$ | $B_2$ | $B_4$ | $B_6$ | $B_8$ | $B_{10}$ | $B_{12}$ | $B_{14}$ | $B_{16}$ | $B_{18}$ | $B_{20}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $-\frac{1}{2}$ | $\frac{1}{6}$ | $-\frac{1}{30}$ | $\frac{1}{42}$ | $-\frac{1}{30}$ | $\frac{5}{66}$ | $-\frac{691}{2730}$ | $\frac{7}{6}$ | $-\frac{3617}{510}$ | $\frac{43867}{798}$ | $-\frac{174611}{330}$ |

We see from the table that increasing of the number of considered terms does not improve accuracy noticeably.

**Summation formula with remainder.** In this lecture we assume that all functions under consideration are differentiable as many times as needed.

**Lemma 1.** *For any function $f(x)$ on $[0,1]$ one has*

$$\frac{1}{2}(f(1)+f(0)) = \int_0^1 f(x)\,dx - \int_0^1 f'(x)B_1(x)\,dx$$

*Proof.* Recall that $B_1(x) = x - \frac{1}{2}$, hence $\int_0^1 f'(x)B_1(x)\,dx = \int_0^1 (x-\frac{1}{2})\,df(x)$. Now, integration by parts gives $\int_0^1 (x-\frac{1}{2})\,df(x) = \frac{1}{2}(f(1)+f(0)) - \int_0^1 f(x)\,dx$. □

Consider the *periodic Bernoulli polynomials* $B_m\{x\} = B_m(x-[x])$. Then $B'_m\{x\} = mB_{m-1}\{x\}$ for non integer $x$.

Let us denote by $\Sigma_m^n a_k$ the sum $\frac{1}{2}a_m + \sum_{k=m-1}^{n-1} a_k + \frac{1}{2}a_n$.

**Lemma 2.** *For any natural $p$, $q$ and a function $f(x)$ one has*

$$\Sigma_p^q f(k) = \int_p^q f(x)\,dx - \int_p^q f'(x)B_1\{x\}\,dx$$

*Proof.* Applying Lemma **1** to $f(x+k)$ one gets $\frac{1}{2}(f(k+1)+f(k)) = \int_0^1 f(x+k)\,dx + \int_0^1 f'(x+k)B_1(x)\,dx = \int_k^{k+1} f(x)\,dx + \int_k^{k+1} f'(x)B_1\{x\}\,dx$. Summing up these equalities for $k$ from $p$ to $q$, one proves the Lemma. □

**Lemma 3.** *For $m > 0$ and a function $f$ one has*

$$\int_p^q f(x)B_m\{x\}\,dx = \frac{B_{m+1}}{m+1}(f(q)-f(p)) - \int_p^q f'(x)B_{m+1}\{x\}\,dx \tag{4.3.5}$$

*Proof.* Since $B_m\{x\}dx = d\frac{B_{m+1}\{x\}}{m+1}$ and $B_{m+1}\{k\} = B_{m+1}$ for any natural $k$, the formula (**3**) is obtained by a simple integration by parts. □

**Theorem 4.** *For any function $f$ and natural numbers $n$ and $m$ one has:*

$$\Sigma_1^n f(k) = \int_1^n f(x)\,dx + \sum_{k=1}^{m-1}\frac{B_{k+1}}{(k+1)!}\left(f^{(k)}(n) - f^{(k)}(1)\right) + \frac{(-1)^{m+1}}{m!}\int_1^n f^{(m)}(x)B_m\{x\}\,dx \tag{4.3.6}$$

*Proof.* The proof is by induction on $m$. For $m = 1$, formula (4.3.6) is just given by Lemma **2**. Suppose (4.3.6) is proved for $m$. The *remainder*

$$\frac{(-1)^{m+1}}{m!} \int_1^n f^{(m)}(x) B_m\{x\}\, dx$$

can be transformed by virtue of Lemma **3** into

$$\frac{(-1)^{m+1} B_{m+1}}{(m+1)!}(f^{(m)}(n) - f^{(m)}(1)) + \frac{(-1)^{m+2}}{(m+1)!} \int_1^n B_{m+1}\{x\} f^{(m+1)}(x)\, dx.$$

Since odd Bernoulli numbers vanish, $(-1)^{m+1} B_{m+1} = B_{m+1}$ for $m > 0$. $\square$

**Estimation of remainder.** For $m = \infty$, (4.3.6) turns into (4.3.4). Denote

$$R_m = \frac{(-1)^{m+1}}{m!} \int_1^n f^{(m)}(x) B_m\{x\}\, dx \tag{4.3.7}$$

This is the so-called *remainder* of Euler-Maclaurin formula.

**Lemma 5.** *$R_{2m} = R_{2m+1}$ for any $m > 1$*

*Proof.* Because $B_{2m+1} = 0$, the only thing which changes in (4.3.6) when one passes from $2m$ to $2m+1$ is the remainder. Hence its value does not change either. $\square$

**Lemma 6.** *If $f(x)$ is monotone on $[0, 1]$ then*

$$\operatorname{sgn} \int_0^1 f(x) B_{2m+1}(x)\, dx = \operatorname{sgn}(f(1) - f(0)) \operatorname{sgn} B_{2m}$$

*Proof.* Since $B_{2m+1}(x) = -B_{2m+1}(1-x)$, the change $x \to 1 - x$ transforms the integral $\int_{0.5}^{1} f(x) B_{2m+1}(x)\, dx$ to $-\int_0^{0.5} f(1-x) B_{2m+1}(x)\, dx$:

$$\int_0^1 f(x) B_{2m+1}(x)\, dx = \int_0^{0.5} f(x) B_{2m+1}(x)\, dx + \int_{0.5}^1 f(x) B_{2m+1}(x)\, dx = \int_0^{0.5} (f(x) - f(1-x)) B_{2m+1}(x)\, dx \tag{4.3.8}$$

$B_{2m+1}(x)$ is equal to 0 at the end-points of $[0, 0.5]$ and has constant sign on $(0, 0.5)$, hence its sign on the interval coincides with the sign of its derivative at 0, that is, it is equal to $\operatorname{sgn} B_{2m}$. The difference $f(x) - f(1-x)$ also has constant sign as $x < 1 - x$ on $(0, 0.5)$ and its sign is $\operatorname{sgn}(f(1) - f(0))$. Hence the integrand in (4.3.8) has constant sign. Consequently the integral itself has the same sign as the integrand has. $\square$

**Lemma 7.** *If $f^{(2m+1)}(x)$ and $f^{(2m+3)}(x)$ are comonotone for $x \geq 1$ then*

$$R_{2m} = \theta_m \frac{B_{2m+2}}{(2m+2)!}(f^{(2m+1)}(n) - f^{(2m+1)}(1)) \quad 0 \leq \theta_m \leq 1 \tag{4.3.9}$$

*Proof.* The signs of $R_{2m+1}$ and $R_{2m+3}$ are opposite. Indeed, by Lemma **5** $\operatorname{sgn} B_{2m} = -\operatorname{sgn} B_{2m+2}$ and $\operatorname{sgn}(f^{(2m+1)}(n) - f^{(2m+1)}(1)) = \operatorname{sgn}(f^{(2m+3)}(n) - f^{(2m+3)}(1)$, owing to comonotonity condition. Hence $\operatorname{sgn} R_{2m+1} = -\operatorname{sgn} R_{2m+3}$ by Lemma **6**.

Set $T_{2m+2} = \frac{B_{2m+2}}{(2m+2)!}(f^{(2m+1)}(n) - f^{(2m+1)}(1))$. Then $T_{2m+2} = R_{2m+1} - R_{2m+2}$. By Lemma **5**, $T_{2m+2} = R_{2m+1} - R_{2m+3}$. Since $R_{2m+3}$ and $R_{2m+1}$ have opposite signs, it follows that $\operatorname{sgn} T_{2m+2} = \operatorname{sgn} R_{2m+1}$ and $|T_{2m+2}| \geq |R_{2m+1}|$. Hence $\theta_m = \frac{R_{2m+1}}{T_{2m+2}} = \frac{R_{2m}}{T_{2m+2}}$ belongs to $[0, 1]$. □

**Theorem 8.** *If $f^{(k)}$ and $f^{(k+2)}$ are comonotone for any $k > 1$, then*

$$\left| \Sigma_1^n f(k) - \int_1^n f(x)\,dx - \sum_{k=1}^m \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(1)\right) \right| \leq \left| \frac{B_{2m+2}}{(2m+2)!}\left(f^{(2m+1)}(n) - f^{(2m+1)}\right) \right| \tag{4.3.10}$$

Hence the value of the error which gives the summation formula (4.3.4) with $m$ terms has the same sign as the first rejected term, and its absolute value does not exceed the absolute value of the term.

**Theorem 9.** *Suppose that $\int_1^\infty |f^{(k)}(x)|\,dx < \infty$, $\lim_{x\to\infty} f^{(k)}(x) = 0$ and $f^{(k)}$ is comonotone with $f^{(k+2)}$ for all $k \geq K$ for some $K$. Then there is a constant $C$ such that for any $m > K$ for some $\theta_m \in [0, 1]$ holds*

$$\sum_{k=1}^n f(k) = C + \frac{f(n)}{2} + \int_1^n f(x)\,dx + \sum_{k=1}^m \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n) + \theta_m \frac{B_{2m+2}}{(2m+2)!} f^{(2m+1)}(n) \tag{4.3.11}$$

**Lemma 10.** *Under the condition of the theorem, for any $m \geq K$,*

$$\frac{(-1)^m}{m!} \int_p^\infty f^{(m)}(x) B_m\{x\}\,dx = -\theta_m \frac{B_{2m+2}}{(2m+2)!} f^{(2m+1)}(p) \tag{4.3.12}$$

*Proof.* By Lemma **7**,

$$\frac{(-1)^{m+1}}{m!} \int_p^q f^{(m)}(x) B_m\{x\}\,dx = \theta_m \frac{B_{2m+2}}{(2m+2)!}(f^{(2m+1)}(q) - f^{(2m+1)}(p)).$$

To get (4.3.12), pass to the limit as $q$ tends to infinity. □

*Proof.* To get (4.3.11) we change the form of the remainder $R_K$ for (4.3.6). Since $\int_1^n B_K\{x\} f^{(K)}\,dx = \int_1^\infty B_K\{x\} f^{(K)}(x)\,dx - \int_n^\infty B_K\{x\} f^{(K)}(x)\,dx$, applying equality 4.3.5 to the interval $[n, \infty)$, one gets

$$-\frac{(-1)^{k+1}B_k}{k!}\int_n^\infty B_k\{x\}f^{(k)}(x)\,dx = \frac{(-1)^{k+1}B_{k+1}}{(k+1)!}f^{(k)}(n) - \frac{(-1)^{k+2}B_{k+1}}{(k+1)!}\int_n^\infty B_{k+1}\{x\}f^{(k+1)}(x)\,dx \tag{4.3.13}$$

Iterating this formula one gets

(4.3.14)

$$R_K = \int_1^\infty B_K\{x\}f^{(K)}\,dx + \sum_{k=K}^{m}\frac{B_{k+1}}{(k+1)!}f^{(k)}(n) + \frac{(-1)^m}{m!}\int_n^\infty B_m\{x\}f^{(m)}(x)\,dx$$

Here we take into account the equalities $(-1)^k B_k = B_k$ and $(-1)^{m+2} = (-1)^m$. Now we substitute this expression of $R_K$ into (4.3.6) and set

$$C = (-1)^{K+1}\int_1^\infty B_K\{x\}f^{(K)}(x)\,dx - \frac{f(1)}{2} - \sum_{k=1}^{K-1}\frac{B_{k+1}}{(k+1)!}f^{(k)}(1). \tag{4.3.15}$$

□

**Stirling formula.** The logarithm satisfies all condition of Theorem **9** with $K = 2$. Its $k$-th derivative at $n$ is equal to $\frac{(-1)^{k+1}(k-1)!}{n^k}$. Thus (4.3.11) for the logarithm turns into

(4.3.16)

$$\sum_{k=1}^{n}\ln k = n\ln n - n + \sigma - \frac{\ln n}{2} + \sum_{k=1}^{m}\frac{B_{2k}}{2k(2k-1)n^{2k-}} + \frac{\theta_m B_{2m+2}}{(2m+2)(2m+1)n^{2k-1}}$$

By (4.3.15), the constant is $\sigma = \int_1^\infty \frac{B_2\{x\}}{x^2}\,dx - \frac{B_2}{2}$. But we already know this constant as $\sigma = \frac{1}{2}\ln 2\pi$. For $m = 0$, the above formula gives the most common form of Stirling formula:

$$n! = \sqrt{2\pi n}\; n^n e^{-n+\frac{\Theta}{12n}} \tag{4.3.17}$$

**Problems.**

1. Write the Euler-Maclaurin series telescoping $\frac{1}{x}$
2. Prove the uniqueness of the constant in Euler-Maclaurin formula.
3. Calculate 10 digit places of $\sum_{k=1}^{\infty}\frac{1}{n^3}$
4. Calculate 8 digit places of $\sum_{k=1}^{1000000}\frac{1}{k}$
5. Evaluate $\ln 1000!$ with accuracy $10^{-4}$.