Chapter XI

Classifications in Low Dimensions

In different geometric subjects there are different ideas which dimensions are low and which high. In topology of manifolds low dimension means at most 4. However, in this chapter only dimensions up to 2 will be considered, and even most of two-dimensional topology will not be touched. Manifolds of dimension 4 are the most mysterious objects of the field. Dimensions higher than 4 are easier: there is enough room for most of the constructions that topology needs.

47. One-Dimensional Manifolds

47°1. Zero-Dimensional Manifolds

This section is devoted to topological classification of manifolds of dimension one. We could skip the case of 0-dimensional manifolds due to triviality of the problem.

47.A. *Two* 0*-dimensional manifolds are homeomorphic iff they have the same number of points.*

The case of 1-dimensional manifolds is also simple, but requires more detailed considerations. Surprisingly, many textbooks manage to ignore 1-dimensional manifolds absolutely.

47°2. Reduction to Connected Manifolds

47.B. *Two manifolds are homeomorphic iff there exists a one-to-one correspondence between their components such that the corresponding components are homeomorphic.*

Thus, for topological classification of n-dimensional manifolds it suffices to classify only *connected* n-dimensional manifolds.

47°3. Examples

47.C. What connected 1-manifolds do you know?

(1) Do you know any *closed* connected 1-manifold?
(2) Do you know a connected *compact* 1-manifold, which is not closed?
(3) What *non-compact* connected 1-manifolds do you know?
(4) Is there a *non-compact* connected 1-manifolds with boundary?

47°4. How to Distinguish Them From Each Other?

47.D. Fill the following table with pluses and minuses.

Manifold X	Is X compact?	Is ∂X empty?
S^1		
$\mathbb{R}^1$		
I		
$\mathbb{R}^1_+$		

47°5. Statements of Main Theorems

47.E. *Any connected manifold of dimension 1 is homeomorphic to one of the following for manifolds:*

- *circle* S^1,
- *line* $\mathbb{R}^1$,
- *interval* I,
- *half-line* $\mathbb{R}^1_+$.

This theorem may be splitted into the following four theorems:

47.F. *Any closed connected manifold of dimension 1 is homeomorphic to circle* S^1.

47.G. *Any non-compact connected manifold of dimension 1 without boundary is homeomorphic to line* $\mathbb{R}^1$.

47.H. *Any compact connected manifold of dimension 1 with nonempty boundary is homeomorphic to interval I.*

47.I. *Any non-compact connected manifold of dimension one with nonempty boundary is homeomorphic to half-line $\mathbb{R}^1_+$.*

47°6. Lemma on 1-Manifold Covered with Two Lines

47.J Lemma. *Any connected manifold of dimension 1 covered with two open sets homeomorphic to $\mathbb{R}^1$ is homeomorphic either to $\mathbb{R}^1$, or S^1.*

Let X be a connected manifold of dimension 1 and $U, V \subset X$ be its open subsets homeomorphic to $\mathbb{R}$. Denote by W the intersection $U \cap V$. Let $\varphi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$ be homeomorphisms.

47.J.1. Prove that each connected component of $\varphi(W)$ is either an open interval, or an open ray, or the whole $\mathbb{R}$.

47.J.2. Prove that a homeomorphism between two open connected subsets of $\mathbb{R}$ is a (strictly) monotone continuous function.

47.J.3. Prove that if a sequence x_n of points of W converges to a point $a \in U \smallsetminus W$ then it does not converge in V.

47.J.4. Prove that if there exists a bounded connected component C of $\varphi(W)$ then $C = \varphi(W)$, $V = W$, $X = U$ and hence X is homeomorphic to $\mathbb{R}$.

47.J.5. In the case of connected W and $U \neq V$, construct a homeomorphism $X \to \mathbb{R}$ which takes:

- W to $(0, 1)$,
- U to $(0, +\infty)$, and
- V to $(-\infty, 1)$.

47.J.6. In the case of W consisting of two connected components, construct a homeomorphism $X \to S^1$, which takes:

- W to $\{z \in S^1 : -1/\sqrt{2} < \mathrm{Im}(z) < 1/\sqrt{2}\}$,
- U to $\{z \in S^1 : -1/\sqrt{2} < \mathrm{Im}(z)\}$, and
- V to $\{z \in S^1 : \mathrm{Im}(z) < 1/\sqrt{2}\}$.

47°7. Without Boundary

47.F.1. Deduce Theorem *47.F* from Lemma *47.I*.

47.G.1. Deduce from Lemma *47.I* that for any connected non-compact one-dimensional manifold X without a boundary there exists an embedding $X \to \mathbb{R}$ with open image.

47.G.2. Deduce Theorem *47.G* from *47.G.1*.

47°8. With Boundary

47.H.1. Prove that any compact connected manifold of dimension 1 can be embedded into S^1.

47.H.2. List all connected subsets of S^1.

47.H.3. Deduce Theorem *47.H* from *47.H.2*, and *47.H.1*.

47.I.1. Prove that any non-compact connected manifold of dimension 1 can be embedded into $\mathbb{R}^1$.

47.I.2. Deduce Theorem *47.I* from *47.I.1*.

47°9. Corollaries of Classification

47.K. Prove that connected sum of closed 1-manifolds is defined up homeomorphism by topological types of summands.

47.L. Which 0-manifolds bound a compact 1-manifold?

47°10. Orientations of 1-manifolds

Orientation of a *connected non-closed* 1-manifold is a linear order on the set of its points such that the corresponding interval topology (see. *7.P.*) coincides with the topology of this manifold.

Orientation of a *connected closed* 1-manifold is a cyclic order on the set of its points such that the topology of this cyclic order (see **??**) coincides with the topology of the 1-manifold.

Orientation of an *arbitrary* 1-manifold is a collection of orientations of its connected components (each component is equipped with an orientation).

47.M. *Any* 1*-manifold admits an orientation.*

47.N. An orientation of 1-manifold induces an orientation (i.e., a linear ordering of points) on each subspace homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$. Vice versa, an orientation of a 1-manifold is determined by a collection of orientations of its open subspaces homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$, if the subspaces cover the manifold and the orientations agree with each other: the orientations of any two subspaces define the same orientation on each connected component of their intersection.

47.O. Let X be a cyclicly ordered set, $a \in X$ and $B \subset X \setminus \{a\}$. Define in $X \setminus \{a\}$ a linear order induced, as in **??**, by the cyclic order on $X \setminus \{a\}$, and equip B with the linear order induced by this linear order on $X \setminus \{a\}$. Prove that if B admits a bijective monotone map onto $\mathbb{R}$, or $[0;1]$, or $[0;1)$, or $(0;1]$, then this linear order on B does not depend on a.

The construction of *47.O* allows one to define an orientation on any 1-manifold which is a subspace of an *oriented closed* 1-manifold. A 1-manifold, which is a subspace of an *oriented non-closed* 1-manifold X, inherits from X an orientation as a linear order. Thus, any 1-manifold, which is a subspace of an *oriented* 1-manifold X, inherits from X an orientation. This orientation is said to be *induced* by the orientation of X.

A topological embedding $X \to Y$ of an oriented 1-manifold to another one is said to *preserve* the orientation if it maps the orientation of X to the orientation induced on the image by the orientation of Y.

47.P. *Any two orientation preserving embeddings of an oriented connected 1-manifold X to an oriented connected 1-manifold Y are isotopic.*

47.Q. *If two embeddings of an oriented 1-manifold X to an oriented 1-manifold Y are isotopic and one of the embeddings preserves the orientation, then the other one also preserves the orientation*

47.R. [Corollary] Orientation of a closed segment is determined by the ordering of its end points.

An orientation of a segment is shown by an arrow directed from the initial end point to the final one.

47.S. *A connected 1-manifold admits two orientations. A 1-manifold consisting of n connected components admits 2^n orientations.*

47°11. Mapping Class Groups

47.T. Find the mapping class groups of

(1) S^1,
(2) $\mathbb{R}^1$,
(3) $\mathbb{R}^1_+$,
(4) $[0, 1]$,
(5) $S^1 \amalg S^1$,
(6) $\mathbb{R}^1_+ \amalg \mathbb{R}^1_+$.

47.1. Find the mapping class group of an arbitrary 1-manifold with finite number of components.

48. Two-Dimensional Manifolds: General Picture

48°1. Examples

48.A. What connected 2-manifolds do you know?

(1) List *closed* connected 2-manifold that you know.
(2) Do you know a connected *compact* 2-manifold, which is not closed?
(3) What *non-compact* connected 2-manifolds do you know?
(4) Is there a *non-compact* connected 2-manifolds with non-empty boundary?

48.1. Construct non-homeomorphic non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group.

For notions relevant to this problem see what follows.

48°2x. Ends and Odds

Let X be a non-compact Hausdorff topological space, which is a union of an increasing sequence of its compact subspaces

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X.$$

Each connected component U of $X \smallsetminus C_n$ is contained in some connected component of $X \smallsetminus C_{n-1}$. A decreasing sequence $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$ of connected components of

$$(X \smallsetminus C_1) \supset (X \smallsetminus C_2) \supset \dots \supset (X \smallsetminus C_n) \supset \dots$$

respectively is called an *end of X with respect to* $C_1 \subset \dots \subset C_n \subset \dots$.

48.Ax. Let X and C_n be as above, D be a compact set in X and V a connected component of $X \smallsetminus D$. Prove that there exists n such that $D \subset C_n$.

48.Bx. Let X and C_n be as above, D_n be an increasing sequence of compact sets of X with $X = \cup_{n=1}^{\infty} D_n$. Prove that for any end $U_1 \supset \dots \supset U_n \supset \dots$ of X with respect to C_n there exists a unique end $V_1 \supset \dots \supset V_n \supset \dots$ of X with respect to D_n such that for any p there exists q such that $V_q \subset U_p$.

48.Cx. Let X, C_n and D_n be as above. Then the map of the set of ends of X with respect to C_n to the set of ends of X with respect to D_n defined by the statement of *48.Bx* is a bijection.

Theorem *48.Cx* allows one to speak about *ends* of X without specifying a system of compact sets

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X$$

with $X = \cup_{n=1}^{\infty} C_n$. Indeed, *48.Bx* and *48.Cx* establish a canonical one-to-one correspondence between ends of X with respect to any two systems of this kind.

48.Dx. Prove that $\mathbb{R}^1$ has two ends, $\mathbb{R}^n$ with $n > 1$ has only one end.

48.Ex. Find the number of ends for the universal covering space of the bouquet of two circles.

48.Fx. Does there exist a 2-manifold with a finite number of ends which cannot be embedded into a compact 2-manifold?

48.Gx. Prove that for any compact set $K \subset S^2$ with connected complement $S^2 \smallsetminus K$ there is a natural map of the set of ends of $S^2 \smallsetminus K$ to the set of connected components of K.

Let W be an open set of X. The set of ends $U_1 \supset \dots \supset U_n \supset \dots$ of X such that $U_n \subset W$ for sufficiently large n is said to be *open*.

48.Hx. Prove that this defines a topological structure in the set of ends of X.

The set of ends of X equipped with this topological structure is called the *space of ends* of X. Denote this space by $\mathcal{E}(X)$.

48.1.1. Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with non-homeomorphic spaces of ends.

48.1.2. Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with different number of ends.

48.1.3. Construct non-compact connected manifolds of dimension two without boundary with isomorphic infinitely generated fundamental group and the same number of ends, but with different topology in the space of ends.

48.1.4. Let K be a completely disconnected closed set in S^2. Prove that the map $\mathcal{E}(S^2 \smallsetminus K) \to K$ defined in *48.Gx* is continuous.

48.1.5. Construct a completely disconnected closed set $K \subset S^2$ such that this map is a homeomorphism.

48.Ix. Prove that there exists an uncountable family of pairwise nonhomeomorphic connected 2-manifolds without boundary.

The examples of non-compact manifolds dimension 2 presented above show that there are too many non-compact connected 2-manifolds. This makes impossible any really useful topological classification of non-compact 2-manifolds. Theorems reducing the homeomorphism problem for 2-manifolds of this type to the homeomorphism problem for their spaces of ends do not seem to be useful: spaces of ends look not much simpler than the surfaces themselves.

However, there is a special class of non-compact 2-manifolds, which admits a simple and useful classification theorem. This is the class of *simply connected* non-compact 2-manifolds without boundary. We postpone its consideration to section 53°4x. Now we turn to the case, which is the simplest and most useful for applications.

48°3. Closed Surfaces

48.B. *Any connected closed manifold of dimension two is homeomorphic either to sphere* S^2*, or sphere with handles, or sphere with crosscaps.*

Recall that according to Theorem *43.O* the basic surfaces represent pairwise distinct topological (and even homotopy) types. Therefore, *43.O* and *48.B* together give topological and homotopy classifications of closed 2-dimensional manifolds.

We do not recommend to have a try at proving Theorem *48.B* immediately and, especially, in the form given above. All known proofs of *48.B* can be decomposed into two main stages: firstly, a manifold under consideration is equipped with some additional structure (like triangulation or smooth structure); then using this structure a required homeomorphism is constructed. Although the first stage appears in the proof necessarily and is rather difficult, it is not useful outside the proof. Indeed, any closed 2-manifold, which we meet in a concrete mathematical context, is either equipped, or can be easily equipped with the additional structure. The methods of imposing the additional structure are much easier, than a general proof of existence for such a structure in an arbitrary 2-manifold.

Therefore, we suggest for the first case to restrict ourselves to the second stage of the proof of Theorem *48.B*, prefacing it with general notions related to the most classical additional structure, which can be used for this purpose.

48°4. Compact Surfaces with Boundary

As in the case of one-dimensional manifolds, classification of compact two-dimensional manifolds with boundary can be easily reduced to the classification of closed manifolds. In the case of one-dimensional manifolds it was very useful to double a manifold. In two-dimensional case there is a construction providing a closed manifold related to a compact manifold with boundary even closer than the double.

48.C. *Contracting to a point each connected component of the boundary of a two-dimensional compact manifold with boundary gives rise to a closed two-dimensional manifold.*

> ***48.2.*** A space homeomorphic to the quotient space of *48.C* can be constructed by attaching copies of D^2 one to each connected component of the boundary.

48.D. *Any connected compact manifold of dimension 2 with nonempty boundary is homeomorphic either to sphere with holes, or sphere with handles and holes, or sphere with crosscaps and holes.*

49. Triangulations

49°1. Triangulations of Surfaces

By an *Euclidean triangle* we mean the convex hall of three non-collinear points of Euclidean space. Of course, it is homeomorphic to disk D^2, but it is not solely the topological structure that is relevant now. The boundary of a triangle contains three distinguished points, its *vertices*, which divide the boundary into three pieces, its *edges*. A *topological triangle* in a topological space X is an embedding of an Euclidean triangle into X. A *vertex* (respectively, *edge*) of a topological triangle $T \to X$ is the image of a vertex (respectively, edge) of T in X.

A set of topological triangles in a 2-manifold X is a *triangulation* of X provided the images of these triangles form a fundamental cover of X and any two of the images either are disjoint or intersect in a common side or in a common vertex.

49.A. Prove that in the case of compact X the former condition (about fundamental cover) means that the number of triangles is finite.

49.B. Prove that the condition about fundamental cover means that the cover is locally finite.

49°2. Triangulation as cellular decomposition

49.C. A triangulation of a 2-manifold turns it into a cellular space, 0-cells of which are the vertices of all triangles of the triangulation, 1-cells are the sides of the triangles, and 2-cells are the interiors of the triangles.

This result allows us to apply all the terms introduced above for cellular spaces. In particular, we can speak about skeletons, cellular subspaces and cells. However,in the latter two cases we rather use terms *triangulated subspace* and *simplex*. Triangulations and terminology related to them appeared long before cellular spaces. Therefore in this context the adjective *cellular* is replaced usually by adjectives *triangulated* or *simplicial*.

49°3. Two Properties of Triangulations of Surfaces

49.D Unramified. Let E be a side of a triangle involved into a triangulation of a 2-manifold X. Prove that there exist at most two triangles of this triangulation for which E is a side. Cf. *44.G*, *44.H* and *44.P*.

49.E Local strong connectedness. Let V be a vertex of a triangle involved into a triangulation of a 2-manifold X and T, T' be two triangles of the triangulation adjacent to V. Prove that there exists a sequence

$T = T_1, T_2, \dots, T_n = T'$ of triangles of the triangulation such that V is a vertex of each of them and triangles T_i, T_{i+1} have common side for each $i = 1, \dots, n-1$.

Triangulations present a surface combinatorially.

49°4x. Scheme of Triangulation

Let X be a 2-manifold and $\mathcal{T}$ a triangulation of X. Denote the set of vertices of $\mathcal{T}$ by V. Denote by Σ_2 the set of triples of vertices, which are vertices of a triangle of $\mathcal{T}$. Denote by Σ_1 the set of pairs of vertices, which are vertices of a side of $\mathcal{T}$. Put $\Sigma_0 = S$. This is the set of vertices of $\mathcal{T}$. Put $\Sigma = \Sigma_2 \cup \Sigma_1 \cup \Sigma_0$. The pair (V, Σ) is called the *(combinatorial) scheme* of $\mathcal{T}$.

49.Ax. Prove that the combinatorial scheme (V, Σ) of a triangulation of a 2-manifold has the following properties:

(1) Σ is a set consisting of subsets of V,
(2) each element of Σ consists of at most 3 elements of V,
(3) three-element elements of Σ cover V,
(4) any subset of an element of Σ belongs to Σ,
(5) intersection of any collection of elements of Σ belongs to Σ,
(6) for any two-element element of Σ there exist exactly two three-element elements of Σ containing it.

Recall that objects of this kind appeared above, in Section 23°3x. Let V be a set and Σ is a set of finite subsets of V. The pair (V, Σ) is called a *triangulation scheme* if

- any subset of an element of Σ belongs to Σ,
- intersection of any collection of elements of Σ belongs to Σ,
- any one element subset of V belongs to Σ.

For any simplicial scheme (V, Σ) in 23°3x a topological space $S(V, \Sigma)$ was constructed. This is, in fact, a cellular space, see *40.Ax*.

49.Bx. Prove that if (V, Σ) is the combinatorial scheme of a triangulation of a 2-manifold X then $S(V, \Sigma)$ is homeomorphic to X.

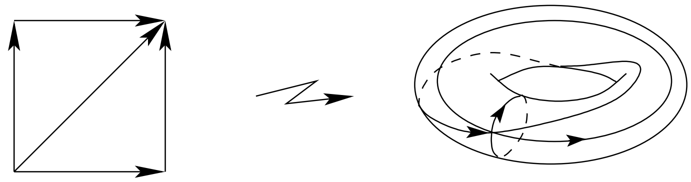
49.Cx. Let (V, Σ) be a triangulation scheme such that

(1) V is countable,
(2) each element of Σ consists of at most 3 elements of V,
(3) three-element elements of Σ cover V,
(4) for any two-element element of Σ there exist exactly two three-element elements of Σ containing it

Prove that (V, Σ) is a combinatorial scheme of a triangulation of a 2-manifold.

49°5. Examples

49.1. Consider the cover of torus obtained in the obvious way from the cover of the square by its halves separated by a diagonal of the square.



Is it a triangulation of torus? Why not?

49.2. Prove that the simplest triangulation of S^2 consists of 4 triangles.

49.3*. Prove that a triangulation of torus $S^1 \times S^1$ contains at least 14 triangles, and a triangulation of the projective plane contains at least 10 triangles.

49°6. Subdivision of a Triangulation

A triangulation $\mathcal{S}$ of a 2-manifold X is said to be a *subdivision* of a triangulation $\mathcal{T}$, if each triangle of $\mathcal{S}$ is contained in some triangle[1] of $\mathcal{T}$. Then $\mathcal{S}$ is also called a *refinement* of $\mathcal{T}$.

There are several standard ways to subdivide a triangulation. Here is one of the simplest of them. Choose a point inside a triangle τ, call it a new vertex, connect it by disjoint arcs with vertices of τ and call these arcs new edges. These arcs divide τ to three new triangles. In the original triangulation replace τ by these three new triangles. This operation is called a *star subdivision centered at* τ. See Figure 1.

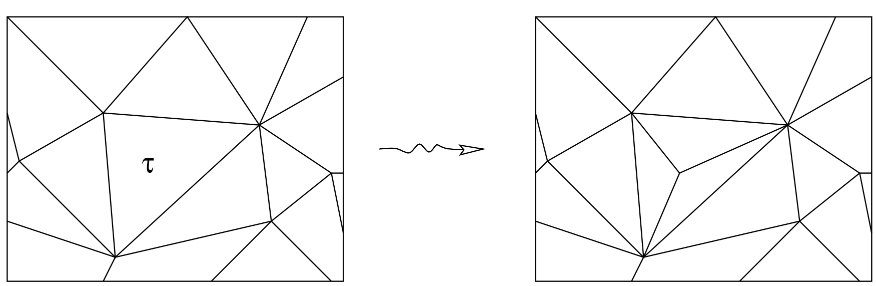


Figure 1. Star subdivision centered at triangle τ

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[1]Although triangles which form a triangulation of X have been defined as topological embeddings, we hope that a reader guess that when one of such triangles is said to be contained in another one this means that the image of the embedding which is the former triangle is contained in the image of the other embedding which is the latter.

49.F. Give a formal description of a star subdivision centered at a triangle τ. I.e., present it as a change of a triangulation thought of as a collection of topological triangles. What three embeddings of Euclidean triangles are to replace τ? Show that the replacement gives rise to a triangulation. Describe the corresponding operation on the combinatorial scheme.

Here is another subdivision defined locally. One adds a new vertex taken on an edge ε of a given triangulation. One connects the new vertex by two new edges to the vertices of the two tringles adjacent to ε. The new edges divide these triangles, each to two new triangles. The rest of triangles of the original triangulation are not affected. This operation is called a ***star subdivision centered at*** ε. See Figure 2.

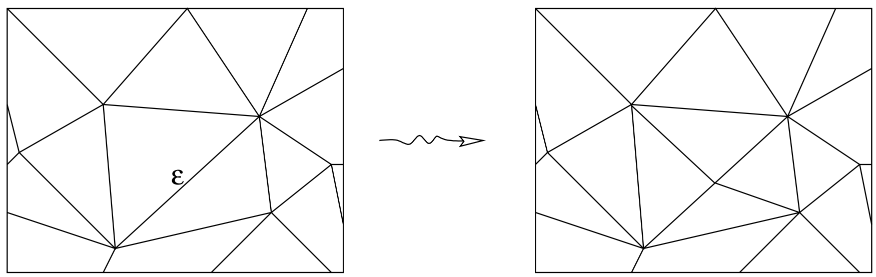

Figure 2. Star subdivision centered at edge ε.

49.G. Give a formal description of a star subdivision centered at edge ε. What four embeddings of Euclidean triangles are to replace the topological triangles with edge ε? Show that the replacement gives rise to a triangulation. Describe the corresponding operation on the combinatorial scheme.

49.4. Find a triangulation and its subdivision, which cannot be presented as a composition of star subdivisions at edges or triangles.

49.5.* Prove that any subdivision of a triangulation of a compact surface can be presented as a composition of a finite sequences of star subdivisions centered at edges or triangles and operations inverse to such subdivisions.

By a ***baricentric subdivision*** of a triangle we call a composition of a star subdivision centered at this tringle followed by star subdivisions at each of its edges. See Figure 3.

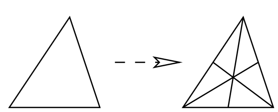

Figure 3. Baricentric subdivision of a triangle.

Baricentric subdivision of a triangulation of 2-manifold is a subdivision which is a simultaneous baricentric subdivision of all triangles of this triangulation. See Figure 4.

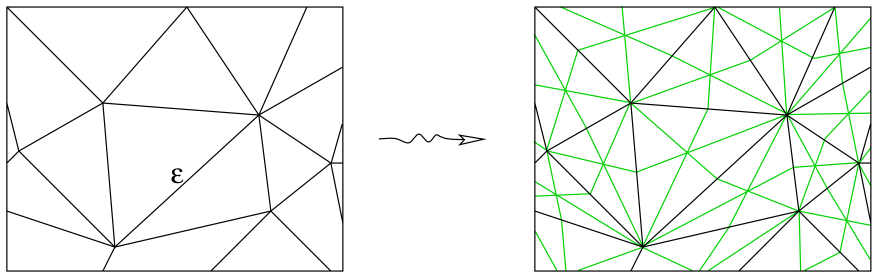

Figure 4. Baricentric subdivision of a triangulation.

49.H. Establish a natural one-to-one correspondence between vertices of a baricentric subdivision a simplices (i.e., vertices, edges and triangles) of the original tringulation.

49.I. Establish a natural one-to-one correspondence between triangles of a baricentric subdivision and triples each of which is formed of a triangle of the original triangulation, an edge of this triangle and a vertex of this edge.

The expression *baricentric subdivision* has appeared in a diiferent context, see Section 20. Let us relate the two notions sharing this name .

49.Dx Baricentric subdivision of a triangulation and its scheme. Prove that the combinatorial scheme of the baricentric subdivision of a triangulation of a 2-manifold coincides with the baricentric subdivision of the scheme of the original triangulation (see 23°4x).

49°7. Homotopy Type of Compact Surface with Non-Empty Boundary

49.J. Any compact connected triangulated 2-manifold with non-empty boundary collapses to a one-dimensional simplicial subspace.

49.K. Any compact connected triangulated 2-manifold with non-empty boundary is homotopy equivalent to a bouquet of circles.

49.L. The Euler characteristic of a triangulated compact connected 2-manifold with non-empty boundary does not depend on triangulation. It is equal to $1-r$, where r is the rank of the one-dimensional homology group of the 2-manifold.

49.M. The Euler characteristic of a triangulated compact connected 2-manifold with non-empty boundary is not greater than 1.

49.N. The Euler characteristic of a triangulated closed connected 2-manifold with non-empty boundary is not greater than 2.

49°8. Triangulations in dimension one

By an *Euclidean segment* we mean the convex hall of two different points of a Euclidean space. It is homeomorphic to I. A *topological segment* or *topological edge* in a topological space X is a topological embedding of an Euclidean segment into X. A set of topological segments in a 1-manifold X is a *triangulation* of X if the images of these topological segments constitute a fundamental cover of X and any two of the images either are disjoint or intersect in one common end point.

Traingulations of 1-manifolds are similar to triangulations of 2-manifolds considered above.

49.O. Find counter-parts for theorems above. Which of them have no counter-parts? What is a counter-part for the property *49.D*? What are counter-parts for star and baricentric subdivisions?

49.P. Find homotopy classification of triangulated compact 1-manifolds using arguments similar to the ones from Section 49°7. Compare with the topological classification of 1-manifolds obtained in Section 47.

49.Q. What values take the Euler characteristic on compact 1-manifolds?

49.R. What is relation of the Euler characteristic of a compact triangulated 1-manifold X and the number of ∂X?

49.S. *Triangulation of a 2-manifold X gives rise to a triangulation of its boundary ∂X. Namely, the edges of the triangualtion of ∂X are the sides of triangles of the original triangulation which lie in ∂X.*

49°9. Triangualtions in higher dimensions

49.T. Generalize everything presented above in this section to the case of manifolds of higher dimensions.

50. Handle Decomposition

50°1. Handles and Their Anatomy

Together with triangulations, it is useful to consider representations of a manifold as a union of balls of the same dimension, but adjacent to each other as if they were thickening of cells of a cellular space

A space $D^p \times D^{n-p}$ is called a *(standard) handle of dimension n and index p*. Its subset $D^p \times \{0\} \subset D^p \times D^{n-p}$ is called the *core* of handle $D^p \times D^{n-p}$, and a subset $\{0\} \times D^{n-p} \subset D^p \times D^{n-p}$ is called its cocore. The boundary $\partial(D^p \times D^{n-p}) =$ of the handle $D^p \times D^{n-p}$ can be presented as union of its *base* $D^p \times S^{n-p-1}$ and *cobase* $S^{p-1} \times D^{n-p}$.

50.A. Draw all standard handles of dimensions ≤ 3.

A topological embedding h of the standard handle $D^p \times D^{n-p}$ of dimension n and index p into a manifold of the same dimansion n is called a *handle of dimension n and index p*. The image under h of $\operatorname{Int} D^p \times \operatorname{Int} D^{n-p}$ is called the *interior* of h, the image of the core $h(D^p \times \{0\})$ of the standard handle is called the *core* of h, the image $h(\{0\} \times D^{n-p})$ of cocore, the *cocore*, etc.

50°2. Handle Decomposition of Manifold

Let X be a manifold of dimension n. A collecton of n-dimensional handles in X is called a *handle decomposition of X*, if

(1) the images of these handles constitute a locally finite cover of X,
(2) the interiors of these handles are pairwise disjoint,
(3) the base of each of the handles is contained in the union of cobases of the handles of smaller indices.

Let X be a manifold of dimension n with boundary. A collection of n-dimensional handles in X is called a *handle decomposion of X modulo boundary*, if

(1) the images of these handles constitute a locally finite cover of X,
(2) the interiors of these handles are pairwise disjoint,
(3) the base of each of the handles is contained in the union of ∂X and cobases of the handles of smaller indices.

A composition of a handle $h : D^p \times D^{n-p} \to X$ with the homeomorphism of transposition of the factors $D^p \times D^{n-p} \to D^{n-p} \times D^p$ turns the handle h of index p into a handle of the same dimension n, but of the complementary index $n - p$. The core of the handle turns into the cocore, while the base, to cobase.

50.B. Composing each handle with the homeomorphism transposing the factors turns a handle decomposition of manifold into a handle decomposition modulo boundary of the same manifold. Vice versa, a handle decomposition modulo boundary turns into a handle decomposition of the same manifold.

Handle decompositions obtained from each other in this way are said to be *dual* to each other.

50.C. Riddle. For n-dimensional manifold with boundary split into two $(n-1)$-dimensional manifolds with disjoint closures, define handle decomposition modulo one of these manifolds so that the dual handle decomposition would be modulo the complementary part of the boundary.

50.1. Find handle decompositions with a minimal number of handles for the following manifolds:

(a) circle S^1;	(b) sphere S^n;	(c) ball D^n
(d) torus $S^1 \times S^1$;	(e) handle;	(f) cylinder $S^1 \times I$;
(g) Möbius band;	(h) projective plane $\mathbb{R}P^2$;	(i) projective space $\mathbb{R}P^n$;
(j) sphere with p handles;	(k) sphere with p cross-caps;	(l) sphere with n holes.

50°3. Handle Decomposition and Triangulation

Let X be a 2-manifold, τ its triangulation, τ' its baricentric subdivision, and τ'' the baricentric subdivision of τ'. For each simplex S of τ denote by H_S the union of all simplices of τ'' which contain the unique vertex of τ' that lies in $\int S$. Thus, if S is a vertex then H_S is the union of all triangles of τ'' containing this vertex, if S is an edge then H_S is the union all of the triangles of τ'' which intersect with S but do not contain any of its vertices, and, finally, if S is a triangle of τ then H_S is the union of all triangles of τ'' which lie in S but do not intersect its boundary.

50.D Handle Decomposition out of a Triangulation. Sets H_S constitute a handle decomposition of X. The index of H_S equals the dimension of S.

50.E. Can every handle decomposition of a 2-manifold be constructed from a triangulation as indicated in *50.D*?

50.F. How to triangulate a 2-manifold which is equipped with a handle decomposition?

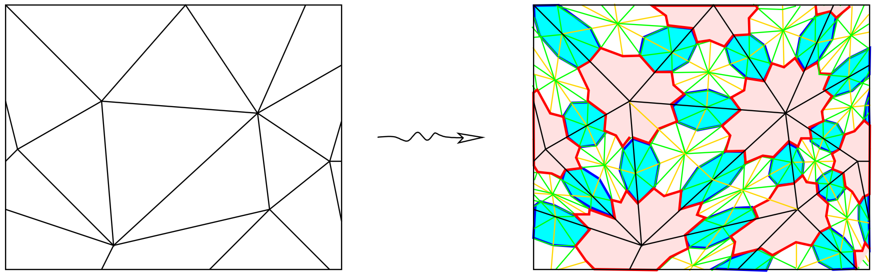

Figure 5. Construction of a handle decomposition from a triangulation.

50°4. Regular Neighborhoods

Let X be a 2-manifold, τ its triangulation, and A be a simplicial subspace of X. The union of all those simplices of the double baricentric subdivision τ'' of τ which intersect A is called the *regular* or *second baricentric neighborhood* of A (with respect to τ).

Of course, usually regular neighborhood is not open in X, since it is the union of simplices, which are closed. So, it is a neighborhood of A only in wide sense (its interior contains A).

50.G. A regular neighborhood of A in X is a 2-manifold. It coincides with the union of handles corresponding to the simplices contained in A. These handles constitute a handle decomposition of the regular neighborhood.

50.H Collaps Induces Homemorphism. Let X be a triangulated 2-manifold and $A \subset X$ be its triangulated subspace. If $X \searrow A$ then X is homeomorphic to a regular neighborhood of A.

50.I. Any triangulated compact connected 2-manifold with non-empty boundary is homeomorphic to a regular neighborhood of some of its 1-dimensional triangulated subspaces.

50.J. In a triangulated 2-manifold, any triangulated subspace which is a tree has regular neighborhood homeomorphic to disk.

50.K. In a triangulated 2-manifold, any triangulated subspace homeomorphic to circle has regular neighborhood homeomorphic either to the Möbius band or cylinder $S^1 \times I$.

In the former case the circle is said to be *one-sided*, in the latter, *two-sided*.

50°5. Cutting 2-Manifold Along a Curve

50.L Cut Along a Curve. Let F be a triangulated surface and $C \subset F$ be a compact one-dimensional manifold contained in the 1-skeleton of F and satisfying condition $\partial C = \partial F \cap C$. Prove that there exists a 2-manifold T and surjective map $p : T \to F$ such that:

(1) $p| : T \smallsetminus p^{-1}(C) \to F \smallsetminus C$ is a homeomorphism,

(2) $p| : p^{-1}(C) \to C$ is a two-fold covering.

50.M Uniqueness of Cut. The 2-manifold T and map p which exist according to Theorem *50.L*, are unique up to homeomorphism: if $\tilde{T}$ and $\tilde{p}$ are other 2-manifold and map satisfying the same hypothesis then there exists a homeomorphism $h : \tilde{T} \to T$ such that $p \circ h = \tilde{p}$.

The 2-manifold T described in *50.L* is called the result of *cutting of F along C*. It is denoted by F ✄ C. This is not at all the complement $F \smallsetminus C$, although a copy of $F \smallsetminus C$ is contained in F ✄ C as a dense subset homotopy equivalent to the whole F ✄ C.

50.N Triangulation of Cut Result. F ✄ C possesses a unique triangulation such that the natural map F ✄ $C \to F$ maps homeomorphically edges and triangles of this triangulation onto edges and, respectivly, triangles of the original triangulation of F.

50.O. Let X be a triangulated 2-manifold, C be its triangulated subspace homeomorphic to circle, and let F be a regular neighborhood of C in X. Prove

(1) F ✄ C consists of two connected components, if C is two-sided on X, it is connected if C is one-sided;

(2) the inverse image of C under the natural map X ✄ $C \to X$ consists of two connected components if C is two-sided on X, it is connected if C is one-sided on X.

This proposition discloses the meaning of words *one-sided and two-sided* circle on a 2-manifold. Indeed, both connected components of the result of cutting of a regular neighborhood, and connected components of the inverse image of the circle can claim its right to be called a *side* of the circle or a *side of the cut*.

50.2. Describe the topological type of F ✄ C for the following F and C:

(1) F is sphere S^2, and C is its equator;
(2) F is a Möbius strip, and C is its middle circle (deformation retract);
(3) $F = S^1 \times S^1$, $C = S^1 \times 1$;
(4) F is torus $S^1 \times S^1$ standardly embedded into $\mathbb{R}^3$, and C is the trefoil knot lying on F, that is $\{(z, w) \in S^1 \times S^1 \mid z^2 = w^3\}$;
(5) F is a Möbius strip, C is a segment: findtwo topologically different position of C on F and describe F ✄ C for each of them;
(6) $F = \mathbb{R}P^2$, $C = \mathbb{R}P^1$.

(7) $F = \mathbb{R}P^2$, C is homeomorphic to circle: find two topologically different position C on F and describe $F \text{✂} C$ for each of them.

50.P Euler Characteristic and Cut. *Let F be a triangulated compact 2-manifold and $C \subset \int F$ be a closed one-dimensional contained in the 1-skeleton of the triangulation of F. Then $\chi(F \text{✂} C) = \chi F$.*

50.Q. Find the Euler characteristic of $F \text{✂} C$, if $\partial C \neq \varnothing$.

50.R Generalized Cut (Incise). Let F be a triangulated 2-manifold and $C \subset F$ be a compact 1-dimensional manifold contained in 1-skeleton of F and satisfying condition $\partial F \cap C \subset \partial C$. Let $D = C \smallsetminus (\partial C \smallsetminus \partial F)$. Prove that there exist a 2-manifold T and sujective continuous map $p : T \to F$ such that:

(1) $p| : T \smallsetminus p^{-1}(D) \to F \smallsetminus D$ is a homeomorphism,
(2) $p| : p^{-1}(D) \to D$ is a two-fold covering.

50.S Uniqueness of Cut. The 2-manifold T and map p, which exist according to Theorem *50.R*, ae unique up to homeomorphism: if $\tilde{T}$ and $\tilde{p}$ are other 2-manifold and map satisfying the same hypothesis then there exists a homeomorphism $h : \tilde{T} \to T$ such that $p \circ h = \tilde{p}$.

The 2-Manifold T described in *50.R* is also called the result of *cutting of F along C* and denoted by $F \text{✂} C$.

50.3. Show that if C is a segment contained in the interior of a 2-manifold F then $F \text{✂} C$ is homeomorphic to $F \smallsetminus \operatorname{Int} B$, where B is the subset of $\int F$ homeomorphic to disk.

50.4. Show that if C is a segment such that one of its end points is in $\int F$ and the other one is on ∂F then $F \text{✂} C$ is homeomorphic to F.

50°6. Orientations

Recall that an *orientation of a segment* is a linear order of the set of its points. It is determined by its restriction to the set of its end points, see *47.R.* To describe an orientation of a segment it suffices to say which of its end points is initial and which is final.

Similarly, orientation of a triangle can be described in a number of ways, each of which can be chosen as the definition. By an *orientation of a triangle* one means a collection of orientations of its edges such that each vertex of the triangle is the final point for one of the edges adjacent to it and initial point for the other edge. Thus, an orientation of a triangle defines an orientation on each of its sides.

A segment admits two orientations. A triangle also admits two orientations: one is obtained from another one by change of the orientation on each side of the triangle. Therefore an orientation of any side of a triangle defines an orientation of the triangle.

Vertices of an oriented triangle are cyclicly ordered: a vertex A follows immediately the vertex B which is the initial vertex of the edge which finishes at A. Similarly the edges of an oriented triangle are cyclicly ordered: a side a follows immediately the side b which final end point is the initial point of a.

Vice versa, each of these cyclic orders defines an orientation of the triangle.

An *orientation of a triangulation* of a 2-manifold is a collection of orientations of all triangles constituting the triangulation such that for each edge the orientations defined on it by the orientations of the two adjacent triangles are opposite to each other. A triangulation is said to be *orientable*, if it admits an orientation.

50.T Number of Orientations. *A triangulation of a connected 2-manifold is either non-orientable or admits exactly two orientations. These two orientations are opposite to each other. Each of them can be recovered from the orientation of any triangle involved in the triangulation.*

50.U Lifting of Triangulation. Let B be a triangulated surface and $p : X \to B$ be a covering. Can you equip X with a triangulation?

50.V Lifting of Orientation. Let B be an oriented triangulated surface and $p : X \to B$ be a covering. Equip X with a triangulation such that p maps each simplex of this triangulation homeomorphically onto a simplex of the original triangulation of B. Is this triangulation orientable?

50.W. Let X be a triangulated surface, $C \subset X$ be a 1-dimensional manifold contained in 1-skeleton of X. If the triangulation of X is orientable, then C is two-sided.

51. Topological Classification of Compact Triangulated 2-Manifolds

51°1. Spines and Their Regular Neighborhoods

Let X be a triangulated compact connected 2-manifold with non-empty boundary. A simplicial subspace S of the 1-skeleton of X is a *spine* of X if X collapses to S.

51.A. *Let X be a triangulated compact connected 2-manifold with non-empty boundary. Then a regular neighborhood of its spine is homeomorphic to X.*

51.B Corollary. *A triangulated compact connected 2-manifold with non-empty boundary admits a handle decomposition without handles of index 2.*

A *spine* of a *closed* connected 2-manifold is a spine of this manifold with an interior of a triangle from the triangulation removed.

51.C. *A triangulated closed connected 2-manifold admits a handle decomposition with exactly one handle of index 2.*

51.D. *A spine of a triangulated closed connected 2-manifold is connected.*

51.E Corollary. *The Euler characteristic of a closed connected triangulated 2-manifold is not greater than 2. If it is equal to 2, then the 2-manifold is homeomorphic to S^2.*

51.F Corollary: Extremal Case. *Let X be a closed connected triangulated 2-manifold X. If $\chi(X) = 2$, then X is homeomorphic to S^2.*

51°2. Simply connected compact 2-manifolds

51.G. *A simply connected compact triangulated 2-manifold with non-empty boundary collapses to a point.*

51.H Corollary. *A simply connected compact triangulated 2-manifold with non-empty boundary is homeomorphic to disk D^2.*

51.I Corollary. *Let X be a compact connected triangulated 2-manifold X with $\partial X \neq \varnothing$. If $\chi(X) = 1$, then X is homeomorphic to D^2.*

51°3. Splitting off crosscaps and handles

51.J. *A non-orientable triangulated 2-manifold X is a connected sum of $\mathbb{R}P^2$ and a triangulated 2-manifold Y. If X is connected, then Y is also connected.*

51.K. Under conditions of Theorem *51.J*, if X is compact then Y is compact and $\chi(Y) = \chi(X) + 1$.

51.L. *If on an orientable connected triangulated* 2*-manifold* X *there is a simple closed curve* C *contained in the* 1*-skeleton of* X *such that* $X \setminus C$ *is connected, then* C *is contained in a simplicial subspace* H *of* X *homeomorphic to torus with a hole and* X *is a connected sum of a torus and a triangulated connected orientable* 2*-manifold* Y*.*

If X *is compact, then* Y *is compact and* $\chi(Y) = \chi(X) + 2$.

51.M. *A compact connected triangulated* 2*-manifold with non-empty connected boundary is a connected sum of a disk and some number of copies of the projective plane and/or torus.*

51.N Corollary. *A simply connected closed triangulated* 2*-manifold is homeomorphic to* S^2*.*

51.O. *A compact connected triangulated* 2*-manifold with non-empty boundary is a connected sum of a sphere with holes and some number of copies of the projective plane and/or torus.*

51.P. *A closed connected triangulated* 2*-manifold is a connected sum of some number of copies of the projective plane and/or torus.*

51°4. Splitting of a Handle on a Non-Orientable 2-Manifold

51.Q. *A connected sum of torus and projective plane is homeomorphic to a connected sum of three copies of the projective plane.*

51.Q.1. *On torus there are 3 simple closed curves which meet at a single point transversal to each other.*

51.Q.2. *A connected sum of a surface* S *with* $\mathbb{R}P^2$ *can be obtained by deleting an open disk from* S *and identifying antipodal points on the boundary of the hole.*

51.Q.3. *On a connected sum of torus and projective plane there exist three disjoint one-sided simple closed curves.*

51°5. Final Formulations

51.R. *Any connected closed triangulated* 2*-manifold is homeomorphic either to sphere, or sphere with handles, or sphere with crosscaps.*

51.S. *Any connected compact triangulated* 2*-manifold with non-empty boundary is homeomorphic either to sphere with holes, or sphere with holes and handles, or sphere with holes and crosscaps.*

51.1. Find the place for the Klein Bottle in the above classification.

51.2. Prove that any closed triangulated surface with non-orientable triangulation is homeomorphic either to projective plane number of handles or Klein bottle with handles. (Here the number of handles is allowed to be null.)

52. Cellular Approach to Topological Classification of Compact surfaces

In this section we consider another, more classical and detailed solution of the same problem. We classify compact triangulated 2-manifolds in a way which provides also an algorithm building a homeomorphism between a given surface and one of the standard surfaces.

52°1. Families of Polygons

Triangulations provide a combinatorial description of 2-dimensional manifolds, but this description is usually too bulky. Here we will study other, more practical way to present 2-dimensional manifolds combinatorially. The main idea is to use larger building blocks.

Let $\mathcal{F}$ be a collection of convex polygons $P_1, P_2, \dots$. Let the sides of these polygons be oriented and paired off. Then we say that this is a *family of polygons.* There is a natural quotient space of the sum of polygons involved in a family: one identifies each side with its pair-mate by a homeomorphism, which respects the orientations of the sides. This quotient space is called just the *quotient of the family.*

52.A. Prove that the quotient of the family of polygons is a 2-manifold without boundary.

52.B. Prove that the topological type of the quotient of a family does not change when the homeomorphism between the sides of a distinguished pair is replaced by other homeomorphism which respects the orientations.

52.C. Prove that any triangulation of a 2-manifold gives rise to a family of polygons whose quotient is homeomorphic to the 2-manifold.

A family of polygons can be described combinatorially: Assign a letter to each distinguished pair of sides. Go around the polygons writing down the letters assigned to the sides and equipping a letter with exponent -1 if the side is oriented against the direction in which we go around the polygon. At each polygon we write a word. The word depends on the side from which we started and on the direction of going around the polygon. Therefore it is defined up to cyclic permutation and inversion. The collection of words assigned to all the polygons of the family is called a *phrase associated with the family of polygons.* It describes the family to the extend sufficient to recovering the topological type of the quotient.

52.1. Prove that the quotient of the family of polygons associated with phrase $aba^{-1}b^{-1}$ is homeomorphic to $S^1 \times S^1$.

52.2. Identify the topological type of the quotient of the family of polygons associated with phrases

(1) aa^{-1};
(2) ab, ab;
(3) aa;
(4) $abab^{-1}$;
(5) $abab$;
(6) $abcabc$;
(7) $aabb$;
(8) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$;
(9) $a_1a_1a_2a_2\dots a_ga_g$.

52.D. A collection of words is a phrase associated with a family of polygons, iff each letter appears twice in the words.

A family of polygons is called *irreducible* if the quotient is connected.

52.E. A family of polygons is irreducible, iff a phrase associated with it does not admit a division into two collections of words such that there is no letter involved in both collections.

52°2. Operations on Family of Polygons

Although any family of polygons defines a 2-manifold, there are many families defining the same 2-manifold. There are simple operations which change a family, but do not change the topological type of the quotient of the family. Here are the most obvious and elementary of these operations.

(1) Simultaneous reversing orientations of sides belonging to one of the pairs.
(2) Select a pair of sides and subdivide each side in the pair into two sides. The orientations of the original sides define the orderings of the halves. Unite the first halves into one new pair of sides, and the second halves into the other new pair. The orientations of the original sides define in an obvious way orientations of their halves. This operation is called 1-*subdivision.* In the quotient it effects in subdivision of a 1-cell (which is the image of the selected pair of sides) into two 1-cells. This 1-cells is replaced by two 1-cells and one 0-cell.
(3) The inverse operation to 1-subdivision. It is called 1-*consolidation.*
(4) Cut one of the polygons along its diagonal into two polygons. The sides of the cut constitute a new pair. They are equipped with an orientation such that gluing the polygons by a homeomorphism respecting these orientations recovers the original polygon. This operation is called 2-*subdivision.* In the quotient it effects in subdivision of a 2-cell into two new 2-cells along an arc whose end-points

are 0-cells (may be coinciding). The original 2-cell is replaced by two 2-cells and one 1-cell.

(5) The inverse operation to 2-subdivision. It is called 2-*consolidation.*

52°3. Topological and Homotopy Classification of Closed Surfaces

52.F Reduction Theorem. *Any finite irreducible family of polygons can be reduced by the five elementary operations to one of the following standard families:*

(1) aa^{-1}
(2) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$
(3) $a_1a_1a_2a_2\dots a_ga_g$ *for some natural* g.

52.G Corollary, see 51.R. *Any triangulated closed connected manifold of dimension 2 is homeomorphic to either sphere, or sphere with handles, or sphere with crosscaps.*

Theorems *52.G* and *43.O* provide classifications of triangulated closed connected 2-manifolds up to homeomorphisms and homotopy equivalence.

52.F.1 Reduction to Single Polygon. Any finite irreducible family of polygons can be reduced by elementary operations to a family consisting of a single polygon.

52.F.2 Cancellation. A family of polygons corresponding to a phrase containing a fragment aa^{-1} or $a^{-1}a$, where a is any letter, can be transformed by elementary operations to a family corresponding to the phrase obtained from the original one by erasing this fragment, unless the latter is the whole original phrase.

52.F.3 Reduction to Single Vertex. An irreducible family of polygons can be turned by elementary transformations to a polygon such that all its vertices are projected to a single point of the quotient.

52.F.4 Separation of Crosscap. A family corresponding to a phrase consisting of a word $XaYa$, where X and Y are words and a is a letter, can be transformed to the family corresponding to the phrase $bbY^{-1}X$.

52.F.5. If a family, whose quotient has a single vertex in the natural cell decomposition, corresponds to a phrase consisting of a word $XaYa^{-1}$, where X and Y are nonempty words and a is a letter, then $X = UbU'$ and $Y = Vb^{-1}V'$.

52.F.6 Separation of Handle. A family corresponding to a phrase consisting of a word $UbU'aVb^{-1}V'a^{-1}$, where U, U', V, and V' are words and a, b are letters, can be transformed to the family presented by phrase $dcd^{-1}c^{-1}UV'VU'$.

52.F.7 Handle plus Crosscap Equals 3 Crosscaps. A family corresponding to phrase $aba^{-1}b^{-1}ccX$ can be transformed by elementary transformations to the family corresponding to phrase $abdbadX$.

53. Recognizing Closed Surfaces

53.A. What is the topological type of the 2-manifold, which can be obtained as follows: Take two disjoint copies of disk. Attach three parallel strips connecting the disks and twisted by π. The resulting surface S has a connected boundary. Attach a copy of disk along its boundary by a homeomorphism onto the boundary of the S. This is the space to recognize.

53.B. *Euler characteristic of the cellular space obtained as quotient of a family of polygons is invariant under homotopy equivalences.*

53.1. How can *53.B* help to solve *53.A*?

53.2. Let X be a closed connected surface. What values of $\chi(X)$ allow to recover the topological type of X? What ambiguity is left for other values of $\chi(X)$?

53°1. Orientations

By an *orientation of a polygon* one means orientation of all its sides such that each vertex is the final end point for one of the adjacent sides and initial for the other one. Thus an orientation of a polygon includes orientation of all its sides. Each segment can be oriented in two ways, and each polygon can be oriented in two ways.

An orientation of a family of polygons is a collection of orientations of all the polygons comprising the family such that for each pair of sides one of the pair-mates has the orientation inherited from the orientation of the polygon containing it while the other pair-mate has the orientation opposite to the inherited orientation. A family of polygons is said to be *orientable* if it admits an orientation.

53.3. Which of the families of polygons from Problem *52.2* are orientable?

53.4. Prove that a family of polygons associated with a word is orientable iff each letter appear in the word once with exponent -1 and once with exponent 1.

53.C. *Orientability of a family of polygons is preserved by the elementary operations.*

A surface is said to be *orientable* if it can be presented as the quotient of an orientable family of polygons.

53.D. A surface S is orientable, iff any family of polygons whose quotient is homeomorphic to S is orientable.

53.E. Spheres with handles are orientable. Spheres with crosscaps are not.

53°2. More About Recognizing Closed Surfaces

53.5. How can the notion of orientability and *53.C* help to solve *53.A*?

53.F. *Two closed connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic and either are both orientable or both non-orientable.*

53°3. Recognizing Compact Surfaces with Boundary

53.G. Riddle. Generalize orientabilty to the case of nonclosed manifolds of dimension two. (Give as many generalization as you can and prove that they are equivalent. The main criterium of success is that the generalized orientability should help to recognize the topological type.)

53.H. *Two compact connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic, are both orientable or both nonorientable and their boundaries have the same number of connected components.*

53°4x. Simply Connected Surfaces

53.Ax Theorem*. *Any simply connected non-compact manifold of dimension two without boundary is homeomorphic to $\mathbb{R}^2$.*

53° 4x.1. Any simply connected triangulated non-compact manifold without boundary can be presented as the union of an increasing sequence of compact simplicial subspaces $C_0 \supset C_1 \supset C_2 \supset \dots \supset C_n \supset \dots$ such that each of them is a 2-manifold with boundary and $\operatorname{Int} C_n \subset C_{n+1}$ for each n.

53° 4x.2. Under conditions of *53° 4x.1* the sequence C_n can be modified in such a way that each C_n becomes simply connected.

53.Bx Corollary. *The universal covering of any surface with empty boundary and infinite fundamental group is homeomorphic to R^2.*

Proofs and Comments

47.A Indeed, any 0-dimensional manifold is just a countable discrete topological space, and the only topological invariant needed for topological classification of 0-manifolds is the number of points.

47.B Each manifold is the sum of its connected components.

47.C

(1) S^1,

(2) I,

(3) $\mathbb{R}$, $\mathbb{R}_+$,

(4) $\mathbb{R}_+$.

47.D

Manifold X	Is X compact?	Is ∂X empty?
S^1	+	+
$\mathbb{R}^1$	−	+
I	+	−
$\mathbb{R}^1_+$	−	−

48.Fx Yes, for example, a plane with infinite number of handles.

49.Q All non-negative inetegers.

49.R $\chi(X) = \frac{1}{2}\chi(\partial X) = \frac{1}{2}\sharp(\partial X)$. To prove this, consider double DX of X, and observe that $\chi(DX) = 2\chi(X) - \chi(\partial X)$, while $\chi(DX) = 0$, since DX is a closed 1-manifold.

50.V Yes, it is orientable. An orientation can be obtain by taking on each triangle of X the orientation which is mapped by p to the orientation of its image.

51.Q.1 Represent the torus as the quotient space of the unit square. Take the images of a diagonal of the square and the two segments connecting the midpoints of the opposite sides of the square.