Continuity

8. Set-Theoretic Digression: Maps

8°1. Maps and Main Classes of Maps

A *map* f of a set X to a set Y is a triple consisting of X, Y, and a rule,[1] which assigns to every element of X exactly one element of Y. There are other words with the same meaning: *mapping*, *function*, etc.

If f is a map of X to Y, then we write $f : X \to Y$, or $X \xrightarrow{f} Y$. The element b of Y assigned by f to an element a of X is denoted by $f(a)$ and called the *image* of a under f, or the f-image of a. We write $b = f(a)$, or $a \overset{f}{\mapsto} b$, or $f : a \mapsto b$.

A map $f : X \to Y$ is a *surjective map*, or just a *surjection* if every element of Y is the image of at least one element of X. A map $f : X \to Y$ is an *injective map*, *injection*, or *one-to-one map* if every element of Y is the image of at most one element of X. A map is a *bijective map*, *bijection*, or *invertible map* if it is both surjective and injective.

[1]Certainly, the rule (as everything in set theory) may be thought of as a set. Namely, we consider the set of the ordered pairs (x, y) with $x \in X$ and $y \in Y$ such that the rule assigns y to x. This is the *graph* of f. It is a subset of $X \times Y$. (Recall that $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.)

8°2. Image and Preimage

The *image* of a set $A \subset X$ under a map $f : X \to Y$ is the set of images of all points of A. It is denoted by $f(A)$. Thus

$$f(A) = \{f(x) \mid x \in A\}.$$

The image of the entire set X (i.e., the set $f(X)$) is the *image* of f, it is denoted by $\operatorname{Im} f$.

The *preimage* of a set $B \subset Y$ under a map $f : X \to Y$ is the set of elements of X whith images in to B. It is denoted by $f^{-1}(B)$. Thus

$$f^{-1}(B) = \{a \in X \mid f(a) \in B\}.$$

Be careful with these terms: their etymology can be misleading. For example, the image of the preimage of a set B can differ from B. And even if it does not differ, it may happen that the preimage is not the only set with this property. Hence, the preimage *cannot* be defined as a set whose image is the given set.

8.A. We have $f\bigl(f^{-1}(B)\bigr) \subset B$ for any map $f : X \to Y$ and any $B \subset Y$.

8.B. $f\bigl(f^{-1}(B)\bigr) = B$ iff $B \subset \operatorname{Im} f$.

8.C. Let $f : X \to Y$ be a map and let $B \subset Y$ be such that $f\bigl(f^{-1}(B)\bigr) = B$. Then the following statements are equivalent:

(1) $f^{-1}(B)$ is the unique subset of X whose image equals B;

(2) for any $a_1, a_2 \in f^{-1}(B)$ the equality $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

8.D. A map $f : X \to Y$ is an injection iff for each $B \subset Y$ such that $f\bigl(f^{-1}(B)\bigr) = B$ the preimage $f^{-1}(B)$ is the unique subset of X with image equal to B.

8.E. We have $f^{-1}\bigl(f(A)\bigr) \supset A$ for any map $f : X \to Y$ and any $A \subset X$.

8.F. $f^{-1}\bigl(f(A)\bigr) = A$ iff $f(A) \cap f(X \smallsetminus A) = \varnothing$.

8.1. Do the following equalities hold true for any $A, B \subset Y$ and $f : X \to Y$:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \tag{10}$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \tag{11}$$

$$f^{-1}(Y \smallsetminus A) = X \smallsetminus f^{-1}(A)? \tag{12}$$

8.2. Do the following equalities hold true for any $A, B \subset X$ and any $f : X \to Y$:

$$f(A \cup B) = f(A) \cup f(B), \tag{13}$$

$$f(A \cap B) = f(A) \cap f(B), \tag{14}$$

$$f(X \smallsetminus A) = Y \smallsetminus f(A)? \tag{15}$$

8.3. Give examples in which two of the above equalities (13)–(15) are false.

8.4. Replace false equalities of *8.2* by correct inclusions.

8.5. Riddle. What simple condition on $f : X \to Y$ should be imposed in order to make correct all equalities of *8.2* for any $A, B \subset X$?

8.6. Prove that for any map $f : X \to Y$ and any subsets $A \subset X$ and $B \subset Y$ we have:

$$B \cap f(A) = f\big(f^{-1}(B) \cap A\big).$$

8°3. Identity and Inclusion

The *identity map* of a set X is the map $\mathrm{id}_X : X \to X : x \mapsto x$. It is denoted just by id if there is no ambiguity. If A is a subset of X, then the map $\mathrm{in} : A \to X : x \mapsto x$ is the *inclusion map*, or just *inclusion*, of A into X. It is denoted just by in when A and X are clear.

8.G. The preimage of a set B under the inclusion $\mathrm{in} : A \to X$ is $B \cap A$.

8°4. Composition

The *composition* of maps $f : X \to Y$ and $g : Y \to Z$ is the map $g \circ f : X \to Z : x \mapsto g\big(f(x)\big)$.

8.H Associativity. $h \circ (g \circ f) = (h \circ g) \circ f$ for any maps $f : X \to Y$, $g : Y \to Z$, and $h : Z \to U$.

8.I. $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for any $f : X \to Y$.

8.J. A composition of injections is injective.

8.K. If the composition $g \circ f$ is injective, then so is f.

8.L. A composition of surjections is surjective.

8.M. If the composition $g \circ f$ is surjective, then so is g.

8.N. A composition of bijections is a bijection.

8.7. Let a composition $g \circ f$ be bijective. Is then f or g necessarily bijective?

8°5. Inverse and Invertible

A map $g : Y \to X$ is *inverse* to a map $f : X \to Y$ if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. A map having an inverse map is *invertible*.

8.O. A map is invertible iff it is a bijection.

8.P. If an inverse map exists, then it is unique.

8°6. Submaps

If $A \subset X$ and $B \subset Y$, then for every $f : X \to Y$ such that $f(A) \subset B$ we have a map $\mathrm{ab}(f) : A \to B : x \mapsto f(x)$, which is called the *abbreviation* of f to A and B, a *submap*, or a *submapping*. If $B = Y$, then $\mathrm{ab}(f) : A \to Y$ is denoted by $f|_A$ and called the *restriction* of f to A. If $B \neq Y$, then $\mathrm{ab}(f) : A \to B$ is denoted by $f|_{A,B}$ or even simply $f|$.

8.Q. The restriction of a map $f : X \to Y$ to $A \subset X$ is the composition of the inclusion $\mathrm{in} : A \to X$ and f. In other words, $f|_A = f \circ \mathrm{in}$.

8.R. Any submap (in particular, any restriction) of an injection is injective.

8.S. If a map possesses a surjective restriction, then it is surjective.

9. Continuous Maps

9°1. Definition and Main Properties of Continuous Maps

Let X and Y be two topological spaces. A map $f : X \to Y$ is *continuous* if the preimage of any open subset of Y is an open subset of X.

9.A. *A map is continuous iff the preimage of each closed set is closed.*

9.B. *The identity map of any topological space is continuous.*

9.1. Let Ω_1 and Ω_2 be two topological structures in a space X. Prove that the identity map

$$\mathrm{id} : (X, \Omega_1) \to (X, \Omega_2)$$

is continuous iff $\Omega_2 \subset \Omega_1$.

9.2. Let $f : X \to Y$ be a continuous map. Find out whether or not it is continuous with respect to

(1) a finer topology in X and the same topology in Y,
(2) a coarser topology in X and the same topology in Y,
(3) a finer topology in Y and the same topology in X,
(4) a coarser topology in Y and the same topology in X.

9.3. Let X be a discrete space and Y an arbitrary space. 1) Which maps $X \to Y$ are continuous? 2) Which maps $Y \to X$ are continuous?

9.4. Let X be an indiscrete space and Y an arbitrary space. 1) Which maps $X \to Y$ are continuous? 2) Which maps $Y \to X$ are continuous?

9.C. *Let A be a subspace of X. The inclusion* $\mathrm{in} : A \to X$ *is continuous.*

9.D. The topology Ω_A induced on $A \subset X$ by the topology of X is the coarsest topology in A with respect to which the inclusion $\mathrm{in} : A \to X$ is continuous.

9.5. Riddle. The statement *9.D* admits a natural generalization with the inclusion map replaced by an arbitrary map $f : A \to X$ of an arbitrary set A. Find this generalization.

9.E. *A composition of continuous maps is continuous.*

9.F. *A submap of a continuous map is continuous.*

9.G. *A map $f : X \to Y$ is continuous iff* $\mathrm{ab}\, f : X \to f(X)$ *is continuous.*

9.H. Any constant map (i.e., a map with image consisting of a single point) is continuous.

9°2. Reformulations of Definition

9.6. Prove that a map $f : X \to Y$ is continuous iff

$$\operatorname{Cl} f^{-1}(A) \subset f^{-1}(\operatorname{Cl} A)$$

for any $A \subset Y$.

9.7. Formulate and prove similar criteria of continuity in terms of $\operatorname{Int} f^{-1}(A)$ and $f^{-1}(\operatorname{Int} A)$. Do the same for $\operatorname{Cl} f(A)$ and $f(\operatorname{Cl} A)$.

9.8. Let Σ be a base for topology in Y. Prove that a map $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open for each $U \in \Sigma$.

9°3. More Examples

9.9. Consider the map

$$f : [0,2] \to [0,2] : f(x) = \begin{cases} x & \text{if } x \in [0,1), \\ 3 - x & \text{if } x \in [1,2]. \end{cases}$$

Is it continuous (with respect to the topology induced from the real line)?

9.10. Consider the map f from the segment $[0,2]$ (with the relative topology induced by the topology of the real line) into the arrow (see Section 2) defined by the formula

$$f(x) = \begin{cases} x & \text{if } x \in [0,1], \\ x + 1 & \text{if } x \in (1,2]. \end{cases}$$

Is it continuous?

9.11. Give an explicit characterization of continuous maps of $\mathbb{R}_{T_1}$ (see Section 2) to $\mathbb{R}$.

9.12. Which maps $\mathbb{R}_{T_1} \to \mathbb{R}_{T_1}$ are continuous?

9.13. Give an explicit characterization of continuous maps of the arrow to itself.

9.14. Let f be a map of the set $\mathbb{Z}_+$ of nonnegative numbers onto $\mathbb{R}$ defined by formula

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Let $g : \mathbb{Z}_+ \to f(\mathbb{Z}_+)$ be its submap. Induce a topology on $\mathbb{Z}_+$ and $f(\mathbb{Z}_+)$ from $\mathbb{R}$. Are f and the map g^{-1} inverse to g continuous?

9°4. Behavior of Dense Sets

9.15. Prove that the image of an everywhere dense set under a surjective continuous map is everywhere dense.

9.16. Is it true that the image of nowhere dense set under a continuous map is nowhere dense?

9.17*. Do there exist a nowhere dense set A of $[0,1]$ (with the topology induced from the real line) and a continuous map $f : [0,1] \to [0,1]$ such that $f(A) = [0,1]$?

9°5. Local Continuity

A map f from a topological space X to a topological space Y is said to be *continuous at a point* $a \in X$ if for every neighborhood V of $f(a)$ there exists a neighborhood U of a such that $f(U) \subset V$.

9.I. *A map $f : X \to Y$ is continuous iff it is continuous at each point of X.*

9.J. *Let X and Y be two metric spaces, $a \in X$. A map $f : X \to Y$ is continuous at a iff for every ball with center at $f(a)$ there exists a ball with center at a whose image is contained in the first ball.*

9.K. *Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$.*

Theorem *9.K* means that the definition of continuity usually studied in Calculus, when applicable, is equivalent to the above definition stated in terms of topological structures.

9°6. Properties of Continuous Functions

9.18. Let $f, g : X \to \mathbb{R}$ be continuous. Prove that the maps $X \to \mathbb{R}$ defined by formulas

$$x \mapsto f(x) + g(x), \tag{16}$$

$$x \mapsto f(x)g(x), \tag{17}$$

$$x \mapsto f(x) - g(x), \tag{18}$$

$$x \mapsto \big|f(x)\big|, \tag{19}$$

$$x \mapsto \max\{f(x), g(x)\}, \tag{20}$$

$$x \mapsto \min\{f(x), g(x)\} \tag{21}$$

are continuous.

9.19. Prove that if $0 \notin g(X)$, then the map

$$X \to \mathbb{R} : x \mapsto \frac{f(x)}{g(x)}$$

is continuous.

9.20. Find a sequence of continuous functions $f_i : \mathbb{R} \to \mathbb{R}$, $(i \in \mathbb{N})$, such that the function

$$\mathbb{R} \to \mathbb{R} : x \mapsto \sup\{\, f_i(x) \mid i \in \mathbb{N}\,\}$$

is not continuous.

9.21. Let X be a topological space. Prove that a function $f : X \to \mathbb{R}^n : x \mapsto (f_1(x), \dots, f_n(x))$ is continuous iff so are all functions $f_i : X \to \mathbb{R}$ with $i = 1, \dots, n$.

Real $p \times q$-matrices form a space $Mat(p \times q, \mathbb{R})$, which differs from $\mathbb{R}^{pq}$ only in the way of numeration of its natural coordinates (they are numerated by pairs of indices).

9.22. Let $f : X \to Mat(p \times q, \mathbb{R})$ and $g : X \to Mat(q \times r, \mathbb{R})$ be continuous maps. Prove that then

$$X \to Mat(p \times r, \mathbb{R}) : x \mapsto g(x)f(x)$$

is a continuous map.

Recall that $GL(n; \mathbb{R})$ is the subspace of $Mat(n \times n, \mathbb{R})$ consisting of all invertible matrices.

9.23. Let $f : X \to GL(n; \mathbb{R})$ be a continuous map. Prove that $X \to GL(n; \mathbb{R}) : x \mapsto (f(x))^{-1}$ is continuous.

9°7. Continuity of Distances

9.L. *For every subset A of a metric space X, the function $X \to \mathbb{R} : x \mapsto \rho(x, A)$ (see Section 4) is continuous.*

9.24. Prove that a topology of a metric space is the coarsest topology with respect to which the function $X \to \mathbb{R} : x \mapsto \rho(x, A)$ is continuous for every $A \subset X$.

9°8. Isometry

A map f of a metric space X into a metric space Y is an *isometric embedding* if $\rho(f(a), f(b)) = \rho(a, b)$ for any $a, b \in X$. A bijective isometric embedding is an *isometry*.

9.M. Every isometric embedding is injective.

9.N. Every isometric embedding is continuous.

9°9. Contractive Maps

A map $f : X \to X$ of a metric space X is *contractive* if there exists $\alpha \in (0, 1)$ such that $\rho(f(a), f(b)) \leq \alpha\rho(a, b)$ for any $a, b \in X$.

9.25. Prove that every contractive map is continuous.

Let X and Y be metric spaces. A map $f : X \to Y$ is a *Hölder* map if there exist $C > 0$ and $\alpha > 0$ such that $\rho(f(a), f(b)) \leq C\rho(a, b)^\alpha$ for any $a, b \in X$.

9.26. Prove that every Hölder map is continuous.

9°10. Sets Defined by Systems of Equations and Inequalities

9.O. *Let f_i $(i = 1, \dots, n)$ be continuous maps $X \to \mathbb{R}$. Then the subset of X consisting of solutions of the system of equations*

$$f_1(x) = 0, \dots, f_n(x) = 0$$

is closed.

9.P. *Let f_i $(i = 1, \dots, n)$ be continuous maps $X \to \mathbb{R}$. Then the subset of X consisting of solutions of the system of inequalities*

$$f_1(x) \geq 0, \dots, f_n(x) \geq 0$$

is closed, while the set consisting of solutions of the system of inequalities

$$f_1(x) > 0, \dots, f_n(x) > 0$$

is open.

9.27. Where in *9.O* and *9.P* a finite system can be replaced by an infinite one?

9.28. Prove that in $\mathbb{R}^n$ $(n \geq 1)$ every proper algebraic set (i.e., a set defined by algebraic equations) is nowhere dense.

9°11. Set-Theoretic Digression: Covers

A collection Γ of subsets of a set X is a *cover* or a *covering* of X if X is the union of sets belonging to Γ, i.e., $X = \bigcup_{A \in \Gamma} A$. In this case, elements of Γ *cover* X.

There is also a more general meaning of these words. A collection Γ of subsets of a set Y is a *cover* or a *covering* of a set $X \subset Y$ if X is contained in the union of the sets in Γ, i.e., $X \subset \bigcup_{A \in \Gamma} A$. In this case, the sets belonging to Γ are also said to *cover* X.

9°12. Fundamental Covers

Consider a cover Γ of a topological space X. Each element of Γ inherits a topological structure from X. When are these structures sufficient for recovering the topology of X? In particular, under what conditions on Γ does the continuity of a map $f : X \to Y$ follow from that of its restrictions to elements of Γ? To answer these questions, solve Problems *9.29–9.30* and *9.Q–9.V*.

9.29. Find out whether or not this is true for the following covers:

(1) $X = [0, 2]$, and $\Gamma = \{[0, 1], (1, 2]\}$;
(2) $X = [0, 2]$, and $\Gamma = \{[0, 1], [1, 2]\}$;
(3) $X = \mathbb{R}$, and $\Gamma = \{\mathbb{Q}, \mathbb{R} \smallsetminus \mathbb{Q}\}$;
(4) $X = \mathbb{R}$, and Γ is a set of all one-point subsets of $\mathbb{R}$.

A cover Γ of a space X is *fundamental* if a set $U \subset X$ is open iff for every $A \in \Gamma$ the set $U \cap A$ is open in A.

9.Q. A cover Γ of a space X is fundamental iff a set $U \subset X$ is open, provided $U \cap A$ is open in A for every $A \in \Gamma$.

9.R. A cover Γ of a space X is fundamental iff a set $F \subset X$ is closed, provided $F \cap A$ is closed A for every $A \in \Gamma$.

9.30. The cover of a topological space by singletons is fundamental iff the space is discrete.

A cover of a topological space is *open* if it consists of open sets, and it is *closed* if it consists of closed sets. A cover of a topological space is *locally finite* if every point of the space has a neighborhood intersecting only a finite number of elements of the cover.

9.S. *Every open cover is fundamental.*

9.T. *A finite closed cover is fundamental.*

9.U. Every locally finite closed cover is fundamental.

9.V. *Let Γ be a fundamental cover of a topological space X, and let $f : X \to Y$ be a map. If the restriction of f to each element of Γ is continuous, then so is f.*

A cover Γ' is a *refinement* of a cover Γ if every element of Γ' is contained in an element of Γ.

9.31. Prove that if a cover Γ' is a refinement of a cover Γ and Γ' is fundamental, then so is Γ.

9.32. Let Δ be a fundamental cover of a topological space X, and Γ be a cover of X such that $\Gamma_A = \{ U \cap A \mid U \in \Gamma \}$ is a fundamental cover for subspace $A \subset X$ for every $A \in \Delta$. Prove that Γ is a fundamental cover.

9.33. Prove that the property of being fundamental is local, i.e., if every point of a space X has a neighborhood V such that $\Gamma_V = \{ U \cap V \mid U \in \Gamma \}$ is fundamental, then Γ is fundamental.

9°13x. Monotone Maps

Let $(X, \prec)$ and $(Y, \prec)$ be posets. A map $f : X \to Y$ is

- (*non-strictly*) *monotonically increasing* or just *monotone* if $f(a) \preceq f(b)$ for any $a, b \in X$ with $a \preceq b$;
- (*non-strictly*) *monotonically decreasing* or *antimonotone* if $f(b) \preceq f(a)$ for any $a, b \in X$ with $a \preceq b$;
- *strictly monotonically increasing* or just *strictly monotone* if $f(a) \prec f(b)$ for any $a, b \in X$ with $a \prec b$;
- *strictly monotonically decreasing* or *strictly antimonotone* if $f(b) \prec f(a)$ for any $a, b \in X$ with $a \prec b$.

9.Ax. *Let X and Y be linearly ordered sets. With respect to the interval topology in X and Y any surjective strictly monotone or antimonotone map $X \to Y$ is continuous.*

9.1x. Show that the surjectivity condition in *9.Ax* is needed.

9.2x. Is it possible to remove the word *strictly* from the hypothesis of Theorem 9.Ax?

9.3x. Under conditions of Theorem 9.Ax, is f continuous with respect to the right-ray or left-ray topologies?

9.Bx. *A map of a poset to a poset is monotone iff it is continuous with respect to the poset topologies.*

9°14x. Gromov–Hausdorff Distance

9.Cx. For any metric spaces X and Y, there exists a metric space Z such that X and Y can be isometrically embedded into Z.

Having isometrically embedded two metric space in a single one, we can consider the Hausdorff distance between their images (see. 4°15x). The infimum of such Hausdorff distances over all pairs of isometric embeddings of metric spaces X and Y into metric spaces is the *Gromov–Hausdorff distance* between X and Y.

9.Dx. Does there exist metric spaces with infinite Gromov–Hausdorff distance?

9.Ex. Prove that the Gromov–Hausdorff distance is symmetric and satisfies the triangle inequality.

9.Fx. *Riddle.* In what sense the Gromov–Hausdorff distance can satisfy the first axiom of metric?

9°15x. Functions on the Cantor Set and Square-Filling Curves

Recall that the Cantor set K is the set of real numbers that can be presented as sums of series of the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n \in \{0, 2\}$.

9.Gx. Consider the map

$$\gamma_1 : K \to [0, 1] : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

Prove that it is a continuous surjection. Sketch the graph of γ_1.

9.Hx. Prove that the function

$$K \to K : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \sum_{n=1}^{\infty} \frac{a_{2n}}{3^n}$$

is continuous.

Denote by K^2 the set $\{(x, y) \in \mathbb{R}^2 \mid x \in K, y \in K\}$.

9.Ix. Prove that the map

$$\gamma_2 : K \to K^2 : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \left(\sum_{n=1}^{\infty} \frac{a_{2n-1}}{3^n}, \sum_{n=1}^{\infty} \frac{a_{2n}}{3^n} \right)$$

is a continuous surjection.

The unit segment $[0,1]$ is denoted by I, the set

$$\{(x_1, \dots, x_n) \subset \mathbb{R}^n \mid 0 \le x_i \le 1 \text{ for each } i\}$$

is denoted by I^n and called the (unit) *n-cube*.

9.Jx. Prove that the map $\gamma_3 : K \to I^2$ defined as the composition of $\gamma_2 : K \to K^2$ and $K^2 \to I^2 : (x, y) \mapsto (\gamma_1(x), \gamma_1(y))$ is a continuous surjection.

9.Kx. Prove that the map $\gamma_3 : K \to I^2$ is a restriction of a continuous map. (Cf. *2.Bx.2*.)

The latter map is a continuous surjection $I \to I^2$. Thus, this is a curve filling the square. A curve with this property was first constructed by G. Peano in 1890. Though the construction sketched above involves the same ideas as the original Peano's construction, the two constructions are slightly different. Since then a lot of other similar examples have been found. You may find a nice survey of them in Hans Sagan's book Space-Filling Curves, Springer-Verlag 1994. Here is a sketch of Hilbert's construction.

9.Lx. Prove that there exists a sequence of polygonal maps $f_n : I \to I^2$ such that

(1) f_n connects all centers of the squares forming the obvious subdivision of I^2 into 4^n equal squares with side $1/2^n$;

(2) $\text{dist}(f_n(x), f_{n-1}(x)) \le \sqrt{2}/2^{n+1}$ for any $x \in I$ (here dist denotes the metric induced on I^2 from the standard Euclidean metric of R^2).

9.Mx. Prove that any sequence of paths $f_n : I \to I^2$ satisfying the conditions of *9.Lx* converges to a map $f : I \to I^2$ (i.e., for any $x \in I$ there exists a limit $f(x) = \lim_{n\to\infty} f_n(x)$), this map is continuous, and its image is dense in I^2.

9.Nx.[2] Prove that any continuous map $I \to I^2$ with dense image is surjective.

9.Ox. Generalize *9.Ix – 9.Kx*, *9.Lx – 9.Nx* to obtain a continuous surjection of I onto I^n.

[2] Although this problem can be solved by using theorems that are well known from Calculus, we have to mention that it would be more appropriate to solve it after Section 16. Cf. Problems *16.P*, *16.U*, and *16.K*.

10. Homeomorphisms

10°1. Definition and Main Properties of Homeomorphisms

An invertible map is a *homeomorphism* if both this map and its inverse are continuous.

10.A. Find an example of a continuous bijection which is not a homeomorphism.

10.B. Find a continuous bijection $[0,1) \to S^1$ which is not a homeomorphism.

10.C. The identity map of a topological space is a homeomorphism.

10.D. *A composition of homeomorphisms is a homeomorphism.*

10.E. *The inverse of a homeomorphism is a homeomorphism.*

10°2. Homeomorphic Spaces

A topological space X is *homeomorphic* to a space Y if there exists a homeomorphism $X \to Y$.

10.F. *Being homeomorphic is an equivalence relation.*

10.1. Riddle. How is Theorem *10.F* related to *10.C–10.E*?

10°3. Role of Homeomorphisms

10.G. Let $f : X \to Y$ be a homeomorphism. Then $U \subset X$ is open (in X) iff $f(U)$ is open (in Y).

10.H. $f : X \to Y$ is a homeomorphism iff f is a bijection and determines a bijection between the topological structures of X and Y.

10.I. Let $f : X \to Y$ be a homeomorphism. Then for every $A \subset X$

(1) A is closed in X iff $f(A)$ is closed in Y;
(2) $f(\operatorname{Cl} A) = \operatorname{Cl}(f(A))$;
(3) $f(\operatorname{Int} A) = \operatorname{Int}(f(A))$;
(4) $f(\operatorname{Fr} A) = \operatorname{Fr}(f(A))$;
(5) A is a neighborhood of a point $x \in X$ iff $f(A)$ is a neighborhood of the point $f(x)$;
(6) etc.

Therefore, from the topological point of view, homeomorphic spaces are completely identical: a homeomorphism $X \to Y$ establishes a one-to-one correspondence between all phenomena in X and Y that can be expressed in terms of topological structures. [3]

10°4. More Examples of Homeomorphisms

10.J. Let $f : X \to Y$ be a homeomorphism. Prove that for every $A \subset X$ the submap $\mathrm{ab}(f) : A \to f(A)$ is also a homeomorphism.

10.K. Prove that every isometry (see Section 9) is a homeomorphism.

10.L. Prove that every nondegenerate affine transformation of $\mathbb{R}^n$ is a homeomorphism.

10.M. *Let X and Y be two linearly ordered sets. Any strictly monotone surjection $f : X \to Y$ is a homeomorphism with respect to the interval topological structures in X and Y.*

10.N Corollary. *Any strictly monotone surjection $f : [a, b] \to [c, d]$ is a homeomorphism.*

10.2. Let R be a positive real. Prove that the inversion

$$\tau : \mathbb{R}^n \smallsetminus 0 \to \mathbb{R}^n \smallsetminus 0 : x \mapsto \frac{Rx}{|x|^2}$$

is a homeomorphism.

10.3. Let $\mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$ be the upper half-plane, let $a, b, c, d \in \mathbb{R}$, and let $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$. Prove that

$$f : \mathcal{H} \to \mathcal{H} : z \mapsto \frac{az + b}{cz + d}$$

is a homeomorphism.

10.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a bijection. Prove that f is a homeomorphism iff f is a monotone function.

10.5. 1) Prove that every bijection of an indiscrete space onto itself is a homeomorphism. Prove the same 2) for a discrete space and 3) $\mathbb{R}_{T_1}$.

10.6. Find all homeomorphisms of the space  (see Section 2) to itself.

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[3]This phenomenon was used as a basis for a definition of the subject of topology in the first stages of its development, when the notion of topological space had not been developed yet. Then mathematicians studied only subspaces of Euclidean spaces, their continuous maps, and homeomorphisms. Felix Klein in his famous Erlangen Program classified various geometries that had emerged up to that time, like Euclidean, Lobachevsky, affine, and projective geometries, and defined topology as a part of geometry that deals with properties preserved by homeomorphisms. In fact, it was not assumed to be a program in the sense of being planned, although it became a kind of program. It was a sort of dissertation presented by Klein for getting a professor position at the Erlangen University.

10.7. Prove that every continuous bijection of the arrow onto itself is a homeomorphism.

10.8. Find two homeomorphic spaces X and Y and a continuous bijection $X \to Y$ which is not a homeomorphism.

10.9. Is $\gamma_2 : K \to K^2$ considered in Problem *9.Ix* a homeomorphism? Recall that K is the Cantor set, $K^2 = \{(x, y) \in \mathbb{R}^2 \mid x \in K, y \in K\}$ and γ_2 is defined by

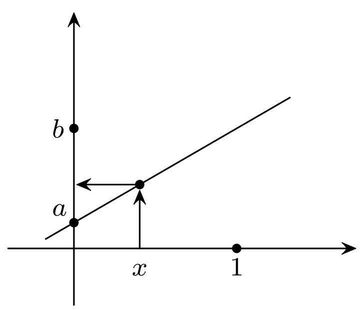
$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{3^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k} \right)$$

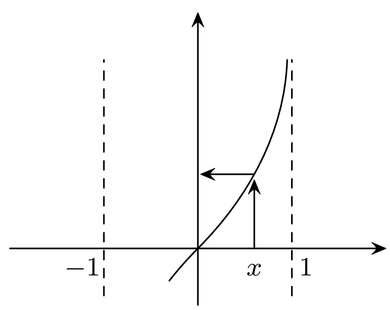
10°5. Examples of Homeomorphic Spaces

Below the homeomorphism relation is denoted by $\cong$. This notation it is not commonly accepted. In other textbooks, any sign close to, but distinct from $=$, e.g., $\sim$, $\simeq$, $\approx$, is used.

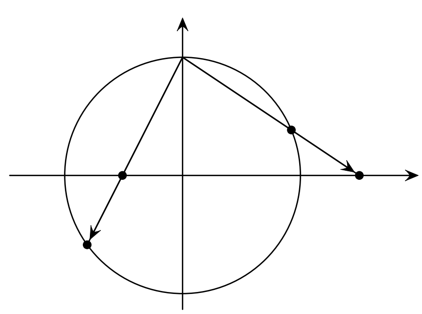
10.O. Prove that

(1) $[0, 1] \cong [a, b]$ for any $a < b$;
(2) $[0, 1) \cong [a, b) \cong (0, 1] \cong (a, b]$ for any $a < b$;
(3) $(0, 1) \cong (a, b)$ for any $a < b$;
(4) $(-1, 1) \cong \mathbb{R}$;
(5) $[0, 1) \cong [0, +\infty)$ and $(0, 1) \cong (0, +\infty)$.





10.P. Let $N = (0, 1) \in S^1$ be the North Pole of the unit circle. Prove that $S^1 \smallsetminus N \cong \mathbb{R}^1$.



10.Q. The graph of a continuous real-valued function defined on an interval is homeomorphic to the interval.

10.R. $S^n \setminus \text{point} \cong \mathbb{R}^n$. (The first space is the "punctured sphere".)

10.10. Prove that the following plane domains are homeomorphic. (Here and below, our notation is sometimes slightly incorrect: in the curly brackets, we drop the initial part "$(x, y) \in \mathbb{R}^2 \mid$".)

(1) The whole plane $\mathbb{R}^2$;
(2) open square $\operatorname{Int} I^2 = \{ x, y \in (0, 1) \}$;
(3) open strip $\{ x \in (0, 1) \}$;
(4) open half-plane $\mathcal{H} = \{ y > 0 \}$;
(5) open half-strip $\{ x > 0,\ y \in (0, 1) \}$;
(6) open disk $B^2 = \{ x^2 + y^2 < 1 \}$;
(7) open rectangle $\{ a < x < b,\ c < y < d \}$;
(8) open quadrant $\{ x, y > 0 \}$;
(9) open angle $\{ x > y > 0 \}$;
(10) $\{ y^2 + |x| > x \}$, i.e., plane without the ray $\{ y = 0 \le x \}$;
(11) open half-disk $\{ x^2 + y^2 < 1,\ y > 0 \}$;
(12) open sector $\{ x^2 + y^2 < 1,\ x > y > 0 \}$.

10.S. Prove that

(1) the closed disk D^2 is homeomorphic to the square $I^2 = \{ (x, y) \in \mathbb{R}^2 \mid x, y \in [0, 1] \}$;
(2) the open disk $B^2 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$ is homeomorphic to the open square $\operatorname{Int} I^2 = \{ (x, y) \in \mathbb{R}^2 \mid x, y \in (0, 1) \}$;
(3) the circle S^1 is homeomorphic to the boundary $\partial I^2 = I^2 \setminus \operatorname{Int} I^2$ of the square.

10.T. Let $\Delta \subset \mathbb{R}^2$ be a planar bounded closed convex set with nonempty interior U. Prove that

(1) Δ is homeomorphic to the closed disk D^2;
(2) U is homeomorphic to the open disk B^2;
(3) $\operatorname{Fr} \Delta = \operatorname{Fr} U$ is homeomorphic to S^1.

10.11. In which of the assertions in *10.T* can we omit the assumption that the closed convex set Δ be bounded?

10.12. Classify up to homeomorphism all (nonempty) closed convex sets in the plane. (Make a list without repeats; prove that every such a set is homeomorphic to one in the list; postpone a proof of nonexistence of homeomorphisms till Section 11.)

10.13*. Generalize the previous three problems to the case of sets in $\mathbb{R}^n$ with arbitrary n.

The latter four problems show that angles are not essential in topology, i.e., for a line or the boundary of a domain the property of having angles is not preserved by homeomorphism. Here are two more problems on this.

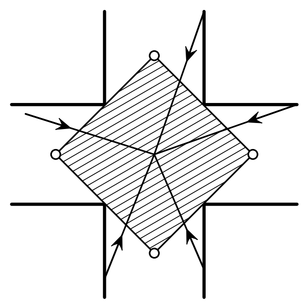
10.14. Prove that every simple (i.e., without self-intersections) closed polygon in $\mathbb{R}^2$ (as well as in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the circle S^1.

10.15. Prove that every nonclosed simple finite unit polyline in $\mathbb{R}^2$ (as well as in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the segment $[0, 1]$.

The following problem generalizes the technique used in the previous two problems and is used more often than it may seem at first glance.

10.16. Let X and Y be two topological spaces equipped with fundamental covers: $X = \bigcup_\alpha X_\alpha$ and $Y = \bigcup_\alpha Y_\alpha$. Suppose $f : X \to Y$ is a map such that $f(X_\alpha) = Y_\alpha$ for each α and the submap $\mathrm{ab}(f) : X_\alpha \to Y_\alpha$ is a homeomorphism. Then f is a homeomorphism.

10.17. Prove that $\mathbb{R}^2 \smallsetminus \{\, |x|, |y| > 1 \,\} \cong I^2 \smallsetminus \{x, y \in \{0, 1\}\}$. (An "infinite cross" is homeomorphic to a square without vertices.)



10.18*. A nonempty set $\Sigma \subset \mathbb{R}^2$ is "star-shaped with respect to a point c" if Σ is a union of segments and rays with an endpoint at c. Prove that if Σ is open, then $\Sigma \cong B^2$. (What can you say about a closed star-shaped set with nonempty interior?)

10.19. Prove that the following plane figures are homeomorphic to each other. (See *10.10* for our agreement about notation.)

(1) A half-plane: $\{\, x \geq 0 \,\}$;
(2) a quadrant: $\{\, x, y \geq 0 \,\}$;
(3) an angle: $\{\, x \geq y \geq 0 \,\}$;
(4) a semi-open strip: $\{\, y \in [0, 1) \,\}$;
(5) a square without three sides: $\{\, 0 < x < 1,\ 0 \leq y < 1 \,\}$;
(6) a square without two sides: $\{\, 0 \leq x,\ y < 1 \,\}$;
(7) a square without a side: $\{\, 0 \leq x \leq 1,\ 0 \leq y < 1 \,\}$;
(8) a square without a vertex: $\{\, 0 \leq x,\ y \leq 1 \,\} \smallsetminus (1, 1)$;
(9) a disk without a boundary point: $\{\, x^2 + y^2 \leq 1,\ y \neq 1 \,\}$;
(10) a half-disk without the diameter: $\{\, x^2 + y^2 \leq 1,\ y > 0 \,\}$;
(11) a disk without a radius: $\{\, x^2 + y^2 \leq 1 \,\} \smallsetminus [0, 1]$;
(12) a square without a half of the diagonal: $\{\, |x| + |y| \leq 1 \,\} \smallsetminus [0, 1]$.

10.20. Prove that the following plane domains are homeomorphic to each other:

(1) punctured plane $\mathbb{R}^2 \smallsetminus (0, 0)$;
(2) punctured open disk $B^2 \smallsetminus (0, 0) = \{\, 0 < x^2 + y^2 < 1 \,\}$;
(3) annulus $\{\, a < x^2 + y^2 < b \,\}$, where $0 < a < b$;
(4) plane without a disk: $\mathbb{R}^2 \smallsetminus D^2$;
(5) plane without a square: $\mathbb{R}^2 \smallsetminus I^2$;

(6) plane without a segment: $\mathbb{R}^2 \smallsetminus [0,1]$;
(7) $\mathbb{R}^2 \smallsetminus \Delta$, where Δ is a closed bounded convex set with $\operatorname{Int}\Delta \neq \varnothing$.

10.21. Let $X \subset \mathbb{R}^2$ be an union of several segments with a common endpoint. Prove that the complement $\mathbb{R}^2 \smallsetminus X$ is homeomorphic to the punctured plane.

10.22. Let $X \subset \mathbb{R}^2$ be a simple nonclosed finite polyline. Prove that its complement $\mathbb{R}^2 \smallsetminus X$ is homeomorphic to the punctured plane.

10.23. Let $K = \{a_1, \dots, a_n\} \subset \mathbb{R}^2$ be a finite set. The complement $\mathbb{R}^2 \smallsetminus K$ is a *plane with n punctures*. Prove that any two planes with n punctures are homeomorphic, i.e., the position of $a_1, \dots, a_n$ in $\mathbb{R}^2$ does not affect the topological type of $\mathbb{R}^2 \smallsetminus \{a_1, \dots, a_n\}$.

10.24. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed disks. Prove that the complement of their union is homeomorphic to a plane with n punctures.

10.25. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed disks. The complement of the union of its interiors is said to be *plane with n holes*. Prove that any two planes with n holes are homeomorphic, i.e., the location of disks $D_1, \dots, D_n$ does not affect the topological type of $\mathbb{R}^2 \smallsetminus \cup_{i=1}^n \operatorname{Int} D_i$.

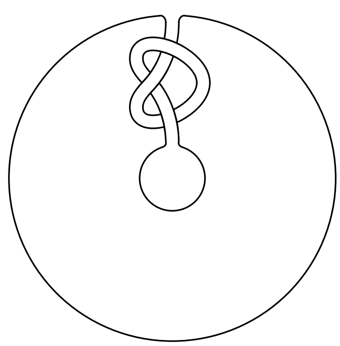
10.26. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions such that $f < g$. Prove that the "strip" $\{(x,y) \in \mathbb{R}^2 \mid f(x) \le y \le g(x)\}$ bounded by their graphs is homeomorphic to the closed strip $\{(x,y) \mid y \in [0,1]\}$.

10.27. Prove that a mug (with a handle) is homeomorphic to a doughnut.

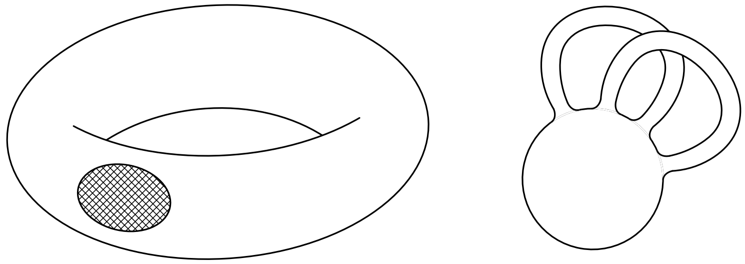
10.28. Arrange the following items to homeomorphism classes: a cup, a saucer, a glass, a spoon, a fork, a knife, a plate, a coin, a nail, a screw, a bolt, a nut, a wedding ring, a drill, a flower pot (with a hole in the bottom), a key.

10.29. In a spherical shell (the space between two concentric spheres), one drilled out a cylindrical hole connecting the boundary spheres. Prove that the rest is homeomorphic to D^3.

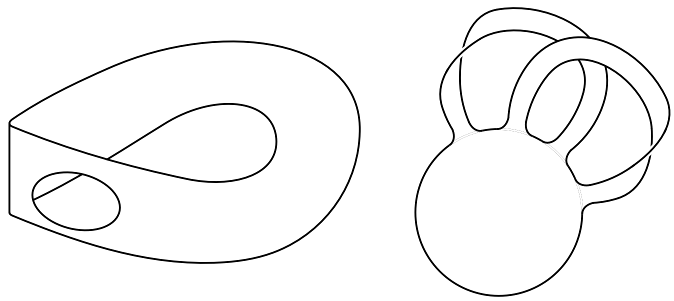
10.30. In a spherical shell, one made a hole connecting the boundary spheres and having the shape of a knotted tube (see Figure).Prove that the rest of the shell is homeomorphic to D^3.



10.31. Prove that surfaces shown in the Figure are homeomorphic (they are called *handles*).



10.32. Prove that surfaces shown in the the Figure are homeomorphic. (They are homeomorphic to a *Klein bottle with two holes*. More details about this is given in Section 21.)



10.33*. Prove that $\mathbb{R}^3 \smallsetminus S^1 \cong \mathbb{R}^3 \smallsetminus \big(\mathbb{R}^1 \cup (0,0,1)\big)$. (What can you say in the case of $\mathbb{R}^n$?)

10.34. Prove that the subset of S^n defined in the standard coordinates in $\mathbb{R}^{n+1}$ by the inequality $x_1^2+x_2^2+\dots+x_k^2 < x_{k+1}^2+\dots+x_n^2$ is homeomorphic to $\mathbb{R}^n \smallsetminus \mathbb{R}^{n-k}$.

10°6. Examples of Nonhomeomorphic Spaces

10.U. Spaces consisting of different number of points are not homeomorphic.

10.V. A discrete space and an indiscrete space (having more than one point) are not homeomorphic.

10.35. Prove that the spaces $\mathbb{Z}$, $\mathbb{Q}$ (with topology induced from $\mathbb{R}$), $\mathbb{R}$, $\mathbb{R}_{T_1}$, and the arrow are pairwise not homeomorphic.

10.36. Find two spaces X and Y that are not homeomorphic, but there exist continuous bijections $X \to Y$ and $Y \to X$.

10°7. Homeomorphism Problem and Topological Properties

One of the classical problems in topology is the *homeomorphism problem*: to find out whether two given topological spaces are homeomorphic. In each special case, the character of solution depends mainly on the answer. In order to prove that two spaces are homeomorphic, it suffices to present a homeomorphism between them. Essentially this is what one usually does in this case (see the examples above). To prove that two spaces are **not** homeomorphic, it does not suffice to consider any special map, and usually it is impossible to review all the maps. Therefore, for proving the nonexistence of a homeomorphism one uses indirect arguments. In particular, we can find a property or a characteristic shared by homeomorphic spaces and such that one of the spaces has it, while the other does not. Properties and characteristics that are shared by homeomorphic spaces are called *topological properties* and *invariants*. Obvious examples are the cardinality (i.e., the number of elements) of the set of points and the set of open sets (cf. Problems *10.34* and *10.U*). Less obvious properties are the main object of the next chapter.

10°8. Information: Nonhomeomorphic Spaces

Euclidean spaces of different dimensions are not homeomorphic. The disks D^p and D^q with $p \neq q$ are not homeomorphic. The spheres S^p, S^q with $p \neq q$ are not homeomorphic. Euclidean spaces are homeomorphic neither to balls, nor to spheres (of any dimension). Letters A and B are not homeomorphic (if the lines are absolutely thin!). The punctured plane $\mathbb{R}^2 \smallsetminus$point is not homeomorphic to the plane with a hole: $\mathbb{R}^2 \smallsetminus \{ x^2+y^2 < 1 \}$.

These statements are of different degrees of difficulty. Some of them will be considered in the next section. However, some of them can not be proved by techniques of this course. (See, e.g., [**6**].)

10°9. Embeddings

A continuous map $f : X \to Y$ is a (*topological*) *embedding* if the submap $\operatorname{ab}(f) : X \to f(X)$ is a homeomorphism.

10.W. The inclusion of a subspace into a space is an embedding.

10.X. Composition of embeddings is an embedding.

10.Y. Give an example of a continuous injection which is not a topological embedding. (Find such an example above and create a new one.)

10.37. Find topological spaces X and Y such that X can be embedded into Y, Y can be embedded into X, but $X \not\cong Y$.

10.38. Prove that $\mathbb{Q}$ cannot be embedded into $\mathbb{Z}$.

10.39. 1) Can a discrete space be embedded into an indiscrete space? 2) How about vice versa?

10.40. Prove that the spaces $\mathbb{R}$, $\mathbb{R}_{T_1}$, and the arrow cannot be embedded into each other.

10.41 Corollary of Inverse Function Theorem. Deduce from the Inverse Function Theorem (see, e.g., any course of advanced calculus) the following statement:

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map whose Jacobian $\det(\partial f_i/\partial x_j)$ does not vanish at the origin $0 \in \mathbb{R}^n$. Then there exists a neighborhood U of the origin such that the restriction $f|_U : U \to \mathbb{R}^n$ is an embedding and $f(U)$ is open.

It is of interest that if $U \subset \mathbb{R}^n$ is an open set, then any continuous injection $f : U \to \mathbb{R}^n$ is an embedding and $f(U)$ is also open in $\mathbb{R}^n$.

10°10. Equivalence of Embeddings

Two embeddings $f_1, f_2 : X \to Y$ are *equivalent* if there exist homeomorphisms $h_X : X \to X$ and $h_Y : Y \to Y$ such that $f_2 \circ h_X = h_Y \circ f_1$. (The latter equality may be stated as follows: the diagram

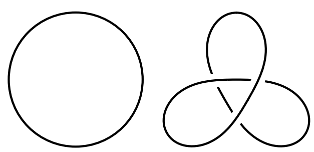
$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ X & \xrightarrow{f_2} & Y \end{array}$$

is commutative.)

An embedding $S^1 \to \mathbb{R}^3$ is called a *knot*.

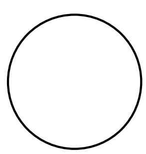
10.42. Prove that knots $f_1, f_2 : S^1 \to \mathbb{R}^3$ with $f_1(S^1) = f_2(S^1)$ are equivalent.

10.43. Prove that knots with images

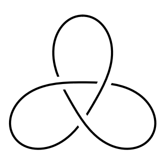


are equivalent.

Information: There are nonequivalent knots. For instance, those with images



and



Proofs and Comments

8.A If $x \in f^{-1}(B)$, then $f(x) \in B$.

8.B $\boxed{\Rightarrow}$ Obvious. $\boxed{\Leftarrow}$ For each $y \in B$, there exists an element x such that $f(x) = y$. By the definition of the preimage, $x \in f^{-1}(B)$, whence $y \in f(f^{-1}(B))$. Thus, $B \subset f(f^{-1}(B))$. The opposite inclusion holds true for any set, see *8.A*.

8.C (a) $\Longrightarrow$ (b) Assume that $f(C) = B$ implies $C = f^{-1}(B)$. If there exist distinct $a_1, a_2 \in f^{-1}(B)$ such that $f(a_1) = f(a_2)$, then also $f(f^{-1}(B) \smallsetminus a_2) = B$, which contradicts the assumption.
(b) $\Longrightarrow$ (a) Assume now that there exists $C \neq f^{-1}(B)$ such that $f(C) = B$. Clearly, $C \subset f^{-1}(B)$. Therefore, C can differ from $f^{-1}(B)$ only if $f^{-1}(B) \smallsetminus C \neq \varnothing$. Take $a_1 \in f^{-1}(B) \smallsetminus C$, let $b = f(a_1)$. Since $f(C) = B$, there exists $a_2 \in C$ with $f(a_2) = f(a_1)$, but $a_2 \neq a_1$ because $a_2 \in C$, while $a_1 \notin C$.

8.D This follows from *8.C*.

8.E Let $x \in A$. Then $f(x) = y \in f(A)$, whence $x \in f^{-1}(f(A))$.

8.F Both equalities are obviously equivalent to the following statement: $f(x) \notin f(A)$ for each $x \notin A$.

8.G $\mathrm{in}^{-1}(B) = \{x \in A \mid x \in B\} = A \cap B$.

8.H Let $x \in X$. Then

$$h \circ (g \circ f)(x) = h(g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = (h \circ g) \circ f(x).$$

8.J Let $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$, because f is injective, and $g(f(x_1)) \neq g(f(x_2))$, because g is injective.

8.K If f is not injective, then there exist $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. However, then $(g \circ f)(x_1) = (g \circ f)(x_2)$, which contradicts the injectivity of $g \circ f$.

8.L Let $f : X \to Y$ and $g : Y \to Z$ be surjective. Then we have $f(X) = Y$, whence $g(f(X)) = g(Y) = Z$.

8.M This follows from the obvious inclusion $\mathrm{Im}(g \circ f) \subset \mathrm{Im}\, g$.

8.N This follows from *8.J* and *8.L*.

8.O $\boxed{\Rightarrow}$ Use *8.K* and *8.M*. $\boxed{\Leftarrow}$ Let $f : X \to Y$ be a bijection. Then, by the surjectivity, for each $y \in Y$ there exists $x \in X$ such that $y = f(x)$, and, by the injectivity, such an element of X is unique. Putting $g(y) = x$, we obtain a map $g : Y \to X$. It is easy to check (please, do it!) that g is inverse to f.

8.P This is actually obvious. On the other hand, it is interesting to look at "mechanical" proof.Let two maps $g, h : Y \to X$ be inverse to a map $f : X \to Y$. Consider the composition $g \circ f \circ h : Y \to X$. On the one hand, $g \circ f \circ h = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$. On the other hand, $g \circ f \circ h = g \circ (f \circ h) = g \circ \mathrm{id}_Y = g$.

9.A Let $f : X \to Y$ be a map. $\boxed{\Longrightarrow}$ If $f : X \to Y$ is continuous, then, for each closed set $F \subset Y$, the set $X \smallsetminus f^{-1}(F) = f^{-1}(Y \smallsetminus F)$ is open, and therefore $f^{-1}(F)$ is closed. $\boxed{\Longleftarrow}$ Exchange the words *open* and *closed* in the above argument.

9.C If a set U is open in X, then its preimage $\mathrm{in}^{-1}(U) = U \cap A$ is open in A by the definition of the relative topology.

9.D If $U \in \Omega_A$, then $U = V \cap A$ for some $V \in \Omega$. If the map $\mathrm{in} : (A, \Omega') \to (X, \Omega)$ is continuous, then the preimage $U = \mathrm{in}^{-1}(V) = V \cap A$ of a set $V \in \Omega$ belongs to Ω'. Thus, $\Omega_A \subset \Omega'$.

9.E Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. We must show that for every $U \subset Z$ which is open in Z its preimage $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. The set $g^{-1}(U)$ is open in Y by continuity of g. In turn, its preimage $f^{-1}(g^{-1}(U))$ is open in X by the continuity of f.

9.F $(f|_{A,B})^{-1}(V) = (f|_{A,B})^{-1}(U \cap B) = A \cap f^{-1}(U)$.

9.G $\boxed{\Longrightarrow}$ Use *9.F*. $\boxed{\Longleftarrow}$ Use the fact that $f = \mathrm{in}_{f(X)} \circ \mathrm{ab}\, f$.

9.H The preimage of any set under a continuous map either is empty or coincides with the whole space.

9.I $\boxed{\Longrightarrow}$ Let $a \in X$. Then for any neighborhood U of $f(a)$ we can construct a desired neighborhood V of a just by putting $V = f^{-1}(U)$: indeed, $f(V) = f(f^{-1}(U)) \subset U$. $\boxed{\Longleftarrow}$ We must check that the preimage of each open set is open. Let $U \subset Y$ be an open set in Y. Take $a \in f^{-1}(U)$. By continuity of f at a, there exists a neighborhood V of a such that $f(V) \subset U$. Then, obviously, $V \subset f^{-1}(U)$. This proves that any point of $f^{-1}(U)$ is internal, and hence $f^{-1}(U)$ is open.

9.J Proving each of the implications, use Theorem *4.I*, according to which any neighborhood of a point in a metric space contains a ball centered at the point.

9.K The condition "for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$" means that $f(B_\delta(a)) \subset B_{Ge}(f(a))$. Now, apply *9.J*.

9.L This immediately follows from the inequality of Problem *4.35*.

9.M If $f(x) = f(y)$, then $\rho(f(x), f(y)) = 0$, whence $\rho(x, y) = 0$.

9.N Use the obvious fact that the primage of any open ball under isometric embedding is an open ball of the same radius.

9.O The set of solutions of the system is the intersection of the preimages of the point $0 \in \mathbb{R}$. As the maps are continuos and the point is closed, the preimages of the point are closed, and hence the intersection of the preimages is closed.

9.P The set of solutions of a system of nonstrict inequalities is the intersection of preimages of closed ray $[0, +\infty)$, the set of solutions of a system of strict inequalities is the intersection of the preimages of open ray $(0, +\infty)$.

9.Q Indeed, it makes no sense to require the necessity: the intersection of an open set with any set A is open in A anyway.

9.R Consider the complement $X \setminus F$ of F and apply *9.Q*.

9.S Let Γ be an open cover of a space X. Let $U \subset X$ be a set such that $U \cap A$ is open in A for any $A \in \Gamma$. By *5.E*, open subset of open subspace is open in the whole space. Therefore, $A \cap U$ is open in X. Then $U = \bigcup_{A \in \Gamma} A \cap U$ is open as a union of open sets.

9.T Argue as in the preceding proof, but instead of the definition of a fundamental cover use its reformulation *9.R*, and instead of Theorem *5.E* use Theorem *5.F*, according to which a closed set of a closed subspace is closed in the entire space.

9.U Denote the space by X and the cover by Γ. As Γ is locally finite, each point $a \in X$ has a neighborhood U_a meeting only a finite number of elements of Γ. Form the cover $\Sigma = \{U_a \mid a \in X\}$ of X. Let $U \subset X$ be a set such that $U \cap A$ is open for each $A \in \Gamma$. By *9.T*, $\{A \cap U_a \mid A \in \Gamma\}$ is a fundamental cover of U_a for any $a \in X$. Hence $U_a \cap U$ is open in U_a. By *9.S*, Σ is fundamental. Hence, U is open.

9.V Let U be a set open in Y. As the restrictions of f to elements of Γ are continuous, the preimage of U under restriction of f to any $A \in \Gamma$ is open. Obviously, $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$. Hence $f^{-1}(U) \cap A$ is open in A for any $A \in \Gamma$. By hypothesis, Γ is fundamental. Therefore $f^{-1}(U)$ is open in X. We have proved that the preimage of any open set under f is open. Thus f is continuous.

***9.A*x** It suffices to prove that the preimage of any base open set is open. The proof is quite straight-forward. For instance, the preimage of $\{x \mid a \prec x \prec b\}$ is $\{x \mid c \prec x \prec d\}$, where $f(c) = a$ and $f(d) = b$, which is a base open set.

***9.B*x** Let X and Y be two posets, $f : X \to Y$ a map. ⟹ Assume that $f : X \to Y$ is monotone. To prove the continuity of f it suffices to prove that the preimage of each base open set is open. Put $U = C_Y^+(b)$ and $V = f^{-1}(U)$. If $x \in V$ (i.e., $b \prec f(x)$), then for any $y \in C_X^+(x)$ (i.e., $x \prec y$)

we have $y \in V$. Therefore, $V = \bigcup_{f(x) \in U} C_X^+(x)$. This set is open as a union of open base sets (in the poset topology of X).

$\Longleftarrow$ Let $a, b \in X$ and $a \prec b$. Then b is contained in any neighborhood of a. The set $C_Y^+(f(a))$ is a neighborhood of $f(a)$ in Y. By continuity of f, a has a neighborhood in X whose f-image is contained in $C_Y^+(f(a))$. However, then the minimal neighborhood of a in X (i.e., $C_X^+(a)$) also has this property. Therefore, $f(b) \in f(C_X^+(a)) \subset C_Y^+(f(a))$, and hence $f(a) \prec f(b)$.

***9.C*x** Construct Z as the disjoint union of X and Y. In the union, put the distance between two points in (the copy of) X (respectively, Y) to be equal to the distance between the corresponding points in X (respectively, Y). To define the distance between points of different copies, choose points $x_0 \in X$ and $y_0 \in Y$, and put $\rho(a, b) = \rho_X(a, x_0) + \rho_Y(y_0, b) + 1$ for $a \in X$ and $b \in Y$. Check (this is easy, really), that this defines a metric.

***9.D*x** Yes. For example, consider a singleton and any unbounded space.

***9.E*x** Although, as we have seen solving the previous problem, the Gromov–Hausdorff distance can be infinite, while symmetricity and the triangle inequality were formulated above only for functions with finite values, these two properties make sense if infinite values are admitted. (The triangle inequality should be considered fulfilled if two or three of the quantities involved are infinite, and not fulfilled if only one of them is infinite.) The following construction helps to prove the triangle inequality. Let metric spaces X and Y are isometrically embedded into a metric space A, and metric spaces Y and Z are isometrically embedded into a metric space B. Construct a new metric space in which A and B would be isometrically embedded meeting in Y. To do this, add to A all points of $B \smallsetminus A$. Put distances between these points to be equal to the distances between them in B. Put the distance between $x \in A \smallsetminus B$ and $z \in B \smallsetminus A$ equal to $\inf\{\rho_A(x, y) + \rho_B(y, z) \mid y \in A \cap B\}$. Compare this construction with the construction from the solution of Problem *9.C*x. Prove that this gives a metric space and use the triangle inequality for the Hausdorff distance between X, Y, and Z in this space.

***9.F*x** Partially, the answer is obvious. Certainly, the Gromov–Hausdorff distance is nonnegative! But what if it is zero, in what sense the spaces should be equal then? First, the most optimistic idea is that then there should exist an isometric bijection between the spaces. But this is not true, as we can see looking at the spaces $\mathbb{Q}$ and $\mathbb{R}$ with standard distances in them. However, it is true for *compact* metric spaces.

10.A For example, consider the identity map of a discrete topological space X onto the same set but equipped with indiscrete topology. For another example, see *10.B*.

10.B Consider the map $x \mapsto (\cos 2\pi x, \sin 2\pi x)$.

10.C This and the next two statements directly follow from the definition of a homeomorphism.

10.F See the solution of *10.1*.

10.G Denote $f(U) \subset Y$ by V. Since f is a bijection, we have $U = f^{-1}(V)$. We also denote $f^{-1} : Y \to X$ by g. $\boxed{\Rightarrow}$ We have $V = g^{-1}(U)$, which is open by continuity of g. $\boxed{\Leftarrow}$ If $V = f(U)$ is open, then $U = g(V)$ is open as the preimage of an open set under a continuous map.

10.H See *10.G*.

10.I (a) A homeomorphism establishes a one-to-one correspondence between open sets of X and Y. Hence, it also establishes a one-to-one correspondence between closed sets of X and Y.
(b)–(f) Use the fact that the definitions of the closure, interior, boundary, etc. can be given in terms of open and closed sets.

10.J Obviously, $\operatorname{ab}(f)$ is a bijection. The continuity of $\operatorname{ab}(f)$ and $(\operatorname{ab} f)^{-1}$ follows from the general theorem *9.F* on the continuity of a submap of a continuous map.

10.K Any isometry is continuous, see *9.N* ; the map inverse to an isometry is an isometry.

10.L Recall that an affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by the formula $y = f(x) = Ax + b$, where A is a matrix and b a vector; f is nondegenerate if A is invertible, whence $x = A^{-1}(y - b) = A^{-1}(y) - A^{-1}(b)$, which means that f is a bijection and f^{-1} is also a nondegenerate affine transformation. Finally, f and f^{-1} are continuous, e.g., because they are given in coordinates by linear formulas (see *9.18* and *9.21*).

10.M Prove that f is invertible and f^{-1} is also strictly monotone. Then apply *9.Ax*.

10.O Homeomorphisms of the form $\langle 0, 1\rangle \to \langle a, b\rangle$ are defined, for example, by the formula $x \mapsto a + (b - a)x$, and homeomorphisms $(-1; 1) \to \mathbb{R}^1$ and $\langle 0, 1) \to \langle 0, +\infty)$ by the formula $x \mapsto \tan(\pi x/2)$. (In the latter case, you can easily find, e.g., a rational formula, but it is of interest that the above homeomorphism also arises quite often!)

10.P Observe that $(1/4, 5/4) \to S^1 \smallsetminus N : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a homeomorphism and use assertions (c) and (d) of the preceding problem. Here is another, more sophisticated construction, which can be of use in higher dimensions. The restriction f of the central projection $\mathbb{R}^2 \smallsetminus N \to \mathbb{R}^1$

(the x axis) to $S^1 \smallsetminus N$ is a homeomorphism. Indeed, f is obviously invertible: f^{-1} is a restriction of the central projection $\mathbb{R}^2 \smallsetminus N \to S^1 \smallsetminus N$. The map $S^1 \smallsetminus N \to \mathbb{R}$ is presented by formula $(x, y) \mapsto \frac{x}{1-y}$, and the inverse map by formula $x \mapsto (\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1})$. (Why are these maps continuous?)

10.Q Check that the vertical projection to the x axis determines a homeomorphism.

10.R As usual, we identify $\mathbb{R}^n$ and $\{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$. Then the restriction of the central projection

$$\mathbb{R}^{n+1} \smallsetminus (0, \dots, 0, 1) \to \mathbb{R}^n$$

to $S^n \smallsetminus (0, \dots, 0, 1)$ is a homeomorphism, which is called the *stereographic projection*. For $n = 2$, it is used in cartography. It is invertible: the inverse map is the restriction of the central projection $\mathbb{R}^{n+1} \smallsetminus (0, \dots, 0, 1) \to S^n \smallsetminus (0, \dots, 0, 1)$ to $\mathbb{R}^n$. The first map is defined by formula

$$x = (x_1, \dots, x_{n+1}) \mapsto \left(\frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right),$$

and the second one by

$$x = (x_1, \dots, x_n) \mapsto \left(\frac{2x_1}{|x|^2 + 1}, \dots, \frac{2x_n}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1} \right).$$

Check this. (Why are these maps continuous?) Explain how we can obtain a solution of this problem geometrically from the second solution to Problem *10.P*.

10.S After reading the proof, you may see that sometimes formulas are cumbersome, while a clearer verbal description is possible.
(a) Instead of I^2 it is convenient to consider the homeomorphic square $K = \{(x, y) \mid |x| \le 1, |y| \le 1\}$ of double size centered at the origin. (There is a linear homeomorphism $I^2 \to K : (x, y) \mapsto (2x - 1, 2y - 1)$.) We have a homeomorphism

$$K \to D^2 : (x, y) \mapsto \left(\frac{x \max\{|x|, |y|\}}{\sqrt{x^2 + y^2}}, \frac{y \max\{|x|, |y|\}}{\sqrt{x^2 + y^2}} \right).$$

Geometrically, this means that each segment joining the origin with a point on the contour of the square is linearly mapped to the part of the segment that lies within the circle.
(b), (c) Take suitable submaps of the above homeomorphism $K \to D^2$. Certainly, assertion (b) follows from the previous problem. It is also of

interest that in case (c) we can use a much simpler formula:

$$\partial K \to S^1 : (x, y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

(This is simply a central projection!) We can also divide the circle into four arcs and map each of them to a side of K, cf. below.

10.T (a) For simplicity, assume that $D^2 \subset \Delta$. For $x \in \mathbb{R}^2 \setminus 0$, let $a(x)$ be the (unique) positive number such that $a(x)\frac{x}{|x|} \in \operatorname{Fr} \Delta$. Then we have a homeomorphism

$$\Delta \to D^2 : x \mapsto \frac{x}{a(x)} \quad \text{if } x \neq 0, \quad \text{while} \quad 0 \mapsto 0.$$

(Observe that in the case where Δ is the square K, we obtain the homeomorphism described in the preceding problem.)
(b), (c) Take suitable submaps of the above homeomorphism $\Delta \to D^2$.

10.U There is no bijection between them.

10.V These spaces have different numbers of open sets.

10.W Indeed, if $\mathrm{in} : A \to X$ is an inclusion, then the submap $\mathrm{ab}(\mathrm{in}) : A \to A$ is the identity homeomorphism.

10.X Let $f : X \to Y$ and $g : Y \to Z$ be two embeddings. Then the submap $\mathrm{ab}(g \circ f) : X \to g(f(X))$ is the composition of the homeomorphisms $\mathrm{ab}(f) : X \to f(X)$ and $\mathrm{ab}(g) : f(X) \to g(f(X))$.

10.Y The previous examples are $[0, 1) \to S^1$ and $\mathbb{Z}_+ \to \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. Here is another one: Let $f : \mathbb{Z} \to \mathbb{Q}$ be a bijection and $\mathrm{in}_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$ the inclusion. Then the composition $\mathrm{in}_{\mathbb{Q}} \circ f : \mathbb{Z} \to \mathbb{R}$ is a continuous injection, but not an embedding.