

Topological Properties

11. Connectedness

11°1. Definitions of Connectedness and First Examples

A topological space X is *connected* if X has only two subsets that are both open and closed: the empty set \emptyset and the entire X . Otherwise, X is *disconnected*.

A *partition* of a set is a cover of this set with pairwise disjoint subsets. To *partition* a set means to construct such a cover.

11.A. *A topological space is connected, iff it has no partition into two nonempty open sets, iff it has no partition into two nonempty closed sets.*

11.1. 1) Is an indiscrete space connected? The same question for 2) the arrow and 3) $\mathbb{R}_{\mathcal{T}_1}$.

11.2. Describe explicitly all connected discrete spaces.

11.3. Describe explicitly all disconnected two-point spaces.

11.4. 1) Is the set \mathbb{Q} of rational numbers (with the relative topology induced from \mathbb{R}) connected? 2) The same question for the set of irrational numbers.

11.5. Let Ω_1 and Ω_2 be two topologies in a set X , and let Ω_2 be finer than Ω_1 (i.e., $\Omega_1 \subset \Omega_2$). 1) If (X, Ω_1) is connected, is (X, Ω_2) connected? 2) If (X, Ω_2) is connected, is (X, Ω_1) connected?

11°2. Connected Sets

When we say that a set A is connected, this means that A lies in some topological space (which should be clear from the context) and, equipped with the relative topology, A a connected space.

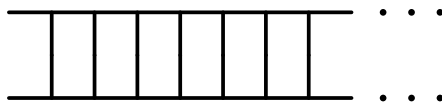
- 11.6.** Characterize disconnected subsets without mentioning the relative topology.
- 11.7.** Is the set $\{0, 1\}$ connected 1) in \mathbb{R} , 2) in the arrow, 3) in \mathbb{R}_{T_1} ?
- 11.8.** Describe explicitly all connected subsets 1) of the arrow, 2) of \mathbb{R}_{T_1} .
- 11.9.** Show that the set $[0, 1] \cup (2, 3]$ is disconnected in \mathbb{R} .
- 11.10.** Prove that every nonconvex subset of the real line is disconnected. (In other words, each connected subset of the real line is a singleton or an interval.)
- 11.11.** Let A be a subset of a space X . Prove that A is disconnected iff A has two nonempty subsets B and C such that $A = B \cup C$, $B \cap \text{Cl}_X C = \emptyset$, and $C \cap \text{Cl}_X B = \emptyset$.
- 11.12.** Find a space X and a disconnected subset $A \subset X$ such that if U and V are any two open sets partitioning X , then we have either $U \supset A$, or $V \supset A$.
- 11.13.** Prove that for every disconnected set A in \mathbb{R}^n there are disjoint open sets $U, V \subset \mathbb{R}^n$ such that $A \subset U \cup V$, $U \cap A \neq \emptyset$, and $V \cap A \neq \emptyset$.

Compare 11.11–11.13 with 11.6.

11°3. Properties of Connected Sets

- 11.14.** Let X be a space. If a set $M \subset X$ is connected and $A \subset X$ is open-closed, then either $M \subset A$, or $M \subset X \setminus A$.
- 11.B.** *The closure of a connected set is connected.*
- 11.15.** Prove that if a set A is connected and $A \subset B \subset \text{Cl} A$, then B is connected.
- 11.C.** Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subsets of a space X . Assume that any two sets in this family intersect. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected. (In other words: the union of pairwise intersecting connected sets is connected.)
- 11.D Special case.** If $A, B \subset X$ are two connected sets with $A \cap B \neq \emptyset$, then $A \cup B$ is also connected.
- 11.E.** Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subsets of a space X . Assume that each set in this family intersects A_{λ_0} for some $\lambda_0 \in \Lambda$. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.
- 11.F.** Let $\{A_k\}_{k \in \mathbb{Z}}$ be a family of connected sets such that $A_k \cap A_{k+1} \neq \emptyset$ for any $k \in \mathbb{Z}$. Prove that $\bigcup_{k \in \mathbb{Z}} A_k$ is connected.
- 11.16.** Let A and B be two connected sets such that $A \cap \text{Cl} B \neq \emptyset$. Prove that $A \cup B$ is also connected.
- 11.17.** Let A be a connected subset of a connected space X , and let $B \subset X \setminus A$ be an open-closed set in the relative topology of $X \setminus A$. Prove that $A \cup B$ is connected.
- 11.18.** Does the connectedness of $A \cup B$ and $A \cap B$ imply that of A and B ?

11.19. Let A and B be two sets such that both their union and intersection are connected. Prove that A and B are connected if both of them are 1) open or 2) closed.



11.20. Let $A_1 \supset A_2 \supset \dots$ be an infinite decreasing sequence of closed connected sets in the plane \mathbb{R}^2 . Is $\bigcap_{k=1}^{\infty} A_k$ a connected set?

11°4. Connected Components

A *connected component* of a space X is a maximal connected subset of X , i.e., a connected subset that is not contained in any other (strictly) larger connected subset of X .

11.G. Every point belongs to some connected component. Furthermore, this component is unique. It is the union of all connected sets containing this point.

11.H. Two connected components either are disjoint or coincide.

A connected component of a space X is also called just a *component* of X . Theorems 11.G and 11.H mean that connected components constitute a partition of the whole space. The next theorem describes the corresponding equivalence relation.

11.I. Prove that two points lie in the same component iff they belong to the same connected set.

11.J Corollary. A space is connected iff any two of its points belong to the same connected set.

11.K. Connected components are closed.

11.21. If each point of a space X has a connected neighborhood, then each connected component of X is open.

11.22. Let x and y belong to the same component. Prove that any open-closed set contains either both x and y , or none of them (cf. 11.36).

11°5. Totally Disconnected Spaces

A topological space is *totally disconnected* if all of its components are singletons.

11.L Obvious Example. Any discrete space is totally disconnected.

11.M. The space \mathbb{Q} (with the topology induced from \mathbb{R}) is totally disconnected.

Note that \mathbb{Q} is not discrete.

11.23. Give an example of an uncountable closed totally disconnected subset of the line.

11.24. Prove that Cantor set (see 2.Bx) is totally disconnected.

11°6. Boundary and Connectedness

11.25. Prove that if A is a proper nonempty subset of a connected space, then $\text{Fr } A \neq \emptyset$.

11.26. Let F be a connected subset of a space X . Prove that if $A \subset X$ and neither $F \cap A$, nor $F \cap (X \setminus A)$ is empty, then $F \cap \text{Fr } A \neq \emptyset$.

11.27. Let A be a subset of a connected space. Prove that if $\text{Fr } A$ is connected, then so is $\text{Cl } A$.

11°7. Connectedness and Continuous Maps

A *continuous image* of a space is its image under a continuous map.

11.N. A continuous image of a connected space is connected. (In other words, if $f : X \rightarrow Y$ is a continuous map and X is connected, then $f(X)$ is also connected.)

11.O Corollary. Connectedness is a topological property.

11.P Corollary. The number of connected components is a topological invariant.

11.Q. A space X is disconnected iff there is a continuous surjection $X \rightarrow S^0$.

11.28. Theorem 11.Q often yields shorter proofs of various results concerning connected sets. Apply it for proving, e.g., Theorems 11.B–11.F and Problems 11.D and 11.16.

11.29. Let X be a connected space and $f : X \rightarrow \mathbb{R}$ a continuous function. Then $f(X)$ is an interval of \mathbb{R} .

11.30. Suppose a space X has a group structure and the multiplication by any element of the group is a continuous map. Prove that the component of unity is a normal subgroup.

11°8. Connectedness on Line

11.R. The segment $I = [0, 1]$ is connected.

There are several ways to prove Theorem 11.R. One of them is suggested by 11.Q, but refers to a famous Intermediate Value Theorem from calculus, see 12.A. However, when studying topology, it would be more natural to find an independent proof and deduce Intermediate Value Theorem from Theorems 11.R and 11.Q. Two problems below provide a sketch of basically the same proof of 11.R. Cf. 2.Ax below.

11.R.1 Bisection Method. Let U, V be subsets of I with $V = I \setminus U$. Let $a \in U, b \in V$, and $a < b$. Prove that there exists a nondecreasing sequence a_n with $a_1 = a, a_n \in U$, and a nonincreasing sequence b_n with $b_1 = b, b_n \in V$, such that $b_n - a_n = \frac{b-a}{2^{n-1}}$.

11.R.2. Under assumptions of 11.R.1, if U and V are closed in I , then which of them contains $c = \sup\{a_n\} = \inf\{b_n\}$?

11.31. Deduce 11.R from the result of Problem 2.Ax.

11.S. Prove that an open set in \mathbb{R} has countably many connected components.

11.T. Prove that \mathbb{R}^1 is connected.

11.U. Each convex set in \mathbb{R}^n is connected. (In particular, so are \mathbb{R}^n itself, the ball B^n , and the disk D^n .)

11.V Corollary. Intervals in \mathbb{R}^1 are connected.

11.W. Every star-shaped set in \mathbb{R}^n is connected.

11.X Connectedness on Line. A subset of a line is connected iff it is an interval.

11.Y. Describe explicitly all nonempty connected subsets of the real line.

11.Z. Prove that the n -sphere S^n is connected. In particular, the circle S^1 is connected.

11.32. Consider the union of spiral

$$r = \exp\left(\frac{1}{1+\varphi^2}\right), \text{ with } \varphi \geq 0$$

(r, φ are the polar coordinates) and circle S^1 . 1) Is this set connected? 2) Will the answer change if we replace the entire circle by some of its subsets? (Cf. 11.15.)

11.33. Are the following subsets of the plane \mathbb{R}^2 connected:

- (1) the set of points with both coordinates rational;
- (2) the set of points with at least one rational coordinate;
- (3) the set of points whose coordinates are either both irrational, or both rational?

11.34. Prove that for any $\varepsilon > 0$ the ε -neighborhood of a connected subset of Euclidean space is connected.

11.35. Prove that each neighborhood U of a connected subset A of Euclidean space contains a connected neighborhood of A .



11.36. Find a space X and two points belonging to distinct components of X such that each simultaneously open and closed set contains either both points, or neither of them. (Cf. 11.22.)

12. Application of Connectedness

12°1. Intermediate Value Theorem and Its Generalizations

The following theorem is usually included in Calculus. You can easily deduce it from the material of this section. In fact, in a sense it is equivalent to connectedness of the segment.

12.A Intermediate Value Theorem. *A continuous function*

$$f : [a, b] \rightarrow \mathbb{R}$$

takes every value between $f(a)$ and $f(b)$.

Many problems that can be solved by using Intermediate Value Theorem can be found in Calculus textbooks. Here are few of them.

12.1. Prove that any polynomial of odd degree in one variable with real coefficients has at least one real root.

12.B Generalization of 12.A. Let X be a connected space and $f : X \rightarrow \mathbb{R}$ a continuous function. Then $f(X)$ is an interval of \mathbb{R} .

12.C Corollary. Let $J \subset \mathbb{R}$ be an interval of the real line, $f : X \rightarrow \mathbb{R}$ a continuous function. Then $f(J)$ is also an interval of \mathbb{R} . (In other words, continuous functions map intervals to intervals.)

12°2. Applications to Homeomorphism Problem

Connectedness is a topological property, and the number of connected components is a topological invariant (see Section 10).

12.D. $[0, 2]$ and $[0, 1] \cup [2, 3]$ are not homeomorphic.

Simple constructions assigning homeomorphic spaces to homeomorphic ones (e.g., deleting one or several points), allow us to use connectedness for proving that some *connected* spaces are not homeomorphic.

12.E. I , $[0, \infty)$, \mathbb{R}^1 , and S^1 are pairwise nonhomeomorphic.

12.2. Prove that a circle is not homeomorphic to a subspace of \mathbb{R}^1 .

12.3. Give a topological classification of the letters of the alphabet: A, B, C, D, ..., regarded as subsets of the plane (the arcs comprising the letters are assumed to have zero thickness).

12.4. Prove that square and segment are not homeomorphic.

Recall that there exist continuous surjections of the segment onto square, which are called *Peano curves*, see Section 9.

12.F. \mathbb{R}^1 and \mathbb{R}^n are not homeomorphic if $n > 1$.

Information. \mathbb{R}^p and \mathbb{R}^q are not homeomorphic unless $p = q$. This follows, for instance, from the Lebesgue–Brouwer Theorem on the invariance of dimension (see, e.g., W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, NJ, 1941).

12.5. The statement “ \mathbb{R}^p is not homeomorphic to \mathbb{R}^q unless $p = q$ ” implies that S^p is not homeomorphic to S^q unless $p = q$.

12°3x. Induction on Connectedness

A map f is *locally constant* if each point of its source space has a neighborhood U such that the restriction of f to U is constant.

12.1x. Prove that any locally constant map is continuous.

12.2x. A locally constant map on a connected set is constant.

12.3x. Riddle. How are 11.26 and 12.2x related?

12.4x. Let G be a group equipped with a topology such that for any $g \in G$ the map $G \rightarrow G : x \mapsto xgx^{-1}$ is continuous, and let G with this topology be connected. Prove that if the topology induced in a normal subgroup H of G is discrete, then H is contained in the center of G (i.e., $hg = gh$ for any $h \in H$ and $g \in G$).

12.5x Induction on Connectedness. Let \mathcal{E} be a property of subsets of a topological space X such that the union of sets with nonempty pairwise intersections inherits this property from the sets involved. Prove that if X is connected and each point in X has a neighborhood with property \mathcal{E} , then X also has property \mathcal{E} .

12.6x. Prove 12.2x and solve 12.4x using 12.5x.

For more applications of induction on connectedness, see 13.T, 13.4x, 13.6x, and 13.8x.

12°4x. Dividing Pancakes

12.7x. Any irregularly shaped pancake can be cut in half by one stroke of the knife made in any prescribed direction. In other words, if A is a bounded open set in the plane and l is a line in the plane, then there exists a line L parallel to l that divides A in half by area.

12.8x. If, under the assumptions of 12.7x, A is connected, then L is unique.

12.9x. Suppose two irregularly shaped pancakes lie on the same platter; show that it is possible to cut both exactly in half by one stroke of the knife. In other words: if A and B are two bounded regions in the plane, then there exists a line in the plane that halves each region by area.

12.10x. Prove that a plane pancake of any shape can be divided to four pieces of equal area by two straight cuts orthogonal to each other. In other words, if A is a bounded connected open set in the plane, then there are two perpendicular lines that divide A into four parts having equal areas.

12.11x. Riddle. What if the knife is curved and makes cuts of a shape different from the straight line? For what shapes of the cuts can you formulate and solve problems similar to 12.7x–12.10x?

12.12x. Riddle. Formulate and solve counterparts of Problems 12.7x–12.10x for regions in three-space. Can you increase the number of regions in the counterpart of 12.7x and 12.9x?

12.13x. Riddle. What about pancakes in \mathbb{R}^n ?

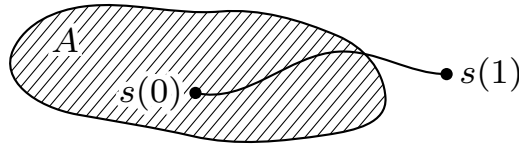
13. Path-Connectedness

13°1. Paths

A *path* in a topological space X is a continuous map of the segment $I = [0, 1]$ to X . The point $s(0)$ is the *initial* point of a path $s : I \rightarrow X$, while $s(1)$ is the *final* point of s . We say that the path s *connects* $s(0)$ with $s(1)$. This terminology is inspired by an image of a moving point: at the moment $t \in [0, 1]$, the point is at $s(t)$. To tell the truth, this is more than what is usually called a path, since besides information on the trajectory of the point it contains a complete account on the movement: the schedule saying when the point goes through each point.

13.1. If $s : I \rightarrow X$ is a path, then the image $s(I) \subset X$ is connected.

13.2. Let $s : I \rightarrow X$ be a path connecting a point in a set $A \subset X$ with a point in $X \setminus A$. Prove that $s(I) \cap \text{Fr}(A) \neq \emptyset$.

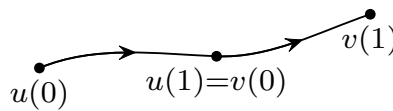


13.3. Let A be a subset of a space X , $\text{in}_A : A \rightarrow X$ the inclusion. Prove that $u : I \rightarrow A$ is a path in A iff the composition $\text{in}_A \circ u : I \rightarrow X$ is a path in X .

A constant map $s_a : I \rightarrow X : x \mapsto a$ is a *stationary* path. For a path s , the *inverse* path is defined by $t \mapsto s(1 - t)$. It is denoted by s^{-1} . Although, strictly speaking, this notation is already used (for the inverse map), the ambiguity of notation usually leads to no confusion: as a rule, inverse maps do not appear in contexts involving paths.

Let $u : I \rightarrow X$ and $v : I \rightarrow X$ be paths such that $u(1) = v(0)$. We define

$$uv : I \rightarrow X : t \mapsto \begin{cases} u(2t) & \text{if } t \in [0, \frac{1}{2}], \\ v(2t - 1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \quad (22)$$



13.A. Prove that the above map $uv : I \rightarrow X$ is continuous (i.e., it is a path). Cf. 9.T and 9.V.

The path uv is the *product* of u and v . Recall that it is defined only if the final point $u(1)$ of u is the initial point $v(0)$ of v .

13°2. Path-Connected Spaces

A topological space is *path-connected* (or *arcwise connected*) if any two points can be connected in it by a path.

13.B. Prove that I is path-connected.

13.C. Prove that the Euclidean space of any dimension is path-connected.

13.D. Prove that the n -sphere S^n with $n > 0$ is path-connected.

13.E. Prove that the 0-sphere S^0 is not path-connected.

13.4. Which of the following spaces are path-connected:

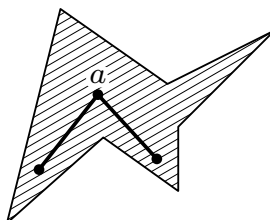
- (a) a discrete space; (b) an indiscrete space;
- (c) the arrow; (d) \mathbb{R}_{T_1} ;
- (e) \mathfrak{Y} ?

13°3. Path-Connected Sets

A *path-connected set* (or *arcwise connected set*) is a subset of a topological space (which should be clear from the context) that is path-connected as a space with the relative topology.

13.5. Prove that a subset A of a space X is path-connected iff any two points in A are connected by a path $s : I \rightarrow X$ with $s(I) \subset A$.

13.6. Prove that a convex subset of Euclidean space is path-connected.



13.7. Every star-shaped set in \mathbb{R}^n is path-connected.

13.8. The image of a path is a path-connected set.

13.9. Prove that the set of plane convex polygons with topology generated by the Hausdorff metric is path-connected. (What can you say about the set of convex n -gons with fixed n ?)

13.10. Riddle. What can you say about the assertion of Problem 13.9 in the case of arbitrary (not necessarily convex) polygons?

13°4. Properties of Path-Connected Sets

Path-connectedness is very similar to connectedness. Further, in some important situations it is even equivalent to connectedness. However, some properties of connectedness do not carry over to the path-connectedness

(see 13.Q and 13.R). For the properties that do carry over, proofs are usually easier in the case of path-connectedness.

13.F. *The union of a family of pairwise intersecting path-connected sets is path-connected.*

13.11. Prove that if two sets A and B are both closed or both open and their union and intersection are path-connected, then A and B are also path-connected.

13.12. 1) Prove that the interior and boundary of a path-connected set may not be path-connected. 2) Connectedness shares this property.

13.13. Let A be a subset of Euclidean space. Prove that if $\text{Fr } A$ is path-connected, then so is $\text{Cl } A$.

13.14. Prove that the same holds true for a subset of an arbitrary path-connected space.

13°5. Path-Connected Components

A *path-connected component* or *arcwise connected component* of a space X is a path-connected subset of X that is not contained in any other path-connected subset of X .

13.G. *Every point belongs to a path-connected component.*

13.H. *Two path-connected components either coincide or are disjoint.*

Theorems 13.G and 13.H mean that path-connected components constitute a partition of the entire space. The next theorem describes the corresponding equivalence relation.

13.I. Prove that two points belong to the same path-connected component iff they can be connected by a path (cf. 11.I).

Unlike to the case of connectedness, path-connected components are not necessarily closed. (See 13.Q, cf. 13.P and 13.R.)

13°6. Path-Connectedness and Continuous Maps

13.J. *A continuous image of a path-connected space is path-connected.*

13.K Corollary. *Path-connectedness is a topological property.*

13.L Corollary. *The number of path-connected components is a topological invariant.*

13°7. Path-Connectedness Versus Connectedness

13.M. *Any path-connected space is connected.*

Put

$$A = \{ (x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin(1/x) \}, \quad X = A \cup (0, 0).$$

13.15. Sketch A .

- 13.N.** Prove that A is path-connected and X is connected.
- 13.O.** Prove that deleting any point from A makes A and X disconnected (and hence, not path-connected).
- 13.P.** X is not path-connected.
- 13.Q.** Find an example of a path-connected set, whose closure is not path-connected.
- 13.R.** Find an example of a path-connected component that is not closed.
- 13.S.** If each point of a space has a path-connected neighborhood, then each path-connected component is open. (Cf. 11.21.)
- 13.T.** Assume that each point of a space X has a path-connected neighborhood. Then X is path-connected iff X is connected.
- 13.U.** For open subsets of Euclidean space connectedness is equivalent to path-connectedness.

13.16. For subsets of the real line path-connectedness and connectedness are equivalent.

13.17. Prove that for any $\varepsilon > 0$ the ε -neighborhood of a connected subset of Euclidean space is path-connected.

13.18. Prove that any neighborhood U of a connected subset A of Euclidean space contains a path-connected neighborhood of A .

13°8x. Polygon-Connectedness

A subset A of Euclidean space is *polygon-connected* if any two points of A are connected by a finite polyline contained in A .

13.1x. Each polygon-connected set in \mathbb{R}^n is path-connected, and thus also connected.

13.2x. Each convex set in \mathbb{R}^n is polygon-connected.

13.3x. Each star-shaped set in \mathbb{R}^n is polygon-connected.

13.4x. Prove that for open subsets of Euclidean space connectedness is equivalent to polygon-connectedness.

13.5x. Construct a path-connected subset A of Euclidean space such that A consists of more than one point and no two distinct points of A can be connected by a polygon in A .

13.6x. Let $X \subset \mathbb{R}^2$ be a countable set. Prove that then $\mathbb{R}^2 \setminus X$ is polygon-connected.

13.7x. Let $X \subset \mathbb{R}^n$ be the union of a countable collection of affine subspaces with dimensions not greater than $n - 2$. Prove that then $\mathbb{R}^n \setminus X$ is polygon-connected.

13.8x. Let $X \subset \mathbb{C}^n$ be the union of a countable collection of algebraic subsets (i.e., subsets defined by systems of algebraic equations in the standard coordinates of \mathbb{C}^n). Prove that then $\mathbb{C}^n \setminus X$ is polygon-connected.

13°9x. Connectedness of Some Sets of Matrices

Recall that real $n \times n$ -matrices constitute a space, which differs from \mathbb{R}^{n^2} only in the way of enumerating its natural coordinates (they are numerated by pairs of indices). The same relation holds true between the set of complex $n \times n$ -matrix and \mathbb{C}^{n^2} (homeomorphic to \mathbb{R}^{2n^2}).

13.9x. Find connected and path-connected components of the following subspaces of the space of real $n \times n$ -matrices:

- (1) $GL(n; \mathbb{R}) = \{A \mid \det A \neq 0\}$;
- (2) $O(n; \mathbb{R}) = \{A \mid A \cdot ({}^t A) = \mathbb{E}\}$;
- (3) $Symm(n; \mathbb{R}) = \{A \mid {}^t A = A\}$;
- (4) $Symm(n; \mathbb{R}) \cap GL(n; \mathbb{R})$;
- (5) $\{A \mid A^2 = \mathbb{E}\}$.

13.10x. Find connected and path-connected components of the following subspaces of the space of complex $n \times n$ -matrices:

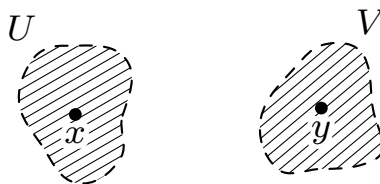
- (1) $GL(n; \mathbb{C}) = \{A \mid \det A \neq 0\}$;
- (2) $U(n; \mathbb{C}) = \{A \mid A \cdot ({}^t \bar{A}) = \mathbb{E}\}$;
- (3) $Herm(n; \mathbb{C}) = \{A \mid {}^t A = \bar{A}\}$;
- (4) $Herm(n; \mathbb{C}) \cap GL(n; \mathbb{C})$.

14. Separation Axioms

The aim of this section is to consider natural restrictions on the topological structure making the structure closer to being metrizable. A lot of separation axioms are known. We restrict ourselves to the five most important of them. They are numerated, and denoted by T_0 , T_1 , T_2 , T_3 , and T_4 , respectively.¹

14°1. The Hausdorff Axiom

We start with the second axiom, which is most important. Besides the notation T_2 , it has a name: the *Hausdorff axiom*. A topological space satisfying T_2 is a *Hausdorff space*. This axiom is stated as follows: *any two distinct points possess disjoint neighborhoods*. We can state it more formally: $\forall x, y \in X, x \neq y \exists U_x, V_y : U_x \cap V_y = \emptyset$.



14.A. Any metric space is Hausdorff.

14.1. Which of the following spaces are Hausdorff:

- (1) a discrete space;
- (2) an indiscrete space;
- (3) the arrow;
- (4) \mathbb{R}_{T_1} ;
- (5) \mathbb{V} ?

If the next problem holds you up even for a minute, we advise you to think over all definitions and solve all simple problems.

14.B. Is the segment $[0, 1]$ with the topology induced from \mathbb{R} a Hausdorff space? Do the points 0 and 1 possess disjoint neighborhoods? Which if any?

14.C. A space X is Hausdorff iff for each $x \in X$ we have $\{x\} = \bigcap_{U \ni x} \text{Cl}U$.

¹Letter T in these notation originates from the German word *Trennungsaxiom*, which means *separation axiom*.

14°2. Limits of Sequence

Let $\{a_n\}$ be a sequence of points of a topological space X . A point $b \in X$ is the *limit* of the sequence if for any neighborhood U of b there exists a number N such that $a_n \in U$ for any $n \geq N$.² In this case, we say that the sequence *converges* or *tends* to b as n tends to infinity.

14.2. Explain the meaning of the statement “ b is not a limit of sequence a_n ”, using as few negations (i.e., the words *no*, *not*, *none*, etc.) as you can.

14.3. The limit of a sequence does not depend on the order of the terms. More precisely, let a_n be a convergent sequence: $a_n \rightarrow b$, and let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Then the sequence $a_{\phi(n)}$ is also convergent and has the same limit: $a_{\phi(n)} \rightarrow b$. For example, if the terms in the sequence are pairwise distinct, then the convergence and the limit depend only on the set of terms, which shows that these notions actually belong to geometry.

14.D. In a Hausdorff space any sequence has at most one limit.

14.E. Prove that in the space \mathbb{R}_{T_1} each point is a limit of the sequence $a_n = n$.

14°3. Coincidence Set and Fixed Point Set

Let $f, g : X \rightarrow Y$ be maps. Then the set $C(f, g) = \{x \in X \mid f(x) = g(x)\}$ is the *coincidence set* of f and g .

14.4. Prove that the coincidence set of two continuous maps from an arbitrary space to a Hausdorff space is closed.

14.5. Construct an example proving that the Hausdorff condition in 14.4 is essential.

A point $x \in X$ is a *fixed point* of a map $f : X \rightarrow X$ if $f(x) = x$. The set of all fixed points of a map f is the *fixed point set* of f .

14.6. Prove that the fixed-point set of a continuous map from a Hausdorff space to itself is closed.

14.7. Construct an example showing that the Hausdorff condition in 14.6 is essential.

14.8. Prove that if $f, g : X \rightarrow Y$ are two continuous maps, Y is Hausdorff, A is everywhere dense in X , and $f|_A = g|_A$, then $f = g$.

14.9. Riddle. How are problems 14.4, 14.6, and 14.8 related to each other?

14°4. Hereditary Properties

A topological property is *hereditary* if it carries over from a space to its subspaces, i.e., if a space X has this property, then each subspace of X also has it.

²You can also rephrase this as follows: each neighborhood of b contains all members of the sequence that have sufficiently large indices.

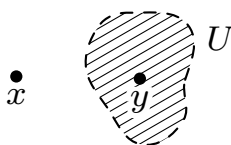
14.10. Which of the following topological properties are hereditary:

- (1) finiteness of the set of points;
- (2) finiteness of the topological structure;
- (3) infiniteness of the set of points;
- (4) connectedness;
- (5) path-connectedness?

14.F. The property of being a Hausdorff space is hereditary.

14°5. The First Separation Axiom

A topological space X satisfies the *first separation axiom* T_1 if each one of any two points of X has a neighborhood that does not contain the other point.³ More formally: $\forall x, y \in X, x \neq y \exists U_y : x \notin U_y$.



14.G. A space X satisfies the first separation axiom,

- iff all one-point sets in X are closed,
- iff all finite sets in X are closed.

14.11. Prove that a space X satisfies the first separation axiom iff every point of X is the intersection of all of its neighborhoods.

14.12. Any Hausdorff space satisfies the first separation axiom.

14.H. In a Hausdorff space any finite set is closed.

14.I. A metric space satisfies the first separation axiom.

14.13. Find an example showing that the first separation axiom does not imply the Hausdorff axiom.

14.J. Show that \mathbb{R}_{T_1} meets the first separation axiom, but is not a Hausdorff space (cf. 14.13).

14.K. The first separation axiom is hereditary.

14.14. Suppose that for any two distinct points a and b of a space X there exists a continuous map f from X to a space with the first separation axiom such that $f(a) \neq f(b)$. Prove that then X also satisfies the first separation axiom.

14.15. Prove that a continuous map of an indiscrete space to a space satisfying axiom T_1 is constant.

14.16. Prove that in every set there exists a coarsest topological structure satisfying the first separation axiom. Describe this structure.

³ T_1 is also called the Tikhonov axiom.

14°6. The Kolmogorov Axiom

The first separation axiom emerges as a weakened Hausdorff axiom.

14.L. Riddle. How can the first separation axiom be weakened?

A topological space satisfies the *Kolmogorov axiom* or the *zeroth separation axiom* T_0 if *at least one* of any two distinct points of this space has a neighborhood that does not contain the other of these points.

14.M. An indiscrete space containing at least two points does not satisfy T_0 .

14.N. The following properties of a space X are equivalent:

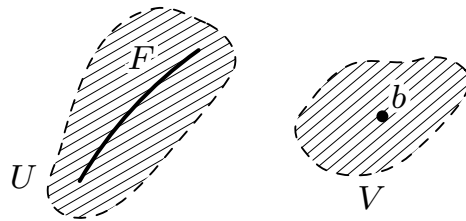
- (1) X satisfies the Kolmogorov axiom;
- (2) any two different points of X has different closures;
- (3) X contains no indiscrete subspace consisting of two points.
- (4) X contains no indiscrete subspace consisting of more than one point;

14.O. A topology is a poset topology iff it is a smallest neighborhood topology satisfying the Kolmogorov axiom.

Thus, on the one hand, posets give rise to numerous examples of topological spaces, among which we see the most important spaces, like the line with the standard topology. On the other hand, all posets are obtained from topological spaces of a special kind, which are quite far away from the class of metric spaces.

14°7. The Third Separation Axiom

A topological space X satisfies the *third separation axiom* if every closed set in X and every point of its complement have disjoint neighborhoods, i.e., for every closed set $F \subset X$ and every point $b \in X \setminus F$ there exist open sets $U, V \subset X$ such that $U \cap V = \emptyset$, $F \subset U$, and $b \in V$.



A space is *regular* if it satisfies the first and third separation axioms.

14.P. A regular space is a Hausdorff space.

14.Q. A space is regular iff it satisfies the second and third separation axioms.

14.17. Find a Hausdorff space which is not regular.

14.18. Find a space satisfying the third, but not the second separation axiom.

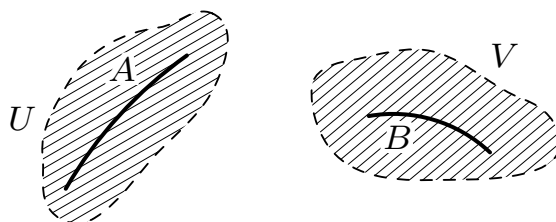
14.19. Prove that a space X satisfies the third separation axiom iff every neighborhood of every point $x \in X$ contains the closure of a neighborhood of x .

14.20. Prove that the third separation axiom is hereditary.

14.R. Any metric space is regular.

14°8. The Fourth Separation Axiom

A topological space X satisfies the *fourth separation axiom* if any two disjoint closed sets in X have disjoint neighborhoods, i.e., for any two closed sets $A, B \subset X$ with $A \cap B = \emptyset$ there exist open sets $U, V \subset X$ such that $U \cap V = \emptyset$, $A \subset U$, and $B \subset V$.



A space is *normal* if it satisfies the first and fourth separation axioms.

14.S. A normal space is regular (and hence Hausdorff).

14.T. A space is normal iff it satisfies the second and fourth separation axioms.

14.21. Find a space which satisfies the fourth, but not second separation axiom.

14.22. Prove that a space X satisfies the fourth separation axiom iff every neighborhood of every closed set $F \subset X$ contains the closure of some neighborhood of F .

14.23. Prove that any closed subspace of a normal space is normal.

14.24. Find two closed disjoint subsets A and B of some metric space such that $\inf\{\rho(a, b) \mid a \in A, b \in B\} = 0$.

14.U. Any metric space is normal.

14.25. Let $f : X \rightarrow Y$ be a continuous surjection such that the image of any closed set is closed. Prove that if X is normal, then so is Y .

14°9x. Niemytski's Space

Denote by \mathcal{H} the open upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the topology generated by the Euclidean metric. Denote by \mathcal{N} the union of \mathcal{H} and the boundary line \mathbb{R}^1 : $\mathcal{N} = \mathcal{H} \cup \mathbb{R}^1$, but equip it with the topology obtained by adjoining to the Euclidean topology the sets of the form $x \cup D$, where $x \in \mathbb{R}^1$ and D is an open disk in \mathcal{H} touching \mathbb{R}^1 at the point x . This is the *Niemytski space*. It can be used to clarify properties of the fourth separation axiom.

14.1x. Prove that the Niemytski space is Hausdorff.

14.2x. Prove that the Niemytski space is regular.

14.3x. What topological structure is induced on \mathbb{R}^1 from \mathcal{N} ?

14.4x. Prove that the Niemytski space is not normal.

14.5x Corollary. There exists a regular space which is not normal.

14.6x. Embed the Niemytski space into a normal space in such a way that the complement of the image would be a single point.

14.7x Corollary. Theorem 14.23 does not extend to nonclosed subspaces, i.e., the property of being normal is not hereditary, is it?

14°10x. Urysohn Lemma and Tietze Theorem

14.8x. Let A and B be two disjoint closed subsets of a metric space X . Then there exists a continuous function $f : X \rightarrow I$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

14.9x. Let F be a closed subset of a metric space X . Then any continuous function $f : X \rightarrow [-1, 1]$ can be extended over the whole X .

14.9x.1. Let F be a closed subset of a metric space X . For any continuous function $f : F \rightarrow [-1, 1]$ there exists a function $g : X \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that $|f(x) - g(x)| \leq \frac{2}{3}$ for each $x \in F$.

14.Ax Urysohn Lemma. Let A and B be two disjoint closed subsets of a normal space X . Then there exists a continuous function $f : X \rightarrow I$ such that $f(A) = 0$ and $f(B) = 1$.

14.Ax.1. Let A and B be two disjoint closed subsets of a normal space X . Consider the set $\Lambda = \{\frac{k}{2^n} \mid k, n \in \mathbb{Z}_+, k \leq 2^n\}$. There exists a collection $\{U_p\}_{p \in \Lambda}$ of open subsets of X such that for any $p, q \in \Lambda$ we have: 1) $A \subset U_0$ and $B \subset X \setminus U_1$ and 2) if $p < q$ then $\text{Cl}U_p \subset U_q$.

14.Bx Tietze Extension Theorem. Let A be a closed subset of a normal space X . Let $f : A \rightarrow [-1, 1]$ be a continuous function. Prove that there exists a continuous function $F : X \rightarrow [-1, 1]$ such that $F|_A = f$.

14.Cx Corollary. Let A be a closed subset of a normal space X . Any continuous function $A \rightarrow \mathbb{R}$ can be extended to a function on the whole space.

14.10x. Will the statement of the Tietze theorem remain true if in the hypothesis we replace the segment $[-1, 1]$ by \mathbb{R} , \mathbb{R}^n , S^1 , or S^2 ?

14.11x. Derive the Urysohn Lemma from the Tietze Extension Theorem.

15. Countability Axioms

In this section, we continue to study topological properties that are additionally imposed on a topological structure to make the abstract situation under consideration closer to special situations and hence richer in contents. The restrictions studied in this section bound a topological structure from above: they require that something be countable.

15°1. Set-Theoretic Digression: Countability

Recall that two sets have equal *cardinality* if there exists a bijection of one of them onto the other. A set of the same cardinality as a subset of the set \mathbb{N} of positive integers is *countable*.

15.1. A set X is countable iff there exists an injection $X \rightarrow \mathbb{N}$ (or, more generally, an injection of X into another countable set).

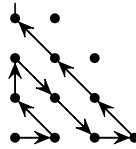
Sometimes this term is used only for infinite countable sets, i.e., for sets of the cardinality of the whole set \mathbb{N} of positive integers, while sets countable in the above sense are said to be *at most countable*. This is less convenient. In particular, if we adopted this terminology, this section would be called “At Most Countability Axioms”. This would also lead to other more serious inconveniences as well. Our terminology has the following advantageous properties.

15.A. Any subset of a countable set is countable.

15.B. The image of a countable set under any map is countable.

15.C. \mathbb{Z} is countable.

15.D. The set $\mathbb{N}^2 = \{(k, n) \mid k, n \in \mathbb{N}\}$ is countable.



15.E. The union of a countable family of countable sets is countable.

15.F. \mathbb{Q} is countable.

15.G. \mathbb{R} is not countable.

15.2. Prove that any set Σ of disjoint figure eight curves in the plane is countable.

15°2. Second Countability and Separability

In this section, we study three restrictions on the topological structure. Two of them have numbers (one and two), the third one has no number. As in the previous section, we start from the restriction having number two.

A topological space X satisfies the *second axiom of countability* or is *second countable* if X has a countable base. A space is *separable* if it contains a countable dense set. (This is the countability axiom without a number that we mentioned above.)

15.H. *The second axiom of countability implies separability.*

15.I. The second axiom of countability is hereditary.

15.3. Are the arrow and \mathbb{R}_{T_1} second countable?

15.4. Are the arrow and \mathbb{R}_{T_1} separable?

15.5. Construct an example proving that separability is not hereditary.

15.J. *A metric separable space is second countable.*

15.K Corollary. *For metrizable spaces, separability is equivalent to the second axiom of countability.*

15.L. (Cf. 15.5.) Prove that for metrizable spaces separability is hereditary.

15.M. Prove that Euclidean spaces and all their subspaces are separable and second countable.

15.6. Construct a metric space which is not second countable.

15.7. Prove that in a separable space any collection of pairwise disjoint open sets is countable.

15.8. Prove that the set of components of an open set $A \subset \mathbb{R}^n$ is countable.

15.N. *A continuous image of a separable space is separable.*

15.9. Construct an example proving that a continuous image of a second countable space may be not second countable.

15.O Lindelöf Theorem. *Any open cover of a second countable space contains a countable part that also covers the space.*

15.10. Prove that each base of a second countable space contains a countable part which is also a base.

15.11 Brouwer Theorem*. Let $\{K_\lambda\}$ be a family of closed sets of a second countable space and assume that for every decreasing sequence $K_1 \supset K_2 \supset \dots$ of sets belonging to this family the intersection $\bigcap_1^\infty K_n$ also belongs to the family. Then the family contains a minimal set A , i.e., a set such that no proper subset of A belongs to the family.

15°3. Bases at a Point

Let X be a space, a a point of X . A *neighborhood base* at a or just a *base of X at a* is a collection Σ of neighborhoods of a such that each neighborhood of a contains a neighborhood from Σ .

15.P. If Σ is a base of a space X , then $\{U \in \Sigma \mid a \in U\}$ is a base of X at a .

15.12. In a metric space the following collections of balls are neighborhood bases at a point a :

- the set of all open balls of center a ;
- the set of all open balls of center a and rational radii;
- the set of all open balls of center a and radii r_n , where $\{r_n\}$ is any sequence of positive numbers converging to zero.

15.13. What are the minimal bases at a point in the discrete and indiscrete spaces?

15°4. First Countability

A topological space X satisfies the *first axiom of countability* or is a *first countable space* if X has a countable neighborhood base at each point.

15.Q. Any metric space is first countable.

15.R. The second axiom of countability implies the first one.

15.S. Find a first countable space which is not second countable. (Cf. 15.6.)

15.14. Which of the following spaces are first countable:

- (a) the arrow; (b) \mathbb{R}_{T_1} ;
 (c) a discrete space; (d) an indiscrete space?

15.15. Find a first countable separable space which is not second countable.

15.16. Prove that if X is a first countable space, then at each point it has a decreasing countable neighborhood base: $U_1 \supset U_2 \supset \dots$

15°5. Sequential Approach to Topology

Specialists in Mathematical Analysis love sequences and their limits. Moreover, they like to talk about all topological notions relying on the notions of sequence and its limit. This tradition has almost no mathematical justification, except for a long history descending from the XIX century studies on the foundations of analysis. In fact, almost always⁴ it is more convenient to avoid sequences, provided you deal with topological notions, except summing of series, where sequences are involved in the underlying

⁴The exceptions which one may find in the standard curriculum of a mathematical department can be counted on two hands.

definitions. Paying a tribute to this tradition, here we explain how and in what situations topological notions can be described in terms of sequences.

Let A be a subset of a space X . The set $\text{Scl} A$ of limits of all sequences a_n with $a_n \in A$ is the *sequential closure* of A .

15.T. Prove that $\text{Scl} A \subset \text{Cl} A$.

15.U. If a space X is first countable, then for any $A \subset X$ the opposite inclusion $\text{Cl} A \subset \text{Scl} A$ also holds true, whence $\text{Scl} A = \text{Cl} A$.

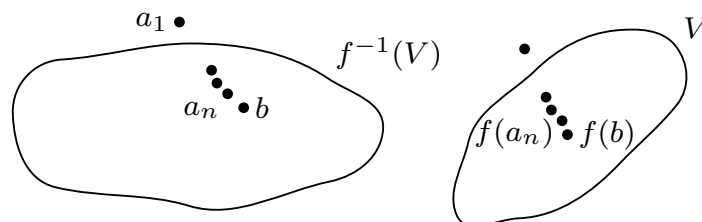
Therefore, in a first countable space (in particular, any metric spaces) we can recover (hence, define) the closure of a set provided it is known which sequences are convergent and what the limits are. In turn, the knowledge of closures allows one to determine which sets are closed. As a consequence, knowledge of closed sets allows one to recover open sets and all other topological notions.

15.17. Let X be the set of real numbers equipped with the topology consisting of \emptyset and complements of all countable subsets. (Check that this is actually a topology.) Describe convergent sequences, sequential closure and closure in X . Prove that in X there exists a set A with $\text{Scl} A \neq \text{Cl} A$.

15°6. Sequential Continuity

Now we consider the continuity of maps along the same lines. A map $f : X \rightarrow Y$ is *sequentially continuous* if for each $b \in X$ and each sequence $a_n \in X$ converging to b the sequence $f(a_n)$ converges to $f(b)$.

15.V. Any continuous map is sequentially continuous.



15.W. The preimage of a sequentially closed set under a sequentially continuous map is sequentially closed.

15.X. If X is a first countable space, then any sequentially continuous map $f : X \rightarrow Y$ is continuous.

Thus for maps of a first countable space continuity and sequential continuity are equivalent.

15.18. Construct a sequentially continuous, but discontinuous map. (Cf. 15.17)

15°7x. Embedding and Metrization Theorems

15.Ax. Prove that the space l_2 is separable and second countable.

15.Bx. Prove that a regular second countable space is normal.

15.Cx. Prove that a normal second countable space can be embedded into l_2 . (Use the Urysohn Lemma 14.Ax.)

15.Dx. Prove that a second countable space is metrizable iff it is regular.

16. Compactness

16°1. Definition of Compactness

This section is devoted to a topological property playing a very special role in topology and its applications. It is a sort of topological counterpart for the property of being finite in the context of set theory. (It seems though that this analogy has never been formalized.)

A topological space X is *compact* if each open cover of X contains a finite part that also covers X .

If Γ is a cover of X and $\Sigma \subset \Gamma$ is a cover of X , then Σ is a *subcover* (or *subcovering*) of Γ . Thus, a space X is compact if every open cover of X contains a finite subcovering.

16.A. Any finite space and indiscrete space are compact.

16.B. Which discrete spaces are compact?

16.1. Let $\Omega_1 \subset \Omega_2$ be two topological structures in X . 1) Does the compactness of (X, Ω_2) imply that of (X, Ω_1) ? 2) And vice versa?

16.C. The line \mathbb{R} is not compact.

16.D. A space X is not compact iff it has an open cover containing no finite subcovering.

16.2. Is the arrow compact? Is \mathbb{R}_{T_1} compact?

16°2. Terminology Remarks

Originally the word *compactness* was used for the following weaker property: any countable open cover contains a finite subcovering.

16.E. For a second countable space, the original definition of compactness is equivalent to the modern one.

The modern notion of compactness was introduced by P. S. Alexandrov (1896–1982) and P. S. Urysohn (1898–1924). They suggested for it the term *bicompactness*. This notion appeared to be so successful that it has displaced the original one and even took its name, i.e., compactness. The term bicompactness is sometimes used (mainly by topologists of Alexandrov's school).

Another deviation from the terminology used here comes from Bourbaki: we do not include the Hausdorff property into the definition of compactness, which Bourbaki includes. According to our definition, \mathbb{R}_{T_1} is compact, according to Bourbaki it is not.

16°3. Compactness in Terms of Closed Sets

A collection of subsets of a set is said to have the *finite intersection property* if the intersection of any finite subcollection is nonempty.

16.F. A collection Σ of subsets of a set X has the finite intersection property iff there exists no finite $\Sigma_1 \subset \Sigma$ such that the complements of the sets in Σ_1 cover X .

16.G. A space is compact iff for every collection of its closed sets having the finite intersection property its intersection is nonempty.

16°4. Compact Sets

A *compact set* is a subset A of a topological space X (the latter must be clear from the context) provided A is compact as a space with the relative topology induced from X .

16.H. A subset A of a space X is compact iff each cover of A with sets open in X contains a finite subcovering.

16.3. Is $[1, 2) \subset \mathbb{R}$ compact?

16.4. Is the same set $[1, 2)$ compact in the arrow?

16.5. Find a necessary and sufficient condition (formulated not in topological terms) for a subset of the arrow to be compact?

16.6. Prove that any subset of \mathbb{R}_{T_1} is compact.

16.7. Let A and B be two compact subsets of a space X . 1) Does it follow that $A \cup B$ is compact? 2) Does it follow that $A \cap B$ is compact?

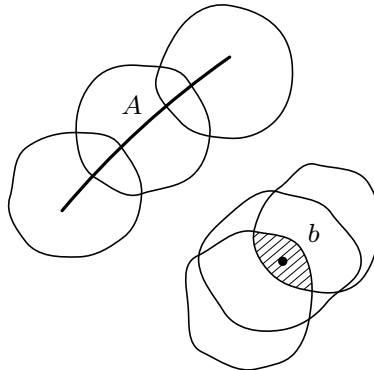
16.8. Prove that the set $A = 0 \cup \{\frac{1}{n}\}_{n=1}^{\infty}$ in \mathbb{R} is compact.

16°5. Compact Sets Versus Closed Sets

16.I. Is compactness hereditary?

16.J. Any closed subset of a compact space is compact.

16.K. Any compact subset of a Hausdorff space is closed.



16.L Lemma to 16.K, but not only Let A be a compact subset of a Hausdorff space X and b a point of X that does not belong to A . Then there exist open sets $U, V \subset X$ such that $b \in V$, $A \subset U$, and $U \cap V = \emptyset$.

16.9. Construct a nonclosed compact subset of some topological space. What is the minimal number of points needed?

16°6. Compactness and Separation Axioms

16.M. A compact Hausdorff space is regular.

16.N. Prove that a compact Hausdorff space is normal.

16.O Lemma to 16.N. In a Hausdorff space, any two disjoint compact subsets possess disjoint neighborhoods.

16.10. Prove that the intersection of any family of compact subsets of a Hausdorff space is compact. (Cf. 16.7.)

16.11. Let X be a Hausdorff space, let $\{K_\lambda\}_{\lambda \in \Lambda}$ be a family of its compact subsets, and let U be an open set containing $\bigcap_{\lambda \in \Lambda} K_\lambda$. Prove that for some finite $A \subset \Lambda$ we have $U \supset \bigcap_{\lambda \in A} K_\lambda$.

16.12. Let $\{K_n\}_1^\infty$ be a decreasing sequence of nonempty compact connected sets in a Hausdorff space. Prove that the intersection $\bigcap_1^\infty K_n$ is nonempty and connected. (Cf. 11.20)

16°7. Compactness in Euclidean Space

16.P. The segment I is compact.

Recall that n -dimensional cube is the set

$$I^n = \{x \in \mathbb{R}^n \mid x_i \in [0, 1] \text{ for } i = 1, \dots, n\}.$$

16.Q. The cube I^n is compact.

16.R. Any compact subset of a metric space is bounded.

Therefore, any compact subset of a metric space is closed and bounded (see Theorems 14.A, 16.K, and 16.R).

16.S. Construct a closed and bounded, but noncompact set in a metric space.

16.13. Are the metric spaces of Problem 4.A compact?

16.T. A subset of a Euclidean space is compact iff it is closed and bounded.

16.14. Which of the following sets are compact:

- | | | |
|--------------------------------|---|----------------|
| (a) $[0, 1]$; | (b) ray $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$; | (c) S^1 ; |
| (d) S^n ; | (e) one-sheeted hyperboloid; | (f) ellipsoid; |
| (g) $[0, 1] \cap \mathbb{Q}$? | | |

An $(n \times k)$ -matrix (a_{ij}) with real entries can be regarded as a point in \mathbb{R}^{nk} . To do this, we only need to enumerate somehow (e.g., lexicographically) the entries of (a_{ij}) by numbers from 1 to nk . This identifies the set $L(n, k)$ of all matrices like that with \mathbb{R}^{nk} and endows it with a topological structure. (Cf. Section 13.)

16.15. Which of the following subsets of $L(n, n)$ are compact:

- (1) $GL(n) = \{A \in L(n, n) \mid \det A \neq 0\}$;
- (2) $SL(n) = \{A \in L(n, n) \mid \det A = 1\}$;
- (3) $O(n) = \{A \in L(n, n) \mid A \text{ is an orthogonal matrix}\}$;
- (4) $\{A \in L(n, n) \mid A^2 = \mathbb{E}\}$, where \mathbb{E} is the unit matrix?

16°8. Compactness and Continuous Maps

16.U. A continuous image of a compact space is compact. (In other words, if X is a compact space and $f : X \rightarrow Y$ is a continuous map, then $f(X)$ is compact.)

16.V. A continuous numerical function on a compact space is bounded and attains its maximal and minimal values. (In other words, if X is a compact space and $f : X \rightarrow \mathbb{R}$ is a continuous function, then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for every $x \in X$.) Cf. 16.U and 16.T.

16.16. Prove that if $f : I \rightarrow \mathbb{R}$ is a continuous function, then $f(I)$ is a segment.

16.17. Let A be a subset of \mathbb{R}^n . Prove that A is compact iff each continuous numerical function on A is bounded.

16.18. Prove that if F and G are disjoint subsets of a metric space, F is closed, and G is compact, then $\rho(G, F) = \inf \{\rho(x, y) \mid x \in F, y \in G\} > 0$.

16.19. Prove that any open set U containing a compact set A of a metric space X contains an ε -neighborhood of A (i.e., the set $\{x \in X \mid \rho(x, A) < \varepsilon\}$) for some $\varepsilon > 0$.

16.20. Let A be a closed connected subset of \mathbb{R}^n and let V be the closed ε -neighborhood of A (i.e., $V = \{x \in \mathbb{R}^n \mid \rho(x, A) \leq \varepsilon\}$). Prove that V is path-connected.

16.21. Prove that if the closure of each open ball in a compact metric space is the closed ball with the same center and radius, then any ball in this space is connected.

16.22. Let X be a compact metric space, and let $f : X \rightarrow X$ be a map such that $\rho(f(x), f(y)) < \rho(x, y)$ for any $x, y \in X$ with $x \neq y$. Prove that f has a unique fixed point. (Recall that a fixed point of f is a point x such that $f(x) = x$, see 14.6.)

16.23. Prove that for any open cover of a compact metric space there exists a (sufficiently small) number $r > 0$ such that each open ball of radius r is contained in an element of the cover.

16.W Lebesgue Lemma. Let $f : X \rightarrow Y$ be a continuous map from a compact metric space X to a topological space Y , and let Γ be an open cover of Y . Then there exists a number $\delta > 0$ such that for any set $A \subset X$ with diameter $\text{diam}(A) < \delta$ the image $f(A)$ is contained in an element of Γ .

16°9. Closed Maps

A continuous map is *closed* if the image of each closed set under this map is closed.

16.24. A continuous bijection is a homeomorphism iff it is closed.

16.X. A continuous map of a compact space to a Hausdorff space is closed.

Here are two important corollaries of this theorem.

16.Y. A continuous bijection of a compact space onto a Hausdorff space is a homeomorphism.

16.Z. A continuous injection of a compact space into a Hausdorff space is a topological embedding.

16.25. Show that none of the assumptions in 16.Y can be omitted without making the statement false.

16.26. Does there exist a noncompact subspace A of the Euclidian space such that any continuous map of A to a Hausdorff space is closed? (Cf. 16.V and 16.X.)

16.27. A restriction of a closed map to a closed subset is a also closed map.

16°10x. Norms in \mathbb{R}^n

16.1x. Prove that each norm $\mathbb{R}^n \rightarrow \mathbb{R}$ (see Section 4) is a continuous function (with respect to the standard topology of \mathbb{R}^n).

16.2x. Prove that any two norms in \mathbb{R}^n are equivalent (i.e., determine the same topological structure). See 4.27, cf. 4.31.

16.3x. Does the same hold true for metrics in \mathbb{R}^n ?

16°11x. Induction on Compactness

A function $f : X \rightarrow \mathbb{R}$ is *locally bounded* if for each point $a \in X$ there exist a neighborhood U and a number $M > 0$ such that $|f(x)| \leq M$ for $x \in U$ (i.e., each point has a neighborhood U such that the restriction of f to U is bounded).

16.4x. Prove that if a space X is compact and a function $f : X \rightarrow \mathbb{R}$ is locally bounded, then f is bounded.

This statement is a simplest application of a general principle formulated below in 16.5x. This principle may be called *induction on compactness* (cf. induction on connectedness, which was discussed in Section 11).

Let X be a topological space, \mathcal{C} a property of subsets of X . We say that \mathcal{C} is *additive* if the union of any finite family of sets having \mathcal{C} also has \mathcal{C} . The space X *possesses \mathcal{C} locally* if each point of X has a neighborhood with property \mathcal{C} .

16.5x. Prove that a compact space which locally possesses an additive property has this property itself.

16.6x. Using induction on compactness, deduce the statements of Problems 16.R, 17.M, and 17.N.

17. Sequential Compactness

17°1. Sequential Compactness Versus Compactness

A topological space is *sequentially compact* if every sequence of its points contains a convergent subsequence.

17.A. *If a first countable space is compact, then it is sequentially compact.*

A point b is an *accumulation point* of a set A if each neighborhood of b contains infinitely many points of A .

17.A.1. Prove that in a space satisfying the first separation axiom a point is an accumulation point iff it is a limit point.

17.A.2. *In a compact space, any infinite set has an accumulation point.*

17.A.3. *A space in which each infinite set has an accumulation point is sequentially compact.*

17.B. *A sequentially compact second countable space is compact.*

17.B.1. *In a sequentially compact space a decreasing sequence of nonempty closed sets has a nonempty intersection.*

17.B.2. Prove that each nested sequence of nonempty closed sets in a space X has nonempty intersection iff each countable collection of closed sets in X the finite intersection property has nonempty intersection.

17.B.3. Derive Theorem 17.B from 17.B.1 and 17.B.2.

17.C. *For second countable spaces, compactness and sequential compactness are equivalent.*

17°2. In Metric Space

A subset A of a metric space X is an ε -*net* (where ε is a positive number) if $\rho(x, A) < \varepsilon$ for each point $x \in X$.

17.D. Prove that in any compact metric space for any $\varepsilon > 0$ there exists a finite ε -net.

17.E. Prove that in any sequentially compact metric space for any $\varepsilon > 0$ there exists a finite ε -net.

17.F. Prove that a subset A of a metric space is everywhere dense iff A is an ε -net for each $\varepsilon > 0$.

17.G. *Any sequentially compact metric space is separable.*

17.H. Any sequentially compact metric space is second countable.

17.I. For metric spaces compactness and sequential compactness are equivalent.

17.1. Prove that a sequentially compact metric space is bounded. (Cf. 17.E and 17.I.)

17.2. Prove that in any metric space for any $\varepsilon > 0$ there exists

- (1) a discrete ε -net and even
- (2) an ε -net such that the distance between any two of its points is greater than ε .

17°3. Completeness and Compactness

A sequence $\{x_n\}_{n \in \mathbb{N}}$ of points of a metric space is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a number N such that $\rho(x_n, x_m) < \varepsilon$ for any $n, m \geq N$. A metric space X is *complete* if every Cauchy sequence in X converges.

17.J. A Cauchy sequence containing a convergent subsequence converges.

17.K. Prove that a metric space M is complete iff every nested decreasing sequence of closed balls in M with radii tending to 0 has nonempty intersection.

17.L. Prove that a compact metric space is complete.

17.M. Prove that a complete metric space is compact iff for each $\varepsilon > 0$ it contains a finite ε -net.

17.N. Prove that a complete metric space is compact iff for any $\varepsilon > 0$ it contains a compact ε -net.

17°4x. Noncompact Balls in Infinite Dimension

By l^∞ denote the set of all bounded sequences of real numbers. This is a vector space with respect to the component-wise operations. There is a natural norm in it: $\|x\| = \sup\{|x_n| \mid n \in \mathbb{N}\}$.

17.1x. Are closed balls of l^∞ compact? What about spheres?

17.2x. Is the set $\{x \in l^\infty \mid |x_n| \leq 2^{-n}, n \in \mathbb{N}\}$ compact?

17.3x. Prove that the set $\{x \in l^\infty \mid |x_n| = 2^{-n}, n \in \mathbb{N}\}$ is homeomorphic to the Cantor set K introduced in Section 2.

17.4x*. Does there exist an infinitely dimensional normed space in which closed balls are compact?

17°5x. p -Adic Numbers

Fix a prime integer p . By \mathbb{Z}_p denote the set of series of the form $a_0 + a_1p + \dots + a_n p^n + \dots$ with $0 \leq a_n < p$, $a_n \in \mathbb{N}$. For $x, y \in \mathbb{Z}_p$, put $\rho(x, y) = 0$ if $x = y$, and $\rho(x, y) = p^{-m}$ if m is the smallest number such that the m th coefficients in the series x and y differ.

17.5x. Prove that ρ is a metric in \mathbb{Z}_p .

This metric space is the *space of integer p -adic numbers*. There is an injection $\mathbb{Z} \rightarrow \mathbb{Z}_p$ assigning to $a_0 + a_1p + \dots + a_n p^n \in \mathbb{Z}$ with $0 \leq a_k < p$ the series

$$a_0 + a_1p + \dots + a_n p^n + 0p^{n+1} + 0p^{n+2} + \dots \in \mathbb{Z}_p$$

and to $-(a_0 + a_1p + \dots + a_n p^n) \in \mathbb{Z}$ with $0 \leq a_k < p$ the series

$$b_0 + b_1p + \dots + b_n p^n + (p-1)p^{n+1} + (p-1)p^{n+2} + \dots,$$

where

$$b_0 + b_1p + \dots + b_n p^n = p^{n+1} - (a_0 + a_1p + \dots + a_n p^n).$$

Cf. 4.Ix.

17.6x. Prove that the image of the injection $\mathbb{Z} \rightarrow \mathbb{Z}_p$ is dense in \mathbb{Z}_p .

17.7x. Is \mathbb{Z}_p a complete metric space?

17.8x. Is \mathbb{Z}_p compact?

17°6x. Spaces of Convex Figures

Let $D \subset \mathbb{R}^2$ be a closed disk of radius p . Consider the set \mathcal{P}_n of all convex polygons P with the following properties:

- the perimeter of P is at most p ;
- P is contained in D ;
- P has at most n vertices (the cases of one and two vertices are not excluded; the perimeter of a segment is twice its length).

See 4.Ax, cf. 4.Cx.

17.9x. Equip \mathcal{P}_n with a natural topological structure. For instance, define a natural metric on \mathcal{P}_n .

17.10x. Prove that \mathcal{P}_n is compact.

17.11x. Prove that there exists a polygon belonging to \mathcal{P}_n and having the maximal area.

17.12x. Prove that this polygon is a regular n -gon.

Consider now the set \mathcal{P}_∞ of all convex polygons that have perimeter at most p and are contained in D . In other words, $\mathcal{P}_\infty = \bigcup_{n=1}^{\infty} \mathcal{P}_n$.

17.13x. Construct a topological structure in \mathcal{P}_∞ inducing the structures introduced above in the spaces \mathcal{P}_n .

17.14x. Prove that the space \mathcal{P}_∞ is not compact.

Consider now the set \mathcal{P} of all convex closed subsets of the plane that have perimeter at most p and are contained in D . (Observe that all sets in \mathcal{P} are compact.)

17.15x. Construct a topological structure in \mathcal{P} that induces the structure introduced above in the space \mathcal{P}_∞ .

17.16x. Prove that the space \mathcal{P} is compact.

17.17x. Prove that there exists a convex plane set with perimeter at most p having a maximal area.

17.18x. Prove that this is a disk of radius $\frac{p}{2\pi}$.

18x. Local Compactness and Paracompactness

18°1x. Local Compactness

A topological space X is *locally compact* if each point of X has a neighborhood with compact closure.

18.1x. Compact spaces are locally compact.

18.2x. Which of the following spaces are locally compact: (a) \mathbb{R} ; (b) \mathbb{Q} ; (c) \mathbb{R}^n ; (d) a discrete space?

18.3x. Find two locally compact sets on the line such that their union is not locally compact.

18.Ax. Is the local compactness hereditary?

18.Bx. A closed subset of a locally compact space is locally compact.

18.Cx. Is it true that an open subset of a locally compact space is locally compact?

18.Dx. A Hausdorff locally compact space is regular.

18.Ex. An open subset of a locally compact Hausdorff space is locally compact.

18.Fx. Local compactness is a local property for a Hausdorff space, i.e., a Hausdorff space is locally compact iff each of its points has a locally compact neighborhood.

18°2x. One-Point Compactification

Let (X, Ω) be a Hausdorff topological space. Let X^* be the set obtained by adding a point x_* to X (of course, x_* does not belong to X). Let Ω^* be the collection of subsets of X^* consisting of

- sets open in X and
- sets of the form $X^* \setminus C$, where $C \subset X$ is a compact set:

$$\Omega^* = \Omega \cup \{X^* \setminus C \mid C \subset X \text{ is a compact set}\}.$$

18.Gx. Prove that Ω^* is a topological structure on X^* .

18.Hx. Prove that the space (X^*, Ω^*) is compact.

18.Ix. Prove that the inclusion $(X, \Omega) \hookrightarrow (X^*, \Omega^*)$ is a topological embedding.

18.Jx. Prove that if X is locally compact, then the space (X^*, Ω^*) is Hausdorff. (Recall that in the definition of X^* we assumed that X is Hausdorff.)

A topological embedding of a space X into a compact space Y is a *compactification* of X if the image of X is dense in Y . In this situation, Y is also called a *compactification* of X . (To simplify the notation, we identify X with its image in Y .)

18.Kx. Prove that if X is a locally compact Hausdorff space and Y is a compactification of X with one-point $Y \setminus X$, then there exists a homeomorphism $Y \rightarrow X^*$ which is the identity on X .

Any space Y of Problem 18.Kx is called a *one-point compactification* or *Alexandrov compactification* of X . Problem 18.Kx says Y is essentially unique.

18.Lx. Prove that the one-point compactification of the plane is homeomorphic to S^2 .

18.4x. Prove that the one-point compactification of \mathbb{R}^n is homeomorphic to S^n .

18.5x. Give explicit descriptions of one-point compactifications of the following spaces:

- (1) annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$;
- (2) square without vertices $\{(x, y) \in \mathbb{R}^2 \mid x, y \in [-1, 1], |xy| < 1\}$;
- (3) strip $\{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1]\}$;
- (4) a compact space.

18.Mx. Prove that a locally compact Hausdorff space is regular.

18.6x. Let X be a locally compact Hausdorff space, K a compact subset of X , U a neighborhood of K . Then there exists a neighborhood V of K such that the closure $\text{Cl} V$ is compact and contained in U .

18°3x. Proper Maps

A continuous map $f : X \rightarrow Y$ is *proper* if each compact subset of Y has compact preimage.

Let X, Y be Hausdorff spaces. Any map $f : X \rightarrow Y$ obviously extends to the map

$$f^* : X^* \rightarrow Y^* : x \mapsto \begin{cases} f(x) & \text{if } x \in X, \\ y^* & \text{if } x = x^*. \end{cases}$$

18.Nx. Prove that f^* is continuous iff f is a proper continuous map.

18.Ox. Prove that any proper map of a Hausdorff space to a Hausdorff locally compact space is closed.

Problem 18.Ox is related to Theorem 16.X.

18.Px. Extend this analogy: formulate and prove statements corresponding to Theorems 16.Z and 16.Y.

18°4x. Locally Finite Collections of Subsets

A collection Γ of subsets of a space X is *locally finite* if each point $b \in X$ has a neighborhood U such that $A \cap U = \emptyset$ for all sets $A \in \Gamma$ except, maybe, a finite number.

18.Qx. A locally finite cover of a compact space is finite.

18.7x. If a collection Γ of subsets of a space X is locally finite, then so is $\{\text{Cl } A \mid A \in \Gamma\}$.

18.8x. If a collection Γ of subsets of a space X is locally finite, then each compact set $A \subset X$ intersects only a finite number of elements of Γ .

18.9x. If a collection Γ of subsets of a space X is locally finite and each $A \in \Gamma$ has compact closure, then each $A \in \Gamma$ intersects only a finite number of elements of Γ .

18.10x. Any locally finite cover of a sequentially compact space is finite.

18.Rx. Find an open cover of \mathbb{R}^n that has no locally finite subcovering.

Let Γ and Δ be two covers of a set X . The cover Δ is a *refinement* of Γ if for each $A \in \Delta$ there exists $B \in \Gamma$ such that $A \subset B$.

18.Sx. Prove that any open cover of \mathbb{R}^n has a locally finite open refinement.

18.Tx. Let $\{U_i\}_{i \in \mathbb{N}}$ be a (locally finite) open cover of \mathbb{R}^n . Prove that there exists an open cover $\{V_i\}_{i \in \mathbb{N}}$ of \mathbb{R}^n such that $\text{Cl } V_i \subset U_i$ for each $i \in \mathbb{N}$.

18°5x. Paracompact Spaces

A space X is *paracompact* if every open cover of X has a locally finite open refinement.

18.Ux. Any compact space is paracompact.

18.Vx. \mathbb{R}^n is paracompact.

18.Wx. Let $X = \bigcup_{i=1}^{\infty} X_i$, where X_i are compact sets such that $X_i \subset \text{Int } X_{i+1}$. Then X is paracompact.

18.Xx. Let X be a locally compact space. If X has a countable cover by compact sets, then X is paracompact.

18.11x. Prove that if a locally compact space is second countable, then it is paracompact.

18.12x. A closed subspace of a paracompact space is paracompact.

18.13x. A disjoint union of paracompact spaces is paracompact.

18°6x. Paracompactness and Separation Axioms

18.14x. Let X be a paracompact topological space, and let F and M be two disjoint subsets of X , where F is closed. Suppose that F is covered by open sets U_α whose closures are disjoint with M : $\text{Cl}U_\alpha \cap M = \emptyset$. Then F and M have disjoint neighborhoods.

18.15x. A Hausdorff paracompact space is regular.

18.16x. A Hausdorff paracompact space is normal.

18.17x. Let X be a Hausdorff locally compact and paracompact space, Γ a locally finite open cover of X . Then X has a locally finite open cover Δ such that the closures $\text{Cl}V$, where $V \in \Delta$, are compact sets and $\{\text{Cl}V \mid V \in \Delta\}$ is a refinement of Γ .

Here is a more general (though formally weaker) fact.

18.18x. Let X be a normal space, Γ a locally finite open cover of X . Then X has a locally finite open cover Δ such that $\{\text{Cl}V \mid V \in \Delta\}$ is a refinement of Γ .

Information. *Metrisable spaces are paracompact.*

18°7x. Partitions of Unity

Let X be a topological space, $f : X \rightarrow \mathbb{R}$ a function. Then the set $\text{supp} f = \text{Cl}\{x \in X \mid f(x) \neq 0\}$ is the *support* of f .

18.19x. Let X be a topological space, and let $\{f_\alpha : X \rightarrow \mathbb{R}\}_{\alpha \in \Lambda}$ be a family of continuous functions whose supports $\text{supp}(f_\alpha)$ constitute a locally finite cover of X . Prove that the formula

$$f(x) = \sum_{\alpha \in \Lambda} f_\alpha(x)$$

determines a continuous function $f : X \rightarrow \mathbb{R}$.

A family of nonnegative functions $f_\alpha : X \rightarrow \mathbb{R}_+$ is a *partition of unity* if the supports $\text{supp}(f_\alpha)$ constitute a locally finite cover of the space X and $\sum_{\alpha \in \Lambda} f_\alpha(x) = 1$.

A partition of unity $\{f_\alpha\}$ is *subordinate to a cover* Γ if $\text{supp}(f_\alpha)$ is contained in an element of Γ for each α . We also say that Γ *dominates* $\{f_\alpha\}$.

18.Yx. Let X be a normal space. Then each locally finite open cover of X dominates a certain partition of unity.

18.20x. Let X be a Hausdorff space. If each open cover of X dominates a certain partition of unity, then X is paracompact.

Information. *A Hausdorff space X is paracompact iff each open cover of X dominates a certain partition of unity.*

18°8x. Application: Making Embeddings From Pieces

18.21x. Let X be a topological space, $\{U_i\}_{i=1}^k$ an open cover of X . If U_i can be embedded in \mathbb{R}^n for each $i = 1, \dots, k$, then X can be embedded in $\mathbb{R}^{k(n+1)}$.

18.21x.1. Let $h_i : U_i \rightarrow \mathbb{R}^n$, $i = 1, \dots, k$, be embeddings, and let $f_i : X \rightarrow \mathbb{R}$ form a partition of unity subordinate to the cover $\{U_i\}_{i=1}^k$.

We put $\hat{h}_i(x) = (h_i(x), 1) \in \mathbb{R}^{n+1}$. Show that the map $X \rightarrow \mathbb{R}^{k(n+1)} : x \mapsto (f_i(x)\hat{h}_i(x))_{i=1}^k$ is an embedding.

18.22x. Riddle. How can you generalize 18.21x?

Proofs and Comments

11.A A set A is open and closed, iff A and $X \setminus A$ are open, iff A and $X \setminus A$ are closed.

11.B It suffices to prove the following apparently less general assertion: *A space having a connected everywhere dense subset is connected.* (See 6.3.) Let $X \supset A$ be the space and the subset. To prove that X is connected, let $X = U \cup V$, where U and V are disjoint sets open in X , and prove that one of them is empty (cf. 11.A). $U \cap A$ and $V \cap A$ are disjoint sets open in A , and

$$A = X \cap A = (U \cup V) \cap A = (U \cap A) \cup (V \cap A).$$

Since A is connected, one of these sets, say $U \cap A$, is empty. Then U is empty since A is dense, see 6.M.

11.C To simplify the notation, we may assume that $X = \bigcup_{\lambda} A_{\lambda}$. By Theorem 11.A, it suffices to prove that if U and V are two open sets partitioning X , then either $U = \emptyset$ or $V = \emptyset$. For each $\lambda \in \Lambda$, since A_{λ} is connected, we have either $A_{\lambda} \subset U$ or $A_{\lambda} \subset V$ (see 11.14). Fix a $\lambda_0 \in \Lambda$. To be definite, let $A_{\lambda_0} \subset U$. Since each of the sets A_{λ} meets A_{λ_0} , all sets A_{λ} also lie in U , and so none of them meets V , whence

$$V = V \cap X = V \cap \bigcup_{\lambda} A_{\lambda} = \bigcup_{\lambda} (V \cap A_{\lambda}) = \emptyset.$$

11.E Apply Theorem 11.C to the family $\{A_{\lambda} \cup A_{\lambda_0}\}_{\lambda \in \Lambda}$, which consists of connected sets by 11.D. (Or just repeat the proof of Theorem 11.C.)

11.F Using 11.D, prove by induction that $\bigcup_{-n}^n A_k$ is connected, and apply Theorem 11.C.

11.G The union of all connected sets containing a given point is connected (by 11.C) and obviously maximal.

11.H Let A and B be two connected components with $A \cap B \neq \emptyset$. Then $A \cup B$ is connected by 11.D. By the maximality of connected components, we have $A \supset A \cup B \subset B$, whence $A = A \cup B = B$.

11.I \Leftrightarrow This is obvious since the component is connected. \Leftarrow Since the components of the points are not disjoint, they coincide.

11.K If A is a connected component, then its closure $\text{Cl } A$ is connected by 11.B. Therefore, $\text{Cl } A \subset A$ by the maximality of connected components. Hence, $A = \text{Cl } A$, because the opposite inclusion holds true for any set A .

11.M See 11.10.

11.N Passing to the map $\text{ab } f : X \rightarrow f(X)$, we see that it suffices to prove the following theorem:

If X is a connected space and $f : X \rightarrow Y$ is a continuous surjection, then Y is also connected.

Consider a partition of Y in two open sets U and V and prove that one of them is empty. The preimages $f^{-1}(U)$ and $f^{-1}(V)$ are open by continuity of f and constitute a partition of X . Since X is connected, one of them, say $f^{-1}(U)$, is empty. Since f is surjective, we also have $U = \emptyset$.

11.Q \Leftrightarrow Let $X = U \cup V$, where U and V are nonempty disjoint sets open in X . Set $f(x) = -1$ for $x \in U$ and $f(x) = 1$ for $x \in V$. Then f is continuous and surjective, is it not? \Leftrightarrow Assume the contrary: let X be connected. Then S^0 is also connected by 11.N, a contradiction.

11.R By Theorem 11.Q, this statement follows from Cauchy Intermediate Value Theorem. However, it is more natural to deduce Intermediate Value Theorem from 11.Q and the connectedness of I .

Thus assume the contrary: let $I = [0, 1]$ be disconnected. Then $[0, 1] = U \cup V$, where U and V are disjoint and open in $[0, 1]$. Suppose $0 \in U$, consider the set $C = \{x \in [0, 1] \mid [0, x] \subset U\}$ and put $c = \sup C$. Show that each of the possibilities $c \in U$ and $c \in V$ gives rise to contradiction. A slightly different proof of Theorem 11.R is sketched in Lemmas 11.R.1 and 11.R.2.

11.R.1 Use induction: for $n = 1, 2, 3, \dots$, put

$$(a_{n+1}, b_{n+1}) := \begin{cases} (\frac{a_n+b_n}{2}, b_n) & \text{if } \frac{a_n+b_n}{2} \in U, \\ (a_n, \frac{a_n+b_n}{2}) & \text{if } \frac{a_n+b_n}{2} \in V. \end{cases}$$

11.R.2 On the one hand, we have $c \in U$ since $c \in \text{Cl}\{a_n \mid n \in \mathbb{N}\}$, and a_n belong to U , which is closed in I . On the other hand, we have $c \in V$ since $c \in \text{Cl}\{b_n \mid n \in \mathbb{N}\}$, and b_n belong to V , which is also closed in I . The contradiction means that U and V cannot be both closed, i.e., I is connected.

11.S Every open set on a line is a union of disjoint open intervals (see 2.Ax), each of which contains a rational point. Therefore each open subset U of a line is a union of a countable collection of open intervals. Each of them is open and connected, and thus is a connected component of U (see 11.T).

11.T Apply 11.R and 11.J. (Cf. 11.U and 11.X.)

11.U Apply 11.R and 11.J. (Recall that a set $K \subset \mathbb{R}^n$ is said to be convex if for any $p, q \in K$ we have $[p, q] \subset K$.)

11.V Combine 11.R and 11.C.

11.X \Leftrightarrow This is 11.10. \Leftarrow This is 11.V.

11.Y Singletons and all kinds of intervals (including open and closed rays and the whole line).

11.Y Use 10.R, 11.U, and, say Theorem 11.B (or 11.I).

12.A Since the segment $[a, b]$ is connected by 11.R, its image is an interval by 11.29. Therefore, it contains all points between $f(a)$ and $f(b)$.

12.B Combine 11.N and 11.10.

12.C Combine 11.V and 11.29.

12.D One of them is connected, while the other one is not.

12.E For each of the spaces, find the number of points with connected complement. (This is obviously a topological invariant.)

12.F Cf. 12.4.

13.A Since the cover $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ of $[0, 1]$ is fundamental and the restriction of uv to each element of the cover is continuous, the entire map uv is also continuous.

13.B If $x, y \in I$, then $I \rightarrow I : t \mapsto (1 - t)x + ty$ is a path connecting x and y .

13.C If $x, y \in \mathbb{R}^n$, then $[0, 1] \rightarrow \mathbb{R}^n : t \mapsto (1 - t)x + ty$ is a path connecting x and y .

13.D Use 10.R and 13.C.

13.E Combine 11.R and 11.Q.

13.7 Use (the formula of) 13.C, 13.A, and 13.5.

13.F Let x and y be two points in the union, and let A and B be the sets in the family that contain x and y . If $A = B$, there is nothing to prove. If $A \neq B$, take $z \in A \cap B$, join x with z in A by a path u , and join y with z in B by a path v . Then the path uv joins x and y in the union, and it remains to use 13.5.

13.G Consider the union of all path-connected sets containing the point and use 13.F. (Cf. 11.G.)

13.H Similarly to 11.H, only instead of 11.D use 13.F.

13.I \Leftrightarrow Recall the definition of a path-connected component. \Leftarrow This follows from (the proof of) 13.G.

13.J Let X be path-connected, let $f : X \rightarrow Y$ be a continuous map, and let $y_1, y_2 \in f(X)$. If $y_i = f(x_i)$, $i = 1, 2$, and u is a path joining x_1 and x_2 , then how can you construct a path joining y_1 and y_2 ?

13.M Combine 13.8 and 11.J.

13.N By 10.Q, A is homeomorphic to $(0, +\infty) \cong \mathbb{R}$, which is path-connected by 13.C, and so A is also path-connected by 13.K. Since A is connected (combine 11.T and 11.O, or use 13.M) and, obviously, $A \subset X \subset \text{Cl}A$ (what is $\text{Cl}A$, by the way?), it follows from 11.15 that X is also connected.

13.O This is especially obvious for A since $A \cong (0, \infty)$ (you can also use 11.2).

13.P Prove that any path in X starting at $(0, 0)$ is constant.

13.Q Let A and X be as above. Check that A is dense in X (cf. the solution to 13.N) and plug in Problems 13.N and 13.P.

13.R See 13.Q.

13.S Let C be a path-connected component of X , $x \in C$ an arbitrary point. If U_x is a path-connected neighborhood of x , then U_x lies entirely in C (by the definition of a path-connected component!), and so x is an interior point of C , which is thus open.

13.T \Leftrightarrow This is 13.M. \Leftarrow Since path-connected components of X are open (see Problem 13.S) and X is connected, there can be only one path-connected component.

13.U This follows from 13.T because spherical neighborhoods in \mathbb{R}^n (i.e., open balls) are path-connected (by 13.6 or 13.7).

14.A If $r_1 + r_2 \leq \rho(x_1, x_2)$, then the balls $B_{r_1}(x_1)$ and $B_{r_2}(x_2)$ are disjoint.

14.B Certainly, I is Hausdorff since it is metrizable. The intervals $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are disjoint neighborhoods of 0 and 1, respectively.

14.C \Rightarrow If $y \neq x$, then there exist disjoint neighborhoods U_x and V_y . Therefore, $y \notin \text{Cl}U_x$, whence $y \notin \bigcap_{U \ni x} \text{Cl}U$.

\Leftarrow If $y \neq x$, then $y \notin \bigcap_{U \ni x} \text{Cl}U$, it follows that there exists a neighborhood U_x such that $y \notin \text{Cl}U_x$. Set $V_y = X \setminus \text{Cl}U_x$.

14.D Assume the contrary: let $x_n \rightarrow a$ and $x_n \rightarrow b$, where $a \neq b$. Let U and V be disjoint neighborhoods of a and b , respectively. Then for sufficiently large n we have $x_n \in U \cap V = \emptyset$, a contradiction.

14.E A neighborhood of a point in \mathbb{R}_{T_1} has the form $U = \mathbb{R} \setminus \{x_1, \dots, x_N\}$, where, say, $x_1 < x_2 < \dots < x_N$. Then, obviously, $a_n \in U$ for each $n > x_N$.

14.F Assume that X is a space, $A \subset X$ is a subspace, and $x, y \in A$ are two distinct points. If X is Hausdorff, then x and y have disjoint neighborhoods U and V in X . In this case, $U \cap A$ and $V \cap A$ are disjoint neighborhoods of x and y in A . (Recall the definition of the relative topology!)

14.G (a) \Leftrightarrow Let X satisfy T_1 and let $x \in X$. By Axiom T_1 , each point $y \in X \setminus x$ has a neighborhood U that does not contain x , i.e., $U \subset X \setminus x$, which means that all points in $X \setminus x$ are inner. Therefore, $X \setminus x$ is open, and so its complement $\{x\}$ is closed. \Leftarrow If singletons in X are closed and $x, y \in X$ are two distinct points, then $X \setminus x$ is a neighborhood of y that does not contain x , as required in T_1 .

(b) \Leftrightarrow If singletons in X are closed, then so are finite subsets of X , which are finite unions of singletons. \Leftarrow Obvious.

14.H Combine 14.12 and 14.G.

14.I Combine 14.A and 14.12.

14.J Each point in \mathbb{R}_{T_1} is closed, as required by T_1 , but any two nonempty sets intersect, which contradicts T_2 .

14.K Combine 14.G and 5.4, and once more use 14.G; or just modify the proof of 14.F.

14.N (a) \Rightarrow (b) Actually, T_0 precisely says that at least one of the points does not lie in the closure of the other (to see this, use Theorem 6.F). (b) \Rightarrow (a) Use the above reformulation of T_0 and the fact that if $x \in \text{Cl}\{y\}$ and $y \in \text{Cl}\{x\}$, then $\text{Cl}\{x\} = \text{Cl}\{y\}$.

(a) \Leftrightarrow (c) This is obvious. (Recall the definition of the relative topology!)

(c) \Leftrightarrow (d) This is also obvious.

14.O \Leftrightarrow This is obvious. \Leftarrow Let X be a T_0 space such that each point $x \in X$ has a smallest neighborhood C_x . Then we say that $x \preceq y$ if $y \in C_x$. Let us verify the axioms of order. Reflexivity is obvious. Transitivity: assume that $x \preceq y$ and $y \preceq z$. Then C_x is a neighborhood of y , whence $C_y \subset C_x$, and so also $z \in C_x$, which means that $x \preceq z$. Antisymmetry: if $x \preceq y$ and $y \preceq x$, then $y \in C_x$ and $x \in C_y$, whence $C_x = C_y$. By T_0 , this is possible only if $x = y$. Verify that this order generates the initial topology.

14.P Let X be a regular space, and let $x, y \in X$ be two distinct points. Since X satisfies T_1 , the singleton $\{y\}$ is closed, and so we can apply T_3 to x and $\{y\}$.

14.Q \Leftrightarrow See Problem 14.P. \Leftarrow See Problem 14.12.

14.R Let X be a metric space, $x \in X$, and $r > 0$. Prove that, e.g., $\text{Cl} B_r(x) \subset B_{2r}(x)$, and use 14.19.

14.S Apply T_4 to a closed set and a singleton, which is also closed by T_1 .

14.T \Leftrightarrow See Problem 14.S. \Leftarrow See Problem 14.12.

14.U Let A and B be two disjoint closed sets in a metric space (X, ρ) . Then, obviously, $A \subset U = \{x \in X \mid \rho(x, A) < \rho(x, B)\}$ and $B \subset V = \{x \in X \mid \rho(x, A) > \rho(x, B)\}$. U and V are open (use 9.L) and disjoint.

14.Ax.1 Put $U_1 = X \setminus B$. Since X is normal, there exists an open neighborhood $U_0 \supset A$ such that $\text{Cl}U_0 \subset U_1$. Let $U_{1/2}$ be an open neighborhood of $\text{Cl}U_0$ such that $\text{Cl}U_{1/2} \subset U_1$. Repeating the process, we obtain the required collection $\{U_p\}_{p \in \Lambda}$.

14.Ax Put $f(x) = \inf\{\lambda \in \Lambda \mid x \in \text{Cl}U_\lambda\}$. We easily see that f is continuous.

14.Bx Slightly modify the proof of 14.9x, using Urysohn Lemma 14.Ax instead of 14.9x.1.

15.A Let $f : X \rightarrow \mathbb{N}$ be an injection and let $A \subset X$. Then the restriction $f|_A : A \rightarrow \mathbb{N}$ is also an injection. Use 15.1.

15.B Let X be a countable set, and let $f : X \rightarrow Y$ be a map. Taking each $y \in f(X)$ to a point in $f^{-1}(y)$, we obtain an injection $f(X) \rightarrow X$. Hence, $f(X)$ is countable by 15.1.

15.D Suggest an algorithm (or even a formula!) for enumerating elements in \mathbb{N}^2 .

15.E Use 15.D.

15.G Derive this from 6.44.

15.H Construct a countable set A intersecting each base set (at least) at one point and prove that A is everywhere dense.

15.I Let X be a second countable space, $A \subset X$ a subspace. If $\{U_i\}_1^\infty$ is a countable base in X , then $\{U_i \cap A\}_1^\infty$ is a countable base in A . (See 5.1.)

15.J Show that if the set $A = \{x_n\}_{n=1}^\infty$ is everywhere dense, then the collection $\{B_r(x) \mid x \in A, r \in \mathbb{Q}, r > 0\}$ is a countable base of X . (Use Theorems 4.I and 3.A to show that this is a base and 15.E to show that it is countable.)

15.L Use 15.K and 15.I.

15.M By 15.K and 15.I (or, more to the point, combine 15.J, 15.I, and 15.H), it is sufficient to find a countable everywhere-dense set in \mathbb{R}^n . For example, take $\mathbb{Q}^n = \{x \in \mathbb{R}^n \mid x_i \in \mathbb{Q}, i = 1, \dots, n\}$. To see that \mathbb{Q}^n is dense in \mathbb{R}^n , use the metric $\rho^{(\infty)}$. To see that \mathbb{Q}^n is countable, use 15.F and 15.E.

15.N Use 9.15.

15.O Let X be the space, let $\{U\}$ be a countable base in X , and let $\Gamma = \{V\}$ be a cover of X . Let $\{U_i\}_{i=1}^{\infty}$ be the base sets that are contained in at least element of the cover: let $U_i \subset V_i$. Using the definition of a base, we easily see that $\{U_i\}_{i=1}^{\infty}$ is a cover of X . Then $\{V_i\}_{i=1}^{\infty}$ is the required countable subcovering of Γ .

15.P Use 3.A.

15.Q Use 15.12

15.R Use 15.P and 15.A.

15.S Consider an uncountable discrete space.

15.T If $x_n \in A$ and $x_n \rightarrow a$, then, obviously, a is an adherent point for A .

15.U Let $a \in \text{Cl} A$, and let $\{U_n\}_{n \in \mathbb{N}}$ be a decreasing neighborhood base at a (see 15.16). For each n , there is $x_n \in U_n \cap A$, and we easily see that $x_n \rightarrow a$.

15.V Indeed, let $f : X \rightarrow Y$ be a continuous map, let $b \in X$, and let $a_n \rightarrow b$ in X . We must prove that $f(a_n) \rightarrow f(b)$ in Y . Let $V \subset Y$ be a neighborhood of $f(b)$. Since f is continuous, $f^{-1}(V) \subset X$ is a neighborhood of b , and since $a_n \rightarrow b$, we have $a_n \in f^{-1}(V)$ for $n > N$. Then also $f(a_n) \in V$ for $n > N$, as required.

15.W Assume that $f : X \rightarrow Y$ is a sequentially continuous map and $A \subset Y$ is a sequentially closed set. To prove that $f^{-1}(A)$ is sequentially closed, we must prove that if $\{x_n\} \subset f^{-1}(A)$ and $x_n \rightarrow a$, then $a \in f^{-1}(A)$. Since f is sequentially continuous, we have $f(x_n) \rightarrow f(a)$, and since A is sequentially closed, we have $f(a) \in A$, whence $a \in f^{-1}(A)$, as required.

15.X It suffices to check that if $F \subset Y$ is a closed set, then so is the preimage $f^{-1}(F) \subset X$, i.e., $\text{Cl}(f^{-1}(F)) \subset f^{-1}(F)$. Let $a \in \text{Cl}(f^{-1}(F))$. Since X is first countable, we also have $a \in \text{SCL}(f^{-1}(F))$ (see 15.U), and so there is a sequence $\{x_n\} \subset f^{-1}(F)$ such that $x_n \rightarrow a$, whence $f(x_n) \rightarrow f(a)$ because f is sequentially continuous. Since F is closed, we have $f(a) \in F$ (by 15.T), i.e., $a \in f^{-1}(F)$, as required.

15.Ax Since l_2 is a metric space, it is sufficient to prove that l_2 is separable (see 15.K), i.e., to find a countable everywhere dense set $A \subset l_2$. The first idea here might be to consider the set of sequences with rational components, but this set is uncountable! Instead of this, let A be the set of all rational sequences $\{x_i\}$ such that $x_i = 0$ for all sufficiently large i . (To show that A is countable, use 15.F and 15.E. To show that A is everywhere dense, use the fact that if a series $\sum x_i^2$ converges, then for each $\varepsilon > 0$ there is k such that $\sum_{i=k}^{\infty} x_i^2 < \varepsilon$.)

16.A Each of the spaces has only a finite number of open sets, and so each open cover is finite.

16.B Only the finite ones. (Consider the cover consisting of all singletons.)

16.C Consider the cover of \mathbb{R} by the open intervals $(-n, n)$, $n \in \mathbb{N}$.

16.D The latter condition is precisely the negation of compactness.

16.E This follows from the Lindelöf theorem 15.O.

16.F This follows from the second De Morgan formula (see 2.E). Indeed, $\bigcap A_\lambda \neq \emptyset$ iff $\bigcup (X \setminus A_\lambda) = X \setminus \bigcap A_\lambda \neq X$.

16.G \Leftrightarrow Let X be a compact space and let $\Gamma = \{F_\lambda\}$ be a family of closed subsets of X with the finite intersection property. Assume the contrary: let $\bigcap F_\lambda = \emptyset$. Then by the second De Morgan formula we have $\bigcup (X \setminus F_\lambda) = X \setminus \bigcap F_\lambda = X$, i.e., $\{X \setminus F_\lambda\}$ is an open cover of X . Since X is compact, this cover contains a finite subcovering: $\bigcup_1^n (X \setminus F_i) = X$, whence $\bigcap_1^n F_i = \emptyset$, which contradicts the finite intersection property of Γ .

\Leftarrow Prove the converse implication on your own.

16.H \Leftrightarrow Let $\Gamma = \{U_\alpha\}$ be a cover of A by open subsets of X . Since A is a compact set, the cover of A with the sets $A \cap U_\alpha$ contains a finite subcovering $\{A \cap U_{\alpha_i}\}_1^n$. Hence $\{U_{\alpha_i}\}$ is a finite subcovering of Γ .

\Leftarrow Prove the converse implication on your own.

16.I Certainly not.

16.J Let X be a compact space, $F \subset X$ a closed subset, and $\{U_\alpha\}$ an open cover of A . Then $\{X \setminus F\} \cup \{U_\alpha\}$ is an open cover of X , which contains a finite subcovering $\{X \setminus F\} \cup \{U_i\}_1^n$. Clearly, $\{U_i\}_1^n$ is a cover of F .

16.K This follows from 16.L.

16.L Since X is Hausdorff, for each $x \in A$ the points x and b possess disjoint neighborhoods U_x and $V_b(x)$. Obviously, $\{U_x\}_{x \in A}$ is an open cover of A . Since A is compact, the cover contains a finite subcovering $\{U_{x_i}\}_1^n$. Put $U = \bigcup_1^n U_{x_i}$ and $V = \bigcap_1^n V_b(x_i)$. Then U and V are the required sets. (Check that they are disjoint.)

16.M Combine 16.J and 16.L.

16.N This follows from 16.O.

16.O (Cf. the proof of Lemma 16.L.) Let X be a Hausdorff space, and let $A, B \subset X$ be two compact sets. By Lemma 16.L, each $x \in B$ has a neighborhood V_x disjoint with a certain neighborhood $U(x)$ of A . Obviously, $\{V_x\}_{x \in B}$ is an open cover of B . Since B is compact, the cover contains a finite subcovering $\{V_{x_i}\}_1^n$. Put $V = \bigcup_1^n V_{x_i}$ and $U = \bigcap_1^n U_b(x_i)$. Then U and V are the required neighborhoods. (Check that they are disjoint.)

16.P Let us argue by contradiction. If I is not compact, then I has a cover Γ_0 such that no finite part of Γ_0 covers I (see 16.D). We bisect I

and denote by I_1 the half that also is not covered by any finite part of Γ_0 . Then we bisect I_1 , etc. As a result, we obtain a sequence of nested segments I_n , where the length of I_n is equal to 2^{-n} . By the completeness axiom, they have a unique point in common: $\bigcap_1^\infty I_n = \{x_0\}$. Consider an element $U_0 \in \Gamma_0$ containing x_0 . Since U_0 is open, we have $I_n \subset U_0$ for sufficiently large n , in contradiction to the fact that, by construction, I_n is covered by no finite part of Γ_0 .

16.Q Repeat the argument used in the proof of Theorem 16.P, only instead of bisecting the segment each time subdivide the current cube into 2^n equal smaller cubes.

16.R Consider the cover by open balls, $\{B_n(x_0)\}_{n=1}^\infty$.

16.S Let, e.g., $X = [0, 1) \cup [2, 3]$. (Or just put $X = [0, 1)$.) The set $[0, 1)$ is bounded, it is also closed in X , but it is not compact.

16.T \Leftrightarrow Combine Theorems 14.A, 16.K, and 16.R.

\Leftrightarrow If a subset $F \subset \mathbb{R}^n$ is bounded, then F lies in a certain cube, which is compact (see Theorem 16.Q). If, in addition, F is closed, then F is also compact by 16.J.

16.U We use Theorem 16.H. Let $\Gamma = \{U_\lambda\}$ be a cover of $f(X)$ by open subsets of Y . Since f is continuous, $\{f^{-1}(U_\lambda)\}$ is an open cover of X . Since X is compact, this cover has a finite subcovering $\{f^{-1}(U_{\lambda_i})\}_{i=1}^n$. Then $\{U_{\lambda_i}\}_{i=1}^n$ is a finite subcovering of Γ .

16.V By 16.U and 16.T, the set $f(X) \subset \mathbb{R}$ is closed and bounded. Since $f(X)$ is bounded, there exist finite numbers $m = \inf f(X)$ and $M = \sup f(X)$, whence, in particular, $m \leq f(x) \leq M$. Since $f(X)$ is closed, we have $m, M \in f(X)$, whence it follows that there are $a, b \in X$ with $f(a) = m$ and $f(b) = M$, as required.

16.W This follows from 16.23: consider the cover $\{f^{-1}(U) \mid U \in \Gamma\}$ of X .

16.X This immediately follows from 16.J, 16.K, and 16.U.

16.Y Combine 16.X and 16.24.

16.Z See Problem 16.Y.

17.A.1 \Leftrightarrow This is obvious. \Leftrightarrow Let x be a limit point. If x is not an accumulation point of A , then x has a neighborhood U_x such that the set $U_x \cap A$ is finite. Show that x has a neighborhood W_x such that $(W_x \setminus x) \cap A = \emptyset$.

17.A.2 Argue by contradiction: consider the cover of the space by neighborhoods having finite intersections with the infinite set.

17.A.3 Let X be a space, and let $\{a_n\}$ be a sequence of points in X . Let A be the set of all points in the sequence. If A is finite, there is not

much to prove. So, we assume that A is infinite. By Theorem 17.A.2, A has an accumulation point x_0 . Let $\{U_n\}$ be a countable neighborhood base of x_0 and $x_{n_1} \in U_1 \cap A$. Since the set $U_2 \cap A$ is infinite, there is $n_2 > n_1$ such that $x_{n_2} \in U_2 \cap A$. Prove that the subsequence $\{x_{n_k}\}$ thus constructed converges to x_0 . If A is finite, then the argument simplifies a great deal.

17.B.1 Consider a sequence $\{x_n\}$, $x_n \in F_n$ and show that if $x_{n_k} \rightarrow x_0$, then $x_n \in F_n$ for all $n \in \mathbb{N}$.

17.B.2 \Leftrightarrow Let $\{F_k\} \subset X$ be a sequence of closed sets the finite intersection property. Then $\{\bigcap_1^n F_k\}$ is a nested sequence of nonempty closed sets, whence $\bigcap_1^\infty F_k \neq \emptyset$. \Leftarrow This is obvious.

17.B.3 By the Lindelöf theorem 15.O, it is sufficient to consider countable covers $\{U_n\}$. If no finite collection of sets in this cover is not a cover, then the closed sets $F_n = X \setminus U_n$ form a collection with the finite intersection property.

17.C This follows from 17.B and 17.A.

17.D Reformulate the definition of an ε -net: A is an ε -net if $\{B_\varepsilon(x)\}_{x \in A}$ is a cover of X . Now the proof is obvious.

17.E We argue by contradiction. If $\{x_i\}_{i=1}^{k-1}$ is not an ε -net, then there is a point x_k such that $\rho(x_i, x_k) \geq \varepsilon$, $i = 1, \dots, k-1$. As a result, we obtain a sequence in which the distance between any two points is at least ε , and so it has no convergent subsequences.

17.F \Leftrightarrow This is obvious because open balls in a metric space are open sets. \Leftarrow Use the definition of the metric topology.

17.G The union of finite $\frac{1}{n}$ -nets of the space is countable and everywhere dense. (see 17.E).

17.H Use 13.82.

17.I If X is compact, then X is sequentially compact by 17.A. If X is sequentially compact, then X is separable, and hence X has a countable base. Then 17.C implies that X is compact.

17.J Assume that $\{x_n\}$ is a Cauchy sequence and its subsequence x_{n_k} converges to a point a . Find a number m such that $\rho(x_l, x_k) < \frac{\varepsilon}{2}$ for $k, l \geq m$, and i such that $n_i > m$ and $\rho(x_{n_i}, a) < \frac{\varepsilon}{2}$. Then for all $l \geq m$ we have the inequality $\rho(x_l, a) \leq \rho(x_l, x_{n_i}) + \rho(x_{n_i}, a) < \varepsilon$.

17.K \Leftarrow Obvious. \Rightarrow Let $\{x_n\}$ be a Cauchy sequence. Let n_1 be such that $\rho(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_1$. Therefore, $x_n \in B_{1/2}(x_{n_1})$ for all $n \geq n_1$. Further, take $n_2 > n_1$ so that $\rho(x_n, x_m) < \frac{1}{4}$ for all $n, m \geq n_2$, then $B_{1/4}(x_{n_2}) \subset B_{1/2}(x_{n_1})$. Proceeding the construction, we obtain a sequence

of decreasing disks such that their unique common point x_0 satisfies $x_n \rightarrow x_0$.

17.L Let $\{x_n\}$ be a Cauchy sequence of points of a compact metric space X . Since X is also sequentially compact, $\{x_n\}$ contains a convergent subsequence, and then the initial sequence also converges.

17.M \Leftrightarrow Each compact space contains a finite ε -net.

\Leftarrow Let us show that the space is sequentially compact. Consider an arbitrary sequence $\{x_n\}$. We denote by A_n a finite $\frac{1}{n}$ -net in X . Since $X = \bigcup_{x \in A_1} B_1(x)$, one of the balls contains infinitely many points of the sequence; let x_{n_1} be the first of them. From the remaining members lying in the first ball, we let x_{n_2} be the first one of those lying in the ball $B_{1/2}(x)$, $x \in A_2$. Proceeding with this construction, we obtain a subsequence $\{x_{n_k}\}$. Let us show that the latter is fundamental. Since by assumption the space is complete, the constructed sequence has a limit. We have thus proved that the space is sequentially compact, hence, it is also compact.

17.N \Leftarrow Obvious. \Rightarrow This follows from assertion 17.M because an $\frac{\varepsilon}{2}$ -net for a $\frac{\varepsilon}{2}$ -net is an ε -net for the entire space.

18.Ax No, it is not: consider $\mathbb{Q} \subset \mathbb{R}$.

18.Bx Let X be a locally compact space, $F \subset X$ a closed subset space, $x \in F$. Let $U_x \subset X$ be a neighborhood of x with compact closure. Then $U_x \cap F$ is a neighborhood of x in F . Since F is closed, the set $\text{Cl}_F(U \cap F) = (\text{Cl}U) \cap F$ (see 6.3) is compact as a closed subset of a compact set.

18.Cx No, this is wrong in general. Take any space (X, Ω) that is not locally compact (e.g., let $X = \mathbb{Q}$). We put $X^* = X \cup x_*$ and $\Omega^* = \{X^*\} \cup \Omega$. The space (X^*, Ω^*) is compact for a trivial reason (which one?), hence, it is locally compact. Now, X is an open subset of X^* , but it is not locally compact by our choice of X .

18.Dx Let X be the space, W be a neighborhood of a point $x \in X$. Let U_0 be a neighborhood of x with compact closure. Since X is Hausdorff, it follows that $\{x\} = \bigcap_{U \ni x} \text{Cl}U$, whence $\{x\} = \bigcap_{U \ni x} (\text{Cl}U_0 \cap \text{Cl}U)$. Since each of the sets $\text{Cl}U_0 \cap \text{Cl}U$ is compact, 16.11 implies that x has neighborhoods U_1, \dots, U_n such that $\text{Cl}U_0 \cap \text{Cl}U_1 \cap \dots \cap \text{Cl}U_n \subset W$. Put $V = U_0 \cap U_1 \cap \dots \cap U_n$. Then $\text{Cl}V \subset W$. Therefore, each neighborhood of x contains the closure of a certain neighborhood (a “closed neighborhood”) of x . By 14.19, X is regular.

18.Ex Let X be the space, $V \subset X$ the open subset, $x \in V$ a point. Let U be a neighborhood of x such that $\text{Cl}U$ is compact. By 18.Dx and 14.19, x has a neighborhood W such that $\text{Cl}W \subset U \cap V$. Therefore, $\text{Cl}_V W = \text{Cl}W$ is compact, and so the space V is locally compact.

18.Fx \Leftrightarrow Obvious. \Leftarrow See the idea used in 18.Ex.

18.Gx Since \emptyset is both open and compact in X , we have $\emptyset, X^* \in \Omega^*$. Let us verify that unions and finite intersections of subsets in Ω^* lie in Ω^* . This is obvious for subsets in Ω . Let $X^* \setminus K_\lambda \in \Omega^*$, where $K_\lambda \subset X$ are compact sets, $\lambda \in \Lambda$. Then we have $\bigcup(X^* \setminus K_\lambda) = X^* \setminus \bigcap K_\lambda \in \Omega^*$ because X is Hausdorff and so $\bigcap K_\lambda$ is compact. Similarly, if Λ is finite, then we also have $\bigcap(X^* \setminus K_\lambda) = X^* \setminus \bigcup K_\lambda \in \Omega^*$. Therefore, it suffices to consider the case where a set in Ω^* and a set in Ω are united (intersected). We leave this as an exercise.

18.Hx Let $U = X^* \setminus K_0$ be an element of the cover that contains the added point. Then the remaining elements of the cover provide an open cover of the compact set K_0 .

18.Ix In other words, the topology of X^* induced on X the initial topology of X (i.e., $\Omega^* \cap 2^X = \Omega$). We must check that there arise no new open sets in X . This is true because compact sets in the Hausdorff space X are closed.

18.Jx If $x, y \in X$, this is obvious. If, say, $y = x_*$ and U_x is a neighborhood of x with compact closure, then U_x and $X \setminus \text{Cl}U_x$ are neighborhoods separating x and x_* .

18.Kx Let $X^* \setminus X = \{x_*\}$ and $Y \setminus X = \{y\}$. We have an obvious bijection

$$f : Y \rightarrow X^* : x \mapsto \begin{cases} x & \text{if } x \in X, \\ x_* & \text{if } x = y. \end{cases}$$

If $U \subset X^*$ and $U = X^* \setminus K$, where K is a compact set in X , then the set $f^{-1}(U) = Y \setminus K$ is open in Y . Therefore, f is continuous. It remains to apply 16.Y.

18.Lx Verify that if an open set $U \subset S^2$ contains the “North Pole” $(0, 0, 1)$ of S^2 , then the complement of the image of U under the stereographic projection is compact in \mathbb{R}^2 .

18.Mx X^* is compact and Hausdorff by 18.Hx and 18.Jx, therefore, X^* is regular by 16.M. Since X is a subspace of X^* by 18.Ix, it remains to use the fact that regularity is hereditary by 14.20. (Also try to prove the required assertion without using the one-point compactification.)

18.Nx \Leftrightarrow If f^* is continuous, then, obviously, so is f (by 18.Ix). Let $K \subset Y$ be a compact set, and let $U = Y \setminus K$. Since f^* is continuous, the set $(f^*)^{-1}(U) = X^* \setminus f^{-1}(K)$ is open in X^* , i.e., $f^{-1}(K)$ is compact in X . Therefore, f is proper. \Leftarrow Use a similar argument.

18.Ox Let $f^* : X^* \rightarrow Y^*$ be the canonical extension of a map $f : X \rightarrow Y$. Prove that if F is closed in X , then $F \cup \{x_*\}$ is closed in X^* , and hence compact. After that, use 18.Nx, 16.X, and 18.Ix.

18.Px A proper injection of a Hausdorff space into a locally compact Hausdorff space is a topological embedding. A proper bijection of a Hausdorff space onto a locally compact Hausdorff space is a homeomorphism.

18.Qx Let Γ be a locally finite cover, and let Δ be a cover of X by neighborhoods each of which meets only a finite number of sets in Γ . Since X is compact, we can assume that Δ is finite. In this case, obviously, Γ is also finite.

18.Rx Cover \mathbb{R}^n by the balls $B_n(0)$, $n \in \mathbb{N}$.

18.Sx Use a locally finite covering of \mathbb{R}^n by equal open cubes.

18.Tx Cf. 18.17x.

18.Ux This is obvious.

18.Vx This is 18.Sx.

18.Wx Let Γ be an open cover of X . Since each of the sets $K_i = X_i \setminus \text{Int } X_{i-1}$ is compact, Γ contains a finite subcovering Γ_i of K_i . Observe that the sets $W_i = \text{Int } X_{i+1} \setminus X_{i-2} \supset K_i$ form a locally finite open cover of X . Intersecting for each i elements of Γ_i with W_i , we obtain a locally finite refinement of Γ .

18.Xx Using assertion 18.6x, construct a sequence of open sets U_i such that for each i the closure $X_i := \text{Cl } U_i$ is compact and lies in $U_{i+1} \subset \text{Int } X_{i+1}$. After that, apply 18.Wx.

18.Yx Let $\Gamma = \{U_\alpha\}$ be the cover. By 18.18x, there exists an open cover $\Delta = \{V_\alpha\}$ such that $\text{Cl } V_\alpha \subset U_\alpha$ for each α . Let $\varphi_\alpha : X \rightarrow I$ be an Urysohn function with $\text{supp } \varphi_\alpha = X \setminus U_\alpha$ and $\varphi_\alpha^{-1}(1) = \text{Cl } V_\alpha$ (see 14.Ax). Put $\varphi(x) = \sum_\alpha \varphi_\alpha(x)$. Then the collection $\{\varphi_\alpha(x)/\varphi(x)\}$ is the required partition of unity.