Topological Constructions

19. Multiplication

19°1. Set-Theoretic Digression: Product of Sets

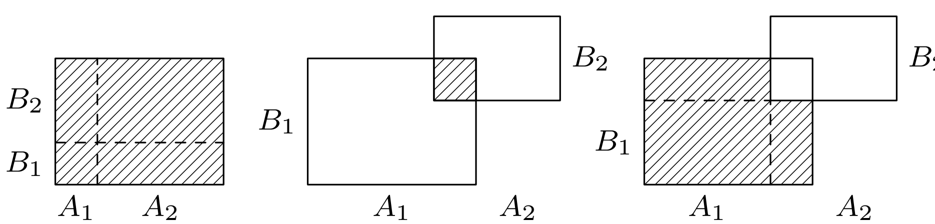
Let X and Y be sets. The set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is called the *direct product* or *Cartesian product* or just *product* of X and Y and denoted by $X \times Y$. If $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$. Sets $X \times b$ with $b \in Y$ and $a \times Y$ with $a \in X$ are *fibers* of the product $X \times Y$.

19.A. Prove that for any $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$ we have

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2),$$

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

$$(A_1 \times B_1) \smallsetminus (A_2 \times B_2) = \big((A_1 \smallsetminus A_2) \times B_1\big) \cap \big(A_1 \times (B_1 \smallsetminus B_2)\big).$$



The natural maps

$$\mathrm{pr}_X : X \times Y \to X : (x, y) \mapsto x \quad \text{and} \quad \mathrm{pr}_Y : X \times Y \to Y : (x, y) \mapsto y$$

are (*natural*) *projections*.

19.B. Prove that $\mathrm{pr}_X^{-1}(A) = A \times Y$ for any $A \subset X$.

> ***19.1.*** Find the corresponding formula for $B \subset Y$.

19°2. Graphs

A map $f : X \to Y$ determines a subset Γ_f of $X \times Y$ defined by $\Gamma_f = \{(x, f(x)) \mid x \in X\}$, it is called the *graph* of f.

19.C. A set $\Gamma \subset X \times Y$ is the graph of a map $X \to Y$ iff for each $a \in X$ the intersection $\Gamma \cap (a \times Y)$ is one-point.

> ***19.2.*** Prove that for any map $f : X \to Y$ and any set $A \subset X$, we have
> $$f(A) = \mathrm{pr}_Y(\Gamma_f \cap (A \times Y)) = \mathrm{pr}_Y(\Gamma_f \cap \mathrm{pr}_X^{-1}(A))$$
> and $f^{-1}(B) = \mathrm{pr}_X(\Gamma \cap (X \times B))$ for any $B \subset Y$.
>
> The set $\Delta = \{(x, x) \mid x \in X\} = \{(x, y) \in X \times X \mid x = y\}$ is the *diagonal* of $X \times X$.
>
> ***19.3.*** Let A and B be two subsets of X. Prove that $(A \times B) \cap \Delta = \varnothing$ iff $A \cap B = \varnothing$.
>
> ***19.4.*** Prove that the map $\mathrm{pr}_X \big|_{\Gamma_f}$ is bijective.
>
> ***19.5.*** Prove that f is injective iff $\mathrm{pr}_Y \big|_{\Gamma_f}$ is injective.
>
> ***19.6.*** Consider the map $T : X \times Y \to Y \times X : (x, y) \mapsto (y, x)$. Prove that $\Gamma_{f^{-1}} = T(\Gamma_f)$ for any invertible map $f : X \to Y$.

19°3. Product of Topologies

Let X and Y be two topological spaces. If U is an open set of X and B is an open set of Y, then we say that $U \times V$ is an *elementary* set of $X \times Y$.

19.D. *The set of elementary sets of $X \times Y$ is a base of a topological structure in $X \times Y$.*

The *product* of two spaces X and Y is the set $X \times Y$ with the topological structure determined by the base consisting of elementary sets.

> ***19.7.*** Prove that for any subspaces A and B of spaces X and Y the product topology on $A \times B$ coincides with the topology induced from $X \times Y$ via the natural inclusion $A \times B \subset X \times Y$.

19.E. $Y \times X$ is canonically homeomorphic to $X \times Y$.

The word *canonically* means here that a homeomorphism between $X \times Y$ and $Y \times X$, which exists according to the statement, can be chosen in a nice special (or even obvious?) way, so that we may expect that it has additional pleasant properties.

19.F. The canonical bijection $X \times (Y \times Z) \to (X \times Y) \times Z$ is a homeomorphism.

19.8. Prove that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

19.9. Prove that $\mathrm{Cl}(A \times B) = \mathrm{Cl}\, A \times \mathrm{Cl}\, B$ for any $A \subset X$ and $B \subset Y$.

19.10. Is it true that $\mathrm{Int}(A \times B) = \mathrm{Int}\, A \times \mathrm{Int}\, B$?

19.11. Is it true that $\mathrm{Fr}(A \times B) = \mathrm{Fr}\, A \times \mathrm{Fr}\, B$?

19.12. Is it true that $\mathrm{Fr}(A \times B) = (\mathrm{Fr}\, A \times B) \cup (A \times \mathrm{Fr}\, B)$?

19.13. Prove that $\mathrm{Fr}(A \times B) = (\mathrm{Fr}\, A \times B) \cup (A \times \mathrm{Fr}\, B)$ for closed A and B.

19.14. Find a formula for $\mathrm{Fr}(A \times B)$ in terms of A, $\mathrm{Fr}\, A$, B, and $\mathrm{Fr}\, B$.

19°4. Topological Properties of Projections and Fibers

19.G. *The natural projections* $\mathrm{pr}_X : X \times Y \to X$ *and* $\mathrm{pr}_Y : X \times Y \to Y$ *are continuous for any topological spaces* X *and* Y*.*

19.H. The topology of product is the coarsest topology with respect to which pr_X and pr_Y are continuous.

19.I. *A fiber of a product is canonically homeomorphic to the corresponding factor. The canonical homeomorphism is the restriction to the fiber of the natural projection of the product onto the factor.*

19.J. Prove that $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$, $(\mathbb{R}^1)^n = \mathbb{R}^n$, and $(I)^n = I^n$. (We remind the reader that I^n is the n-dimensional unit cube in $\mathbb{R}^n$.)

19.15. Let Σ_X and Σ_Y be bases of spaces X and Y. Prove that the sets $U \times V$ with $U \in \Sigma_X$ and $V \in \Sigma_Y$ constitute a base for $X \times Y$.

19.16. Prove that a map $f : X \to Y$ is continuous iff $\mathrm{pr}_X|_{\Gamma_f} : \Gamma_f \to X$ is a homeomorphism.

19.17. Prove that if W is open in $X \times Y$, then $\mathrm{pr}_X(W)$ is open in X.

A map from a space X to a space Y is *open* (*closed*) if the image of any open set under this map is open (respectively, closed). Therefore, *19.17* states that $\mathrm{pr}_X : X \times Y \to X$ is an open map.

19.18. Is pr_X a closed map?

19.19. Prove that for each space X and each compact space Y the map $\mathrm{pr}_X : X \times Y \to X$ is closed.

19°5. Cartesian Products of Maps

Let X, Y, and Z be three sets. A map $f : Z \to X \times Y$ determines the compositions $f_1 = \mathrm{pr}_X \circ f : Z \to X$ and $f_2 = \mathrm{pr}_Y \circ f : Z \to Y$, which are called the *factors* (or *components*) of f. Indeed, f can be recovered from them as a sort of product.

19.K. Prove that for any maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ there exists a unique map $f : Z \to X \times Y$ with $\mathrm{pr}_X \circ f = f_1$ and $\mathrm{pr}_Y \circ f = f_2$.

19.20. Prove that $f^{-1}(A \times B) = f_1^{-1}(A) \cap f_2^{-1}(B)$ for any $A \subset X$ and $B \subset Y$.

19.L. Let X, Y, and Z be three spaces. Prove that $f : Z \to X \times Y$ is continuous iff so are f_1 and f_2.

Any two maps $g_1 : X_1 \to Y_1$ and $g_2 : X_2 \to Y_2$ determine a map

$$g_1 \times g_2 : X_1 \times X_2 \to Y_1 \times Y_2 : (x_1, x_2) \mapsto (g_1(x_1), g_2(x_2)),$$

which is their (*Cartesian*) *product*.

19.21. Prove that $(g_1 \times g_2)(A_1 \times A_2) = g_1(A_1) \times g_2(A_2)$ for any $A_1 \subset X_1$ and $A_2 \subset X_2$.

19.22. Prove that $(g_1 \times g_2)^{-1}(B_1 \times B_2) = g_1^{-1}(B_1) \times g_2^{-1}(B_2)$ for any $B_1 \subset Y_1$ and $B_2 \subset Y_2$.

19.M. Prove that the Cartesian product of continuous maps is continuous.

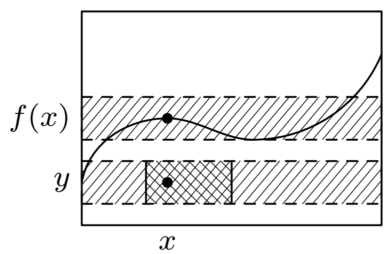
19.23. Prove that the Cartesian product of open maps is open.

19.24. Prove that a metric $\rho : X \times X \to \mathbb{R}$ is continuous with respect to the topology generated by the metric.

19.25. Let $f : X \to Y$ be a map. Prove that the graph Γ_f is the preimage of the diagonal $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$ under the map $f \times \mathrm{id}_Y : X \times Y \to Y \times Y$.

19°6. Properties of Diagonal and Other Graphs

19.26. Prove that a space X is Hausdorff iff the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.



19.27. Prove that if Y is a Hausdorff space and $f : X \to Y$ is a continuous map, then the graph Γ_f is closed in $X \times Y$.

19.28. Let Y be a compact space. Prove that if a map $f : X \to Y$ has closed graph Γ_f, then f is continuous.

19.29. Prove that the hypothesis on compactness in *19.28* is necessary.

19.30. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that its graph is:

(1) closed;
(2) connected;
(3) path connected;
(4) locally connected;
(5) locally compact.

19.31. Consider the following functions

1) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x}, & \text{otherwise.} \end{cases}$; 2) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \sin \frac{1}{x}, & \text{otherwise.} \end{cases}$ Do their graphs possess the properties listed in *19.30*?

19.32. Does any of the properties of the graph of a function f that are mentioned in *19.30* imply that f is continuous?

19.33. Let Γ_f be closed. Then the following assertions are equivalent:

(1) f is continuous;
(2) f is locally bounded;
(3) the graph Γ_f of f is connected;
(4) the graph Γ_f of f is path-connected.

19.34. Prove that if Γ_f is connected and locally connected, then f is continuous.

19.35. Prove that if Γ_f is connected and locally compact, then f is continuous.

19.36. Are some of the assertions in Problems *19.33–19.35* true for maps $f : \mathbb{R}^2 \to \mathbb{R}$?

19°7. Topological Properties of Products

19.N. *The product of Hausdorff spaces is Hausdorff.*

19.37. Prove that the product of regular spaces is regular.

19.38. The product of normal spaces is not necessarily normal.

19.38.1*. Prove that the space $\mathcal{R}$ formed by real numbers with the topology determined by the base consisting of all semi-open intervals $[a, b)$ is normal.

19.38.2. Prove that in the Cartesian square of the space introduced in *19.38.1* the subspace $\{(x, y) \mid x = -y\}$ is closed and discrete.

19.38.3. Find two disjoint subsets of $\{(x, y) \mid x = -y\}$ that have no disjoint neighborhoods in the Cartesian square of the space of *19.38.1*.

19.O. *The product of separable spaces is separable.*

19.P. *First countability of factors implies first countability of the product.*

19.Q. *The product of second countable spaces is second countable.*

19.R. *The product of metrizable spaces is metrizable.*

19.S. *The product of connected spaces is connected.*

19.39. Prove that for connected spaces X and Y and any proper subsets $A \subset X$, $B \subset Y$ the set $X \times Y \smallsetminus A \times B$ is connected.

19.T. *The product of path-connected spaces is path-connected.*

19.U. *The product of compact spaces is compact.*

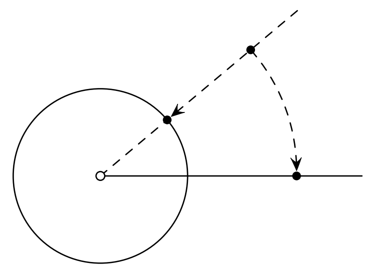
19.40. Prove that the product of locally compact spaces is locally compact.

19.41. If X is a paracompact space and Y is compact, then $X \times Y$ is paracompact.

19.42. For which of the topological properties studied above is it true that if $X \times Y$ possesses the property, then so does X?

19°8. Representation of Special Spaces as Products

19.V. Prove that $\mathbb{R}^2 \smallsetminus 0$ is homeomorphic to $S^1 \times \mathbb{R}$.



19.43. Prove that $\mathbb{R}^n \smallsetminus \mathbb{R}^k$ is homeomorphic to $S^{n-k-1} \times \mathbb{R}^{k+1}$.

19.44. Prove that $S^n \cap \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_k^2 \leq x_{k+1}^2 + \dots + x_{n+1}^2\}$ is homeomorphic to $S^{k-1} \times D^{n-k+1}$.

19.45. Prove that $O(n)$ is homeomorphic to $SO(n) \times O(1)$.

19.46. Prove that $GL(n)$ is homeomorphic to $SL(n) \times GL(1)$.

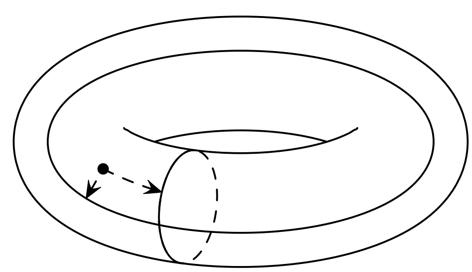
19.47. Prove that $GL_+(n)$ is homeomorphic to $SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$, where

$$GL_+(n) = \{A \in L(n, n) \mid \det A > 0\}.$$

19.48. Prove that $SO(4)$ is homeomorphic to $S^3 \times SO(3)$.

The space $S^1 \times S^1$ is a *torus*.

19.W. Construct a topological embedding of the torus to $\mathbb{R}^3$.



The product $S^1 \times \dots \times S^1$ of k factors is the k-*dimensional torus*.

19.X. Prove that the k-dimensional torus can be topologically embedded into $\mathbb{R}^{k+1}$.

19.Y. Find topological embeddings of $S^1 \times D^2$, $S^1 \times S^1 \times I$, and $S^2 \times I$ into $\mathbb{R}^3$.

20. Quotient Spaces

20°1. Set-Theoretic Digression: Partitions and Equivalence Relations

Recall that a *partition* of a set A is a cover of A consisting of pairwise disjoint sets.

Each partition of a set X determines an *equivalence relation* (i.e., a relation, which is reflexive, symmetric, and transitive): two elements of X are said to be equivalent if they belong to the same element of the partition. Vice versa, each equivalence relation in X determines the partition of X into classes of equivalent elements. Thus, partitions of a set into nonempty subsets and equivalence relations in the set are essentially the same. More precisely, they are two ways of describing the same phenomenon.

Let X be a set, S a partition. The set whose elements are members of the partition S (which are subsets of X) is the *quotient set* or *factor set* of X by S, it is denoted by $X/_S$. [1]

20.1. Riddle. How does this operation relate to division of numbers? Why is there a similarity in terminology and notation?

The set $X/_S$ is also called the *set of equivalence classes* for the equivalence relation corresponding to the partition S.

The map $\mathrm{pr} : X \to X/_S$ that maps $x \in X$ to the element of S containing x is the (*canonical*) *projection* or *factorization map*. A subset of X which is a union of elements of a partition is *saturated*. The smallest saturated set containing a subset A of X is the *saturation* of A.

20.2. Prove that $A \subset X$ is an element of a partition S of X iff $A = \mathrm{pr}^{-1}(\text{point})$, where $\mathrm{pr} : X \to X/_S$ is the natural projection.

20.A. Prove that the saturation of a set A equals $\mathrm{pr}^{-1}\big(\mathrm{pr}(A)\big)$.

20.B. Prove that a set is saturated iff it is equal to its saturation.

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[1] At first glance, the definition of a quotient set contradicts one of the very profound principles of the set theory, which states that a set is determined by its elements. Indeed, according to this principle, we have $X/_S = S$ since S and $X/_S$ have the same elements. Hence, there seems to be no need to introduce $X/_S$. The real sense of the notion of quotient set is not in its literal set-theoretic meaning, but in our way of thinking of elements of partitions. If we remember that they are subsets of the original set and want to keep track of their internal structure (at least, of their elements), then we speak of a partition. If we think of them as atoms, getting rid of their possible internal structure, then we speak about the quotient set.

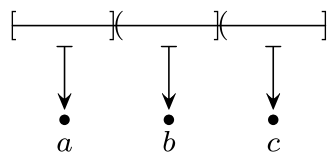
20°2. Quotient Topology

A quotient set $X/_S$ of a topological space X with respect to a partition S into nonempty subsets is provided with a natural topology: a set $U \subset X/_S$ is said to be open in $X/_S$ if its preimage $\mathrm{pr}^{-1}(U)$ under the canonical projection $\mathrm{pr} : X \to X/_S$ is open.

20.C. *The collection of these sets is a topological structure in the quotient set* $X/_S$.

This topological structure is the ***quotient topology***. The set $X/_S$ with this topology is the ***quotient space*** of X by partition S.

> ***20.3.*** Give an explicit description of the quotient space of the segment $[0, 1]$ by the partition consisting of $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3}]$, $(\frac{2}{3}, 1]$.



> ***20.4.*** What can you say about a partition S of a space X if the quotient space $X/_S$ is known to be discrete?

20.D. *A subset of a quotient space* $X/_S$ *is open iff it is the image of an open saturated set under the canonical projection* pr.

20.E. A subset of a quotient space $X/_S$ is closed, iff its preimage under pr is closed in X, iff it is the image of a closed saturated set.

20.F. *The canonical projection* $\mathrm{pr} : X \to X/_S$ *is continuous.*

20.G. Prove that the quotient topology is the finest topology in $X/_S$ such that the canonical projection pr is continuous with respect to it.

20°3. Topological Properties of Quotient Spaces

20.H. *A quotient space of a connected space is connected.*

20.I. *A quotient space of a path-connected space is path-connected.*

20.J. *A quotient space of a separable space is separable.*

20.K. *A quotient space of a compact space is compact.*

20.L. The quotient space of the real line by partition $\mathbb{R}_+$, $\mathbb{R} \smallsetminus \mathbb{R}_+$ is not Hausdorff.

20.M. The quotient space of a space X by a partition S is Hausdorff iff any two elements of S have disjoint saturated neighborhoods.

20.5. Formulate similar necessary and sufficient conditions for a quotient space to satisfy other separation axioms and countability axioms.

20.6. Give an example showing that the second countability can may get lost when we pass to a quotient space.

20°4. Set-Theoretic Digression: Quotients and Maps

Let S be a partition of a set X into nonempty subsets. Let $f : X \to Y$ be a map which is constant on each element of S. Then there is a map $X/_S \to Y$ which sends each element A of S to the element $f(a)$, where $a \in A$. This map is denoted by $f/_S$ and called the *quotient map* or *factor map* of f (by the partition S).

20.N. 1) Prove that a map $f : X \to Y$ is constant on each element of a partition S of X iff there exists a map $g : X/_S \to Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \text{pr}}\downarrow & \nearrow g & \\ X/_S & & \end{array}$$

2) Prove that such a map g coincides with $f/_S$.

More generally, if S and T are partitions of sets X and Y, then every map $f : X \to Y$ that maps each element of S to an element of T determines a map $X/_S \to Y/_T$ which sends an element A of partition S to the element of partition T containing $f(A)$. This map is denoted by $f/_{S,T}$ and called the *quotient map* or *factor map* of f (*with respect to* S and T).

20.O. Formulate and prove for $f/_{S,T}$ a statement generalizing *20.N*.

A map $f : X \to Y$ determines a partition of the set X into nonempty preimages of the elements of Y. This partition is denoted by $S(f)$.

20.P. The map $f/_{S(f)} : X/_{S(f)} \to Y$ is injective.

This map is the *injective factor* (or *injective quotient*) of f.

20°5. Continuity of Quotient Maps

20.Q. *Let X and Y be two spaces, S a partition of X into nonempty sets, and $f : X \to Y$ a continuous map constant on each element of S. Then the factor $f/_S$ of f is continuous.*

20.7. If the map f is open, then so is the quotient map $f/_S$.

20.8. Let X and Y be two spaces, S a partition of X into nonempty sets. Prove that the formula $f \mapsto f/_S$ determines a bijection from the set of all continuous maps $X \to Y$ that are constant on each element of S onto the set of all continuous maps $X/_S \to Y$.

20.R. Let X and Y be two spaces, S and T partitions of X and Y, respectively, and $f : X \to Y$ a continuous map which maps each element of S into an element of T. Then the map $f/_{S,T} : X/_S \to Y/_T$ is continuous.

20°6x. Closed Partitions

A partition S of a space X is *closed* if the saturation of each closed set is closed.

***20.1*x.** Prove that a partition is closed iff the canonical projection $X \to X/_S$ is a closed map.

***20.2*x.** Prove that if a partition S contains only one element consisting of more than one point, then S is closed if this element is a closed set.

***20.A*x.** Let X be a space satisfying the first separation axiom, S a closed partition of X. Then the quotient space $X/_S$ also satisfies the first separation axiom.

***20.B*x.** *The quotient space of a normal space with respect to a closed partition is normal.*

20°7x. Open Partitions

A partition S of a space X is *open* if the saturation of each open set is open.

***20.3*x.** Prove that a partition S is open iff the canonical projection $X \to X/_S$ is an open map.

***20.4*x.** Prove that if a set A is saturated with respect to an open partition, then $\operatorname{Int} A$ and $\operatorname{Cl} A$ are also saturated.

***20.C*x.** *The quotient space of a second countable space with respect to an open partition is second countable.*

***20.D*x.** *The quotient space of a first countable space with respect to an open partition is first countable.*

***20.E*x.** *Let X and Y be two spaces, and let S and T be their open partitions. Denote by $S \times T$ the partition of $X \times Y$ consisting of $A \times B$ with $A \in S$ and $B \in T$. Then the injective factor $X \times Y/_{S \times T} \to X/_S \times Y/_T$ of* $\operatorname{pr} \times \operatorname{pr} X \times Y \to X/_S \times Y/_T$ *is a homeomorphism.*

21. Zoo of Quotient Spaces

21°1. Tool for Identifying a Quotient Space with a Known Space

21.A. *If X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is a continuous map, then the injective factor $f/_{S(f)} : X/_{S(f)} \to Y$ is a homeomorphism.*

21.B. *The injective factor of a continuous map from a compact space to a Hausdorff one is a topological embedding.*

> ***21.1.*** Describe explicitly partitions of a segment such that the corresponding quotient spaces are all letters of the alphabet.
>
> ***21.2.*** Prove that there exists a partition of a segment I with the quotient space homeomorphic to square $I \times I$.

21°2. Tools for Describing Partitions

An accurate literal description of a partition can often be somewhat cumbersome, but usually it can be shortened and made more understandable. Certainly, this requires a more flexible vocabulary with lots of words having almost the same meanings. For instance, such words as *factorize* and *pass to a quotient* can be replaced by *attach*, *glue together*, *identify*, *contract*, *paste*, and other words accompanying these ones in everyday life.

Some elements of this language are easy to formalize. For instance, factorization of a space X with respect to a partition consisting of a set A and one-point subsets of the complement of A is the *contraction* (of the subset A to a point), and the result is denoted by $X/_A$.

> ***21.3.*** Let $A, B \subset X$ form a fundamental cover of a space X. Prove that the quotient map $A/_{A \cap B} \to X/_B$ of the inclusion $A \hookrightarrow X$ is a homeomorphism.

If A and B are two disjoint subspaces of a space X and $f : A \to B$ is a homeomorphism, then passing to the quotient of X by the partition into singletons in $X \smallsetminus (A \cup B)$ and two-point sets $\{x, f(x)\}$, where $x \in A$, we *glue* or *identify* the sets A and B via the homeomorphism f.

A rather convenient and flexible way for describing partitions is to describe the corresponding equivalence relations. The main advantage of this approach is that, by transitivity, it suffices to specify only some pairs of equivalent elements: if one states that $x \sim y$ and $y \sim z$, then it is not necessary to state that $x \sim z$ since this already follows.

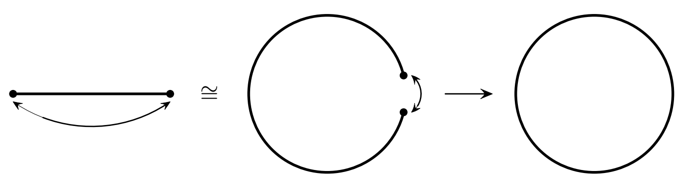
Hence, a partition is represented by a list of statements of the form $x \sim y$ that are sufficient for recovering the equivalence relation. We denote the corresponding partition by such a list enclosed into square brackets. For example, the quotient of a space X obtained by identifying subsets A and B by a homeomorphism $f : A \to B$ is denoted by $X/_{[a \sim f(a) \text{ for any } a \in A]}$ or just $X/_{[a \sim f(a)]}$.

Some partitions are easily described by a picture, especially if the original space can be embedded in the plane. In such a case, as in the pictures below, we draw arrows on the segments to be identified to show the directions to be identified.

Below we introduce all these kinds of descriptions for partitions and give examples of their usage, simultaneously providing literal descriptions. The latter are not that nice, but they may help the reader to remain confident about the meaning of the new words. On the other hand, the reader will appreciate the improvement the new words bring in.

21°3. Welcome to the Zoo

21.C. Prove that $I/_{[0 \sim 1]}$ is homeomorphic to S^1.



In other words, the quotient space of segment I by the partition consisting of $\{0, 1\}$ and $\{a\}$ with $a \in (0, 1)$ is homeomorphic to a circle.

21.C.1. Find a surjective continuous map $I \to S^1$ such that the corresponding partition into preimages of points consists of one-point subsets of the interior of the segment and the pair of boundary points of the segment.

21.D. Prove that $D^n/_{S^{n-1}}$ is homeomorphic to S^n.

In *21.D*, we deal with the quotient space of the n-disk D^n by the partition $\{S^{n-1}\} \cup \{\{x\} \mid x \in B^n\}$.

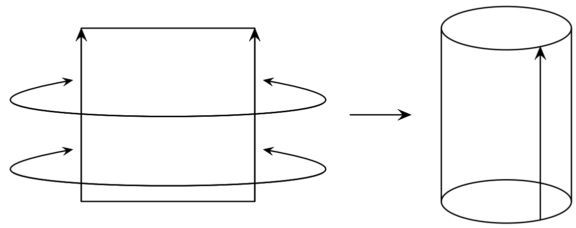
Here is a reformulation of *21.D*: *Contracting* the boundary of an n-dimensional ball to a point, we obtain gives rise an n-dimensional sphere.

21.D.1. Find a continuous map of the n-disk D^n to the n-sphere S^n that maps the boundary of the disk to a single point and bijectively maps the interior of the disk onto the complement of this point.

21.E. Prove that $I^2/_{[(0,t) \sim (1,t) \text{ for } t \in \mathrm{I}]}$ is homeomorphic to $S^1 \times I$.

Here the partition consists of pairs of points $\{(0, t), (1, t)\}$ where $t \in I$, and one-point subsets of $(0, 1) \times I$.

Reformulation of *21.E*: If we ***glue*** the side edges of a square by identifying points on the same hight, then we obtain a cylinder.



21.F. $S^1 \times I/_{[(z,0) \sim (z,1) \text{ for } z \in S^1]}$ is homeomorphic to $S^1 \times S^1$.

Here the partition consists of one-point subsets of $S^1 \times (0,1)$, and pairs of points of the basis circles lying on the same generatrix of the cylinder.

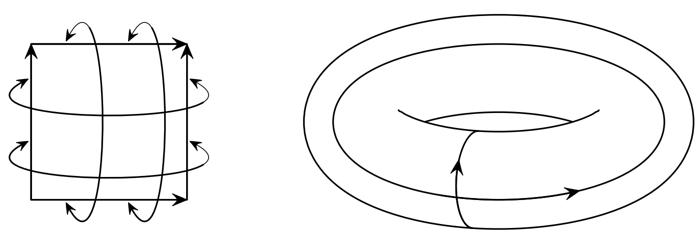
Here is a reformulation of *21.F*: If we ***glue*** the base circles of a cylinder by identifying points on the same generatrix, then we obtain a torus.

21.G. $I^2/_{[(0,t) \sim (1,t),\ (t,0) \sim (t,1)]}$ is homeomorphic to $S^1 \times S^1$.

In *21.G* , the partition consists of

- one-point subsets of the interior $(0,1) \times (0,1)$ of the square,
- pairs of points on the vertical sides that are the same distance from the bottom side (i.e., pairs $\{(0,t),(1,t)\}$ with $t \in (0,1)$),
- pairs of points on the horizontal sides that lie on the same vertical line (i.e., pairs $\{(t,0),(t,1)\}$ with $t \in (0,1)$),
- the four vertices of the square

Reformulation of *21.G*: Identifying the sides of a square according to the picturewe obtain a torus.



21°4. Transitivity of Factorization

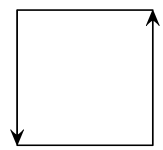
A solution of Problem *21.G* can be based on Problems *21.E* and *21.F* and the following general theorem.

21.H Transitivity of Factorization. *Let S be a partition of a space X, and let S' be a partition of the space $X/_S$. Then the quotient space*

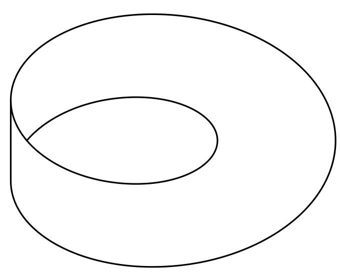
$(X/_S)/_{S'}$ is canonically homeomorphic to $X/_T$, where T is the partition of X into preimages of elements of S' under the projection $X \to X/_S$.

21°5. Möbius Strip

The *Möbius strip* or *Möbius band* is defined as $I^2/_{[(0,t) \sim (1, 1-t)]}$. In other words, this is the quotient space of the square I^2 by the partition into centrally symmetric pairs of points on the vertical edges of I^2, and singletons that do not lie on the vertical edges. The Möbius strip is obtained, so to speak, by identifying the vertical sides of a square in such a way that the directions shown on them by arrows are superimposed:



21.I. Prove that the Möbius strip is homeomorphic to the surface that is swept in $\mathbb{R}^3$ by a segment rotating in a half-plane around the midpoint, while the half-plane rotates around its boundary line. The ratio of the angular velocities of these rotations is such that the rotation of the half-plane through 360° takes the same time as the rotation of the segment through 180°. See Figure.



21°6. Contracting Subsets

21.4. Prove that $[0,1]/_{[\frac{1}{3},\frac{2}{3}]}$ is homeomorphic to $[0,1]$, and $[0,1]/_{\{\frac{1}{3},1\}}$ is homeomorphic to letter P.

21.5. Prove that the following spaces are homeomorphic:
(a) $\mathbb{R}^2$; (b) $\mathbb{R}^2/_I$; (c) $\mathbb{R}^2/_{D^2}$; (d) $\mathbb{R}^2/_{I^2}$;

(e) $\mathbb{R}^2/_A$, where A is a union of several segments with a common end point;
(f) $\mathbb{R}^2/_B$, where B is a simple finite polygonal line, i.e., a union of a finite sequence of segments $I_1, \dots, I_n$ such that the initial point of I_{i+1} is the final point of I_i.

21.6. Prove that if $f : X \to Y$ is a homeomorphism, then the quotient spaces $X/_A$ and $Y/_{f(A)}$ are homeomorphic.

21.7. Let $A \subset \mathbb{R}^2$ be a ray $\{(x, y) \mid x \geq 0, y = 0\}$. Is $\mathbb{R}^2/_A$ homeomorphic to $\operatorname{Int} D^2 \cup \{(0, 1)\}$?

21°7. Further Examples

21.8. Prove that $S^1/_{[z \sim e^{2\pi i/3} z]}$ is homeomorphic to S^1.

The partition in *21.8* consists of triples of points that are vertices of equilateral inscribed triangles.

21.9. Prove that the following quotient spaces of the disk D^2 are homeomorphic to D^2:

(1) $D^2/_{[(x, y) \sim (-x, -y)]}$,

(2) $D^2/_{[(x, y) \sim (x, -y)]}$,

(3) $D^2/_{[(x, y) \sim (-y, x)]}$.

21.10. Find a generalization of *21.9* with D^n substituted for D^2.

21.11. Describe explicitly the quotient space of line $\mathbb{R}^1$ by equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

21.12. Represent the Möbius strip as a quotient space of cylinder $S^1 \times I$.

21°8. Klein Bottle

Klein bottle is $I^2/_{[(t, 0) \sim (t, 1),\ (0, t) \sim (1, 1 - t)]}$. In other words, this is the quotient space of square I^2 by the partition into

- one-point subsets of its interior,
- pairs of points $(t, 0), (t, 1)$ on horizontal edges that lie on the same vertical line,
- pairs of points $(0, t), (1, 1 - t)$ symmetric with respect to the center of the square that lie on the vertical edges, and
- the quadruple of vertices.

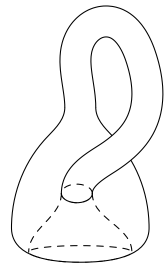
21.13. Present the Klein bottle as a quotient space of

(1) a cylinder;
(2) the Möbius strip.

21.14. Prove that $S^1 \times S^1/_{[(z, w) \sim (-z, \bar{w})]}$ is homeomorphic to the Klein bottle. (Here $\bar{w}$ denotes the complex number conjugate to w.)

21.15. Embed the Klein bottle into $\mathbb{R}^4$ (cf. *21.I* and *19.W*).

21.16. Embed the Klein bottle into $\mathbb{R}^4$ so that the image of this embedding under the orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ would look as follows:



21°9. Projective Plane

Let us identify each boundary point of the disk D^2 with the antipodal point, i.e., factorize the disk by the partition consisting of one-point subsets of the interior of the disk and pairs of points on the boundary circle symmetric with respect to the center of the disk. The result is the *projective plane*. This space cannot be embedded in $\mathbb{R}^3$, too. Thus we are not able to draw it. Instead, we present it in other way.

21.J. A projective plane is a result of gluing together a disk and a Möbius strip via a homeomorphism between their boundary circles.

21°10. You May Have Been Provoked to Perform an Illegal Operation

Solving the previous problem, you did something that did not fit into the theory presented above. Indeed, the operation with two spaces called *gluing* in *21.J* has not appeared yet. It is a combination of two operations: first, we make a single space consisting of disjoint copies of the original spaces, and then we factorize this space by identifying points of one copy with points of another. Let us consider the first operation in detail.

21°11. Set-Theoretic Digression: Sums of Sets

The (*disjoint*) *sum* of a family of sets $\{X_\alpha\}_{\alpha\in A}$ is the set of pairs (x_α, α) such that $x_\alpha \in X_\alpha$. The sum is denoted by $\bigsqcup_{\alpha\in A} X_\alpha$. So, we can write

$$\bigsqcup_{\alpha\in A} X_\alpha = \bigcup_{\alpha\in A} (X_\alpha \times \{\alpha\}).$$

For each $\beta \in A$, we have a natural injection

$$\mathrm{in}_\beta : X_\beta \to \bigsqcup_{\alpha\in A} X_\alpha : x \mapsto (x, \beta).$$

If only two sets X and Y are involved and they are distinct, then we can avoid indices and define the sum by setting

$$X \sqcup Y = \{(x, X) \mid x \in X\} \cup \{(y, Y) \mid y \in Y\}.$$

21°12. Sums of Spaces

21.K. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces. Then the collection of subsets of $\bigsqcup_{\alpha\in A} X_\alpha$ whose preimages under all inclusions in_α, $\alpha \in A$, are open is a topological structure.

The sum $\bigsqcup_{\alpha\in A} X_\alpha$ with this topology is the (*disjoint*) *sum of the topological spaces* X_α $(\alpha \in A)$.

21.L. The topology described in *21.K* is the finest topology with respect to which all inclusions in_α are continuous.

21.17. The maps $\mathrm{in}_\beta : X_\beta \to \bigsqcup_{\alpha\in A} X_\alpha$ are topological embedding, and their images are both open and closed in $\bigsqcup_{\alpha\in A} X_\alpha$.

21.18. Which of the standard topological properties are inherited from summands X_α by the sum $\bigsqcup_{\alpha\in A} X_\alpha$? Which are not?

21°13. Attaching Space

Let X and Y be two spaces, A a subset of Y, and $f : A \to X$ a continuous map. The quotient space $X \cup_f Y = (X \sqcup Y)/_{[a \sim f(a) \text{ for } a \in A]}$ is said to be the result of *attaching* or *gluing* the space Y to the space X via f. The map f is the *attaching map*.

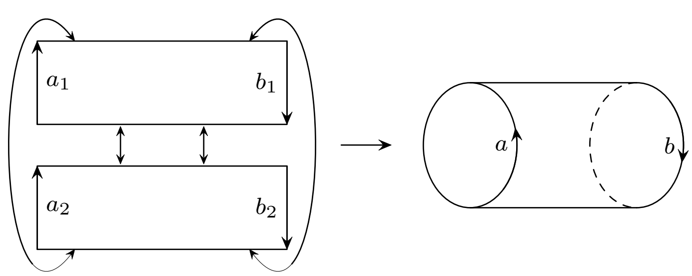
Here the partition of $X \sqcup Y$ consists of one-point subsets of $\mathrm{in}_2(Y \smallsetminus A)$ and $\mathrm{in}_1(X \smallsetminus f(A))$, and sets $\mathrm{in}_1(x) \cup \mathrm{in}_2\big(f^{-1}(x)\big)$ with $x \in f(A)$.

21.19. Prove that the composition of inclusion $X \to X \sqcup Y$ and projection $X \sqcup Y \to X \cup_f Y$ is a topological embedding.

21.20. Prove that if X is a point, then $X \cup_f Y$ is $Y/_A$.

21.M. Prove that attaching the n-disk D^n to its copy via the identity map of the boundary sphere S^{n-1} we obtain a space homeomorphic to S^n.

21.21. Prove that the Klein bottle is a result of gluing together two copies of the Möbius strip via the identity map of the boundary circle.



21.22. Prove that the result of gluing together two copies of a cylinder via the identity map of the boundary circles (of one copy to the boundary circles of the other) is homeomorphic to $S^1 \times S^1$.

21.23. Prove that the result of gluing together two copies of the solid torus $S^1 \times D^2$ via the identity map of the boundary torus $S^1 \times S^1$ is homeomorphic to $S^1 \times S^2$.

21.24. Obtain the Klein bottle by gluing two copies of the cylinder $S^1 \times I$ to each other.

21.25. Prove that the result of gluing together two copies of the solid torus $S^1 \times D^2$ via the map

$$S^1 \times S^1 \to S^1 \times S^1 : (x, y) \mapsto (y, x)$$

of the boundary torus to its copy is homeomorphic to S^3.

21.N. Let X and Y be two spaces, A a subset of Y, and $f, g : A \to X$ two continuous maps. Prove that if there exists a homeomorphism $h : X \to X$ such that $h \circ f = g$, then $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

21.O. Prove that $D^n \cup_h D^n$ is homeomorphic to S^n for any homeomorphism $h : S^{n-1} \to S^{n-1}$.

21.26. Classify up to homeomorphism those spaces which can be obtained from a square by identifying a pair of opposite sides by a homeomorphism.

21.27. Classify up to homeomorphism the spaces that can be obtained from two copies of $S^1 \times I$ by identifying the copies of $S^1 \times \{0, 1\}$ by a homeomorphism.

21.28. Prove that the topological type of the space resulting from gluing together two copies of the Möbius strip via a homeomorphism of the boundary circle does not depend on the homeomorphism.

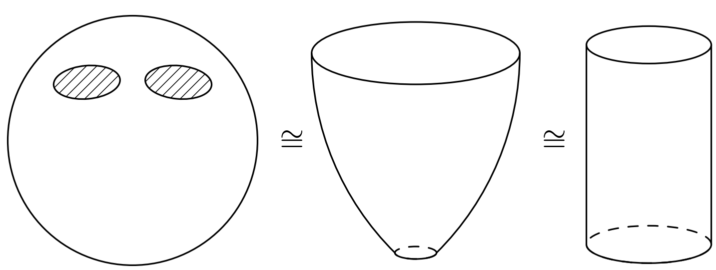
21.29. Classify up to homeomorphism the spaces that can be obtained from $S^1 \times I$ by identifying $S^1 \times 0$ and $S^1 \times 1$ via a homeomorphism.

21°14. Basic Surfaces

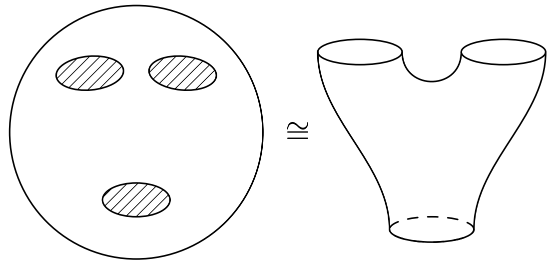
A torus $S^1 \times S^1$ with the interior of an embedded disk deleted is a *handle*. A two-sphere with the interior of n disjoint embedded disks deleted is a *sphere with n holes*.

21.P. A sphere with a hole is homeomorphic to the disk D^2.

21.Q. A sphere with two holes is homeomorphic to the cylinder $S^1 \times I$.



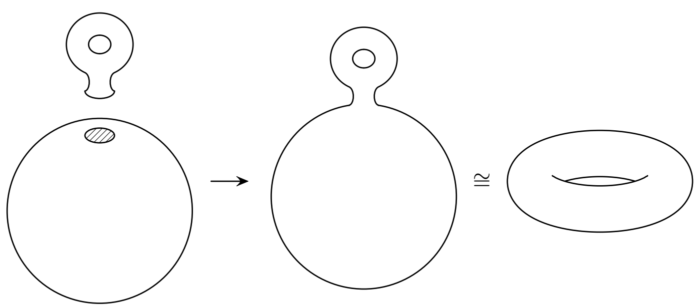
A sphere with three holes has a special name. It is called *pantaloons* or just *pants* .



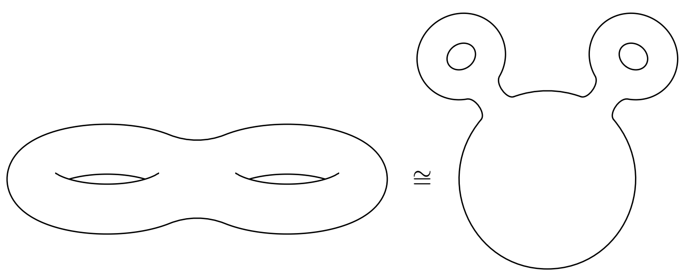
The result of attaching p copies of a handle to a sphere with p holes via embeddings homeomorphically mapping the boundary circles of the handles onto those of the holes is a ***sphere with p handles***, or, in a more ceremonial way (and less understandable, for a while), an ***orientable connected closed surface of genus p***.

> ***21.30.*** Prove that a sphere with p handles is well defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

21.R. A sphere with one handle is homeomorphic to the torus $S^1 \times S^1$.



21.S. A sphere with two handles is homeomorphic to the result of gluing together two copies of a handle via the identity map of the boundary circle.



A sphere with two handles is a ***pretzel***. Sometimes, this word also denotes a sphere with more handles.

The space obtained from a sphere with q holes by attaching q copies of the Möbius strip via embeddings of the boundary circles of the Möbius strips onto the boundary circles of the holes (the boundaries of the holes) is a *sphere with q crosscaps*, or a *nonorientable connected closed surface of genus q*.

21.31. Prove that a sphere with q crosscaps is well defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

21.T. A sphere with a crosscap is homeomorphic to the projective plane.

21.U. A sphere with two crosscaps is homeomorphic to the Klein bottle.

A sphere, spheres with handles, and spheres with crosscaps are *basic surfaces*.

21.V. Prove that a sphere with p handles and q crosscaps is homeomorphic to a sphere with $2p + q$ crosscaps (here $q > 0$).

21.32. Classify up to homeomorphism those spaces which are obtained by attaching p copies of $S^1 \times I$ to a sphere with $2p$ holes via embeddings of the boundary circles of the cylinders onto the boundary circles of the sphere with holes.

22. Projective Spaces

This section can be considered as a continuation of the previous one. The quotient spaces described here are of too great importance to regard them just as examples of quotient spaces.

22°1. Real Projective Space of Dimension n

This space is defined as the quotient space of the sphere S^n by the partition into pairs of antipodal points, and denoted by $\mathbb{R}P^n$.

22.A. *The space $\mathbb{R}P^n$ is homeomorphic to the quotient space of the n-disk D^n by the partition into one-point subsets of the interior of D^n, and pairs of antipodal point of the boundary sphere S^{n-1}.*

22.B. $\mathbb{R}P^0$ is a point.

22.C. The space $\mathbb{R}P^1$ is homeomorphic to the circle S^1.

22.D. The space $\mathbb{R}P^2$ is homeomorphic to the projective plane defined in the previous section.

22.E. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the quotient space of $\mathbb{R}^{n+1} \setminus 0$ by the partition into one-dimensional vector subspaces of $\mathbb{R}^{n+1}$ punctured at 0.*

A point of the space $\mathbb{R}^{n+1} \setminus 0$ is a sequence of real numbers, which are not all zeros. These numbers are the *homogeneous coordinates* of the corresponding point of $\mathbb{R}P^n$. The point with homogeneous coordinates x_0, x_1, ..., x_n is denoted by $(x_0 : x_1 : \dots : x_n)$. Homogeneous coordinates determine a point of $\mathbb{R}P^n$, but are not determined by this point: proportional vectors of coordinates $(x_0, x_1, \dots, x_n)$ and $(\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ determine the same point of $\mathbb{R}P^n$.

22.F. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the metric space, whose points are lines of $\mathbb{R}^{n+1}$ through the origin $0 = (0, \dots, 0)$ and the metric is defined as the angle between lines (which takes values in $[0, \frac{\pi}{2}]$). Prove that this is really a metric.*

22.G. Prove that the map

$$i : \mathbb{R}^n \to \mathbb{R}P^n : (x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$$

is a topological embedding. What is its image? What is the inverse map of its image onto $\mathbb{R}^n$?

22.H. Construct a topological embedding $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$ with image $\mathbb{R}P^n \setminus i(\mathbb{R}^n)$, where i is the embedding from Problem *22.G*.

Therefore the projective space $\mathbb{R}P^n$ can be considered as the result of extending $\mathbb{R}^n$ by adjoining "improper" or "infinite" points, which constitute a projective space $\mathbb{R}P^{n-1}$.

22.1. Introduce a natural topological structure in the set of all lines on the plane and prove that the resulting space is homeomorphic to a) $\mathbb{R}P^2 \smallsetminus \{\text{pt}\}$; b) open Möbius strip (i.e., a Möbius strip with the boundary circle removed).

22.2. Prove that the set of all rotations of the space $\mathbb{R}^3$ around lines passing through the origin equipped with the natural topology is homeomorphic to $\mathbb{R}P^3$.

22°2x. Complex Projective Space of Dimension n

This space is defined as the quotient space of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ by the partition into circles cut by (complex) lines of $\mathbb{C}^{n+1}$ passing through the point 0. It is denoted by $\mathbb{C}P^n$.

22.Ax. $\mathbb{C}P^n$ is homeomorphic to the quotient space of the unit $2n$-disk D^{2n} of the space $\mathbb{C}^n$ by the partition whose elements are one-point subsets of the interior of D^{2n} and circles cut on the boundary sphere S^{2n-1} by (complex) lines of $\mathbb{C}^n$ passing through the origin $0 \in \mathbb{C}^n$.

22.Bx. $\mathbb{C}P^0$ is a point.

The space $\mathbb{C}P^1$ is a *complex projective line*.

22.Cx. The complex projective line $\mathbb{C}P^1$ is homeomorphic to S^2.

22.Dx. The space $\mathbb{C}P^n$ is canonically homeomorphic to the quotient space of the space $\mathbb{C}^{n+1} \smallsetminus 0$ by the partition into complex lines of $\mathbb{C}^{n+1}$ punctured at 0.

Hence, $\mathbb{C}P^n$ can be regarded as the space of complex-proportional nonzero complex sequences $(x_0, x_1, \dots, x_n)$. The notation $(x_0 : x_1 : \dots : x_n)$ and term homogeneous coordinates introduced for the real case are used in the same way for the complex case.

22.Ex. The space $\mathbb{C}P^n$ is canonically homeomorphic to the metric space, whose points are the (complex) lines of $\mathbb{C}^{n+1}$ passing through the origin 0, and the metric is defined as the angle between lines (which takes values in $[0, \frac{\pi}{2}]$).

22°3x. Quaternionic Projective Spaces

Recall that $\mathbb{R}^4$ bears a remarkable multiplication, which was discovered by R. W. Hamilton in 1843. It can be defined by the formula

$$(x_1, x_1, x_3, x_4) \times (y_1, y_2, y_3, y_4) =$$
$$(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4, \quad x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3,$$
$$x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2, \quad x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1).$$

It is bilinear, and to describe it in a shorter way it suffices to specify the products of the basis vectors. The latter are traditionally denoted in this case, following Hamilton, as follows:

$$1 = (1,0,0,0), \quad i = (0,1,0,0), \quad j = (0,0,1,0) \quad \text{and} \quad k = (0,0,0,1).$$

In this notation, 1 is really a unity: $(1,0,0,0) \times x = x$ for any $x \in \mathbb{R}^4$. The rest of multiplication table looks as follows:

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i \quad \text{and} \quad ik = -j.$$

Together with coordinate-wise addition, this multiplication determines a structure of algebra in $\mathbb{R}^4$. Its elements are *quaternions*.

22.Fx. Check that the quaternion multiplication is associative.

It is not commutative (e.g., $ij = k \neq -k = ji$). Otherwise, quaternions are very similar to complex numbers. As in $\mathbb{C}$, there is a transformation called *conjugation* acting in the set of quaternions. As the conjugation of complex numbers, it is also denoted by a bar: $x \mapsto \overline{x}$. It is defined by the formula $(x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, -x_3, -x_4)$ and has two remarkable properties:

22.Gx. We have $\overline{ab} = \overline{b}\,\overline{a}$ for any two quaternions a and b.

22.Hx. We have $a\overline{a} = |a|^2$, i.e., the product of any quaternion a by the conjugate quaternion $\overline{a}$ equals $(|a|^2, 0, 0, 0)$.

The latter property allows us to define, for any $a \in \mathbb{R}^4$, the inverse quaternion

$$a^{-1} = |a|^{-2}\overline{a}$$

such that $aa^{-1} = 1$.

Hence, the quaternion algebra is a *division algebra* or a *skew field*. It is denoted by $\mathbb{H}$ after Hamilton, who discovered it.

In the space $\mathbb{H}^n = \mathbb{R}^{4n}$, there are right quaternionic lines, i.e., subsets $\{(a_1\xi, \dots, a_n\xi) \mid \xi \in \mathbb{H}\}$, and similar left quaternionic lines $\{(\xi a_1, \dots, \xi a_n) \mid \xi \in \mathbb{H}\}$. Each of them is a real 4-dimensional subspace of $\mathbb{H}^n = \mathbb{R}^{4n}$.

***22.I*x.** Find a right quaternionic line that is not a left quaternionic line.

***22.J*x.** Prove that two right quaternionic lines in $\mathbb{H}^n$ either meet only at 0, or coincide.

The quotient space of the unit sphere S^{4n+3} of the space $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4}$ by the partition into its intersections with right quaternionic lines is the (*right*) *quaternionic projective space of dimension* n. Similarly, but with left quaternionic lines, we define the (*left*) *quaternionic projective space of dimension* n.

***22.K*x.** Are the right and left quaternionic projective space of the same dimension homeomorphic?

The left quaternionic projective space of dimension n is denoted by $\mathbb{H}P^n$.

***22.L*x.** $\mathbb{H}P^0$ consists of a single point.

***22.M*x.** $\mathbb{H}P^n$ is homeomorphic to the quotient space of the closed unit disk D^{4n} in $\mathbb{H}^n$ by the partition into points of the interior of D^{4n} and the 3-spheres that are intersections of the boundary sphere S^{4n-1} with (left quaternionic) lines of $\mathbb{H}^n$.

The space $\mathbb{H}P^1$ is the *quaternionic projective line*.

***22.N*x.** Quaternionic projective line $\mathbb{H}P^1$ is homeomorphic to S^4.

***22.O*x.** $\mathbb{H}P^n$ is canonically homeomorphic to the quotient space of $\mathbb{H}^{n+1} \smallsetminus 0$ by the partition to left quaternionic lines of $\mathbb{H}^{n+1}$ passing through the origin and punctured at it.

Hence, $\mathbb{H}P^n$ can be presented as the space of classes of left proportional (in the quaternionic sense) nonzero sequences $(x_0, \dots, x_n)$ of quaternions. The notation $(x_0 : x_1 : \dots : x_n)$ and the term homogeneous coordinates introduced above in the real case are used in the same way in the quaternionic situation.

***22.P*x.** $\mathbb{H}P^n$ is canonically homeomorphic to the set of (left quaternionic) lines of $\mathbb{H}^{n+1}$ equipped with the topology generated by the angular metric (which takes values in $\left[0, \frac{\pi}{2}\right]$).

23x. Finite Topological Spaces

23°1x. Set-Theoretic Digression: Splitting a Transitive Relation Into Equivalence and Partial Order

In the definitions of equivalence and partial order relations, the condition of transitivity seems to be the most important. Below, we supply a formal justification of this feeling by showing that the other conditions are natural companions of transitivity, although they are not its consequences.

23.Ax. *Let $\prec$ be a transitive relation in a set X. Then the relation $\precsim$ defined by*

$$a \precsim b \text{ if } a \prec b \text{ or } a = b$$

is also transitive (and, furthermore, it is certainly reflexive, i.e., $a \precsim a$ for each $a \in X$).

A binary relation $\precsim$ in a set X is a ***preorder*** if it is transitive and reflective, i.e., satisfies the following conditions:

- *Transitivity*. If $a \precsim b$ and $b \precsim c$, then $a \precsim c$.
- *Reflexivity*. We have $a \precsim a$ for any a.

A set X equipped with a preorder is ***preordered***.

If a preorder is antisymmetric, then this is a nonstrict order.

23.1x. Is the relation $a|b$ a preorder in the set $\mathbb{Z}$ of integers?

23.Bx. *If $(X, \precsim)$ is a preordered set, then the relation $\sim$ defined by*

$$a \sim b \text{ if } a \precsim b \text{ and } b \precsim a$$

is an equivalence relation (i.e., it is symmetric, reflexive, and transitive) in X.

23.2x. What equivalence relation is defined in $\mathbb{Z}$ by the preorder $a|b$?

23.Cx. *Let $(X, \precsim)$ be a preordered set and $\sim$ be an equivalence relation defined in X by $\precsim$ according to 23.Bx. Then $a' \sim a$, $a \precsim b$ and $b \sim b'$ imply $a' \precsim b'$ and in this way $\precsim$ determines a relation in the set of equivalence classes $X/_\sim$. This relation is a nonstrict partial order.*

Thus any transitive relation generates an equivalence relation and a partial order in the set of equivalence classes.

23.Dx. How this chain of constructions would degenerate if the original relation was

(1) an equivalence relation, or

(2) nonstrict partial order?

23.Ex. *In any topological space, the relation $\precsim$ defined by*

$$a \precsim b \text{ if } a \in \mathrm{Cl}\{b\}$$

is a preorder.

23.3x. In the set of all subsets of an arbitrary topological space the relation

$$A \precsim B \text{ if } A \subset \mathrm{Cl}\, B$$

is a preorder. This preorder determines the following equivalence relation: sets are equivalent iff they have the same closure.

23.Fx. *The equivalence relation defined by the preorder of Theorem 23.Ex determines the partition of the space into maximal (with respect to inclusion) indiscrete subspaces. The quotient space satisfies the Kolmogorov separation axiom T_0.*

The quotient space of Theorem *23.Fx* is the *maximal T_0-quotient of X*.

23.Gx. A continuous image of an indiscrete space is indiscrete.

23.Hx. Prove that any continuous map $X \to Y$ induces a continuous map of the maximal T_0-quotient of X to the maximal T_0-quotient of Y.

23°2x. The Structure of Finite Topological Spaces

The results of the preceding subsection provide a key to understanding the structure of finite topological spaces. Let X be a finite space. By Theorem *23.Fx*, X is partitioned to indiscrete clusters of points. By *23.Gx*, continuous maps between finite spaces respect these clusters and, by *23.Hx*, induce continuous maps between the maximal T_0-quotient spaces.

This means that we can consider a finite topological space as its maximal T_0-quotient whose points are equipped with multiplicities, that are positive integers: the numbers of points in the corresponding clusters of the original space.

The maximal T_0-quotient of a finite space is a smallest neighborhood space (as a finite space). By Theorem *14.O*, its topology is determined by a partial order. By Theorem *9.Bx*, homeomorphisms between spaces with poset topologies are monotone bijections.

Thus, a finite topological space is characterized up to homeomorphism by a finite poset whose elements are equipped with multiplicities (positive integers). Two such spaces are homeomorphic iff there exists a monotone bijection between the corresponding posets that preserves the multiplicities. To recover the topological space from the poset with multiplicities, we must equip the poset with the poset topology and then replace each of its elements by an indiscrete cluster of points, the number points in which is the multiplicity of the element.

23°3x. Simplicial schemes

Let V be a set, Σ a set of some of subsets of V. A pair (V, Σ) is a *simplicial scheme* with set of *vertices* V and set of *simplices* Σ if

- each subset of any element of Σ belongs to Σ,
- the intersection of any collection of elements of Σ belongs to Σ,
- each one-element subset of V belongs to Σ.

The set Σ is partially ordered by inclusion. When equipped with the poset topology of this partial order, it is called *the space of simplices* of the simplicial scheme (X, Σ).

A simplicial scheme gives rise also to another topological space. Namely, for a simplicial scheme (V, Σ) consider the set $S(V, \Sigma)$ of all functions $c : V \to [0, 1]$ such that

$$\operatorname{Supp}(c) = \{v \in V \mid c(v) \neq 0\} \in \Sigma$$

and $\sum_{v \in V} c(v) = 1$. Equip $S(V, \Sigma)$ with the topology generated by metric

$$\rho(c_1, c_2) = \sup_{v \in V} |c_1(v) - c_2(v)|.$$

The space $S(V, \Sigma)$ is a *simplicial* or *triangulated* space. It is covered by the sets $\{c \in S \mid \operatorname{Supp}(c) = \sigma\}$, where $\sigma \in \Sigma$, which are called its (*open*) *simplices*.

23.4x. Which open simplices of a simplicial space are open sets, which are closed, and which are neither closed nor open?

23.Ix. For each $\sigma \in \Sigma$, find a homeomorphism of the space

$$\{c \in S \mid \operatorname{Supp}(c) = \sigma\} \subset S(V, \Sigma)$$

onto an open simplex whose dimension is one less than the number of vertices belonging to σ. (Recall that the open n-simplex is the set $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_j > 0 \text{ for } j = 1, \dots, n+1 \text{ and } \sum_{i=1}^{n+1} x_i = 1\}$.)

23.Jx. Prove that for any simplicial scheme (V, Σ) the quotient space of the simplicial space $S(V, \Sigma)$ by its partition to open simplices is homeomorphic to the space Σ of simplices of the simplicial scheme (V, Σ).

23°4x. Barycentric Subdivision of a Poset

23.Kx. Find a poset which is not isomorphic to the set of simplices (ordered by inclusion) of whatever simplicial scheme.

Let $(X, \prec)$ be a poset. Consider the set X' of all nonempty finite strictly increasing sequences $a_1 \prec a_2 \prec \dots \prec a_n$ of elements of X. It can also be described as the set of all nonempty finite subsets of X in each of which $\prec$ determines a linear order. It is naturally ordered by inclusion.

The poset $(X', \subset)$ is the *barycentric subdivision of* $(X, \prec)$.

***23.L*x.** For any poset $(X, \prec)$, pair (X, X') is a simplicial scheme.

There is a natural map $X' \to X$ that maps an element of X' (i.e., a nonempty finite linearly ordered subset of X) to its greatest element.

***23.M*x.** Is this map monotone? Strictly monotone? The same questions concerning a similar map that maps a nonempty finite linearly ordered subset of X to its smallest element.

Let (V, Σ) be a simplicial scheme and Σ' be the barycentric subdivision of Σ (ordered by inclusion). The simplicial scheme (Σ, Σ') is the *barycentric subdivision* of the simplicial scheme (V, Σ).

There is a natural mapping $\Sigma \to S(V, \Sigma)$ that maps a simplex $\sigma \in \Sigma$ (i.e., a subset $\{v_0, v_1, \dots, v_n\}$ of V) to the function $b_\sigma : V \to \mathbb{R}$ with $b_\sigma(v_i) = \frac{1}{n+1}$ and $b_\sigma(v) = 0$ for any $v \notin \sigma$.

Define a map $\beta : S(\Sigma, \Sigma') \to S(V, \Sigma)$ that maps a function $\varphi : \Sigma \to \mathbb{R}$ to the function

$$V \to \mathbb{R} : v \mapsto \sum_{\sigma \in \Sigma} \varphi(\sigma) b_\sigma(v).$$

***23.N*x.** Prove that the map $\beta : S(\Sigma, \Sigma') \to S(V, \Sigma)$ is a homeomorphism and constitutes, together with projections $S(V, \Sigma) \to \Sigma$ and $S(\Sigma, \Sigma') \to \Sigma'$ and the natural map $\Sigma' \to \Sigma$ a commutative diagram

$$\begin{array}{ccc} S(\Sigma, \Sigma') & \xrightarrow{\beta} & S(V, \Sigma) \\ \downarrow & & \downarrow \\ \Sigma' & \longrightarrow & \Sigma \end{array}$$

24x. Spaces of Continuous Maps

24°1x. Sets of Continuous Mappings

By $\mathcal{C}(X, Y)$ we denote the set of all continuous maps of a space X to a space Y.

24.1x. Let X be non empty. Prove that $\mathcal{C}(X, Y)$ consists of a single element iff so does Y.

24.2x. Let X be non empty. Prove that there exists an injection $Y \to \mathcal{C}(X, Y)$. In other words, the cardinality $\operatorname{card} \mathcal{C}(X, Y)$ of $\mathcal{C}(X, Y)$ is greater than or equal to $\operatorname{card} Y$.

24.3x. Riddle. Find natural conditions implying that $\mathcal{C}(X, Y) = Y$.

24.4x. Let $Y = \{0, 1\}$ equipped with topology $\{\varnothing, \{0\}, Y\}$. Prove that there exists a bijection between $\mathcal{C}(X, Y)$ and the topological structure of X.

24.5x. Let X be a set of n points with discrete topology. Prove that $\mathcal{C}(X, Y)$ can be identified with $Y \times \dots \times Y$ (n times).

24.6x. Let Y be a set of k points with discrete topology. Find necessary and sufficient condition for the set $\mathcal{C}(X, Y)$ contain k^2 elements.

24°2x. Topologies on Set of Continuous Mappings

Let X and Y be two topological spaces, $A \subset X$, and $B \subset Y$. We define $W(A, B) = \{f \in \mathcal{C}(X, Y) \mid f(A) \subset B\}$,

$$\Delta^{(pw)} = \{W(a, U) \mid a \in X,\ U \text{ is open in } Y\},$$

and

$$\Delta^{(co)} = \{W(C, U) \mid C \subset X \text{ is compact},\ U \text{ is open in } Y\}.$$

24.Ax. $\Delta^{(pw)}$ is a subbase of a topological structure on $\mathcal{C}(X, Y)$.

The topological structure generated by $\Delta^{(pw)}$ is the *topology of pointwise convergence*. The set $\mathcal{C}(X, Y)$ equipped with this structure is denoted by $\mathcal{C}^{(pw)}(X, Y)$.

24.Bx. $\Delta^{(co)}$ is a subbase of a topological structures on $\mathcal{C}(X, Y)$.

The topological structure determined by $\Delta^{(co)}$ is the *compact-open topology*. Hereafter we denote by $\mathcal{C}(X, Y)$ the space of all continuous maps $X \to Y$ with the compact-open topology, unless the contrary is specified explicitly.

24.Cx Compact-Open Versus Pointwise. The compact-open topology is finer than the topology of pointwise convergence.

24.7x. Prove that $\mathcal{C}(I, I)$ is not homeomorphic to $\mathcal{C}^{(pw)}(I, I)$.

Denote by $Const(X, Y)$ the set of all constant maps $f : X \to Y$.

24.8x. Prove that the topology of pointwise convergence and the compact-open topology of $\mathcal{C}(X, Y)$ induce the same topological structure on $Const(X, Y)$, which, with this topology, is homeomorphic Y.

24.9x. Let X be a discrete space of n points. Prove that $\mathcal{C}^{(pw)}(X, Y)$ is homeomorphic $Y \times \ldots \times Y$ (n times). Is this true for $\mathcal{C}(X, Y)$?

24°3x. Topological Properties of Mapping Spaces

24.Dx. Prove that if Y is Hausdorff, then $\mathcal{C}^{(pw)}(X, Y)$ is Hausdorff for any space X. Is this true for $\mathcal{C}(X, Y)$?

24.10x. Prove that $\mathcal{C}(I, X)$ is path connected iff X is path connected.

24.11x. Prove that $\mathcal{C}^{(pw)}(I, I)$ is not compact. Is the space $\mathcal{C}(I, I)$ compact?

24°4x. Metric Case

24.Ex. If Y is metrizable and X is compact, then $\mathcal{C}(X, Y)$ is metrizable.

Let (Y, ρ) be a metric space and X a compact space. For continuous maps $f, g : X \to Y$ put

$$d(f, g) = \max\{\rho(f(x), g(x)) \mid x \in X\}.$$

24.Fx This is a Metric. If X is a compact space and Y a metric space, then d is a metric on the set $\mathcal{C}(X, Y)$.

Let X be a topological space, Y a metric space with metric ρ. A sequence f_n of maps $X \to Y$ *uniformly converges* to $f : X \to Y$ if for any $\varepsilon > 0$ there exists a positive integer N such that $\rho(f_n(x), f(x)) < \varepsilon$ for any $n > N$ and $x \in X$. This is a straightforward generalization of the notion of uniform convergence which is known from Calculus.

24.Gx Metric of Uniform Convergence. Let X be a compact space, (Y, d) a metric space. A sequence f_n of maps $X \to Y$ converges to $f : X \to Y$ in the topology generated by d iff f_n uniformly converges to f.

24.Hx Completeness of $\mathcal{C}(X, Y)$. Let X be a compact space, (Y, ρ) a complete metric space. Then $\big(\mathcal{C}(X, Y), d\big)$ is a complete metric space.

24.Ix Uniform Convergence Versus Compact-Open. Let X be a compact space and Y a metric space. Then the topology generated by d on $\mathcal{C}(X, Y)$ is the compact-open topology.

24.12x. Prove that the space $\mathcal{C}(\mathbb{R}, I)$ is metrizable.

24.13x. Let Y be a bounded metric space, X a topological space admitting a presentation $X = \bigcup_{i=1}^{\infty} X_i$, where X_i is compact and $X_i \subset \operatorname{Int} X_{i+1}$ for each $i = 1, 2, \ldots$. Prove that $\mathcal{C}(X, Y)$ is metrizable.

Denote by $\mathcal{C}_b(X,Y)$ the set of all continuous bounded maps from a topological space X to a metric space Y. For maps $f,g \in \mathcal{C}_b(X,Y)$, put

$$d^\infty(f,g) = \sup\{\rho(f(x),g(x)) \mid x \in X\}.$$

24.Jx Metric on Bounded Maps. This is a metric in $\mathcal{C}_b(X,Y)$.

24.Kx d^∞ ***and Uniform Convergence.*** Let X be a topological space and Y a metric space. A sequence f_n of bounded maps $X \to Y$ converges to $f : X \to Y$ in the topology generated by d^∞ iff f_n uniformly converge to f.

24.Lx When Uniform Is Not Compact-Open. Find X and Y such that the topology generated by d^∞ on $\mathcal{C}_b(X,Y)$ is not the compact-open topology.

24°5x. Interactions With Other Constructions

24.Mx. For any continuous maps $\varphi : X' \to X$ and $\psi : Y \to Y'$ the map $\mathcal{C}(X,Y) \to \mathcal{C}(X',Y') : f \mapsto \psi \circ f \circ \varphi$ is continuous.

24.Nx Continuity of Restricting. Let X and Y be two spaces, $A \subset X$. Prove that the map $\mathcal{C}(X,Y) \to \mathcal{C}(A,Y) : f \mapsto f|_A$ is continuous.

24.Ox Extending Target. For any spaces X and Y and any $B \subset Y$, the map $\mathcal{C}(X,B) \to \mathcal{C}(X,Y) : f \mapsto i_B \circ f$ is a topological embedding.

24.Px Maps to Product. For any three spaces X, Y, and Z, the space $\mathcal{C}(X, Y \times Z)$ is canonically homeomorphic to $\mathcal{C}(X,Y) \times \mathcal{C}(X,Z)$.

24.Qx Restricting to Sets Covering Source. Let $\{X_1, \dots, X_n\}$ be a closed cover of X. Prove that for any space Y

$$\phi : \mathcal{C}(X,Y) \to \prod_{i=1}^{n} \mathcal{C}(X_i,Y) \ : \ f \mapsto (f|_{X_1}, \dots, f|_{X_n})$$

is a topological embedding. What if the cover is not fundamental?

24.Rx. Riddle. Can you generalize assertion *24.Qx*?

24.Sx Continuity of Composing. Let X be a space and Y a locally compact Hausdorff space. Prove that the map

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \ : \ (f,g) \mapsto g \circ f$$

is continuous.

24.14x. Is local compactness of Y necessary in *24.Sx*?

24.T*x *Factorizing Source. Let S be a closed partition[2] of a Hausdorff compact space X. Prove that for any space Y the map

$$\phi : \mathcal{C}(X/S, Y) \to \mathcal{C}(X, Y)$$

is a topological embedding.

***24.15*x.** Are the conditions imposed on S and X in *24.T*x necessary?

24.U*x *The Evaluation Map. Let X and Y be two spaces. Prove that if X is locally compact and Hausdorff, then the map

$$\phi : \mathcal{C}(X, Y) \times X \to Y \ : \ (f, x) \mapsto f(x)$$

is continuous.

***24.16*x.** Are the conditions imposed on X in *24.U*x necessary?

24°6x. Mappings $X \times Y \to Z$ and $X \to \mathcal{C}(Y, Z)$

***24.V*x.** Let X, Y, and Z be three topological spaces, $f : X \times Y \to Z$ a continuous map. Then the map

$$F : X \to \mathcal{C}(Y, Z) \ : \ F(x) : y \mapsto f(x, y),$$

is continuous.

The converse assertion is also true under certain additional assumptions.

***24.W*x.** Let X and Z be two spaces, Y a Hausdorff locally compact space, $F : X \to \mathcal{C}(Y, Z)$ a continuous map. Then the map $f : X \times Y \to Z : (x, y) \mapsto F(x)(y)$ is continuous.

***24.X*x.** If X is a Hausdorff space and the collection $\Sigma_Y = \{U_\alpha\}$ is a subbase of the topological structure of Y, then the collection $\{W(K, U) \mid U \in \Sigma\}$ is a subbase of the compact-open topology in $\mathcal{C}(X, Y)$.

***24.Y*x.** Let X, Y, and Z be three spaces. Let

$$\Phi : \mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$$

be defined by the relation

$$\Phi(f)(x) : y \mapsto f(x, y).$$

Then

(1) if X is a Hausdorff space, then Φ is continuous;
(2) if X is a Hausdorff space, while Y is locally compact and Hausdorff, then Φ is a homeomorphism.

[2]Recall that a partition is *closed* if the saturation of each closed set is closed.

24.Zx. *Let* S *be a partition of a space* X*, and let* $\operatorname{pr} : X \to X/_S$ *be the projection. The space* $X \times Y$ *bears a natural partition* $S' = \{A \times y \mid A \in S,\ y \in Y\}$*. If the space* Y *is Hausdorff and locally compact, then the natural quotient map* $f : (X \times Y)/_{S'} \to X/_S \times Y$ *of the projection* $\operatorname{pr} \times \operatorname{id}_Y$ *is a homeomorphism.*

> ***24.17x.*** Try to prove Theorem *24.Zx* directly.

Proofs and Comments

19.A For example, let us prove the second relation:

$$(x, y) \in (A_1 \times B_1) \cap (A_2 \times B_2) \iff x \in A_1,\ y \in B_1,\ x \in A_2,\ y \in B_2$$

$$\iff x \in A_1 \cap A_2,\ y \in B_1 \cap B_2 \iff (x, y) \in (A_1 \cap A_2) \times (B_1 \cap B_2).$$

19.B Indeed,

$$\mathrm{pr}_X^{-1}(A) = \{z \in X \times Y \mid \mathrm{pr}_X(z) \in A\} = \{(x, y) \in X \times Y \mid x \in A\} = A \times Y.$$

19.C $\boxed{\Rightarrow}$ Indeed, $\Gamma_f \cap (x \times Y) = \{(x, f(x))\}$ is a singleton. $\boxed{\Leftarrow}$ If $\Gamma \cap (x \times Y)$ is a singleton $\{(x, y)\}$, then we can put $f(x) = y$.

19.D This follows from *3.A* because the intersection of elementary sets is an elementary set.

19.E Verify that $X \times Y \to Y \times X : (x, y) \mapsto (y, x)$ is a homeomorphism.

19.F In view of a canonical bijection, we can identify two sets and write

$$(X \times Y) \times Z = X \times (Y \times Z) = \{(x, y, z) \mid x \in X, y \in Y, z \in Z\}.$$

However, elementary sets in the spaces $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are different. Check that the collection $\{U \times V \times W \mid U \in \Omega_X,\ V \in \Omega_Y,\ W \in \Omega_Z\}$ is a base of the topological structures in both spaces.

19.G Indeed, for each open set $U \subset X$ the preimage $\mathrm{pr}_X^{-1}(U) = U \times Y$ is an elementary open set in $X \times Y$.

19.H Let Ω' be a topology in $X \times Y$ such that the projections pr_X and pr_Y are continuous. Then, for any $U \in \Omega_X$ and $V \in \Omega_Y$, we have

$$\mathrm{pr}_X^{-1}(U) \cap \mathrm{pr}_Y^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V \in \Omega'.$$

Therefore, each base set of the product topology lies in Ω', whence it follows that Ω' contains the product topology of X and Y.

19.I Clearly, $\mathrm{ab}(\mathrm{pr}_X) : X \times y_0 \to X$ is a continuous bijection. To see that the inverse map is continuous, we must show that each set open in $X \times y_0$ as in a subspace of $X \times Y$ has the form $U \times y_0$. Indeed, if W is open in $X \times Y$, then

$$W \cap (X \times y_0) = \bigcup_\alpha (U_\alpha \times V_\alpha) \cap (X \times y_0) = \bigcup_{\alpha\,:\, y_0 \in V_\alpha} (U_\alpha \times y_0) = \Big(\bigcup_{\alpha\,:\, y_0 \in V_\alpha} U_\alpha\Big) \times y_0.$$

19.J From the point of view of set theory, we have $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$. The collection of open rectangles is a base of topology in $\mathbb{R}^1 \times \mathbb{R}^1$ (show this), therefore, the topologies in $\mathbb{R}^1 \times \mathbb{R}^1$ and $\mathbb{R}^2$ have one and the same base, and so they coincide. The second assertion is proved by induction and, in turn, implies the third one by *19.7*.

19.K Set $f(z) = (f_1(z), f_2(z))$. If $f(z) = (x, y) \in X \times Y$, then $x = (\mathrm{pr}_X \circ f)(z) = f_1(z)$. We similarly have $y = f_2(z)$.

19.L $\boxed{\Rightarrow}$ The maps $f_1 = \mathrm{pr}_X \circ f$ and $f_2 = \mathrm{pr}_Y \circ f$ are continuous as compositions of continuous maps (use *19.G*).
$\boxed{\Leftarrow}$ Recall the definition of the product topology and use *19.20*.

19.M Recall the definition of the product topology and use *19.22*.

19.N Let X and Y be two Hausdorff spaces, $(x_1, y_1), (x_2, y_2) \in X \times Y$ two distinct points. Let, for instance, $x_1 \neq x_2$. Since X is Hausdorff, x_1 and x_2 have disjoint neighborhoods: $U_{x_1} \cap U_{x_2} = \varnothing$. Then, e.g., $U_{x_1} \times Y$ and $U_{x_2} \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2) in $X \times Y$.

19.O If A and B are countable and dense in X and Y, respectively, then $A \times B$ is a dense countable set in $X \times Y$.

19.P See the proof of Theorem *19.Q* below.

19.Q If $\Sigma_X$and Σ_Y are countable bases in X and Y, respectively, then $\Sigma = \{U \times V \mid U \in \Sigma_X,\ V \in \Sigma_Y\}$ is a base in $X \times Y$ by *19.15*.

19.R Show that if ρ_1 and ρ_2 are metrics on X and Y, respectively, then $\rho\big((x_1, y_1), (x_2, y_2)\big) = \max\{\rho_1(x_1, x_2), \rho_2(y_1, y_2)\}$ is a metric in $X \times Y$ generating the product topology. What form have the balls in the metric space $(X \times Y, \rho)$?

19.S For any two points $(x_1, y_1), (x_2, y_2) \in X \times Y$, the set $(X \times y_2) \cup (x_1 \times Y)$ is connected and contains these points.

19.T If u are v are paths joining x_1 with x_2 and y_1 with y_2, respectively, then the path $u \times v$ joins (x_1, y_1) with (x_2, y_2).

19.U It is sufficient to consider a cover consisting of elementary sets. Since Y is compact, each fiber $x \times Y$ has a finite subcovering $\{U_i^x \times V_i^x\}$. Put $W^x = \cap U_i^x$. Since X is compact, the cover $\{W^x\}_{x \in X}$ has a finite subcovering W^{x_j}. Then $\{U_i^{x_j} \times V_i^{x_j}\}$ is the required finite subcovering.

19.V Consider the map $(x, y) \mapsto \Big(\big(\frac{x}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\big), \ln(\sqrt{x^2 + y^2})\Big)$.

20.A First, the preimage $\mathrm{pr}^{-1}\big(\mathrm{pr}(A)\big)$ is saturated, secondly, it is the least because if $B \supset A$ is a saturated set, then $B = \mathrm{pr}^{-1}\big(\mathrm{pr}(B)\big) \supset \mathrm{pr}^{-1}\big(\mathrm{pr}(A)\big)$.

20.C Put $\Omega' = \{U \subset X/_S \mid \mathrm{pr}^{-1}(U) \in \Omega\}$. Let $U_\alpha \in \Omega'$. Since the sets $p^{-1}(U_\alpha)$ are open, the set $p^{-1}(\cup U_\alpha) = \cup p^{-1}(U_\alpha)$ is also open, whence

$\cup U_\alpha \in \Omega'$. Verify the remaining axioms of topological structure on your own.

20.D $\boxed{\Longrightarrow}$ If a set $V \subset X$ is open and saturated, then $V = \operatorname{pr}^{-1}\big(p(V)\big)$, hence, the set $U = \operatorname{pr}(V)$ is open in $X/_S$.

$\boxed{\Longleftarrow}$ Conversely, if $U \subset X/_S$ is open, then $U = \operatorname{pr}\big(\operatorname{pr}^{-1}(U)\big)$, where $V = \operatorname{pr}^{-1}(U)$ is open and saturated.

20.E The set F closed, iff $X/_S \smallsetminus F$ is open, iff $\operatorname{pr}^{-1}(X/_S \smallsetminus F) = X \smallsetminus \operatorname{pr}^{-1}(F)$ is open, iff $p^{-1}(F)$ is closed.

20.F This immediately follows from the definition of the quotient topology.

20.G We must prove that if Ω' is a topology in $X/_S$ such that the factorization map is continuous, then $\Omega' \subset \Omega_{X/S}$. Indeed, if $U \in \Omega'$, then $p^{-1}(U) \in \Omega_X$, whence $U \in \Omega_{X/S}$ by the definition of the quotient topology.

20.H It is connected as a continuous image of a connected space.

20.I It is path-connected as a continuous image of a path-connected space.

20.J It is separable as a continuous image of a separable space.

20.K It is compact as a continuous image of a compact space.

20.L This quotient space consists of two points, one of which is not open in it.

20.M $\boxed{\Longrightarrow}$ Let $a, b \in X/_S$, and let $A, B \subset X$ be the corresponding elements of the partition. If U_a and U_b are disjoint neighborhoods of a and b, then $p^{-1}(U_a)$ and $p^{-1}(U_b)$ are disjoint saturated neighborhoods of A and B. $\boxed{\Longleftarrow}$ This follows from *20.D*.

20.N 1) $\boxed{\Longrightarrow}$ Put $g = f/_S$. $\boxed{\Longleftarrow}$ The set $f^{-1}(y) = p^{-1}(g^{-1}(y))$ is saturated, i.e., it consists of elements of the partition S. Therefore, f is constant at each of the elements of the partition. 2) If A is an element of S, a is the point of the quotient set corresponding to A, and $x \in A$, then $f/_S(a) = f(A) = g(p(x)) = g(a)$.

20.O The map f maps elements of S to those of T iff there exists a map $g : X/_S \to Y/_T$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \operatorname{pr}_X}\big\downarrow & & {\scriptstyle \operatorname{pr}_Y}\big\downarrow \\ X/_S & \xrightarrow{g} & Y/_T \end{array}$$

is commutative. Then we have $f/_{(S,T)} = g$.

20.P This is so because distinct elements of the partition $S(f)$ are preimages of distinct points in Y.

20.Q Since $p^{-1}((f/_S)^{-1}(U)) = (f/_S \circ p)^{-1}(U) = f^{-1}(U)$, the definition of the quotient topology implies that for each $U \in \Omega_Y$ the set $(f/_S)^{-1}(U)$ is open, i.e., the map $f/_S$ is continuous.

20.R See *20.O* and *20.8*.

***20.A*x** Each singleton in $X/_S$ is the image of a singleton in X. Since X satisfies T_1, each singleton in X is closed, and its image, by *20.1*x, is also closed. Consequently, the quotient space also satisfies T_1.

***20.B*x** This follows from *14.25*.

***20.C*x** Let $U_n = p(V_n)$, $n \in \mathbb{N}$, where $\{V_n\}_{n\in\mathbb{N}}$ is a base X. Consider an open set W in the quotient space. Since $\mathrm{pr}^{-1}(W) = \bigcup_{n\in A} V_n$, we have $W = \mathrm{pr}\big(\mathrm{pr}^{-1}(W)\big) = \bigcup_{n\in A} U_n$, i.e., the collection $\{U_n\}$ is a base in the quotient space.

***20.D*x** For an arbitrary point $y \in X/_S$, consider the image of a countable neighborhood base at a certain point $x \in \mathrm{pr}^{-1}(y)$.

***20.E*x** Since the injective factor of a continuous surjection is a continuous bijection, it only remains to prove that the factor is an open map, which follows by *20.7* from the fact that the map $X \times Y \to X/_S \times Y/_T$ is open (see *19.23*).

21.A This follows from *20.P*, *20.Q*, *20.K*, and *16.Y*.

21.B Use *16.Z* instead of *16.Y*.

21.C.1 If $f : t \in [0,1] \mapsto (\cos 2\pi t, \sin 2\pi t) \in S^1$, then $f/_{S(f)}$ is a homeomorphism as a continuous bijection of a compact space onto a Hausdorff space, and the partition $S(f)$ is the initial one.

21.D.1 If $f : x \in \mathbb{R}^n \mapsto (\frac{x}{r}\sin \pi r, -\cos \pi r) \in S^n \subset \mathbb{R}^{n+1}$, then the partition $S(f)$ is the initial one and $f/_{S(f)}$ is a homeomorphism.

21.E Consider the map $g = f \times \mathrm{id} : I^2 = I \times I \to S^1 \times I$ (f is defined as in *21.C.1*). The partition $S(g)$ is the initial one, so that $g/_{S(g)}$ a homeomorphism.

21.F Check that the partition $S(\mathrm{id}_{S^1} \times f)$ is the initial one.

21.G The partition $S(f \times f)$ is the initial one.

21.H Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{p_1} & X/_S \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p_2}\big\downarrow \\ X/_T & \xrightarrow{q} & X/_{S}/_{S'} \end{array}$$

where the map q is obviously a bijection. The assertion of the problem follows from the fact that a set U is open in $X/_S/_{S'}$ iff $p_1^{-1}\big(p_2^{-1}(U)\big) = p^{-1}\big(q^{-1}(U)\big)$ is open in X iff $q^{-1}(U)$ is open in $X/_T$.

21.I To simplify the formulas, we replace the square I^2 ba a rectangle. Here is a formal argument: consider the map

$$\varphi : [0, 2\pi] \times [-\tfrac{1}{2}, \tfrac{1}{2}] \to \mathbb{R}^3 : (x, y) \mapsto \Big((1 + y \sin \tfrac{x}{2}) \cos x, (1 + y \sin \tfrac{x}{2}) \sin x, y \sin x\Big).$$

Check that φ really maps the square onto the Möbius strip and that $S(\varphi)$ is the given partition. Certainly, the starting point of the argument is not a specific formula. First of all, you should imagine the required map. We map the horizontal midline of the unit square onto the mid-circle of the Möbius strip, and we map each of the vertical segments of the square onto a segment of the strip orthogonal to the the mid-circle. This mapping maps the vertical sides of the square to one and the same segment, but here the opposite vertices of the square are identified with each other (check this).

21.J See the following section.

21.K Actually, it is easier to prove a more general assertion. Assume that we are given topological spaces X_α and maps $f_\alpha : X_\alpha \to Y$. Then $\Omega = \{U \subset Y \mid f_\alpha^{-1}(U) \text{ is open in } X_\alpha\}$ is the finest topological structure in Y with respect to which all maps f_α are continuous.

21.L See the hint to *21.K*.

21.M We map $D^n_1 \sqcup D^n_2$ to S^n so that the images of D^n_1 and D^n_2 are the upper and the lower hemisphere, respectively. The partition into the preimages is the partition with quotient space $D^n \cup_{\mathrm{id}|_{S^{n-1}}} D^n$. Consequently, the corresponding quotient map is a homeomorphism.

21.N Consider the map $F : X \sqcup Y \to X \sqcup Y$ such that $F|_X = \mathrm{id}_X$ and $F|_Y = h$. This mapping maps an element of the partition corresponding to the equivalence relation $z \sim f(x)$ to an element of the partition corresponding to the equivalence relation $x \sim g(x)$. Consequently, there exists a continuous bijection $H : X \cup_f Y \to X \cup_g Y$. Since h^{-1} also is a homeomorphism, H^{-1} is also continuous.

21.O By *21.N*, it is sufficient to prove that any homeomorphism $f : S^{n-1} \to S^{n-1}$ can be extended to a homeomorphism $F : D^n \to D^n$, which is obvious.

21.P For example, the stereographic projection from an inner point of the hole maps the sphere with a hole onto a disk homeomorphically.

21.Q The stereographic projection from an inner point of one of the holes homeomorphically maps the sphere with two holes onto a "disk with a hole". Prove that the latter is homeomorphic to a cylinder. (Another option: if we take the center of the projection in the hole in an appropriate way, then the projection maps the sphere with two holes onto a circular ring, which is obviously homeomorphic to a cylinder.)

21.R By definition, the handle is homeomorphic to a torus with a hole, while the sphere with a hole is homeomorphic to a disk, which precisely fills in the hole.

21.S Cut a sphere with two handles into two symmetric parts each of which is homeomorphic to a handle.

21.T Combine the results of *21.P 21.J.*

21.U Consider the Klein bottle as a quotient space of a square and cut the square into 5 horizontal (rectangular) strips of equal width. Then the quotient space of the middle strip will be a Möbius band, the quotient space of the union of the two extreme strips will be one more Möbius band, and the quotient space of the remaining two strips will be a ring, i.e., precisely a sphere with two holes. (Here is another, maybe more visual, description. Look at the picture of the Klein bottle: it has a horizontal plane of symmetry. Two horizontal planes close to the plane of symmetry cut the Klein bottle into two Möbius bands and a ring.)

21.V The most visual approach here is as follows: single out one of the handles and one of the films. Replace the handle by a "tube" whose boundary circles are attached to those of two holes on the sphere, which should be sufficiently small and close to each other. After that, start moving one of the holes. (The topological type of the quotient space does not change in the course of such a motion.) First, bring the hole to the boundary of the film, then shift it onto the film, drag it once along the film, shift it from the film, and, finally, return the hole to the initial spot. As a result, we transform the initial handle (a torus with a hole) into a Klein bottle with a hole, which splits into two Möbius bands (see Problem *21.U*), i.e., into two films.

22.A Consider the composition f of the embedding D^n in S^n onto a hemisphere and of the projection $\text{pr} : S^n \to \mathbb{R}P^n$. The partition $S(f)$ is that described in the formulation. Consequently, $f/_{S(f)}$ is a homeomorphism.

22.C Consider $f : S^1 \to S^1 : \; z \mapsto z^2 \in \mathbb{C}$. Then $S^1/_{S(f)} \cong \mathbb{R}P^1$.

22.D See *22.A*.

22.E Consider the composition f of the embedding of S^n in $\mathbb{R}^n \setminus 0$ and of the projection onto the quotient space by the described the partition. It is clear that the partition $S(f)$ is the partition factorizing by which we obtain the projective space. Therefore, $f/_{S(f)}$ is a homeomorphism.

22.F To see that the described function is a metric, use the triangle inequality between the plane angles of a trilateral angle. Now, take each point $x \in S^n$ the line $l(x)$ through the origin with direction vector x. We have thus defined a continuous (check this) map of S^n to the indicated space of lines, whose injective factor is a homeomorphism.

22.G The image of this map is the set $U_0 = \{(x_0 : x_1 : \ldots : x_n) \mid x_0 \neq 0\}$, and the inverse map $j : U_0 \to \mathbb{R}^n$ is defined by the formula

$$(x_0 : x_1 : \ldots : x_n) \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Since both i and j are continuous, i is a topological embedding.

22.H Consider the embedding $S^{n-1} = S^n \cap \{x_{n+1} = 0\} \to S^n \subset \mathbb{R}^{n+1}$ and the induced embedding $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$.

***23.A*x** If $a \precsim b \precsim c$, then we have $a \prec b \prec c$, $a = b = c$, $a \prec b = c$, or $a = b \prec c$. In all four cases, we have $a \precsim c$.

***23.B*x** The relation $\sim$ is obviously reflexive, symmetric, and also transitive.

***23.C*x** Indeed, if $a' \sim a$, $a \precsim b$, and $b \sim b'$, then $a' \precsim a \precsim b \precsim b'$, whence $a' \precsim b'$. Clearly, the relation defined on the equivalence classes is transitive and reflexive. Now, if two equivalence classes $[a]$ and $[b]$ satisfy both $a \precsim b$ and $b \precsim a$, then $[a] = [b]$, i.e., the relation is anti-symmetric, hence, it is a nonstrict order.

***23.D*x** (a) In this case, we obtain the trivial nonstrict order on a singleton; (b) In this case, we obtain the same nonstrict order on the same set.

***23.E*x** The relation is obviously reflexive. Further, if $a \precsim b$, then each neighborhood U of a contains b, and so U also is a neighborhood of b, hence, if $b \precsim c$, then $c \in U$. Therefore, $a \in \operatorname{Cl}\{c\}$, whence $a \precsim c$, and thus the relation is also transitive.

***23.F*x** Consider the element of the partition that consists by definition of points each of which lies in the closure of any other point, so that each open set in X containing one of the points also contains any other. Therefore, the topology induced on each element of the partition is indiscrete. It is also clear that each element of the partition is a maximal subset which is an indiscrete subspace. Now consider two points in the quotient space and two points $x, y \in X$ lying in the corresponding elements of the partition. Since $x \not\sim y$, there is an open set containing exactly one of these points. Since each open set U in X is saturated with respect to the partition, the image of U in $X/_S$ is the required neighborhood.

***23.G*x** Obvious.

***23.H*x** This follows from *23.F*x, *23.G*x, and *20.R*.

***24.A*x** It is sufficient to observe that the sets in $\Delta^{(pw)}$ cover the entire set $\mathcal{C}(X, Y)$. (Actually, $\mathcal{C}(X, Y) \in \Delta^{(pw)}$.)

***24.B*x** Similarly to *24.A*x

***24.C*x** Since each one-point subset is compact, it follows that $\Delta^{(pw)} \subset \Delta^{(co)}$, whence $\Omega^{(pw)} \subset \Omega^{(co)}$.

***24.D*x** If $f \neq g$, then there is $x \in X$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, $f(x)$ and $g(x)$ have disjoint neighborhoods U and V, respectively. The subbase elements $W(x, U)$ and $W(x, V)$ are disjoint neighborhoods of f and g in the space $\mathcal{C}^{(pw)}(X, Y)$. They also are disjoint neighborhoods of f and g in $\mathcal{C}(X, Y)$.

***24.E*x** See assertion *24.I*x.

***24.H*x** Consider functions $f_n \in \mathcal{C}(X, Y)$ such that $\{f_n\}_1^\infty$ is a Cauchy sequence. For every point $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in Y. Therefore, since Y is a complete space, this sequence converges. Put $f(x) = \lim f_n(x)$. We have thus defined a function $f : X \to Y$.
Since $\{f_n\}$ is a Cauchy sequence, for each $\varepsilon > 0$ there exists a positive integer N such that $\rho\big(f_n(x), f_k(x)\big) < \frac{\varepsilon}{4}$ for any $n, k \geq N$ and $x \in X$. Passing to the limit as $k \to \infty$, we see that $\rho\big(f_n(x), f(x)\big) \leq \frac{\varepsilon}{4} < \frac{\varepsilon}{3}$ for any $n \geq N$ and $x \in X$. Thus, to prove that $f_n \to f$ as $n \to \infty$, it remains to show that $f \in \mathcal{C}(X, Y)$. For each $a \in X$, there exists a neighborhood U_a such that $\rho\big(f_N(x), f_N(a)\big) < \frac{\varepsilon}{3}$ for every $x \in U_a$. The triangle inequality implies that for every $x \in U_a$ we have

$$\rho\big(f(x), f(a)\big) \leq \rho\big(f(x), f_N(x)\big) + \rho\big(f_N(x), f_N(a)\big) + \rho\big(f_N(a), f(a)\big) < \varepsilon.$$

Therefore, the function f is a continuous limit of the considered Cauchy sequence.

***24.I*x** Take an arbitrary set $W(K, U)$ in the subbase. Let $f \in W(K, U)$. If $r = \rho(f(K), Y \smallsetminus U)$, then $D_r(f) \subset W(K, U)$. As a consequence, we see that each open set in the compact-open topology is open in the topology generated by the metric of uniform convergence. To prove the converse assertion, it suffices to show that for each map $f : X \to Y$ and each $r > 0$ there are compact sets $K_1, K_2, \dots, K_n \subset X$ and open sets $U_1, U_2, \dots, U_n \subset Y$ such that

$$f \in \bigcap_{i=1}^{n} W(K_i, U_i) \subset D_r(f).$$

Cover $f(X)$ by a finite number of balls with radius $r/4$ centered at certain points $f(x_1), f(x_2), \dots, f(x_n)$. Let K_i be the f-preimage of a closed disk in Y with radius $r/4$, and let U_i be the open ball with radius $r/2$. By construction, we have $f \in W(K_1, U_1) \cap \dots \cap W(K_n, U_n)$. Consider an arbitrary map g in this intersection. For each $x \in K_1$, we see that $f(x)$ and $g(x)$ lie in one and the same open ball with radius $r/2$, whence $\rho(f(x), g(x)) < r$. Since, by construction, the sets $K_1, \dots, K_n$ cover X, we have $\rho(f(x), g(x)) < r$ for all $x \in X$, whence $d(f, g) < r$, and, therefore, $g \in D_r(f)$.

***24.M*x** This follows from the fact that for each compact $K \subset X'$ and $U \subset Y'$ the preimage of the subbase set $W(K, U) \in \Delta^{(co)}(X', Y')$ is the subbase set $W(\varphi(K), \psi^{-1}(U)) \in \Delta^{(co)}(X, Y)$.

***24.N*x** This immediately follows from *24.M*x.

***24.O*x** It is clear that the indicated map is an injection. To simplify the notation, we identify the space $\mathcal{C}(X, B)$ with its image under this injection. for each compact set $K \subset X$ and $U \in \Omega_B$ we denote by $W^B(K, U)$ the corresponding subbase set in $\mathcal{C}(X, B)$. If $V \in \Omega_Y$ and $U = B \cap V$, then we have $W^B(K, U) = \mathcal{C}(X, B) \cap W(K, V)$, whence it follows that $\mathcal{C}(X, Y)$ induces the compact-open topology on $\mathcal{C}(X, B)$.

***24.P*x** Verify that the natural mapping $f \mapsto (\mathrm{pr}_Y \circ f, \mathrm{pr}_Z \circ f)$ is a homeomorphism.

***24.Q*x** The injectivity of ϕ follows from the fact that $\{X_i\}$ is a cover, while the continuity of ϕ follows from assertion *24.N*x. Once more, to simplify the notation, we identify the set $\mathcal{C}(X, Y)$ with its image under the injection ϕ. Let $K \subset X$ be a compact set, $U \in \Omega_Y$. Put $K_i = K \cap X_i$ and denote by $W^i(K_i, U)$ the corresponding element in the subbase $\Delta^{(co)}(X_i, Y)$. Since, obviously,

$$W(K, U) = \mathcal{C}(X, Y) \cap \big(W^1(K_1, U) \times \dots \times W^n(K_n, U)\big),$$

the continuous injection ϕ is indeed a topological embedding.

***24.S*x** Consider maps $f : X \to Y$, $g : Y \to Z$, a compact set $K \subset X$ and $V \in \Omega_Z$ such that $g(f(K)) \subset V$, i.e., $\phi(f, g) \in W(K, V)$. Then we have an inclusion $f(K) \subset g^{-1}(V) \in \Omega_Y$. Since Y is Hausdorff and locally compact and the set $f(K)$ is compact, $f(K)$ has a neighborhood U whose closure is compact and also contained in $g^{-1}(V)$ (see, *18.6*x.) In this case, we have $\phi(W(K,U) \times W(\operatorname{Cl} U, V)) \subset W(K,V)$, and, consequently, the map ϕ is continuous.

***24.T*x** The continuity of ϕ follows from *24.Mx*, and its injectivity is obvious. Let $K \subset X/_S$ be a compact set, $U \in \Omega_Y$. The image of the open subbase set $W(K,U) \subset \mathcal{C}(X/_S, Y)$ is the set of all maps $g : X \to Y$ constant on all elements of the partitions and such that $g(\mathrm{pr}^{-1}(K)) \subset U$. It remains to show that the set $W(\mathrm{pr}^{-1}(K), U)$ is open in $\mathcal{C}(X,Y)$. Since the quotient space $X/_S$ is Hausdorff, it follows that the set K is closed. Therefore, the preimage $\mathrm{pr}^{-1}(K)$ is closed, and hence also compact. Consequently, $W(\mathrm{pr}^{-1}(K), U)$ is a subbase set in $\mathcal{C}(X,Y)$.

***24.U*x** Let $f_0 \in \mathcal{C}(X,Y)$ and $x_0 \in X$. To prove that ϕ is continuous at the point (f_0, x_0), consider a neighborhood V of $f_0(x_0)$ in Y. Since the map f_0 is continuous, the point x_0 has a neighborhood U' such that $f_0(U') \subset V$. Since the space X is Hausdorff and locally compact, it follows that x_0 has a neighborhood U such that the closure $\operatorname{Cl} U$ is a compact subset of U'. Since, obviously, $f(x) \in V$ for any map $f \in W = W(\operatorname{Cl} U, V)$ and any point $x \in U$, we see that $\phi(W \times U) \subset V$.

***24.V*x** Assume that $x_0 \in X$, $K \subset Y$ be a compact set, $V \subset \Omega_Z$, and $F(x_0) \in W(K,V)$, i.e., $f(\{x_0\} \times K) \subset V$. Let us show that the map F is continuous. For this purpose, let us find a neighborhood U_0 of x_0 in X such that $F(U_0) \subset W(K,V)$. The latter inclusion is equivalent to the fact that $f(U_0 \times K) \in V$. We cover the set $\{x_0\} \times K$ by a finite number of neighborhoods $U_i \times V_i$ such that $f(U_i \times V_i) \subset V$. It remains to put $U_0 = \bigcap_i U_i$.

***24.W*x** Let $(x_0, y_0) \in X \times Y$, and let G be a neighborhood of the point $z_0 = f(x_0, y_0) = F(x_0)(y_0)$. Since the map $F(x_0) : Y \to Z$ is continuous, y_0 has a neighborhood W such that $F(W) \subset G$. Since Y is Hausdorff and locally compact, y_0 has a neighborhood V with compact closure such that $\operatorname{Cl} V \subset W$ and, consequently, $F(x_0)(\operatorname{Cl} V) \subset G$, i.e., $F(x_0) \in W(\operatorname{Cl} V, G)$. Since the map F is continuous, x_0 has a neighborhood U such that $F(U) \subset W(\operatorname{Cl} V, G)$. Then, if $(x,y) \in U \times V$, we have $F(x) \in W(\operatorname{Cl} V, G)$, whence $f(x,y) = F(x)(y) \in G$. Therefore, $f(U \times V) \subset G$, i.e., f is continuous.

***24.X*x** It suffices to show that for each compact set $K \subset X$, each open set $U \subset Y$, and each $f \in W(K,U)$ there are compact sets $K_1, K_2, \dots, K_m \subset K$ and open sets $U_1, U_2, \dots, U_m \in \Sigma_Y$ such that

$$f \in W(K_1, U_1) \cap W(K_2, U_2) \cap \dots \cap W(K_m, U_m) \subset W(K,U).$$

Let $x \in K$. Since $f(x) \in U$, there are sets $U_1^x, U_2^x, \dots, U_{n_x}^x \in \Sigma_Y$ such that $f(x) \in U_1^x \cap U_2^x \cap \dots \cap U_{n_x} \subset U$. Since f is continuous, x has a neighborhood G_x such that $f(x) \in U_1^x \cap U_2^x \cap \dots \cap U_{n_x}$. Since X is locally compact and Hausdorff, X is regular, consequently, x has a neighborhood

V_x such that $\operatorname{Cl} V_x$ is compact and $\operatorname{Cl} V_x \in G_x$. Since the set K is compact, K is covered by a finite number of neighborhoods V_{x_i}, $i = 1, 2, \dots, n$. We put $K_i = K \cap \operatorname{Cl} V_{x_i}$, $i = 1, 2, \dots, n$, and $U_{ij} = U_j^{x_i}$, $j = 1, 2, \dots, n_{x_i}$. Then the set

$$\bigcap_{i=1}^{n} \bigcap_{j=1}^{n_i} W(K_j, U_{ij})$$

is the required one.

***24.Y*x** First of all, we observe that assertion *24.V*x implies that the map Φ is well defined (i.e., for $f \in \mathcal{C}(X, \mathcal{C}(Y, Z))$ we indeed have $\Phi(f) \in \mathcal{C}(X, \mathcal{C}(Y, Z)))$, while assertion *24.W*x implies that if Y is locally compact and Hausdorff, then Φ is invertible.
1) Let $K \subset X$ and $L \subset Y$ be compact sets, $V \in \Omega_Z$. The sets of the form $W(L, V)$ constitute a subbase in $\mathcal{C}(Y, Z)$. By *24.X*x, the sets of the form $W(K, W(L, V))$ constitute a subbase in $\mathcal{C}(X, \mathcal{C}(Y, Z))$. It remains to observe that $\Phi^{-1}(W(K, W(L, V))) = W(K \times L, V) \in \Delta^{(co)}(X \times Y, Z)$. Therefore, the map Φ is continuous.
2) Let $Q \subset X \times Y$ be a compact set and $G \subset\in \Omega_Z$. Let $\varphi \in \Phi(W(Q, G))$, so that $\varphi(x) : y \mapsto f(x, y)$ for a certain map $f \in W(Q, G)$. For each $q \in Q$, take a neighborhood $U_q \times V_q$ of q such that: the set $\operatorname{Cl} V_q$ is compact and $f(U_q \times \operatorname{Cl} V_q) \subset G$. Since Q is compact, we have $Q \subset \bigcup_{i=1}^{n}(U_{q_i} \times V_{q_i})$. The sets $W_i = W(\operatorname{Cl} V_{q_i}, G)$ are open in $\mathcal{C}(Y, Z)$, hence, the sets $T_i = W(p_X(Q) \cap \operatorname{Cl} U_{q_i}, W_i)$ are open in $\mathcal{C}(X, \mathcal{C}(Y, Z))$. Therefore, $T = \bigcap_{i=1}^{n} T_i$ is a neighborhood of φ. Let us show that $T \subset \Phi(W(Q, G))$. Indeed, if $\psi \in T$, then $\psi = \Phi(g)$, and we have $g(x, y) \in G$ for $(x, y) \in Q$, so that $g \in W(Q, G)$, whence $\psi \in \Phi(W(Q, G))$. Therefore, the set $\Phi(W(Q, G))$ is open, and so Φ is a homeomorphism.

***24.Z*x** It is obvious that the quotient map f is a continuous bijection. Consider the factorization map $p : X \times Y \to (X \times Y)/_{S'}$. By *24.V*x, the map $\Phi : X \to \mathcal{C}(Y, (X \times Y)/_{S'})$, where $\Phi(x)(y) = p(x, y)$, is continuous. We observe that Φ is constant on elements of the partition S, consequently, the quotient map $\widetilde{\Phi} : X/_S \to \mathcal{C}(Y, (X \times Y)/_{S'})$ is continuous. By *24.W*x, the map $g : X/_S \times Y \to (X \times Y)/_{S'}$, where $g(z, y) = \widetilde{\Phi}(z)(y)$, is also continuous. It remains to observe that g and f are mutually inverse maps.