Chapter V

Topological Algebra

In this chapter, we study topological spaces strongly related to groups: either the spaces themselves are groups in a nice way (so that all the maps coming from group theory are continuous), or groups act on topological spaces and can be thought of as consisting of homeomorphisms.

This material has interdisciplinary character. Although it plays important roles in many areas of Mathematics, it is not so important in the framework of general topology. Quite often, this material can be postponed till the introductory chapters of the mathematical courses that really require it (functional analysis, Lie groups, etc.). In the framework of general topology, this material provides a great collection of exercises.

In the second part of the book, which is devoted to algebraic topology, groups appear in a more profound way. So, sooner or later, the reader will meet groups. At latest in the next chapter, when studying fundamental groups.

Groups are attributed to Algebra. In the mathematics built on sets, main objects are sets with additional structure. Above, we met a few of the most fundamental of these structures: topology, metric, partial order. Topology and metric evolved from geometric considerations. Algebra studied algebraic operations with numbers and similar objects and introduced into the set-theoretic Mathematics various structures based on operations. One of the simplest (and most versatile) of these structures is the structure of a group. It emerges in an overwhelming majority of mathematical environments. It often appears together with topology and in a nice interaction with it. This interaction is a subject of Topological Algebra.

The second part of this book is called Algebraic Topology. It also treats interaction of Topology and Algebra, spaces and groups. But this is a completely different interaction. The structures of topological space and group do not live there on the same set, but the group encodes topological properties of the space.

25x. Digression. Generalities on Groups

This section is included mainly to recall the most elementary definitions and statements concerning groups. We do not mean to present a self-contained outline of the group theory. The reader is actually assumed to be familiar with groups, homomorphisms, subgroups, quotient groups, etc.

If this is not yet so, we recommend to read one of the numerous algebraic textbooks covering the elementary group theory. The mathematical culture, which must be acquired for mastering the material presented above in this book, would make this an easy and pleasant exercise.

As a temporary solution, the reader can read few definitions and prove few theorems gathered in this section. They provide a sufficient basis for most of what follows.

25°1x. The Notion of Group

Recall that a *group* is a set G equipped with a group operation. A *group operation* in a set G is a map $\omega : G \times G \to G$ satisfying the following three conditions (known as *group axioms*):

- **Associativity.** $\omega(a, \omega(b, c)) = \omega(\omega(a, b), c)$ for any $a, b, c \in G$.
- **Existence of Neutral Element.** There exists $e \in G$ such that $\omega(e, a) = \omega(a, e) = a$ for every $a \in G$.
- **Existence of Inverse Element.** For any $a \in G$, there exists $b \in G$ such that $\omega(a, b) = \omega(b, a) = e$.

25.Ax Uniqueness of Neutral Element. *A group contains a unique neutral element.*

25.Bx Uniqueness of Inverse Element. *Each element of a group has a unique inverse element.*

25.Cx First Examples of Groups. In each of the following situations, check if we have a group. What is its neutral element? How to calculate the element inverse to a given one?

- The set G is the set $\mathbb{Z}$ of integers, and the group operation is addition: $\omega(a, b) = a + b$.
- The set G is the set $\mathbb{Q}_{>0}$ of positive rational numbers, and the group operation is multiplication: $\omega(a, b) = ab$.
- $G = \mathbb{R}$, and $\omega(a, b) = a + b$.
- $G = \mathbb{C}$, and $\omega(a, b) = a + b$.
- $G = \mathbb{R} \smallsetminus 0$, and $\omega(a, b) = ab$.

- G is the set of all bijections of a set A onto itself, and the group operation is composition: $\omega(a, b) = a \circ b$.

25.1x Simplest Group. 1) Can a group be empty? 2) Can it consist of one element?

A group consisting of one element is *trivial*.

25.2x Solving Equations. Let G be a set with an associative operation $\omega : G \times G \to G$. Prove that G is a group iff for any $a, b \in G$ the set G contains a unique element x such that $\omega(a, x) = b$ and a unique element y such that $\omega(y, a) = b$.

25°2x. Additive Versus Multiplicative

The notation above is never used! (The only exception may happen, as here, when the definition of group is discussed.) Instead, one uses either *multiplicative* or *additive* notation.

Under multiplicative notation, the group operation is called *multiplication* and denoted as multiplication: $(a, b) \mapsto ab$. The neutral element is called *unity* and denoted by 1 or 1_G (or e). The element inverse to a is denoted by a^{-1}. This notation is borrowed, say, from the case of nonzero rational numbers with the usual multiplication.

Under additive notation, the group operation is called *addition* and denoted as addition: $(a, b) \mapsto a + b$. The neutral element is called *zero* and denoted by 0. The element inverse to a is denoted by $-a$. This notation is borrowed, say, from the case of integers with the usual addition.

An operation $\omega : G \times G \to G$ is *commutative* if $\omega(a, b) = \omega(b, a)$ for any $a, b \in G$. A group with commutative group operation is *commutative* or *Abelian*. Traditionally, the additive notation is used only in the case of commutative groups, while the multiplicative notation is used both in the commutative and noncommutative cases. Below, we mostly use the multiplicative notation.

25.3x. In each of the following situations, check if we have a group:

(1) a singleton $\{a\}$ with multiplication $aa = a$,
(2) the set $\mathbb{S}_n$ of bijections of the set $\{1, 2, \dots, n\}$ of the first n positive integers onto itself with multiplication determined by composition (the *symmetric group of degree* n),
(3) the sets $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ with coordinate-wise addition,
(4) the set $Homeo(X)$ of all homeomorphisms of a topological space X with multiplication determined by composition,
(5) the set $GL(n, \mathbb{R})$ of invertible real $n \times n$ matrices equipped with matrix multiplication,
(6) the set $M_n(\mathbb{R})$ of all real $n \times n$ matrices with addition determined by addition of matrices,

(7) the set of all subsets of a set X with multiplication determined by the symmetric difference:

$$(A, B) \mapsto A \triangle B = (A \cup B) \smallsetminus (A \cap B),$$

(8) the set $\mathbb{Z}_n$ of classes of positive integers congruent modulo n with addition determined by addition of positive integers,
(9) the set of complex roots of unity of degree n equipped with usual multiplication of complex numbers,
(10) the set $\mathbb{R}_{>0}$ of positive reals with usual multiplication,
(11) $S^1 \subset \mathbb{C}$ with standard multiplication of complex numbers,
(12) the set of translations of a plane with multiplication determined by composition.

Associativity implies that every finite sequence of elements in a group has a well-defined product, which can be calculated by a sequence of pairwise multiplications determined by any placement of parentheses, say, $abcde = (ab)(c(de))$. The distribution of the parentheses is immaterial. In the case of a sequence of three elements, this is precisely the associativity: $(ab)c = a(bc)$.

25.Dx. Derive from the associativity that the product of any length does not depend on the position of the parentheses.

For an element a of a group G, the powers a^n with $n \in \mathbb{Z}$ are defined by the following formulas: $a^0 = 1$, $a^{n+1} = a^n a$, and $a^{-n} = (a^{-1})^n$.

25.Ex. Prove that raising to a power has the following properties: $a^p a^q = a^{p+q}$ and $(a^p)^q = a^{pq}$.

25°3x. Homomorphisms

Recall that a map $f : G \to H$ of a group to another one is a *homomorphism* if $f(xy) = f(x)f(y)$ for any $x, y \in G$.

25.4x. In the above definition of a homomorphism, the multiplicative notation is used. How does this definition look in the additive notation? What if one of the groups is multiplicative, while the other is additive?

25.5x. Let a be an element of a multiplicative group G. Is the map $\mathbb{Z} \to G : n \mapsto a^n$ a homomorphism?

25.Fx. Let G and H be two groups. Is the constant map $G \to H$ mapping the entire G to the neutral element of H a homomorphism? Is any other constant map $G \to H$ a homomorphism?

25.Gx. *A homomorphism maps the neutral element to the neutral element, and it maps mutually inverse elements to mutually inverse elements.*

25.Hx. The identity map of a group is a homomorphism. The composition of homomorphisms is a homomorphism.

Recall that a homomorphism f is an *epimorphism* if f is surjective, f is a *monomorphism* if f is injective, and f is an *isomorphism* if f is bijective.

25.Ix. *The map inverse to an isomorphism is also an isomorphism.*

Two groups are *isomorphic* if there exists an isomorphism of one of them onto another one.

25.Jx. *Isomorphism is an equivalence relation.*

25.6x. Show that the additive group $\mathbb{R}$ is isomorphic to the multiplicative group $\mathbb{R}_{>0}$.

25°4x. Subgroups

A subset A of a group G is a *subgroup* of G if A is invariant under the group operation of G (i.e., for any $a, b \in A$ we have $ab \in A$) and A equipped with the group operation induced by that in G is a group.

For two subsets A and B of a multiplicative group G, we put $AB = \{ab \mid a \in A,\ b \in B\}$ and $A^{-1} = \{a^{-1} \mid a \in A\}$.

25.Kx. A subset A of a multiplicative group G is a subgroup of G iff $AA \subset G$ and $A^{-1} \subset A$.

25.7x. The singleton consisting of the neutral element is a subgroup.

25.8x. Prove that a subset A of a *finite* group is a subgroup if $AA \subset A$. (The condition $A^{-1} \subset A$ is superfluous in this case.)

25.9x. List all subgroups of the additive group $\mathbb{Z}$.

25.10x. Is $GL(n, \mathbb{R})$ a subgroup of $M_n(\mathbb{R})$? (See *25.3x* for notation.)

25.Lx. *The image of a group homomorphism $f : G \to H$ is a subgroup of H.*

25.Mx. *Let $f : G \to H$ be a group homomorphism, K a subgroup of H. Then $f^{-1}(K)$ is a subgroup of G. In short:*
The preimage of a subgroup under a group homomorphism is a subgroup.

The preimage of the neutral element under a group homomorphism $f : G \to H$ is called the *kernel* of f and denoted by $\operatorname{Ker} f$.

25.Nx Corollary of 25.Mx. *The kernel of a group homomorphism is a subgroup.*

25.Ox. *A group homomorphism is a monomorphism iff its kernel is trivial.*

25.Px. *The intersection of any collection of subgroups of a group is also a subgroup.*

A subgroup H of a group G is *generated* by a subset $S \subset G$ if H is the smallest subgroup of G containing S.

25.Qx. The subgroup H generated by S is the intersection of all subgroups of G that contain S. On the other hand, H is the set of all elements that are products of elements in S and elements inverse to elements in S.

The elements of a set that generates G are *generators* of G. A group generated by one element is *cyclic*.

25.Rx. A cyclic (multiplicative) group consists of powers of its generator. (I.e., if G is a cyclic group and a generates G, then $G = \{a^n \mid n \in \mathbb{Z}\}$.) Any cyclic group is commutative.

25.11x. A group G is cyclic iff there exists an epimorphism $f : \mathbb{Z} \to G$.

25.Sx. *A subgroup of a cyclic group is cyclic.*

The number of elements in a group G is the *order* of G. It is denoted by $|G|$.

25.Tx. Let G be a finite cyclic group, d a positive divisor of $|G|$. Then there exists a unique subgroup H of G with $|H| = d$.

Each element of a group generates a cyclic subgroup, which consists of all powers of this element. The order of the subgroup generated by a (nontrivial) element $a \in G$ is the *order* of a. It can be a positive integer or the infinity.

For each subgroup H of a group G, the *right cosets* of H are the sets $Ha = \{xa \mid x \in H\}$, $a \in G$. Similarly, the sets aH are the *left cosets* of H. The number of distinct right (or left) cosets of H is the *index* of H.

25.Ux Lagrange theorem. *If H is a subgroup of a finite group G, then the order of H divides that of G.*

A subgroup H of a group G is *normal* if for any $h \in H$ and $a \in G$ we have $aha^{-1} \in H$. Normal subgroups are also called *normal divisors* or *invariant subgroups*.

In the case where the subgroup is normal, left cosets coincide with right cosets, and the set of cosets is a group with multiplication defined by the formula $(aH)(bH) = abH$. The group of cosets of H in G is called the *quotient group* or *factor group* of G by H and denoted by G/H.

25.Vx. The kernel $\operatorname{Ker} f$ of a homomorphism $f : G \to H$ is a normal subgroup of G.

25.Wx. *The image $f(G)$ of a homomorphism $f : G \to H$ is isomorphic to the quotient group $G/_{\operatorname{Ker} f}$ of G by the kernel of f.*

25.Xx. The quotient group $\mathbb{R}/_{\mathbb{Z}}$ is canonically isomorphic to the group S^1. Describe the image of the group $\mathbb{Q} \subset \mathbb{R}$ under this isomorphism.

25.Yx. Let G be a group, A a normal subgroup of G, and B an arbitrary subgroup of G. Then AB also is a normal subgroup of G, while $A \cap B$ is a normal subgroup of B. Furthermore, we have $AB/_A \cong B/_{A \cap B}$.

26x. Topological Groups

26°1x. Notion of Topological Group

A *topological group* is a set G equipped with both a topological structure and a group structure such that the maps $G \times G \to G : (x, y) \mapsto xy$ and $G \to G : x \mapsto x^{-1}$ are continuous.

26.1x. Let G be a group and a topological space simultaneously. Prove that the maps $\omega : G \times G \to G : (x, y) \mapsto xy$ and $\alpha : G \to G : x \mapsto x^{-1}$ are continuous iff so is the map $\beta : G \times G \to G : (x, y) \mapsto xy^{-1}$.

26.2x. Prove that if G is a topological group, then the inversion $G \to G : x \mapsto x^{-1}$ is a homeomorphism.

26.3x. Let G be a topological group, X a topological space, $f, g : X \to G$ two maps continuous at a point $x_0 \in X$. Prove that the maps $X \to G : x \mapsto f(x)g(x)$ and $X \to G : x \mapsto (f(x))^{-1}$ are continuous at x_0.

26.Ax. A group equipped with the discrete topology is a topological group.

26.4x. Is a group equipped with the indiscrete topology a topological group?

26°2x. Examples of Topological Groups

26.Bx. The groups listed in *25.Cx* equipped with standard topologies are topological groups.

26.5x. The unit circle $S^1 = \{|z| = 1\} \subset \mathbb{C}$ with the standard multiplication is a topological group.

26.6x. In each of the following situations, check if we have a topological group.

(1) The spaces $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ with coordinate-wise addition. ($\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$, while $\mathbb{H}^n$ is isomorphic to $\mathbb{C}^{2n}$.)
(2) The sets $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, and $M_n(\mathbb{H})$ of all $n \times n$ matrices with real, complex, and, respectively, quaternion elements, equipped with the product topology and element-wise addition. (We identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$, and $M_n(\mathbb{H})$ with $\mathbb{H}^{n^2}$.)
(3) The sets $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, and $GL(n, \mathbb{H})$ of invertible $n \times n$ matrices with real, complex, and quaternionic entries, respectively, under the matrix multiplication.
(4) $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $O(n)$, $O(n, \mathbb{C})$, $U(n)$, $SO(n)$, $SO(n, \mathbb{C})$, $SU(n)$, and other subgroups of $GL(n, K)$ with $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

26.7x. Introduce a topological group structure on the additive group $\mathbb{R}$ that would be distinct from the usual, discrete, and indiscrete topological structures.

26.8x. Find two nonisomorphic connected topological groups that are homeomorphic as topological spaces.

26.9x. On the set $G = [0, 1)$ (equipped with the standard topology), we define addition as follows: $\omega(x, y) = x + y \pmod 1$. Is (G, ω) a topological group?

26°3x. Translations and Conjugations

Let G be a group. Recall that the maps $L_a : G \to G : x \mapsto ax$ and $R_a : G \to G : x \mapsto xa$ are *left* and *right translations through* a, respectively. Note that $L_a \circ L_b = L_{ab}$, while $R_a \circ R_b = R_{ba}$. (To "repair" the last relation, some authors define right translations by $x \mapsto xa^{-1}$.)

26.Cx. *A translation of a topological group is a homeomorphism.*

Recall that the *conjugation* of a group G by an element $a \in G$ is the map $G \to G : x \mapsto axa^{-1}$.

26.Dx. *The conjugation of a topological group by any of its elements is a homeomorphism.*

The following simple observation allows a certain "uniform" treatment of the topology in a group: neighborhoods of distinct points can be compared.

26.Ex. If U is an open set in a topological group G, then for any $x \in G$ the sets xU, Ux, and U^{-1} are open.

26.10x. Does the same hold true for closed sets?

26.11x. Prove that if U and V are subsets of a topological group G and U is open, then UV and VU are open.

26.12x. Will the same hold true if we replace everywhere the word *open* by the word *closed*?

26.13x. Are the following subgroups of the additive group $\mathbb{R}$ closed?

(1) $\mathbb{Z}$,

(2) $\sqrt{2}\,\mathbb{Z}$,

(3) $\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$?

26.14x. Let G be a topological group, $U \subset G$ a compact subset, $V \subset G$ a closed subset. Prove that UV and VU are closed.

26.14x.1. Let F and C be two disjoint subsets of a topological group G. If F is closed and C is compact, then 1_G has a neighborhood V such that $CV \cup VC$ does not meet F. If G is locally compact, then V can be chosen so that $\mathrm{Cl}(CV \cup VC)$ be compact.

26°4x. Neighborhoods

26.Fx. *Let* Γ *be a neighborhood base of a topological group* G *at* 1_G. *Then* $\Sigma = \{aU \mid a \in G,\ U \in \Gamma\}$ *is a base for topology of* G.

A subset A of a group G is *symmetric* if $A^{-1} = A$.

26.Gx. Any neighborhood of 1 in a topological group contains a symmetric neighborhood of 1.

26.Hx. For any neighborhood U of 1 in a topological group, 1 has a neighborhood V such that $VV \subset U$.

26.15x. Let G be a topological group, U a neighborhood of 1_G, and n a positive integer. Then 1_G has a symmetric neighborhood V such that $V^n \subset U$.

26.16x. Let V be a symmetric neighborhood of 1_G in a topological group G. Then $\bigcup_{n=1}^{\infty} V^n$ is an open-closed subgroup.

26.17x. Let G be a group, Σ be a collection of subsets of G. Prove that G carries a unique topology Ω such that Σ is a neighborhood base for Ω at 1_G and (G, Ω) is a topological group, iff Σ satisfies the following five conditions:

(1) each $U \in \Sigma$ contains 1_G,
(2) for every $x \in U \in \Sigma$, there exists $V \in \Sigma$ such that $xV \subset U$,
(3) for each $U \in \Sigma$, there exists $V \in \Sigma$ such that $V^{-1} \subset U$,
(4) for each $U \in \Sigma$, there exists $V \in \Sigma$ such that $VV \subset U$,
(5) for any $x \in G$ and $U \in \Sigma$, there exists $V \in \Sigma$ such that $V \subset x^{-1}Ux$.

26.Ix. Riddle. In what sense *26.Hx* is similar to the triangle inequality?

26.Jx. Let C be a compact subset of G. Prove that for every neighborhood U of 1_G the unity 1_G has a neighborhood V such that $V \subset xUx^{-1}$ for every $x \in C$.

26°5x. Separation Axioms

26.Kx. A topological group G is Hausdorff, iff G satisfies the first separation axiom, iff the unity 1_G (or, more precisely, the singleton $\{1_G\}$) is closed.

26.Lx. A topological group G is Hausdorff iff the unity 1_G is the intersection of its neighborhoods.

26.Mx. If the unity of a topological group G is closed, then G is regular (as a topological space).

Use the following fact.

26.Mx.1. Let G be a topological group, $U \subset G$ a neighborhood of 1_G. Then 1_G has a neighborhood V with closure contained in U: $\operatorname{Cl} V \subset U$.

26.Nx Corollary. *For topological groups, the first three separation axioms are equivalent.*

26.18x. Prove that a finite group carries as many topological group structures as there are normal subgroups. Namely, each finite topological group G contains a normal subgroup N such that the sets gN with $g \in G$ form a base for the topology of G.

26°6x. Countability Axioms

26.Ox. If Γ is a neighborhood base at 1_G in a topological group G and $S \subset G$ is a dense set, then $\Sigma = \{aU \mid a \in S, U \in \Gamma\}$ is a base for the topology of G. (Cf. *26.Fx* and *15.J.*)

26.Px. A first countable separable topological group is second countable.

26.19x*. (Cf. *15.Dx*) A first countable Hausdorff topological group G is metrizable. Furthermore, G can be equipped with a right (left) invariant metric.

27x. Constructions

27°1x. Subgroups

27.Ax. Let H be a subgroup of a topological group G. Then the topological and group structures induced from G make H a topological group.

27.1x. Let H be a subgroup of an Abelian group G. Prove that, given a structure of topological group in H and a neighborhood base at 1, G carries a structure of topological group with the same neighborhood base at 1.

27.2x. Prove that a subgroup of a topological group is open iff it contains an interior point.

27.3x. Prove that every open subgroup of a topological group is also closed.

27.4x. Prove that every closed subgroup of finite index is also open.

27.5x. Find an example of a subgroup of a topological group that

(1) is closed, but not open;
(2) is neither closed, nor open.

27.6x. Prove that a subgroup H of a topological group is a discrete subspace iff H contains an isolated point.

27.7x. Prove that a subgroup H of a topological group G is closed, iff there exists an open set $U \subset G$ such that $U \cap H = U \cap \operatorname{Cl} H \neq \varnothing$, i.e., iff $H \subset G$ is locally closed at one of its points.

27.8x. Prove that if H is a non-closed subgroup of a topological group G, then $\operatorname{Cl} H \smallsetminus H$ is dense in $\operatorname{Cl} H$.

27.9x. *The closure of a subgroup of a topological group is a subgroup.*

27.10x. Is it true that the interior of a subgroup of a topological group is a subgroup?

27.Bx. A connected topological group is generated by any neighborhood of 1.

27.Cx. Let H be a subgroup of a group G. Define a relation: $a \sim b$ if $ab^{-1} \in H$. Prove that this is an equivalence relation, and the right cosets of H in G are the equivalence classes.

27.11x. What is the counterpart of *27.Cx* for left cosets?

Let G be a topological group, $H \subset G$ a subgroup. The set of left (respectively, right) cosets of H in G is denoted by $G/_H$ (respectively, $_H\backslash G$). The sets $G/_H$ and $_H\backslash G$ carry the quotient topology. Equipped with these topologies, they are called *spaces of cosets*.

27.Dx. For any topological group G and its subgroup H, the natural projections $G \to G/_H$ and $G \to {}_H\backslash G$ are open (i.e., the image of every open set is open).

27.Ex. The space of left (or right) cosets of a closed subgroup in a topological group is regular.

27.Fx. The group G is compact (respectively, connected) if so are H and $G/_H$.

27.12x. If H is a connected subgroup of a group G, then the preimage of any connected component of $G/_H$ is a connected component of G.

27.13x. Let us regard the group $SO(n-1)$ as a subgroup of $SO(n)$. If $n \geq 2$, then the space $SO(n)/_{SO(n-1)}$ is homeomorphic to S^{n-1}.

27.14x. The groups $SO(n)$, $U(n)$, $SU(n)$, and $Sp(n)$ are 1) compact and 2) connected for any $n \geq 1$. 3) How many connected components do the groups $O(n)$ and $O(p,q)$ have? (Here, $O(p,q)$ is the group of linear transformations in $\mathbb{R}^{p+q}$ preserving the quadratic form $x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2$.)

27°2x. Normal Subgroups

27.Gx. Prove that the closure of a normal subgroup of a topological group is a normal subgroup.

27.Hx. The connected component of 1 in a topological group is a closed normal subgroup.

27.15x. The path-connected component of 1 in a topological group is a normal subgroup.

27.Ix. The quotient group of a topological group is a topological group (provided that it is equipped with the quotient topology).

27.Jx. The natural projection of a topological group onto its quotient group is open.

27.Kx. If a topological group G is first (respectively, second) countable, then so is any quotient group of G.

27.Lx. Let H be a normal subgroup of a topological group G. Then the quotient group G/H is regular iff H is closed.

27.Mx. Prove that a normal subgroup H of a topological group G is open iff the quotient group G/H is discrete.

The *center* of a group G is the set $C(G) = \{x \in G \mid xg = gx \text{ for each } g \in G\}$.

27.16x. Each discrete normal subgroup H of a connected group G is contained in the center of G.

27°3x. Homomorphisms

For topological groups, by a *homomorphism* one means a group homomorphism which is *continuous*.

27.Nx. *Let G and H be two topological groups. A group homomorphism $f : G \to H$ is continuous iff f is continuous at 1_G.*

Besides similar modifications, which can be summarized by the following principle: *everything is assumed to respect the topological structures*, the terminology of group theory passes over without changes. In particular, an *isomorphism* in group theory is an invertible homomorphism. Its inverse is a homomorphism (and hence an isomorphism) automatically. In the theory of topological groups, this must be included in the definition: an *isomorphism* of topological groups is an invertible homomorphism whose inverse is also a homomorphism. In other words, an isomorphism of topological groups is a map that is both a group isomorphism and a homeomorphism. Cf. Section 10.

27.17x. Prove that the map $[0,1) \to S^1 : \ x \mapsto e^{2\pi i x}$ is a topological group homomorphism.

27.Ox. An epimorphism $f : G \to H$ is an open map iff the injective factor $f/S(f) : G/\operatorname{Ker} f \to H$ of f is an isomorphism.

27.Px. An epimorphism of a compact topological group onto a topological group with closed unity is open.

27.Qx. Prove that the quotient group $\mathbb{R}/\mathbb{Z}$ of the additive group $\mathbb{R}$ by the subgroup $\mathbb{Z}$ is isomorphic to the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with absolute value 1.

27°4x. Local Isomorphisms

Let G and H be two topological groups. A *local isomorphism* from G to H is a homeomorphism f of a neighborhood U of 1_G in G onto a neighborhood V of 1_H in H such that

- $f(xy) = f(x)f(y)$ for any $x, y \in U$ such that $xy \in U$,
- $f^{-1}(zt) = f^{-1}(z)f^{-1}(t)$ for any $z, t \in V$ such that $zt \in V$.

Two topological groups G and H are *locally isomorphic* if there exists a local isomorphism from G to H.

27.Rx. Isomorphic topological groups are locally isomorphic.

27.Sx. The additive group $\mathbb{R}$ and the multiplicative group $S^1 \subset \mathbb{C}$ are locally isomorphic, but not isomorphic.

***27.18*x.** Prove that local isomorphism of topological groups is an equivalence relation.

***27.19*x.** Find neighborhoods of unities in $\mathbb{R}$ and S^1 and a homeomorphism between them that satisfies the first condition in the definition of local isomorphism, but does not satisfy the second one.

***27.20*x.** Prove that if a homeomorphism between neighborhoods of unities in two topological groups satisfies only the first condition in the definition of local isomorphism, then it has a submap that is a local isomorphism between these topological groups.

27°5x. Direct Products

Let G and H be two topological groups. In group theory, the product $G \times H$ is given a group structure.[1] In topology, it is given a topological structure (see Section 19).

***27.T*x.** These two structures are compatible: the group operations in $G \times H$ are continuous with respect to the product topology.

Thus, $G \times H$ is a topological group. It is called the *direct product* of the topological groups G and H. There are canonical homomorphisms related to this: the inclusions $i_G : G \to G \times H : x \mapsto (x, 1)$ and $i_H : H \to G \times H : x \mapsto (1, x)$, which are monomorphisms, and the projections $\mathrm{pr}_G : G \times H \to G : (x, y) \mapsto x$ and $\mathrm{pr}_H : G \times H \to H : (x, y) \mapsto y$, which are epimorphisms.

***27.21*x.** Prove that the topological groups $(G \times H)/_{i_H(H)}$ and G are isomorphic.

***27.22*x.** The product operation is both commutative and associative: $G \times H$ is (canonically) isomorphic to $H \times G$, while $G \times (H \times K)$ is canonically isomorphic to $(G \times H) \times K$.

A topological group G *decomposes into a direct product of two subgroups* A *and* B if the map $A \times B \to G : (x, y) \mapsto xy$ is a topological group isomorphism. If this is the case, the groups G and $A \times B$ are usually identified via this isomorphism.

Recall that a similar definition exists in ordinary group theory. The only difference is that there an isomorphism is just an algebraic isomorphism. Furthermore, in that theory, G decomposes into a direct product of its subgroups A and B iff A and B generate G, A and B are normal subgroups, and $A \cap B = \{1\}$. Therefore, if these conditions are fulfilled in the case of topological groups, then $A \times B \to G : (x, y) \mapsto xy$ is a group isomorphism.

***27.23*x.** Prove that in this situation the map $A \times B \to G : (x, y) \mapsto xy$ is continuous. Find an example where the inverse group isomorphism is not continuous.

[1] Recall that the multiplication in $G \times H$ is defined by the formula $(x, u)(y, v) = (xy, uv)$.

***27.U*x.** Prove that if a compact Hausdorff group G decomposes algebraically into a direct product of two closed subgroups, then G also decomposes into a direct product of these subgroups as a topological group.

***27.24*x.** Prove that the multiplicative group $\mathbb{R}\smallsetminus 0$ of nonzero reals is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^0 = \{1, -1\}$ and $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$.

***27.25*x.** Prove that the multiplicative group $\mathbb{C} \smallsetminus 0$ of nonzero complex numbers is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{R}_{>0}$.

***27.26*x.** Prove that the multiplicative group $\mathbb{H} \smallsetminus 0$ of nonzero quaternions is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ and $\mathbb{R}_{>0}$.

***27.27*x.** Prove that the subgroup $S^0 = \{1, -1\}$ of $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ is not a direct factor.

***27.28*x.** Find a topological group homeomorphic to $\mathbb{R}P^3$ (the three-dimensional real projective space).

Let a group G contain a normal subgroup A and a subgroup B such that $AB = G$ and $A \cap B = \{1_G\}$. If B is also normal, then G is the direct product $A \times B$. Otherwise, G is a *semidirect product* of A and B.

***27.V*x.** Let a topological group G be a semidirect product of its subgroups A and B. If for any neighborhoods of unity, $U \subset A$ and $V \subset B$, their product UV contains a neighborhood of 1_G, then G is homeomorphic to $A \times B$.

27°6x. Groups of Homeomorphisms

For any topological space X, the auto-homeomorphisms of X form a group under composition as the group operation. We denote this group by $\operatorname{Top} X$. To make this group topological, we slightly enlarge the topological structure induced on $\operatorname{Top} X$ by the compact-open topology of $\mathcal{C}(X, X)$.

***27.W*x.** The collection of the sets $W(C, U)$ and $(W(C, U))^{-1}$ taken over all compact $C \subset X$ and open $U \subset X$ is a subbase for the topological structure on $\operatorname{Top} X$.

In what follows, we equip $\operatorname{Top} X$ with this topological structure.

***27.X*x.** If X is Hausdorff and locally compact, then $\operatorname{Top} X$ is a topological group.

27.X*x*.1. If X is Hausdorff and locally compact, then the map $\operatorname{Top} X \times \operatorname{Top} X \to \operatorname{Top} X : (g, h) \mapsto g \circ h$ is continuous.

28x. Actions of Topological Groups

28°1x. Action of a Group on a Set

A *left action* of a group G on a set X is a map $G \times X \to X : (g, x) \mapsto gx$ such that $1x = x$ for any $x \in X$ and $(gh)x = g(hx)$ for any $x \in X$ and $g, h \in G$. A set X equipped with such an action is a *left G-set*. Right G-sets are defined in a similar way.

***28.A*x.** If X is a left G-set, then $G \times X \to X : (x, g) \mapsto g^{-1}x$ is a right action of G on X.

***28.B*x.** If X is a left G-set, then for any $g \in G$ the map $X \to X : x \mapsto gx$ is a bijection.

A left action of G on X is *effective* (or *faithful*) if for each $g \in G \smallsetminus 1$ the map $G \to G : x \mapsto gx$ is not equal to id_G. Let X_1 and X_2 be two left G-sets. A map $f : X_1 \to X_2$ is *G-equivariant* if $f(gx) = gf(x)$ for any $x \in X$ and $g \in G$.

We say that X is a *homogeneous left G-set*, or, what is the same, that G acts on X *transitively* if for any $x, y \in X$ there exists $g \in G$ such that $y = gx$.

The same terminology applies to right actions with obvious modifications.

***28.C*x.** The natural actions of G on $G/_H$ and $_H\backslash G$ transform $G/_H$ and $_H\backslash G$ into homogeneous left and, respectively, right G-sets.

Let X be a homogeneous left G-set. Consider a point $x \in X$ and the set $G^x = \{g \in G \mid gx = x\}$. We easily see that G^x is a subgroup of G. It is called the *isotropy subgroup* of x.

***28.D*x.** Each homogeneous left (respectively, right) G-set X is isomorphic to $G/_H$ (respectively, $_H\backslash G$), where H is the isotropy group of a certain point in X.

***28.D*x.1.** All isotropy subgroups G^x, $x \in G$, are pairwise conjugate.

Recall that the *normalizer* $Nr(H)$ of a subgroup H of a group G consists of all elements $g \in G$ such that $gHg^{-1} = H$. This is the largest subgroup of G containing H as a normal subgroup.

***28.E*x.** The group of all automorphisms of a homogeneous G-set X is isomorphic to $N(H)/_H$, where H is the isotropy group of a certain point in X.

***28.E*x.1.** If two points $x, y \in X$ have the same isotropy group, then there exists an automorphism of X that sends x to y.

28°2x. Continuous Action

We speak about a *left G-space* X if X is a topological space, G is a topological group acting on X, and the action $G \times X \to X$ is continuous (as a map). All terminology (and definitions) concerning G-sets extends to G-spaces literally.

Note that if G is a discrete group, then any action of G by homeomorphisms is continuous and thus provides a G-space.

28.Fx. Let X be a left G-space. Then the natural map $\phi : G \to \operatorname{Top} X$ induced by this action is a group homomorphism.

28.Gx. If in the assumptions of Problem *28.Fx* the G-space X is Hausdorff and locally compact, then the induced homomorphism $\phi : G \to \operatorname{Top} X$ is continuous.

> ***28.1x.*** In each of the following situations, check if we have a continuous action and a continuous homomorphism $G \to \operatorname{Top} X$:
>
> (1) G is a topological group, $X = G$, and G acts on X by left (or right) translations, or by conjugation;
> (2) G is a topological group, $H \subset G$ is a subgroup, $X = G/H$, and G acts on X via $g(aH) = (ga)H$;
> (3) $G = GL(n, K)$ (where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$)), and G acts on K^n via matrix multiplication;
> (4) $G = GL(n, K)$ (where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), and G acts on KP^{n-1} via matrix multiplication;
> (5) $G = O(n, \mathbb{R})$, and G acts on S^{n-1} via matrix multiplication;
> (6) the (additive) group $\mathbb{R}$ acts on the torus $S^1 \times \dots \times S^1$ according to formula $(t, (w_1, \dots, w_r)) \mapsto (e^{2\pi i a_1 t} w_1, \dots, e^{2\pi i a_r t} w_r)$; this action is an *irrational flow* if $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$.

If the action of G on X is not effective, then we can consider its kernel

$$G^{\mathrm{Ker}} = \{g \in G \mid gx = x \text{ for all } x \in X\}.$$

This kernel is a closed normal subgroup of G, and the topological group G/G^{Ker} acts naturally and effectively on X.

28.Hx. The formula $gG^{\mathrm{Ker}}(x) = gx$ determines an effective continuous action of G/G^{Ker} on X.

A group G acts *properly discontinuously* on X if for any compact set $C \subset X$ the set $\{g \in G \mid (gC) \cap C \neq \varnothing\}$ is finite.

28.Ix. If G acts properly discontinuously and effectively on a Hausdorff locally compact space X, then $\phi(G)$ is a discrete subset of $\operatorname{Top} X$. (Here, as before, $\phi : G \to \operatorname{Top} X$ is the monomorphism induced by the G-action.) In particular, G is a discrete group.

> ***28.2x.*** List, up to similarity, all triangles $T \subset \mathbb{R}^2$ such that the reflections in the sides of T generate a group acting on $\mathbb{R}^2$ properly discontinuously.

28°3x. Orbit Spaces

Let X be a G-space. For $x \in X$, the set $G(x) = \{gx \mid g \in G\}$ is the *orbit* of x. In terms of orbits, the action of G on X is transitive iff it has only one orbit. For $A \subset X$ and $E \subset G$, we put $E(A) = \{ga \mid g \in E,\ a \in A\}$.

28.Jx. *Let G be a compact topological group acting on a Hausdorff space X. Then for any $x \in X$ the canonical map $G/_{G^x} \to G(x)$ is a homeomorphism.*

> ***28.3x.*** Give an example where X is Hausdorff, but $G/_{G_x}$ is not homeomorphic to $G(x)$.

28.Kx. If a compact topological group G acts on a compact Hausdorff space X, then $X/_G$ is a compact Hausdorff space.

> ***28.4x.*** Let G be a compact group, X a Hausdorff G-space, $A \subset X$. If A is closed (respectively, compact), then so is $G(A)$.
>
> ***28.5x.*** Consider the canonical action of $G = \mathbb{R} \smallsetminus 0$ on $X = \mathbb{R}$ (by multiplication). Find all orbits and all isotropy subgroups of this action. Recognize $X/_G$ as a topological space.
>
> ***28.6x.*** Let G be the group generated by reflections in the sides of a rectangle in $\mathbb{R}^2$. Recognize the quotient space $\mathbb{R}^2/_G$ as a topological space. Recognize the group G.
>
> ***28.7x.*** Let G be the group from Problem *28.6x*, and let $H \subset G$ be the subgroup of index 2 constituted by the orientation-preserving elements in G. Recognize the quotient space $\mathbb{R}^2/_H$ as a topological space. Recognize the groups G and H.
>
> ***28.8x.*** Consider the (diagonal) action of the torus $G = (S^1)^{n+1}$ on $X = \mathbb{C}P^n$ via $(z_0, z_1, \dots, z_n) \mapsto (\theta_0 z_0, \theta_1 z_1, \dots, \theta_n z_n)$. Find all orbits and isotropy subgroups. Recognize $X/_G$ as a topological space.
>
> ***28.9x.*** Consider the canonical action (by permutations of coordinates) of the symmetric group $G = \mathbb{S}_n$ on $X = \mathbb{R}^n$ and $X = \mathbb{C}^n$, respectively. Recognize $X/_G$ as a topological space.
>
> ***28.10x.*** Let $G = SO(3)$ act on the space X of symmetric 3×3 real matrices with trace 0 by conjugation $x \mapsto gxg^{-1}$. Recognize $X/_G$ as a topological space. Find all orbits and isotropy groups.

28°4x. Homogeneous Spaces

A G-space is *homogeneous* it the action of G is transitive.

28.Lx. Let G be a topological group, $H \subset G$ a subgroup. Then G is a homogeneous H-space under the translation action of H. The quotient space $G/_H$ is a homogeneous G-space under the induced action of G.

28.Mx. *Let X be a Hausdorff homogeneous G-space. If X and G are locally compact and G is second countable, then X is homeomorphic to $G/_{G^x}$ for any $x \in X$.*

28.Nx. Let X be a homogeneous G-space. Then the canonical map $G/_{G^x} \to X$, $x \in X$, is a homeomorphism iff it is open.

***28.11*x.** Show that $O(n+1)/O(n) = S^n$ and $U(n)/U(n-1) = S^{2n-1}$.

***28.12*x.** Show that $O(n+1)/O(n) \times O(1) = \mathbb{R}P^n$ and $U(n)/U(n-1) \times U(1) = \mathbb{C}P^n$.

***28.13*x.** Show that $Sp(n)/Sp(n-1) = S^{4n-1}$, where

$$Sp(n) = \{A \in GL(\mathbb{H}) \mid AA^* = I\}.$$

***28.14*x.** Represent the torus $S^1 \times S^1$ and the Klein bottle as homogeneous spaces.

***28.15*x.** Give a geometric interpretation of the following homogeneous spaces: 1) $O(n)/O(1)^n$, 2) $O(n)/O(k) \times O(n-k)$, 3) $O(n)/SO(k) \times O(n-k)$, and 4) $O(n)/O(k)$.

***28.16*x.** Represent $S^2 \times S^2$ as a homogeneous space.

***28.17*x.** Recognize $SO(n,1)/SO(n)$ as a topological space.

Proofs and Comments

***26.A*x** Use the fact that any auto-homeomorphism of a discrete space is continuous.

***26.C*x** Any translation is continuous, and the translations by a and a^{-1} are mutually inverse.

***26.D*x** Any conjugation is continuous, and the conjugations by g and g^{-1} are mutually inverse.

***26.E*x** The sets xU, Ux, and U^{-1} are the images of U under the homeomorphisms L_x and R_x of the left and right translations through x and passage to the inverse element (i.e., reversing), respectively.

***26.F*x** Let $V \subset G$ be an open set, $a \in V$. If a neighborhood $U \in \Gamma$ is such that $U \subset a^{-1}V$, then $aU \subset V$. By Theorem *3.A*, Σ is a base for topology of G.

***26.G*x** If U is a neighborhood of 1, then $U \cap U^{-1}$ is a symmetric neighborhood of 1.

***26.H*x** By the continuity of multiplication, 1 has two neighborhoods V_1 and V_2 such that $V_1V_2 \subset U$. Put $V = V_1 \cap V_2$.

***26.J*x** Let W be a symmetric neighborhood such that $1_G \in W$ and $W^3 \subset U$. Since C is compact, C is covered by finitely many sets of the form $W_1 = x_1W, \dots, W_n = x_nW$ with $x_1, \dots, x_n \in C$. Put $V = \bigcap(x_iWx_i^{-1})$. Clearly, V is a neighborhood of 1_G. If $x \in C$, then $x = x_iw_i$ for suitable $i, w_i \in W$. Finally, we have

$$x^{-1}Vx = w_i^{-1}x_i^{-1}Vx_iw_i \subset w_i^{-1}Ww_i \subset W^3 \subset U.$$

***26.K*x** If 1_G is closed, then all singletons in G are closed. Therefore, G satisfies T_1 iff 1_G is closed. Let us prove that in this case the group G is also Hausdorff. Consider $g \neq 1$ and take a neighborhood U of 1_G not containing g. By *26.15x*, 1_G has a symmetric neighborhood V such that $V^2 \subset U$. Verify that gV and V are disjoint, whence it follows that G is Hausdorff.

***26.L*x** $\boxed{\Longrightarrow}$ Use *14.C* $\boxed{\Longleftarrow}$ In this case, each element of G is the intersection of its neighborhoods. Hence, G satisfies the first separation axiom, and it remains to apply *26.Kx*.

26.M*x*.1 It suffices to take a symmetric neighborhood V such that $V^2 \subset U$. Indeed, then for any $g \notin U$ the neighborhoods gV and V are disjoint, whence $\operatorname{Cl} V \subset U$.

***26.O*x** Let W be an open set, $g \in W$. Let V be a symmetric neighborhood of 1_G with $V^2 \subset W$. There 1_G has a neighborhood $U \in \Gamma$ such that $U \subset V$. There exists $a \in S$ such that $a \in gU^{-1}$. Then $g \in aU$ and $a \in gU^{-1} \subset gV^{-1} = gV$. Therefore, $aU \subset aV \subset gV^2 \subset W$.

***26.P*x** This immediately follows from *26.O*x.

***27.B*x** This follows from *26.16*x.

***27.D*x** If U is open, then UH (respectively, HU) is open, see *26.11*x.

***27.E*x** Let G be the group, $H \subset G$ the subgroup. The space $G/_H$ of left cosets satisfies the first separation axiom since gH is closed in G for any $g \in G$. Observe that every open set in $G/_H$ has the form $\{gH \mid g \in U\}$, where U is an open set in G. Hence, it is sufficient to check that for every open neighborhood U of 1_G in G the unity 1_G has a neighborhood V in G such that $\operatorname{Cl} VH \subset UH$. Pick a symmetric neighborhood V with $V^2 \subset U$, see *26.15*x. Let $x \in G$ belong to $\operatorname{Cl} VH$. Then Vx contains a point vh with $v \in V$ and $h \in H$, so that there exists $v' \in V$ such that $v'x = vh$, whence $x \in V^{-1}VH = V^2H \subset UH$.

***27.F*x** (*Compactness*) First, we check that if H is compact, then the projection $G \to G/_H$ is a closed map. Let $F \subset G$ be a closed set, $x \notin FH$. Since FH is closed (see *26.14*x), x has a neighborhood U disjoint with FH. Then UH is disjoint with FH. Hence, the projection is closed. Now, consider a family of closed sets in G with finite intersection property. Their images also form a family of closed sets in $G/_H$ with finite intersection property. Since $G/_H$ is compact, the images have a nonempty intersection. Therefore, there is $g \in G$ such that the traces of the closed sets in the family on gH have finite intersection property. Finally, since gH is compact, the closed sets in the family have a nonempty intersection.
(*Connectedness*) Let $G = U \cup V$, where U and V are disjoint open subsets of G. Since all cosets gH, $g \in G$, are connected, each of them is contained either in U or in V. Hence, G is decomposed into UH and VH, which yields a decomposition of $G/_H$ in two disjoint open subsets. Since $G/_H$ is connected, either UH or VH is empty. Therefore, either U or V is empty.

***27.H*x** Let C be the connected component of 1_G in a topological group G. Then C^{-1} is connected and contains 1_G, whence $C^{-1} \subset C$. For any $g \in C$, the set gC is connected and meets C, whence $gC \subset C$. Therefore, C is a subgroup of G. C is closed since connected components are closed. C is normal since gCg^{-1} is connected and contains 1_G, whatever $g \in G$ is.

***27.I*x** Let G be a topological group, H a normal subgroup of G, $a, b \in G$ two elements. Let $\overline{W}$ be a neighborhood of the coset abH in $G/_H$. The preimage of $\overline{W}$ in G is an open set W consisting of cosets of H and containing ab. In particular, W is a neighborhood of ab. Since the multiplication in G is continuous, a and b have neighborhoods U and V, respectively, such that $UV \subset W$. Then $(UH)(VH) = (UV)H \subset WH$. Therefore, multiplication of elements in the quotient group determines a continuous map $G/_H \times G/_H \to G/_H$. Prove on your own that the map $G/_H \times G/_H : aH \to a^{-1}H$ is also continuous.

***27.J*x** This is special case of *27.D*x.

***27.K*x** If $\{U_i\}$ is a countable (neighborhood) base in G, then $\{U_iH\}$ is a countable (neighborhood) base in $G/_H$.

***27.L*x** This is a special case of *27.E*x.

***27.M*x** $\Rightarrow$ In this case, all cosets of H are also open. Therefore, each singleton in $G/_H$ is open. $\Leftarrow$ If $1_{G/H}$ is open in G/H, then H is open in G by the definition of the quotient topology.

***27.N*x** $\Rightarrow$ Obvious. $\Leftarrow$ Let $a \in G$, and let $b = f(a) \in H$. For any neighborhood U of b, the set $b^{-1}U$ is a neighborhood of 1_H in H. Therefore, 1_G has a neighborhood V in G such that $f(V) \subset b^{-1}U$. Then aV is a neighborhood of a, and we have $f(aV) = f(a)f(V) = bf(V) \subset bb^{-1}U = U$. Hence, f is continuous at each point $a \in G$, i.e., f is a topological group homomorphism.

***27.O*x** $\Rightarrow$ Each open subset of $G/_{\operatorname{Ker} f}$ has the form $U \cdot \operatorname{Ker} f$, where U is an open subset of G. Since $f/_{S(f)}(U \cdot \operatorname{Ker} f) = f(U)$, the map $f/_{S(f)}$ is open.

$\Leftarrow$ Since the projection $G \to G/_{\operatorname{Ker} f}$ is open (see *27.D*x), the map f is open if so is $f/_{S(f)}$.

***27.P*x** Combine *27.O*x, *26.K*x, and *16.Y*.

***27.Q*x** This follows from *27.O*x since the exponential map $\mathbb{R} \to S^1 : x \mapsto e^{2\pi xi}$ is open.

***27.S*x** The groups are not isomorphic since only one of them is compact. The exponential map $x \mapsto e^{2\pi xi}$ determines a local isomorphism from $\mathbb{R}$ to S^1.

***27.V*x** The map $A \times B \to G : (a, b) \mapsto ab$ is a continuous bijection. To see that it is a homeomorphism, observe that it is open since for any neighborhoods of unity, $U \subset A$ and $V \subset B$, and any points $a \in A$ and $b \in B$, the product $UaVb = abU'V'$, where $U' = b^{-1}a^{-1}Uab$ and $V' = b^{-1}Vb$, contains abW', where W' is a neighborhood of 1_G contained in $U'V'$.

***27.W*x** This immediately follows from *3.8*.

***27.X*x** The map $\operatorname{Top} X \to \operatorname{Top} X : g \mapsto g^{-1}$ is continuous because it preserves the subbase for the topological structure on $\operatorname{Top} X$. It remains to apply *27.Xx.1*.

27.Xx.1 It suffices to check that the preimage of every element of a subbase is open. For $W(C,U)$, this is a special case of *24.Sx*, where we showed that for any $gh \in W(C,U)$ there is an open U', $h(C) \subset U' \subset g^{-1}(U)$, such that $\operatorname{Cl} U'$ is compact, $h \in W(C,U')$, $g \in W(\operatorname{Cl} U', U)$, and

$$gh \in W(\operatorname{Cl} U', U) \circ W(C, U') \subset W(C,U).$$

The case of $(W(C,U))^{-1}$ reduces to the previous one because for any $gh \in (W(C,U))^{-1}$ we have $h^{-1}g^{-1} \in W(C,U)$, and so, applying the above construction, we obtain an open U' such that $g^{-1}(C) \subset U' \subset h(U)$, $\operatorname{Cl} U'$ is compact, $g^{-1} \in W(C,U')$, $h^{-1} \in W(\operatorname{Cl} U', U)$, and

$$h^{-1}g^{-1} \in W(\operatorname{Cl} U', U) \circ W(C,U') \subset W(C,U).$$

Finally, we have $g \in (W(C,U'))^{-1}$, $h \in (W(\operatorname{Cl} U', U))^{-1}$, and

$$gh \in (W(C,U'))^{-1} \circ (W(\operatorname{Cl} U', U))^{-1} \subset (W(C,U))^{-1}.$$

We observe that the above map is continuous even for the pure compact-open topology on $\operatorname{Top} X$.

28.Gx It suffices to check that the preimage of every element of a subbase is open. For $W(C,U)$, this is a special case of *24.Vx*. Let $\phi(g) \in (W(C,U))^{-1}$. Then $\phi(g^{-1}) \in W(C,U)$, and therefore g^{-1} has an open neighborhood V in G with $\phi(V) \subset W(C,U)$. It follows that V^{-1} is an open neighborhood of g in G and $\phi(V^{-1}) \subset (W(C,U))^{-1}$. (The assumptions about X are needed only to ensure that $\operatorname{Top} X$ is a topological group.)

28.Ix Let us check that 1_G is an isolated point of G. Consider an open set V with compact closure. Let $U \subset V$ be an open subset with compact closure $\operatorname{Cl} U \subset V$. Then, for each of finitely many $g_k \in G$ with $g_k(U) \cap V \neq \varnothing$, let $x_k \in X$ be a point with $g_k(x_k) \neq x_k$, and let U_k be an open neighborhood of x_k disjoint with $g_k(x_k)$. Finally, $G \cap W(\operatorname{Cl} U, V) \cap \bigcap W(x_k, U_k)$ contains only 1_G.

28.Jx The space $G/_{G^x}$ is compact, the orbit $G(x) \subset X$ is Hausdorff, and the map $G/_{G^x} \to G(x)$ is a continuous bijection. It remains to apply *16.Y*.

28.Kx To prove that $X/_G$ is Hausdorff, consider two disjoint orbits, $G(x)$ and $G(y)$. Since $G(y)$ is compact, there are disjoint open sets $U \ni x$ and $V \supset G(y)$. Since $G(x)$ is compact, there is a finite number of elements $g_k \in G$ such that $\bigcup g_k U$ covers $G(x)$. Then $\operatorname{Cl}(\bigcup g_k U) = \bigcup \operatorname{Cl} g_k U = \bigcup g_k \operatorname{Cl} U$ is disjoint with $G(y)$, which shows that $X/_G$ is Hausdorff. (Note that this part of the proof does not involve the compactness of X.) Finally, $X/_G$ is compact as a quotient of the compact space X.

28.Mx It suffices to prove that the canonical map $f : G/_{G^x} \to X$ is open (see *28.Nx*).

Take a neighborhood $V \subset G$ of 1_G with compact closure and a neighborhood $U \subset G$ of 1_G with $\operatorname{Cl} U \cdot \operatorname{Cl} U \subset V$. Since G contains a dense countable set, it follows that there is a sequence $g_n \in G$ such that $\{g_n U\}$ is an open cover of G. It remains to prove that at least one of the sets $f(g_n U) = g_n f(U) = g_n U(x)$ has nonempty interior.
Assume the contrary. Then, using the local compactness of X, its Hausdorff property, and the compactness of $f(g_n \operatorname{Cl} U)$, we construct by induction a sequence $W_n \subset X$ of nested open sets with compact closure such that W_n is disjoint with $g_k U x$ with $k < n$ and $g_n U x \cap W_n$ is closed in W_n. Finally, we obtain nonempty $\bigcap W_n$ disjoint with $G(x)$, a contradiction.

***28.N*x** The canonical map $G/G^x \to X$ is continuous and bijective. Hence, it is a homeomorphism iff it is open (and iff it is closed).