Fundamental Group

29. Homotopy

29°1. Continuous Deformations of Maps

29.A. Is it possible to deform continuously:

(1) the identity map $\mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ to the constant map $\mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto 0$,

(2) the identity map $\mathrm{id} : S^1 \to S^1$ to the symmetry $S^1 \to S^1 : x \mapsto -x$ (here x is considered a complex number because the circle S^1 is $\{x \in \mathbb{C} : |x| = 1\}$),

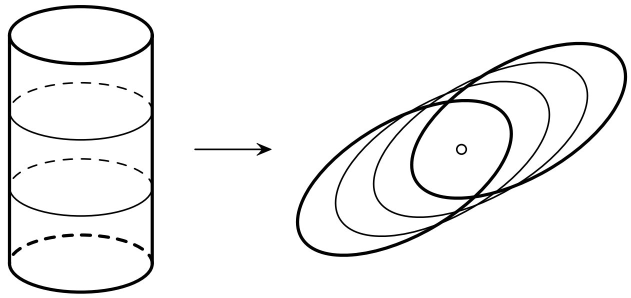
(3) the identity map $\mathrm{id} : S^1 \to S^1$ to the constant map $S^1 \to S^1 : x \mapsto 1$,

(4) the identity map $\mathrm{id} : S^1 \to S^1$ to the two-fold wrapping $S^1 \to S^1 : x \mapsto x^2$,

(5) the inclusion $S^1 \to \mathbb{R}^2$ to a constant map,

(6) the inclusion $S^1 \to \mathbb{R}^2 \smallsetminus 0$ to a constant map?

29.B. Riddle. When you (tried to) solve the previous problem, what did you mean by "*deform continuously*"?



The present section is devoted to the notion of *homotopy* formalizing the naive idea of continuous deformation of a map.

29°2. Homotopy as Map and Family of Maps

Let f and g be two continuous maps of a topological space X to a topological space Y, and $H : X \times I \to Y$ a continuous map such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for any $x \in X$. Then f and g are *homotopic*, and H is a *homotopy* between f and g.

For $x \in X$, $t \in I$ denote $H(x,t)$ by $h_t(x)$. This change of notation results in a change of the point of view of H. Indeed, for a fixed t the formula $x \mapsto h_t(x)$ determines a map $h_t : X \to Y$, and H becomes a family of maps h_t enumerated by $t \in I$.

29.C. Each h_t is continuous.

29.D. Does continuity of all h_t imply continuity of H?

The conditions $H(x,0) = f(x)$ and $H(x,1) = g(x)$ in the above definition of a homotopy can be reformulated as follows: $h_0 = f$ and $h_1 = g$. Thus a homotopy between f and g can be regarded as a family of continuous maps that connects f and g. Continuity of a homotopy allows us to say that it is a *continuous family of continuous maps* (see 29°10).

29°3. Homotopy as Relation

29.E. *Homotopy of maps is an equivalence relation.*

29.E.1. *If $f : X \to Y$ is a continuous map, then $H : X \times I \to Y : (x,t) \mapsto f(x)$ is a homotopy between f and f.*

29.E.2. *If H is a homotopy between f and g, then H' defined by $H'(x,t) = H(x,1-t)$ is a homotopy between g and f.*

29.E.3. *If H is a homotopy between f and f' and H' is a homotopy between f' and f'', then H'' defined by*

$$H''(x,t) = \begin{cases} H(x,2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ H'(x,2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is a homotopy between f and f''.

Homotopy, being an equivalence relation by *29.E*, splits the set $\mathcal{C}(X, Y)$ of all continuous maps from a space X to a space Y into equivalence classes. The latter are ***homotopy classes***. The set of homotopy classes of all continuous maps $X \to Y$ is denoted by $\pi(X, Y)$. Map homotopic to a constant map are said to be ***null-homotopic*** .

29.1. Prove that for any X, the set $\pi(X, I)$ has a single element.

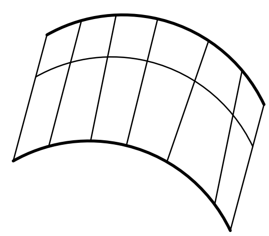
29.2. Prove that two constant maps $Z \to X$ are homotopic iff their images lie in one path-connected component of X.

29.3. Prove that the number of elements of $\pi(I, Y)$ is equal to the number of path connected components of Y.

29°4. Rectilinear Homotopy

29.F. Any two continuous maps of the same space to $\mathbb{R}^n$ are homotopic.

29.G. Solve the preceding problem by proving that for continuous maps $f, g : X \to \mathbb{R}^n$ formula $H(x, t) = (1-t)f(x) + tg(x)$ determines a homotopy between f and g.



The homotopy defined in *29.G* is a ***rectilinear*** homotopy.

29.H. Any two continuous maps of an arbitrary space to a convex subspace of $\mathbb{R}^n$ are homotopic.

29°5. Maps to Star-Shaped Sets

A set $A \subset \mathbb{R}^n$ is ***star-shaped*** if there exists a point $b \in A$ such that for any $x \in A$ the whole segment $[a, x]$ connecting x to a is contained in A. The point a is the ***center*** of the star. (Certainly, the center of the star is not uniquely determined.)

29.4. Prove that any two continuous maps of a space to a star-shaped subspace of $\mathbb{R}^n$ are homotopic.

29°6. Maps of Star-Shaped Sets

29.5. Prove that any continuous map of a star-shaped set $C \subset \mathbb{R}^n$ to any space is null-homotopic.

29.6. Under what conditions (formulated in terms of known topological properties of a space X) any two continuous maps of any star-shaped set to X are homotopic?

29°7. Easy Homotopies

29.7. Prove that each non-surjective map of any topological space to S^n is null-homotopic.

29.8. Prove that any two maps of a one-point space to $\mathbb{R}^n \smallsetminus 0$ with $n > 1$ are homotopic.

29.9. Find two nonhomotopic maps from a one-point space to $\mathbb{R} \smallsetminus 0$.

29.10. For various m, n, and k, calculate the number of homotopy classes of maps $\{1, 2, \dots, m\} \to \mathbb{R}^n \smallsetminus \{x_1, x_2, \dots, x_k\}$, where $\{1, 2, \dots, m\}$ is equipped with discrete topology.

29.11. Let f and g be two maps from a topological space X to $\mathbb{C} \smallsetminus 0$. Prove that if $|f(x) - g(x)| < |f(x)|$ for any $x \in X$, then f and g are homotopic.

29.12. Prove that for any polynomials p and q over $\mathbb{C}$ of the same degree in one variable there exists $r > 0$ such that for any $R > r$ formulas $z \mapsto p(z)$ and $z \mapsto q(z)$ determine maps of the circle $\{z \in \mathbb{C} : |z| = R\}$ to $\mathbb{C} \smallsetminus 0$ and these maps are homotopic.

29.13. Let f, g be maps of an arbitrary topological space X to S^n. Prove that if $|f(a) - g(a)| < 2$ for each $a \in X$, then f is homotopic to g.

29.14. Let $f : S^n \to S^n$ be a continuous map. Prove that if it is fixed-point-free, i.e., $f(x) \neq x$ for every $x \in S^n$, then f is homotopic to the symmetry $x \mapsto -x$.

29°8. Two Natural Properties of Homotopies

29.I. Let $f, f' : X \to Y$, $g : Y \to B$, $h : A \to X$ be continuous maps and $F : X \times I \to Y$ a homotopy between f and f'. Prove that then $g \circ F \circ (h \times \mathrm{id}_I)$ is a homotopy between $g \circ f \circ h$ and $g \circ f' \circ h$.

29.J. Riddle. Under conditions of *29.I*, define a natural map

$$\pi(X, Y) \to \pi(A, B).$$

How does it depend on g and h? Write down all nice properties of this construction.

29.K. Prove that two maps $f_0, f_1 : X \to Y \times Z$ are homotopic iff $\mathrm{pr}_Y \circ f_0$ is homotopic to $pr_Y \circ f_1$ and $\mathrm{pr}_Z \circ f_0$ is homotopic to $pr_Z \circ f_1$.

29°9. Stationary Homotopy

Let A be a subset of X. A homotopy $H : X \times I \to Y$ is *fixed* or *stationary* on A, or, briefly, an A-*homotopy* if $H(x, t) = H(x, 0)$ for all $x \in A$, $t \in I$. Two maps connected by an A-homotopy are A-*homotopic*.

Certainly, any two A-homotopic maps coincide on A. If we want to emphasize that a homotopy is not assumed to be fixed, then we say that it is *free*. If we want to emphasize the opposite (that the homotopy is fixed), then we say that it is *relative*.[1]

[1] Warning: there is a similar, but different kind of homotopy, which is also called relative.

29.L. Prove that, like free homotopy, A-homotopy is an equivalence relation.

The classes into which A-homotopy splits the set of continuous maps $X \to Y$ that agree on A with a map $f : A \to Y$ are *A-homotopy classes of continuous extensions of f to X*.

29.M. For what A is a rectilinear homotopy fixed on A?

29°10. Homotopies and Paths

Recall that a *path* in a space X is a continuous map from the segment I to X. (See Section 13.)

29.N. Riddle. In what sense is any path a homotopy?

29.O. Riddle. In what sense does any homotopy consist of paths?

29.P. Riddle. In what sense is any homotopy a path?

Recall that the *compact-open topology* in $\mathcal{C}(X, Y)$ is the topology generated by the sets $\{\varphi \in \mathcal{C}(X, Y) \mid \varphi(A) \subset B\}$ for compact $A \subset X$ and open $B \subset Y$.

29.15. Prove that any homotopy $h_t : X \to Y$ determines (see 29°2) a path in $\mathcal{C}(X, Y)$ with compact-open topology.

29.16. Prove that if X is locally compact and regular, then any path in $\mathcal{C}(X, Y)$ with compact-open topology determines a homotopy.

29°11. Homotopy of Paths

29.Q. Prove that two paths in a space X are freely homotopic iff their images belong to the same path-connected component of X.

This shows that the notion of free homotopy in the case of paths is not interesting. On the other hand, there is a sort of relative homotopy playing a very important role. This is $(0 \cup 1)$-homotopy. This causes the following commonly accepted deviation from the terminology introduced above: homotopy of paths always means not a free homotopy, but a homotopy fixed on the endpoints of I (i.e., on $0 \cup 1$).

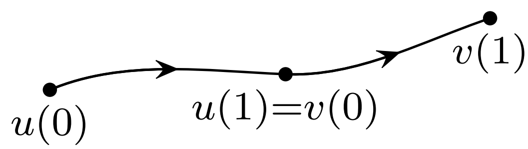
Notation: a homotopy class of a path s is denoted by $[s]$.

30. Homotopy Properties of Path Multiplication

30°1. Multiplication of Homotopy Classes of Paths

Recall (see Section 13) that two paths u and v in a space X can be multiplied, provided the initial point $v(0)$ of v is the final point $u(1)$ of u. The product uv is defined by

$$uv(t) = \begin{cases} u(2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ v(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$



30.A. *If a path u is homotopic to u', a path v is homotopic to v', and there exists the product uv, then $u'v'$ exists and is homotopic to uv.*

Define the product of homotopy classes of paths u and v as the homotopy class of uv. So, $[u][v]$ is defined as $[uv]$, provided uv is defined. This is a definition requiring a proof.

30.B. *The product of homotopy classes of paths is well defined.*[2]

30°2. Associativity

30.C. Is multiplication of paths associative?

Certainly, this question might be formulated in more detail as follows.

30.D. Let u, v, and w be paths in a certain space such that products uv and vw are defined (i.e., $u(1) = v(0)$ and $v(1) = w(0)$). Is it true that $(uv)w = u(vw)$?

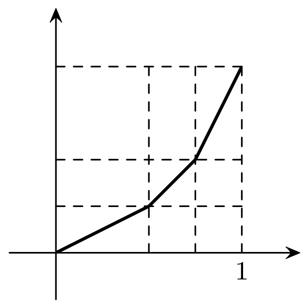
> ***30.1.*** Prove that for paths in a metric space $(uv)w = u(vw)$ implies that u, v, and w are constant maps.
>
> ***30.2. Riddle.*** Find nonconstant paths u, v, and w in an indiscrete space such that $(uv)w = u(vw)$.

30.E. *Multiplication of homotopy classes of paths is associative.*

[2] Of course, when the initial point of paths in the first class is the final point of paths in the second class.

30.E.1. Reformulate Theorem *30.E* in terms of paths and their homotopies.

30.E.2. Find a map $\varphi : I \to I$ such that if u, v, and w are paths with $u(1) = v(0)$ and $v(1) = w(0)$, then $((uv)w) \circ \varphi = u(vw)$.



30.E.3. Any path in I starting at 0 and ending at 1 is homotopic to $\mathrm{id} : I \to I$.

30.E.4. Let u, v and w be paths in a space such that products uv and vw are defined (thus, $u(1) = v(0)$ and $v(1) = w(0)$). Then $(uv)w$ is homotopic to $u(vw)$.

If you want to understand the essence of *30.E*, then observe that the paths $(uv)w$ and $u(vw)$ have the same trajectories and differ only by the time spent in different fragments of the path. Therefore, in order to find a homotopy between them, we must find a continuous way to change one schedule to the other. The lemmas above suggest a formal way of such a change, but the same effect can be achieved in many other ways.

30.3. Present explicit formulas for the homotopy H between the paths $(uv)w$ and $u(vw)$.

30°3. Unit

Let a be a point of a space X. Denote by e_a the path $I \to X : t \mapsto a$.

30.F. Is e_a a unit for multiplication of paths?

The same question in more detailed form:

30.G. For a path u with $u(0) = a$ is $e_a u = u$? For a path v with $v(1) = a$ is $ve_a = v$?

30.4. Prove that if $e_a u = u$ and the space satisfies the first separation axiom, then $u = e_a$.

30.H. *The homotopy class of e_a is a unit for multiplication of homotopy classes of paths.*

30°4. Inverse

Recall that for a path u there is the inverse path $u^{-1} : t \mapsto u(1-t)$ (see Section 13).

30.I. Is the inverse path inverse with respect to multiplication of paths?

In other words:

30.J. For a path u beginning in a and finishing in b, is it true that $uu^{-1} = e_a$ and $u^{-1}u = e_b$?

30.5. Prove that for a path u with $u(0) = a$ equality $uu^{-1} = e_a$ implies $u = e_a$.

30.K. *For any path u, the homotopy class of the path u^{-1} is inverse to the homotopy class of u.*

30.K.1. Find a map $\varphi : I \to I$ such that $uu^{-1} = u \circ \varphi$ for any path u.

30.K.2. Any path in I that starts and finishes at 0 is homotopic to the constant path $e_0 : I \to I$.

We see that from the algebraic point of view multiplication of paths is terrible, but it determines multiplication of homotopy classes of paths, which has nice algebraic properties. The only unfortunate property is that the multiplication of homotopy classes of paths is defined not for any two classes.

30.L. Riddle. How to select a subset of the set of homotopy classes of paths to obtain a group?

31. Fundamental Group

31°1. Definition of Fundamental Group

Let X be a topological space, x_0 its point. A path in X which starts and ends at x_0 is a *loop* in X at x_0. Denote by $\Omega_1(X, x_0)$ the set of loops in X at x_0. Denote by $\pi_1(X, x_0)$ the set of homotopy classes of loops in X at x_0.

Both $\Omega_1(X, x_0)$ and $\pi_1(X, x_0)$ are equipped with a multiplication.

31.A. *For any topological space X and a point $x_0 \in X$ the set $\pi_1(X, x_0)$ of homotopy classes of loops at x_0 with multiplication defined above in Section 30 is a group.*

$\pi_1(X, x_0)$ is the *fundamental group* of the space X with base point x_0. It was introduced by Poincaré, and this is why it is also called the *Poincaré group*. The letter π in this notation is also due to Poincaré.

31°2. Why Index 1?

The index 1 in the notation $\pi_1(X, x_0)$ appeared later than the letter π. It is related to one more name of the fundamental group: the first (or one-dimensional) homotopy group. There is an infinite series of groups $\pi_r(X, x_0)$ with $r = 1, 2, 3, \dots$ the fundamental group being one of them. The higher-dimensional homotopy groups were defined by Witold Hurewicz in 1935, thirty years after the fundamental group was defined. Roughly speaking, the general definition of $\pi_r(X, x_0)$ is obtained from the definition of $\pi_1(X, x_0)$ by replacing I with the cube I^r.

31.B. Riddle. How to generalize problems of this section in such a way that in each of them I would be replaced by I^r?

There is even a "zero-dimensional homotopy group" $\pi_0(X, x_0)$, but it is not a group, as a rule. It is the set of path-connected components of X. Although there is no natural multiplication in $\pi_0(X, x_0)$, unless X is equipped with some special additional structures, there is a natural unit in $\pi_0(X, x_0)$. This is the component containing x_0.

31°3. Circular loops

Let X be a topological space, x_0 its point. A continuous map $l : S^1 \to X$ such that[3] $l(1) = x_0$ is a (*circular*) *loop* at x_0. Assign to each circular loop l the composition of l with the exponential map $I \to S^1 : t \mapsto e^{2\pi it}$. This is a usual loop at the same point.

[3] Recall that S^1 is regarded as a subset of the plane R^2, and the latter is identified in a canonical way with $\mathbb{C}$. Hence, $1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

31.C. Prove that any loop can be obtained in this way from a circular loop.

Two circular loops l_1 and l_2 are ***homotopic*** if they are 1-homotopic. A homotopy of a circular loop not fixed at x_0 is a ***free*** homotopy.

31.D. Prove that two circular loops are homotopic iff the corresponding ordinary loops are homotopic.

31.1. What kind of homotopy of loops corresponds to free homotopy of circular loops?

31.2. Describe the operation with circular loops corresponding to the multiplication of paths.

31.3. Let U and V be the circular loops with common base point $U(1) = V(1)$ corresponding to the loops u and v. Prove that the circular loop

$$z \mapsto \begin{cases} U(z^2) \text{ if } Im(z) \geq 0, \\ V(z^2) \text{ if } Im(z) \leq 0 \end{cases}$$

corresponds to the product of u and v.

31.4. Outline a construction of fundamental group using circular loops.

31°4. The Very First Calculations

31.E. Prove that $\pi_1(\mathbb{R}^n, 0)$ is a trivial group (i.e., consists of one element).

31.F. Generalize *31.E* to the situations suggested by *29.H* and *29.4*.

31.5. Calculate the fundamental group of an indiscrete space.

31.6. Calculate the fundamental group of the quotient space of disk D^2 obtained by identification of each $x \in D^2$ with $-x$.

31.7. Prove that if a two-point space X is path-connected, then X is simply connected.

31.G. Prove that $\pi_1(S^n, (1, 0, \dots, 0))$ with $n \geq 2$ is a trivial group.

Whether you have solved *31.G* or not, we recommend you to consider problems *31.G.1*, *31.G.2*, *31.G.4*, *31.G.5*, and *31.G.6* designed to give an approach to *31.G*, warn about a natural mistake and prepare an important tool for further calculations of fundamental groups.

31.G.1. Prove that any loop $s : I \to S^n$ that does not fill the entire S^n (i.e., $s(I) \neq S^n$) is null-homotopic, provided $n \geq 2$. (Cf. Problem *29.7*.)

Warning: for any n there exists a loop filling S^n. See *9.Ox*.

31.G.2. Can a loop filling S^2 be null-homotopic?

31.G.3 Corollary of Lebesgue Lemma 16.W. *Let $s : I \to X$ be a path, and Γ be an open cover of a topological space X. There exists a sequence of points $a_1, \dots, a_N \in I$ with $0 = a_1 < a_2 < \dots < a_{N-1} < a_N = 1$ such that $s([a_i, a_{i+1}])$ is contained in an element of Γ for each i.*

31.G.4. Prove that if $n \geq 2$, then for any path $s : I \to S^n$ there exists a subdivision of I into a finite number of subintervals such that the restriction of s to each of the subintervals is homotopic to a map with nowhere-dense image via a homotopy fixed on the endpoints of the subinterval.

31.G.5. Prove that if $n \geq 2$, then any loop in S^n is homotopic to a non-surjective loop.

31.G.6. 1) Deduce *31.G* from *31.G.1* and *31.G.5*. 2) Find all points of the proof of *31.G* obtained in this way, where the condition $n \geq 2$ is used.

31°5. Fundamental Group of Product

31.H. *The fundamental group of the product of topological spaces is canonically isomorphic to the product of the fundamental groups of the factors:*

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

31.8. Consider a loop $u : I \to X$ at x_0, a loop $v : I \to Y$ at y_0, and the loop $w = u \times v : I \to X \times Y$. We introduce the loops $u' : I \to X \times Y : t \mapsto (u(t), y_0)$ and $v' : I \to X \times Y : t \mapsto (x_0, v(t)$. Prove that $u'v' \sim w \sim v'u'$.

31.9. Prove that $\pi_1(\mathbb{R}^n \smallsetminus 0, (1, 0, \dots, 0))$ is trivial if $n \geq 3$.

31°6. Simply-Connectedness

A nonempty topological space X is *simply connected* (or *one-connected*) if X is path-connected and every loop in X is null-homotopic.

31.I. *For a path-connected topological space X, the following statements are equivalent:*

(1) *X is simply connected,*

(2) *each continuous map $f : S^1 \to X$ is (freely) null-homotopic,*

(3) *each continuous map $f : S^1 \to X$ extends to a continuous map $D^2 \to X$,*

(4) *any two paths $s_1, s_2 : I \to X$ connecting the same points x_0 and x_1 are homotopic.*

Theorem *31.I* is closely related to Theorem *31.J* below. Notice that since Theorem *31.J* concerns not all loops, but an individual loop, it is applicable in a broader range of situations.

31.J. *Let X be a topological space and $s : S^1 \to X$ be a circular loop. Then the following statements are equivalent:*

(1) *s is null-homotopic,*

(2) *s is* freely *null-homotopic,*

(3) *s extends to a continuous map $D^2 \to X$,*

(4) *the paths* $s_+, s_- : I \to X$ *defined by formula* $s_\pm(t) = s(e^{\pm\pi i t})$ *are homotopic.*

31.J.1. Riddle. To prove that 4 statements are equivalent, we must prove at least 4 implications. What implications would you choose for the easiest proof of Theorem *31.J*?

31.J.2. Does homotopy of circular loops imply that these circular loops are free homotopic?

31.J.3. A homotopy between a map of the circle and a constant map possesses a quotient map whose source space is homoeomorphic to disk D^2.

31.J.4. Represent the problem of constructing of a homotopy between paths s_+ and s_- as a problem of extension of a certain continuous map of the boundary of a square to a continuous of the whole square.

31.J.5. When we solve the extension problem obtained as a result of Problem *31.J.4*, does it help to know that the circular loop $S^1 \to X : t \mapsto s(e^{2\pi i t})$ extends to a continuous map of a disk?

31.10. Which of the following spaces are simply connected:

(a) a discrete space; (b) an indiscrete space; (c) $\mathbb{R}^n$; (d) a convex set; (e) a star-shaped set; (f) S^n; (g) $\mathbb{R}^n \setminus 0$?

31.11. Prove that if a topological space X is the union of two open simply connected sets U and V with path-connected intersection $U \cap V$, then X is simply connected.

31.12. Show that the assumption in *31.11* that U and V are open is necessary.

31.13*. Let X be a topological space, U and V its open sets. Prove that if $U \cup V$ and $U \cap V$ are simply connected, then so are U and V.

31°7x. Fundamental Group of a Topological Group

Let G be a topological group. Given loops $u, v : I \to G$ starting at the unity $1 \in G$, let us define a loop $u \odot v : I \to G$ by the formula $u \odot v(t) = u(t) \cdot v(t)$, where $\cdot$ denotes the group operation in G.

31.Ax. Prove that the set $\Omega(G, 1)$ of all loops in G starting at 1 equipped with the operation $\odot$ is a group.

31.Bx. Prove that the operation $\odot$ on $\Omega(G, 1)$ determines a group operation on $\pi_1(G, 1)$, which coincides with the standard group operation (determined by multiplication of paths).

31.Bx.1. For loops $u, v \to G$ starting at 1, find $(ue_1) \odot (e_1 v)$.

31.Cx. The fundamental group of a topological group is Abelian.

31°8x. High Homotopy Groups

Let X be a topological space and x_0 its point. A continuous map $I^r \to X$ mapping the boundary ∂I^r of I^r to x_0 is a *spheroid of dimension* r of X at x_0, or just an r-*spheroid*. Two r-spheroids are *homotopic* if they are ∂I^r-homotopic. For two r-spheroids u and v of X at x_0, $r \geq 1$, define the product uv by the formula

$$uv(t_1, t_2, \dots, t_r) = \begin{cases} u(2t_1, t_2, \dots, t_r) & \text{if } t_1 \in \left[0, \frac{1}{2}\right], \\ v(2t_1 - 1, t_2, \dots, t_r) & \text{if } t_1 \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The set of homotopy classes of r-spheroids of a space X at x_0 is the rth (or r-dimensional) homotopy group $\pi_r(X, x_0)$ of X at x_0. Thus,

$$\pi_r(X, x_0) = \pi(I^r, \partial I^r;\ X, x_0).$$

Multiplication of spheroids induces multiplication in $\pi_r(X, x_0)$, which makes $\pi_r(X, x_0)$ a group.

31.Dx. Find $\pi_r(\mathbb{R}^n, 0)$.

31.Ex. *For any X and x_0 the group $\pi_r(X, x_0)$ with $r \geq 2$ is Abelian.*

Similar to 31°3, higher-dimensional homotopy groups can be constructed not out of homotopy classes of maps $(I^r, \partial I^r) \to (X, x_0)$, but as

$$\pi(S^r, (1, 0, \dots, 0);\ X, x_0).$$

Another, also quite a popular way, is to define $\pi_r(X, x_0)$ as

$$\pi(D^r, \partial D^r;\ X, x_0).$$

31.Fx. Construct natural bijections

$$\pi(I^r, \partial I^r;\ X, x_0) \to \pi(D^r, \partial D^r;\ X, x_0) \to \pi(S^r, (1, 0, \dots, 0);\ X, x_0)$$

31.Gx. ***Riddle.*** For any X, x_0 and $r \geq 2$ present group $\pi_r(X, x_0)$ as the fundamental group of some space.

31.Hx. Prove the following generalization of *31.H*:

$$\pi_r(X \times Y, (x_0, y_0)) = \pi_r(X, x_0) \times \pi_r(Y, y_0).$$

31.Ix. Formulate and prove analogs of Problems *31.Ax* and *31.Bx* for higher homotopy groups and $\pi_0(G, 1)$.

32. The Role of Base Point

32°1. Overview of the Role of Base Point

Sometimes the choice of the base point does not matter, sometimes it is obviously crucial, sometimes this is a delicate question. In this section, we have to clarify all subtleties related to the base point. We start with preliminary formulations describing the subject in its entirety, but without some necessary details.

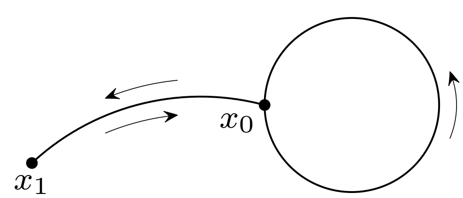
The role of the base point may be roughly described as follows:

- As the base point changes within the same path-connected component, the fundamental group remains in the same class of isomorphic groups.
- However, if the group is non-Abelian, it is impossible to find a natural isomorphism between the fundamental groups at different base points even in the same path-connected component.
- Fundamental groups of a space at base points belonging to different path-connected components have nothing to do to each other.

In this section these will be demonstrated. The proof involves useful constructions, whose importance extends far outside of the frameworks of our initial question on the role of base point.

32°2. Definition of Translation Maps

Let x_0 and x_1 be two points of a topological space X, and let s be a path connecting x_0 with x_1. Denote by σ the homotopy class $[s]$ of s. Define a map $T_s : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by the formula $T_s(\alpha) = \sigma^{-1}\alpha\sigma$.



32.1. Prove that for any loop $a : I \to X$ representing $\alpha \in \pi_1(X, x_0)$ and any path $s : I \to X$ with $s(0) = x_0$ there exists a free homotopy $H : I \times I \to X$ between a and a loop representing $T_s(\alpha)$ such that $H(0, t) = H(1, t) = s(t)$ for $t \in I$.

32.2. Let $a, b : I \to X$ be loops homotopic via a homotopy $H : I \times I \to X$ such that $H(0, t) = H(1, t)$ (i.e., H is a free homotopy of loops: at each moment $t \in I$, it keeps the endpoints of the path coinciding). Set $s(t) = H(0, t)$ (hence, s is the path run through by the initial point of the loop under the homotopy).

Prove that the homotopy class of b is the image of the homotopy class of a under $T_s : \pi_1(X, s(0)) \to \pi_1(X, s(1))$.

32°3. Properties of T_s

32.A. *T_s is a (group) homomorphism.*[4]

32.B. *If u is a path connecting x_0 to x_1 and v is a path connecting x_1 with x_2, then $T_{uv} = T_v \circ T_u$. In other words, the diagram*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{T_u} & \pi_1(X, x_1) \\ & {\scriptstyle T_{uv}} \searrow & \Big\downarrow {\scriptstyle T_v} \\ & & \pi_1(X, x_2) \end{array}$$

is commutative.

32.C. *If paths u and v are homotopic, then $T_u = T_v$.*

32.D. $T_{e_a} = \mathrm{id} : \pi_1(X, a) \to \pi_1(X, a)$

32.E. $T_{s^{-1}} = T_s^{-1}$.

32.F. *T_s is an isomorphism for any path s.*

32.G. *For any points x_0 and x_1 lying in the same path-connected component of X groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

In spite of the result of Theorem *32.G*, we cannot write $\pi_1(X)$ even if the topological space X is path-connected. The reason is that although the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic, there may be no canonical isomorphism between them (see *32.J* below).

32.H. *The space X is simply connected iff X is path-connected and the group $\pi_1(X, x_0)$ is trivial for a certain point $x_0 \in X$.*

32°4. Role of Path

32.I. If a loop s represents an element σ of the fundamental group $\pi_1(X, x_0)$, then T_s is the inner automorphism of $\pi_1(X, x_0)$ defined by $\alpha \mapsto \sigma^{-1}\alpha\sigma$.

32.J. *Let x_0 and x_1 be points of a topological space X belonging to the same path-connected component. The isomorphisms $T_s : \pi_1(X, x_0) \to \pi_1(X, x_1)$ do not depend on s iff $\pi_1(X, x_0)$ is an Abelian group.*

Theorem *32.J* implies that if the fundamental group of a topological space X is Abelian, we may simply write $\pi_1(X)$.

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[4] Recall that this means that $T_s(\alpha\beta) = T_s(\alpha)T_s(\beta)$.

32°5x. In Topological Group

In a topological group G there is another way to relate $\pi_1(G, x_0)$ with $\pi_1(G, x_1)$: there are homeomorphisms $L_g : G \to G : x \mapsto xg$ and $R_g : G \to G : x \mapsto gx$, so that there are the induced isomorphisms $(L_{x_0^{-1}x_1})_* : \pi_1(G, x_0) \to \pi_1(G, x_1)$ and $(R_{x_1 x_0^{-1}})_* : \pi_1(G, x_0) \to \pi_1(G, x_1)$.

***32.A*x.** Let G be a topological group, $s\, I \to G$ be a path. Prove that

$$T_s = (L_{s(0)^{-1}s(1)})_* = (R_{s(1)s(0)^{-1}}) : \pi_1(G, s(0)) \to \pi_1(G, s(1)).$$

***32.B*x.** Deduce from *32.A*x that the fundamental group of a topological group is Abelian (cf. *31.C*x).

***32.1*x.** Prove that the following spaces have Abelian fundamental groups:

(1) the space of nondegenerate real $n \times n$ matrices $GL(n, \mathbb{R}) = \{A \mid \det A \neq 0\}$;
(2) the space of orthogonal real $n \times n$ matrices $O(n, \mathbb{R}) = \{A \mid A \cdot ({}^t A) = \mathbb{E}\}$;
(3) the space of special unitary complex $n \times n$ matrices $SU(n) = \{A \mid A \cdot ({}^t \bar{A}) = 1, \det A = 1\}$.

32°6x. In High Homotopy Groups

32.C*x. *Riddle. Guess how T_s is generalized to $\pi_r(X, x_0)$ with any r.

Here is another form of the same question. We put it because its statement contains a greater piece of an answer.

32.D*x. *Riddle. Given a path $s : I \to X$ with $s(0) = x_0$ and a spheroid $f : I^r \to X$ at x_0, how to cook up a spheroid at $x_1 = s(1)$ out of these?

***32.E*x.** Let $s : I \to X$ be a path, $f : I^r \to X$ a spheroid with $f(\operatorname{Fr} I^r) = s(0)$. Prove that there exists a homotopy $H : I^r \times I \to X$ of f such that $H(\operatorname{Fr} I^r \times t) = s(t)$ for any $t \in I$. Furthermore, the spheroid obtained by such a homotopy is unique up to homotopy and determines an element of $\pi_r(X, s(1))$, which is uniquely determined by the homotopy class of s and the element of $\pi_r(X, s(0))$ represented by f.

Certainly, a solution of *32.E*x gives an answer to *32.D*x and *32.C*x. The map $\pi_r(X, s(0)) \to \pi_r(X, s(1))$ defined by *32.E*x is denoted by T_s. By *32.2*, this T_s generalizes T_s defined in the beginning of the section for the case $r = 1$.

***32.F*x.** Prove that the properties of T_s formulated in Problems *32.A* – *32.F* hold true in all dimensions.

32.G*x. *Riddle. What are the counterparts of *32.A*x and *32.B*x for higher homotopy groups?

Proofs and Comments

29.A (a), (b), (e): yes; (c), (d), (f): no. See *29.B*.

29.B See 29°2.

29.C The map h_t is continuous as the restriction of the homotopy H to the fiber $X \times t \subset X \times I$.

29.D Certainly, no, it does not.

29.E See *29.E.1*, *29.E.2*, and *29.E.3*.

29.E.1 The map H is continuous as the composition of the projection $p : X \times I \to X$ and the map f, and, furthermore, $H(x, 0) = f(x) = H(x, 1)$. Consequently, H is a homotopy.

29.E.2 The map H' is continuous as the composition of the homeomorphism $X \times I \to X \times I : (x, t) \mapsto (x, 1 - t)$ and the homotopy H, and, furthermore, $H'(x, 0) = H(x, 1) = g(x)$ and $H'(x, 1) = H(x, 0) = f(x)$. Therefore, H' is a homotopy.

29.E.3 Indeed, $H''(x, 0) = f(x)$ and $H''(x, 1) = H'(x, 1) = f''(x)$. H'' is continuous since the restriction of H'' to each of the sets $X \times \left[0, \frac{1}{2}\right]$ and $X \times \left[\frac{1}{2}, 1\right]$ is continuous and these sets constitute a fundamental cover of $X \times I$.
Below we do not prove that the homotopies are continuous because this always follows from explicit formulas.

29.F Each of them is homotopic to the constant map mapping the entire space to the origin, for example, if $H(x, t) = (1-t)f(x)$, then $H : X \times I \to \mathbb{R}^n$ is a homotopy between f and the constant map $x \mapsto 0$. (There is a more convenient homotopy between arbitrary maps to $\mathbb{R}^n$, see *29.G*.)

29.G Indeed, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The map H is obviously continuous. For example, this follows from the inequality

$$\left|H(x, t) - H(x', t')\right| \le |f(x) - f(x')| + |g(x) - g(x')| + \big(|f(x)| + |g(x)|\big)|t - t'|.$$

29.H Let K be a convex subset of $\mathbb{R}^n$, $f, g : X \to K$ two continuous maps, and H the rectilinear homotopy between f and g. Then $H(x, t) \in K$ for all $(x, t) \in X \times I$, and we obtain a homotopy $H : X \times I \to K$.

29.I The map $H = g \circ F \circ (h \times \mathrm{id}_I) : A \times I \to B$ is continuous, $H(a, 0) = g(F(h(a), 0)) = g(f(h(a)))$, and $H(a, 1) = g(F(h(a), 1)) = g(f'(h(a)))$. Consequently, H is a homotopy.

29.J Take $f : X \to Y$ to $g \circ f \circ h : A \to B$. Assertion *29.I* shows that this correspondence preserves the homotopy relation, and, hence, it can be transferred to homotopy classes of maps. Thus, a map $\pi(X,Y) \to \pi(A,B)$ is defined.

29.K Any map $f : X \to Y \times Z$ is uniquely determined by its components $\mathrm{pr}_X \circ f$ and $\mathrm{pr}_Y \circ f$. $\Longrightarrow$ If H is a homotopy between f and g, then $\mathrm{pr}_Y \circ H$ is a homotopy between $\mathrm{pr}_Y \circ f$ and $\mathrm{pr}_Y \circ g$, and $\mathrm{pr}_Z \circ H$ is a homotopy between $\mathrm{pr}_Z \circ f$ and $\mathrm{pr}_Z \circ g$.
$\Longleftarrow$ If H_Y is a homotopy between $\mathrm{pr}_Y \circ f$ and $\mathrm{pr}_Y \circ g$ and H_Z is a homotopy between $\mathrm{pr}_Z \circ f$ and $\mathrm{pr}_Z \circ g$, then a homotopy between f and g is determined by the formula $H(x,t) = (H_Y(x,t), H_Z(x,t))$.

29.L The proof does not differ from that of assertion *29.E*.

29.M For the sets A such that $f|_A = g|_A$ (i.e., for the sets contained in the coincidence set of f and g).

29.N A path is a homotopy of a map of a point, cf. *29.8*.

29.O For each point $x \in X$, the map $u_x : I \to X : \ t \mapsto h(x,t)$ is a path.

29.P If H is a homotopy, then for each $t \in I$ the formula $h_t = H(x,t)$ determines a continuous map $X \to Y$. Thus, we obtain a map $\mathcal{H} : I \to \mathcal{C}(X,Y)$ of the segment to the set of all continuous maps $X \to Y$. After that, see *29.15* and *29.16*.

29.15 This follows from *24.Vx*.

29.16 This follows from *24.Wx*.

29.Q This follows from the solution of Problem *29.3*.

30.A 1) We start with a visual description of the required homotopy. Let $u_t : I \to X$ be a homotopy joining u and u', and $v_t : I \to X$ a homotopy joining v and v'. Then the paths $u_t v_t$ with $t \in [0,1]$ form a homotopy between uv and $u'v'$.
2) Now we present a more formal argument. Since the product uv is defined, we have $u(1) = v(0)$. Since $u \sim u'$, we have $u(1) = u'(1)$, we similarly have $v(0) = v'(0)$. Therefore, the product $u'v'$ is defined. The homotopy between uv and $u'v'$ is the map

$$H : I \times I \to X : (s,t) \mapsto \begin{cases} H'(2s,t) & \text{if } s \in \left[0, \frac{1}{2}\right], \\ H''(2s-1,t) & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

(H is continuous because the sets $\left[0, \frac{1}{2}\right] \times I$ and $\left[\frac{1}{2}, 1\right] \times I$ constitute a fundamental cover of the square $I \times I$, and the restriction of H to each of these sets is continuous.)

30.B This is a straight-forward reformulation of *30.A*.

30.C No; see *30.D*, cf. *30.1*.

30.D No, this is almost always wrong (see *30.1* and *30.2*). Here is the simplest example. Let $u(s) = 0$ and $w(s) = 1$ for all $s \in [0, 1]$ and $v(s) = s$. Then $(uv)w(s) = 0$ only for $s \in \left[0, \frac{1}{4}\right]$, and $u(vw)(s) = 0$ for $s \in \left[0, \frac{1}{2}\right]$.

30.E.1 Reformulation: for any three paths u, v, and w such that the products uv and vw are defined, the paths $(uv)w$ and $u(vw)$ are homotopic.

30.E.2 Let

$$\varphi(s) = \begin{cases} \frac{s}{2} & \text{if } s \in \left[0, \frac{1}{2}\right], \\ s - \frac{1}{4} & \text{if } s \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ 2s - 1 & \text{if } s \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Verify that φ is the required function, i.e., $((uv)w)(\varphi(s)) = u(vw)(s)$.

30.E.3 Consider the rectilinear homotopy, which is in addition fixed on $\{0, 1\}$.

30.E.4 This follows from *29.I*, *30.E.2*, and *30.E.3*.

30.F See *30.G*.

30.G Generally speaking, no; see *30.4*.

30.H Let

$$\varphi(s) = \begin{cases} 0 & \text{if } s \in \left[0, \frac{1}{2}\right], \\ 2s - 1 & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Verify that $e_a u = u \circ \varphi$. Since $\varphi \sim \mathrm{id}_I$, we have $u \circ \varphi \sim u$, whence

$$[e_a][u] = [e_a u] = [u \circ \varphi] = [u].$$

30.I See *30.J*.

30.J Certainly not.

30.K.1 Consider the map

$$\varphi(s) = \begin{cases} 2s & \text{if } s \in \left[0, \frac{1}{2}\right], \\ 2 - 2s & \text{if } s \in \left[\frac{1}{2}, 1\right], \end{cases}$$

30.K.2 Consider the rectilinear homotopy.

30.L Groups are the sets of classes of paths u with $u(0) = u(1) = x_0$, where x_0 is a certain marked point of X, as well as their subgroups.

31.A This immediately follows from *30.B*, *30.E*, *30.H*, and *30.K*.

31.B See 31°8x.

31.C If $u : I \to X$ is a loop, then there exists a quotient map $\tilde{u} : I/\{0, 1\} \to X$. It remains to observe that $I/\{0, 1\} \cong S^1$.

31.D ⟹ If $H : S^1 \times I \to X$ is a homotopy of circular loops, then the formula $H'(s,t) = H(e^{2\pi is}, t)$ determines a homotopy H' between ordinary loops.
⟸ Homotopies of circular loops are quotient maps of homotopies of ordinary loops by the partition of the square induced by the relation $(0,t) \sim (1,t)$.

31.E This is true because there is a rectilinear homotopy between any loop in $\mathbb{R}^n$ at the origin and a constant loop.

31.F Here is a possible generalization: for each convex (and even star-shaped) set $V \subset \mathbb{R}^n$ and any point $x_0 \in V$, the fundamental group $\pi_1(V, x_0)$ is trivial.

31.G.1 Let $p \in S^n \smallsetminus u(I)$. Consider the stereographic projection $\tau : S^n \smallsetminus p \to \mathbb{R}^n$. The loop $v = \tau \circ u$ is null-homotopic, let h be the corresponding homotopy. Then $H = \tau^{-1} \circ h$ is a homotopy joining the loop u and a constant loop on the sphere.

31.G.2 Such loops certainly exist. Indeed, if a loop u fills the entire sphere, then so does the loop uu^{-1}, which, however, is null-homotopic.

31.G.4 Let x be an arbitrary point of the sphere. We cover the sphere by two open sets $U = S^n \smallsetminus x$ and $V = S^n \smallsetminus \{-x\}$. By Lemma *31.G.3*, there is a sequence of points $a_1, \dots, a_N \in I$, where $0 = a_1 < a_2 < \dots < a_{N-1} < a_N = 1$, such that for each i the image $u([a_i, a_{i+1}])$ is entirely contained in U or in V. Since each of these sets is homeomorphi to $\mathbb{R}^n$, where any two paths with the same starting and ending points are homotopic, it follows that each of the restrictions $u|_{[a_i,a_{i+1}]}$ is homotopic to a path the image of which is, e.g., an "arc of a great circle" of S^n. Thus, the path u is homotopic to a path the image of which does not fill the sphere, and even is nowhere dense.

31.G.5 This immediately follows from Lemma *31.G.4*.

31.G.6 1) This is immediate. 2) The assumption $n \geq 2$ was used only in Lemma *31.G.4*.

31.H Take a loop $u : I \to X \times Y$ at the point (x_0, y_0) to the pair of loops in X and Y that are the components of u: $u_1 = \operatorname{pr}_X \circ u$ and $u_2 = \operatorname{pr}_Y \circ u$. By assertion *29.I*, the loops u and v are homotopic iff $u_1 \sim v_1$ and $u_2 \sim v_2$. Consequently, taking the class of the loop u to the pair $([u_1], [u_2])$, we obtain a bijection between the fundamental group $\pi_1(X \times Y, (x_0, y_0))$ of the product of the spaces and the product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ of the fundamental groups of the factors. It remains to verify that the bijection constructed is a homomorphism, which is also obvious because $\operatorname{pr}_X \circ (uv) = (\operatorname{pr}_X \circ u)(\operatorname{pr}_X \circ v)$.

31.I (a) $\Longrightarrow$ (b): The space X is simply connected $\Rightarrow$ each loop in X is null-homotopic $\Rightarrow$ each circular loop in X is relatively null-homotopic $\Rightarrow$ each circular loop in X is freely null-homotopic.
(b) $\Longrightarrow$ (c): By assumption, for an arbitrary map $f : S^1 \to X$ there is a homotopy $h : S^1 \times I \to X$ such that $h(p, 0) = f(p)$ and $h(p, 1) = x_0$. Consequently, there is a continuous map $h' : S^1 \times I/_{(S^1 \times 1)} \to X$ such that $h = h' \circ \mathrm{pr}$. It remains to observe that $S^1 \times I/_{(S^1 \times 1)} \cong D^2$.
(c) $\Longrightarrow$ (d): Put $g(t, 0) = u_1(t)$, $g(t, 1) = u_2(t)$, $g(0, t) = x_0$, and $g(1, t) = x_1$ for $t \in I$. Thus, we mapped the boundary of the square $I \times I$ to X. Since the square is homeomorphi to a disk and its boundary is homeomorphi to a circle, it follows that the map extends from the boundary to the entire square. The extension obtained is a homotopy between u_1 and u_2.
(d) $\Longrightarrow$ (a): This is obvious.

31.J.1 It is reasonable to consider the following implications: (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a).

31.J.2 It certainly does. Furthermore, since s is null-homotopic, it follows that the circular loop f is also null-homotopic, and the homotopy is even fixed at the point $1 \in S^1$. Thus, (a) $\Longrightarrow$ (b).

31.J.3 The assertion suggests the main idea of the proof of the implication (b) $\Longrightarrow$ (c). A null-homotopy of a certain circular loop f is a map $H : S^1 \times I \to X$ constant on the upper base of the cylinder. Consequently, there is a quotient map $S^1 \times I/_{S^1 \times 1} \to X$. It remains to observe that the quotient space of the cylinder by the upper base is homeomorphi to a disk.

31.J.4 By the definition of a homotopy $H : I \times I \to X$ between two paths, the restriction of H to the contour of the square is given. Consequently, the problem of constructing a homotopy between two paths is the problem of extending a map from the contour of the square to the entire square.

31.J.5 All that remains to observe for the proof of the implication (c) $\Longrightarrow$ (d), is the following fact: if $F : D^2 \to X$ is an extension of the circular loop f, then the formula $H(t, \tau) = F\big(\cos \pi t, (2\tau - 1) \sin \pi t\big)$ determines a homotopy between s_+ and s_-.

31.J In order to prove the theorem, it remains to prove the implication (d) $\Longrightarrow$ (a). Let us state this assertion without using the notion of circular loop. Let $s : I \to X$ be a loop. Put $s_+(t) = s(2t)$ and $s_-(t) = s(1 - 2t)$. Thus, we must prove that if the paths s_+ and s_- are homotopic, then the loop s is null-homotopic. Try to prove this on your own.

31.Ax The associativity of $\odot$ follows from that of the multiplication in G; the unity in the set $\Omega(G, 1)$ of all loops is the constant loop at the unity of the group; the element inverse to the loop u is the path v, where $v(s) = \big(u(s)\big)^{-1}$.

31.Bx.1 Verify that $(ue_1) \odot (e_1 v) = uv$.

31.Bx We prove that if $u \sim u_1$, then $u \odot v \sim u_1 \odot v$. For this purpose it suffices to check that if h is a homotopy between u and u_1, then the formula $H(s,t) = h(s,t)v(s)$ determines a homotopy between $u \odot v$ and $u_1 \odot v$. Further, since $ue_1 \sim u$ and $e_1 v \sim v$, we have $uv = (ue_1) \odot (e_1 v) \sim u \odot v$, therefore, the paths uv and $u \odot v$ lie in one homotopy class. Consequently, the operation $\odot$ induces the standard group operation in the set of homotopy classes of paths.

31.Cx It is sufficient to prove that $uv \sim vu$, which fact follows from the following chain:

$$uv = (ue_1) \odot (e_1 v) \sim u \odot v \sim (e_1 u) \odot (ve_1) = vu.$$

31.Dx This group is also trivial. The proof is similar to that of assertion *31.E*.

32.A Indeed, if $\alpha = [u]$ and $\beta = [v]$, then

$$T_s(\alpha\beta) = \sigma^{-1}\alpha\beta\sigma = \sigma^{-1}\alpha\sigma\sigma^{-1}\beta\sigma = T_s(\alpha)T_s(\beta).$$

32.B Indeed,

$$T_{uv}(\alpha) = [uv]^{-1}\alpha[uv] = [v]^{-1}[u]^{-1}\alpha[u][v] = T_v\big(T_u(\alpha)\big).$$

32.C By the definition of translation along a path, the homomorphism T_s depends only on the homotopy class of s.

32.D This is so because $T_{e_a}([u]) = [e_a u e_a] = [u]$.

32.E Since $s^{-1}s \sim e_{x_1}$, *32.B*–*32.D* imply that

$$T_{s^{-1}} \circ T_s = T_{s^{-1}s} = T_{e_{x_1}} = \mathrm{id}_{\pi_1(X,x_1)}\,.$$

Similarly, we have $T_s \circ T_{s^{-1}} = \mathrm{id}_{\pi_1(X,x_0)}$, whence $T_{s^{-1}} = T_s^{-1}$.

32.F By *32.E*, the homomorphism T_s has an inverse and, consequently, is an isomorphism.

32.G If x_0 and x_1 lie in one path-connected component, then they are joined by a path s. By *32.F*, $T_s : \pi_1(X,x_0) \to \pi_1(X,x_1)$ is an isomorphism.

32.H This immediately follows from Theorem *32.G*.

32.I This directly follows from the definition of T_s.

32.J $\boxed{\Longrightarrow}$ Assume that the translation isomorphism does not depend on the path. In particular, the isomorphism of translation along any loop at x_0 is trivial. Consider an arbitrary element $\beta \in \pi_1(X,x_0)$ and a loop

s in the homotopy class β. By assumption, $\beta^{-1}\alpha\beta = T_s(\alpha) = \alpha$ for each $\alpha \in \pi_1(X, x_0)$. Therefore, $\alpha\beta = \beta\alpha$ for any elements $\alpha, \beta \in \pi_1(X, x_0)$, which precisely means that the group $\pi_1(X, x_0)$ is Abelian.

$\Leftarrow$ Consider two paths s_1 and s_2 joining x_0 and x_1. Since $T_{s_1 s_2^{-1}} = T_{s_2}^{-1} \circ T_{s_1}$, it follows that $T_{s_1} = T_{s_2}$ iff $T_{s_1 s_2^{-1}} = \mathrm{id}_{\pi_1(X,x_0)}$. Let $\beta \in \pi_1(X, x_0)$ be the class of the loop $s_1 s_2^{-1}$. If the group $\pi_1(X, x_0)$ is Abelian, then $T_{s_1 s_2^{-1}}(\alpha) = \beta^{-1}\alpha\beta = \alpha$, whence $T_{s_1 s_2^{-1}} = \mathrm{id}$, and so $T_{s_1} = T_{s_2}$.

***32.A*x** Let u be a loop at $s(0)$. The formula $H(\tau, t) = u(\tau)s(0)^{-1}s(1)$ determines a free homotopy between u and the loop $L_{s(0)^{-1}s(1)}(u)$ such that $H(0, t) = H(1, t) = s(t)$. Therefore, by *32.2*, the loops $L_{s(0)^{-1}s(1)}(u)$ and $s^{-1}us$ are homotopic, whence $T_s = \big(L_{s(0)^{-1}s(1)}\big)_*$. The equality for $R_{s(0)^{-1}s(1)}$ is proved in a similar way.

***32.B*x** By *32.A*x, we have $T_s = (L_e)_* = \mathrm{id}_{\pi_1(X,x_0)}$ for each loop s at x_0. Therefore, if β is the class of the loop s, then $T_s(\alpha) = \beta^{-1}\alpha\beta = \alpha$, whence $\alpha\beta = \beta\alpha$.