Covering Spaces and Calculation of Fundamental Groups

33. Covering Spaces

33°1. Definition of Covering

Let X, B topological spaces, $p : X \to B$ a continuous map. Assume that p is surjective and each point of B possesses a neighborhood U such that the preimage $p^{-1}(U)$ of U is a disjoint union of open sets V_α and p maps each V_α homeomorphically onto U. Then $p : X \to B$ is a *covering* (of B), the space B is the *base* of this covering, X is the *covering space* for B and the *total space* of the covering. Neighborhoods like U are said to be *trivially covered*. The map p is a *covering map* or *covering projection*.

33.A. Let B be a topological space and F be a discrete space. Prove that the projection $\mathrm{pr}_B : B \times F \to B$ is a covering.

> ***33.1.*** If $U' \subset U \subset B$ and the neighborhood U is trivially covered, then the neighborhood U' is also trivially covered.

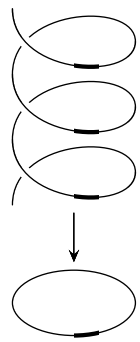
The following statement shows that in a certain sense any covering locally is organized as the covering of *33.A*.

33.B. A continuous surjective map $p : X \to B$ is a covering iff for each point a of B the preimage $p^{-1}(a)$ is discrete and there exist a neighborhood U of a

and a homeomorphism $h : p^{-1}(U) \to U \times p^{-1}(a)$ such that $p|_{p^{-1}(U)} = \mathrm{pr}_U \circ h$. Here, as usual, $\mathrm{pr}_U : U \times p^{-1}(a) \to U$.

However, the coverings of *33.A* are not interesting. They are said to be ***trivial***. Here is the first really interesting example.

33.C. Prove that $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$ is a covering.



To distinguish the most interesting examples, a covering with a connected total space is called a covering in a ***narrow sense***. Of course, the covering of *33.C* is a covering in a narrow sense.

33°2. More Examples

33.D. $\mathbb{R}^2 \to S^1 \times \mathbb{R} : (x, y) \mapsto (e^{2\pi i x}, y)$ is a covering.

33.E. Prove that if $p : X \to B$ and $p' : X' \to B'$ are coverings, then so is $p \times p' : X \times X' \to B \times B'$.

If $p : X \to B$ and $p' : X' \to B'$ are two coverings, then $p \times p' : X \times X' \to B \times B'$ is the ***product of the coverings*** p and p'. The first example of the product of coverings is presented in *33.D*.

33.F. $\mathbb{C} \to \mathbb{C} \smallsetminus 0 : z \mapsto e^z$ is a covering.

33.2. Riddle. In what sense the coverings of *33.D* and *33.F* are the same? Define an appropriate equivalence relation for coverings.

33.G. $\mathbb{R}^2 \to S^1 \times S^1 : (x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$ is a covering.

33.H. For any positive integer n, the map $S^1 \to S^1 : z \mapsto z^n$ is a covering.

33.3. Prove that for each positive integer n the map $\mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0 : z \mapsto z^n$ is a covering.

33.I. For any positive integers p and q, the map $S^1 \times S^1 \to S^1 \times S^1 : (z, w) \mapsto (z^p, w^q)$ is a covering.

33.J. The natural projection $S^n \to \mathbb{R}P^n$ is a covering.

33.K. Is $(0, 3) \to S^1 : x \mapsto e^{2\pi ix}$ a covering? (Cf. *33.14*.)

33.L. Is the projection $\mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ a covering? Indeed, why is not an open interval $(a, b) \subset \mathbb{R}$ a trivially covered neighborhood: its preimage $(a, b) \times \mathbb{R}$ is the union of open intervals $(a, b) \times \{y\}$, which are homeomorphically projected onto (a, b) by the projection $(x, y) \mapsto x$?

33.4. Find coverings of the Möbius strip by a cylinder.

33.5. Find nontrivial coverings of Möbius strip by itself.

33.6. Find a covering of the Klein bottle by a torus. Cf. Problem *21.14*.

33.7. Find coverings of the Klein bottle by the plane $\mathbb{R}^2$ and the cylinder $S^1 \times \mathbb{R}$, and a nontrivial covering of the Klein bottle by itself.

33.8. Describe explicitly the partition of $\mathbb{R}^2$ into preimages of points under this covering.

33.9*. Find a covering of a sphere with any number of crosscaps by a sphere with handles.

33°3. Local Homeomorphisms versus Coverings

33.10. Any covering is an open map.[1]

A map $f : X \to Y$ is a *local homeomorphism* if each point of X has a neighborhood U such that the image $f(U)$ is open in Y and the submap $\text{ab}(f) : U \to f(U)$ is a homeomorphism.

33.11. Any covering is a local homeomorphism.

33.12. Find a local homeomorphism which is not a covering.

33.13. Prove that the restriction of a local homeomorphism to an open set is a local homeomorphism.

33.14. For which subsets of $\mathbb{R}$ is the restriction of the map of Problem *33.C* a covering?

33.15. Find a nontrivial covering $X \to B$ with X homeomorphic to B and prove that it satisfies the definition of a covering.

33°4. Number of Sheets

Let $p : X \to B$ be a covering. The cardinality (i.e., the number of points) of the preimage $p^{-1}(a)$ of a point $a \in B$ is the *multiplicity* of the covering at a or the *number of sheets of the covering over* a.

33.M. *If the base of a covering is connected, then the multiplicity of the covering at a point does not depend on the point.*

[1]We remind that a map is *open* if the image of any open set is open.

In the case of covering with connected base, the multiplicity is called the ***number of sheets*** of the covering. If the number of sheets is n, then the covering is n-***sheeted***, and we talk about an n-***fold*** covering. Of course, unless the covering is trivial, it is impossible to distinguish the sheets of it, but this does not prevent us from speaking about the number of sheets. On the other hand, we adopt the following agreement. By definition, the preimage $p^{-1}(U)$ of any trivially covered neighborhood $U \subset B$ splits into open subsets: $p^{-1}(U) = \cup V_\alpha$, such that the restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Each of the subsets V_α is a ***sheet*** over U.

33.16. What are the numbers of sheets for the coverings from Section $33^\circ 2$?

In problems *33.17–33.19* we did not assume that you would rigorously justify your answers. This will be done below, see problems *39.3–39.6*.

33.17. What numbers can you realize as the number of sheets of a covering of the Möbius strip by the cylinder $S^1 \times I$?

33.18. What numbers can you realize as the number of sheets of a covering of the Möbius strip by itself?

33.19. What numbers can you realize as the number of sheets of a covering of the Klein bottle by a torus?

33.20. What numbers can you realize as the number of sheets of a covering of the Klein bottle by itself?

33.21. Construct a d-fold covering of a sphere with p handles by a sphere with $1 + d(p - 1)$ handles.

33.22. Let $p : X \to Y$ and $q : Y \to Z$ be coverings. Prove that if q has finitely many sheets, then $q \circ p : x \to Y$ is a covering.

33.23*. Is the hypothesis of finiteness of the number of sheets in Problem *33.22* necessary?

33.24. Let $p : X \to B$ be a covering with compact base B. 1) Prove that if X is compact, then the covering is finite-sheeted. 2) If B is Hausdorff and the covering is finite-sheeted, then X is compact.

33.25. Let X be a topological space presentable as the union of two open connected sets U and V. Prove that if the intersection $U \cap V$ is disconnected, then X has a connected infinite-sheeted covering.

$33^\circ 5$. Universal Coverings

A covering $p : X \to B$ is ***universal*** if X is simply connected. The appearance of the word *universal* in this context is explained below in Section 39.

33.N. Which coverings of the problems stated above in this section are universal?

34. Theorems on Path Lifting

34°1. Lifting

Let $p : X \to B$ and $f : A \to B$ be arbitrary maps. A map $g : A \to X$ such that $p \circ g = f$ is said to *cover* f or be a *lifting* of f. Various topological problems can be phrased in terms of finding a continuous lifting of some continuous map. Problems of this sort are called *lifting problems*. They may involve additional requirements. For example, the desired lifting must coincide with a lifting already given on some subspace.

34.A. The identity map $S^1 \to S^1$ does not admit a continuous lifting with respect to the covering $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$. (In other words, there exists no continuous map $g : S^1 \to \mathbb{R}$ such that $e^{2\pi i g(x)} = x$ for $x \in S^1$.)

34°2. Path Lifting

34.B Path Lifting Theorem. *Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Then for any path $s : I \to B$ starting at b_0 there exists a unique path $\tilde{s} : I \to X$ starting at x_0 and being a lifting of s. (In other words, there exists a unique path $\tilde{s} : I \to X$ with $\tilde{s}(0) = x_0$ and $p \circ \tilde{s} = s$.)*

We can also prove a more general assertion than Theorem *34.B*: see Problems *34.1–34.3*.

34.1. Let $p : X \to B$ be a trivial covering. Then for any continuous map f of any space A to B there exists a continuous lifting $\tilde{f} : A \to X$.

34.2. Let $p : X \to B$ be a trivial covering and $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Then for any continuous map f of a space A to B mapping a point a_0 to b_0, a continuous lifting $\tilde{f} : A \to X$ with $\tilde{f}(a_0) = x_0$ is unique.

34.3. Let $p : X \to B$ be a covering, A a connected and locally connected space. If $f, g : A \to X$ are two continuous maps coinciding at some point and $p \circ f = p \circ g$, then $f = g$.

34.4. If we replace x_0, b_0, and a_0 in Problem *34.2* by pairs of points, then the lifting problem may happen to have no solution $\tilde{f}$ with $\tilde{f}(a_0) = x_0$. Formulate a condition necessary and sufficient for existence of such a solution.

34.5. What goes wrong with the Path Lifting Theorem *34.B* for the local homeomorphism of Problem *33.K*?

34.6. Consider the covering $\mathbb{C} \to \mathbb{C} \smallsetminus 0 : z \mapsto e^z$. Find liftings of the paths $u(t) = 2 - t$ and $v(t) = (1+t)e^{2\pi i t}$ and their products uv and vu.

34°3. Homotopy Lifting

34.C Path Homotopy Lifting Theorem. *Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Let $u, v : I \to B$ be paths starting at b_0 and $\tilde{u}, \tilde{v} : I \to X$ be the lifting paths for u, v starting at x_0. If the paths u and v are homotopic, then the covering paths $\tilde{u}$ and $\tilde{v}$ are homotopic.*

34.D Corollary. *Under the assumptions of Theorem 34.C, the covering paths $\tilde{u}$ and $\tilde{v}$ have the same final point (i.e., $\tilde{u}(1) = \tilde{v}(1)$).*

Notice that the paths in *34.C* and *34.D* are assumed to share the initial point x_0. In the statement of *34.D*, we emphasize that then they also share the final point.

34.E Corollary of 34.D. *Let $p : X \to B$ be a covering and $s : I \to B$ be a loop. If there exists a lifting $\tilde{s} : I \to X$ of s with $\tilde{s}(0) \neq \tilde{s}(1)$ (i.e., there exists a covering path which is not a loop), then s is not null-homotopic.*

34.F. If a path-connected space B has a nontrivial path-connected covering space, then the fundamental group of B is nontrivial.

> ***34.7.*** Prove that any covering $p : X \to B$ with simply connected B and path connected X is a homeomorphism.
>
> ***34.8.*** What corollaries can you deduce from *34.F* and the examples of coverings presented above in Section 33?
>
> ***34.9. Riddle.*** Is it really important in the hypothesis of Theorem *34.C* that u and v are paths? To what class of maps can you generalize this theorem?

35. Calculation of Fundamental Groups Using Universal Coverings

35°1. Fundamental Group of Circle

For an integer n, denote by s_n the loop in S^1 defined by the formula $s_n(t) = e^{2\pi i n t}$. The initial point of this loop is 1. Denote the homotopy class of s_1 by α. Thus, $\alpha \in \pi_1(S^1, 1)$.

35.A. The loop s_n represents $\alpha^n \in \pi_1(S^1, 1)$.

35.B. Find the paths in $\mathbb{R}$ starting at $0 \in \mathbb{R}$ and covering the loops s_n with respect to the universal covering $\mathbb{R} \to S^1$.

35.C. *The homomorphism $\mathbb{Z} \to \pi_1(S^1, 1) : n \mapsto \alpha^n$ is an isomorphism.*

35.C.1. The formula $n \mapsto \alpha^n$ determines a homomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$.

35.C.2. Prove that a loop $s : I \to S^1$ starting at 1 is homotopic to s_n if the path $\tilde{s} : I \to \mathbb{R}$ covering s and starting at $0 \in \mathbb{R}$ ends at $n \in \mathbb{R}$ (i.e., $\tilde{s}(1) = n$).

35.C.3. Prove that if the loop s_n is null-homotopic, then $n = 0$.

35.1. Find the image of the homotopy class of the loop $t \mapsto e^{2\pi i t^2}$ under the isomorphism of Theorem *35.C*.

Denote by deg the isomorphism inverse to the isomorphism of Theorem *35.C*.

35.2. For any loop $s : I \to S^1$ starting at $1 \in S^1$, the integer $\deg([s])$ is the final point of the path starting at $0 \in \mathbb{R}$ and covering s.

35.D Corollary of Theorem 35.C. *The fundamental group of $(S^1)^n$ is a free Abelian group of rank n (i.e., isomorphic to $\mathbb{Z}^n$).*

35.E. On torus $S^1 \times S^1$ find two loops whose homotopy classes generate the fundamental group of the torus.

35.F Corollary of Theorem 35.C. *The fundamental group of punctured plane $\mathbb{R}^2 \smallsetminus 0$ is an infinite cyclic group.*

35.3. Solve Problems *35.D* – *35.F* without reference to Theorems *35.C* and *31.H*, but using explicit constructions of the corresponding universal coverings.

35°2. Fundamental Group of Projective Space

The fundamental group of the projective line is an infinite cyclic group. It is calculated in the previous subsection since the projective line is a circle. The zero-dimensional projective space is a point, hence its fundamental group is trivial. Now we calculate the fundamental groups of projective spaces of all other dimensions.

Let $n \geq 2$, and let and $l : I \to \mathbb{R}P^n$ be a loop covered by a path $\tilde{l} : I \to S^n$ which connects two antipodal points of S^n, say the poles $P_+ = (1, 0, \dots, 0)$ and $P_- = (-1, 0, \dots, 0)$. Denote by λ the homotopy class of l. It is an element of $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$.

35.G. *For any $n \geq 2$ group $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$ is a cyclic group of order 2. It consists of two elements: λ and 1.*

35.G.1 Lemma. *Any loop in $\mathbb{R}P^n$ at $(1 : 0 : \dots : 0)$ is homotopic either to l or constant. This depends on whether the covering path of the loop connects the poles P_+ and P_-, or is a loop.*

35.4. Where did we use the assumption $n \geq 2$ in the proofs of Theorem *35.G* and Lemma *35.G.1* ?

35°3. Fundamental Group of Bouquet of Circles

Consider a family of topological spaces $\{X_\alpha\}$. In each of the spaces, let a point x_α be marked. Take the disjoint sum $\bigsqcup_\alpha X_\alpha$ and identify all marked points. The resulting quotient space $\bigvee_\alpha X_\alpha$ is the ***bouquet*** of $\{X_\alpha\}$. Hence a ***bouquet of q circles*** is a space which is a union of q copies of circle. The copies meet at a single common point, and this is the only common point for any two of them. The common point is the ***center*** of the bouquet.

Denote the bouquet of q circles by B_q and its center by c. Let $u_1, \dots,$ u_q be loops in B_q starting at c and parameterizing the q copies of circle comprising B_q. Denote by α_i the homotopy class of u_i.

35.H. *$\pi_1(B_q, c)$ is a free group freely generated by $\alpha_1, \dots, \alpha_q$.*

35°4. Algebraic Digression: Free Groups

Recall that a group G is a free group freely generated by its elements $a_1, \dots, a_q$ if:

- each element $x \in G$ is a product of powers (with positive or negative integer exponents) of $a_1, \dots, a_q$, i.e.,
$$x = a_{i_1}^{e_1} a_{i_2}^{e_2} \dots a_{i_n}^{e_n}$$
and
- this expression is unique up to the following trivial ambiguity: we can insert or delete factors $a_i a_i^{-1}$ and $a_i^{-1} a_i$ or replace a_i^m by $a_i^r a_i^s$ with $r + s = m$.

35.I. *A free group is determined up to isomorphism by the number of its free generators.*

The number of free generators is the *rank* of the free group. For a standard representative of the isomorphism class of free groups of rank q, we can take the group of words in an alphabet of q letters $a_1, \dots, a_q$ and their inverses $a_1^{-1}, \dots, a_q^{-1}$. Two words represent the same element of the group iff they can be obtained from each other by a sequence of insertions or deletions of fragments $a_i a_i^{-1}$ and $a_i^{-1} a_i$. This group is denoted by $\mathbb{F}(a_1, \dots, a_q)$, or just $\mathbb{F}_q$, when the notation for the generators is not to be emphasized.

35.J. *Each element of $\mathbb{F}(a_1, \dots, a_q)$ has a unique shortest representative. This is a word without fragments that could have been deleted.*

The number $l(x)$ of letters in the shortest representative of an element $x \in \mathbb{F}(a_1, \dots, a_q)$ is the *length* of x. Certainly, this number is not well defined unless the generators are fixed.

> ***35.5.*** Show that an automorphism of $\mathbb{F}_q$ can map $x \in \mathbb{F}_q$ to an element with different length. For what value of q does such an example not exist? Is it possible to change the length in this way arbitrarily?

35.K. *A group G is a free group freely generated by its elements $a_1, \dots, a_q$ iff every map of the set $\{a_1, \dots, a_q\}$ to any group X extends to a unique homomorphism $G \to X$.*

Theorem *35.K* is sometimes taken as a definition of a free group. (Definitions of this sort emphasize relations among different groups, rather than the internal structure of a single group. Of course, relations among groups can tell everything about "internal affairs" of each group.)

Now we can reformulate Theorem *35.H* as follows:

35.L. *The homomorphism*

$$\mathbb{F}(a_1, \dots, a_q) \to \pi_1(B_q, c)$$

taking a_i to α_i for $i = 1, \dots, q$ is an isomorphism.

First, for the sake of simplicity we restrict ourselves to the case where $q = 2$. This will allow us to avoid superfluous complications in notation and pictures. This is the simplest case, which really represents the general situation. The case $q = 1$ is too special.

To take advantages of this, let us change the notation. Put $B = B_2$, $u = u_1$, $v = u_2$, $\alpha = \alpha_1$, and $\beta = \alpha_2$.

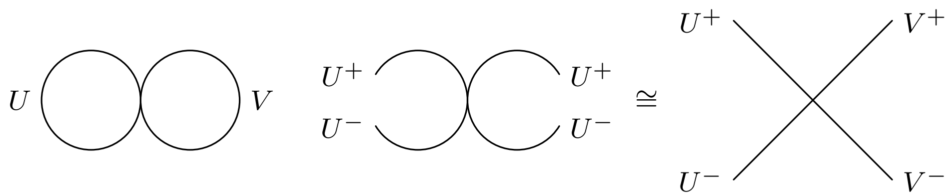
Now Theorem *35.L* looks as follows:

The homomorphism $\mathbb{F}(a, b) \to \pi(B, c)$ taking a to α and b to β is an isomorphism.

This theorem can be proved like Theorems *35.C* and *35.G*, provided the universal covering of B is known.

35°5. Universal Covering for Bouquet of Circles

Denote by U and V the points antipodal to c on the circles of B. Cut B at these points, removing U and V and putting instead each of them two new points. Whatever this operation is, its result is a cross K, which is the union of four closed segments with a common endpoint c. There appears a natural map $P : K \to B$ that takes the center c of the cross to the center c of B and homeomorphically maps the rays of the cross onto half-circles of B. Since the circles of B are parameterized by loops u and v, the halves of each of the circles are ordered: the corresponding loop passes first one of the halves and then the other one. Denote by U^+ the point of $P^{-1}(U)$ belonging to the ray mapped by P onto the second half of the circle, and by U^- the other point of $P^{-1}(U)$. We similarly denote points of $P^{-1}(V)$ by V^+ and V^-.



The restriction of P to $K \smallsetminus \{U^+, U^-, V^+, V^-\}$ maps this set homeomorphically onto $B \smallsetminus \{U, V\}$. Therefore P provides a covering of $B \smallsetminus \{U, V\}$. However, it fails to be a covering at U and V: none of these points has a trivially covered neighborhood. Furthermore, the preimage of each of these points consists of 2 points (the endpoints of the cross), where P is not even a local homeomorphism. To eliminate this defect, we can attach a copy of K at each of the 4 endpoints of K and extend P in a natural way to the result. But then 12 new endpoints appear at which the map is not a local homeomorphism. Well, we repeat the trick and recover the property of being a local homeomorphism at each of the 12 new endpoints. Then we do this at each of the 36 new points, etc. But if we repeat this infinitely many times, all bad points become nice ones.[2]

35.M. Formalize the construction of a covering for B described above.

[2]This sounds like a story about a battle with Hydra, but the happy ending demonstrates that modern mathematicians have a magic power of the sort that the heros of myths and tales could not even dream of. Indeed, we meet a Hydra K with 4 heads, chop off all the heads, but, according to the old tradition of the genre, 3 new heads appear in place of each of the original heads. We chop them off, and the story repeats. We do not even try to prevent this multiplication of heads. We just chop them off. But contrary to the real heros of tales, we act outside of Time and hence have no time limitations. Thus after infinite repetitions of the exercise with an exponentially growing number of heads we succeed! No heads left!

This is a typical success story about an infinite construction in mathematics. Sometimes, as in our case, such a construction can be replaced by a finite one, but dealing with infinite objects. However, there are important constructions in which an infinite fragment is unavoidable.

Consider $\mathbb{F}(a,b)$ as a discrete topological space. Take $K \times \mathbb{F}(a,b)$. It can be thought of as a collection of copies of K enumerated by elements of $\mathbb{F}(a,b)$. Topologically this is a disjoint sum of the copies because $\mathbb{F}(a,b)$ is equipped with discrete topology. In $K \times \mathbb{F}(a,b)$, we identify points (U^-, g) with (U^+, ga) and (V^-, g) with (V^+, gb) for each $g \in \mathbb{F}(a,b)$. Denote the resulting quotient space by X.

35.N. The composition of the projection $K \times \mathbb{F}(a,b) \to K$ and $P : K \to B$ determines a continuous quotient map $p : X \to B$.

35.O. $p : X \to B$ is a covering.

35.P. X is path-connected. For any $g \in \mathbb{F}(a,b)$, there exists a path connecting $(c, 1)$ with (c, g) and covering the loop obtained from g by replacing a with u and b with v.

35.Q. X is simply connected.

35°6. Fundamental Groups of Finite Topological Spaces

35.6. Prove that if a three-point space X is path-connected, then X is simply connected (cf. *31.7*).

35.7. Consider a topological space $X = \{a, b, c, d\}$ with topology determined by the base $\{\{a\}, \{c\}, \{a, b, c\}, \{c, d, a\}\}$. Prove that X is path-connected, but not simply connected.

35.8. Calculate $\pi_1(X)$.

35.9. Let X be a finite topological space with nontrivial fundamental group. Let n_0 be the least possible cardinality of X. 1) Find n_0. 2) What nontrivial groups arise as fundamental groups of n_0-point spaces?

35.10. 1) Find a finite topological space with non-Abelian fundamental group. 2) What is the least possible cardinality of such a space?

35.11*. Let a topological space X be the union of two open path-connected sets U and V. Prove that if $U \cap V$ has at least three connected components, then the fundamental group of X is non-Abelian and, moreover, admits an epimorphism onto a free group of rank 2.

35.12*. Find a finite topological space with fundamental group isomorphic to $\mathbb{Z}_2$.

Proofs and Comments

33.A Let us show that the set B itself is trivially covered. Indeed, $(\mathrm{pr}_B)^{-1}(B) = X = \bigcup_{y \in F}(B \times y)$, and since the topology in F is discrete, it follows that each of the sets $B \times y$ is open in the total space of the covering, and the restriction of pr_B to each of them is a homeomorphism.

33.B $\boxed{\Rightarrow}$ We construct a homeomorphism $h : p^{-1}(U) \to U \times p^{-1}(a)$ for an arbitrary trivially covered neighborhood $U \subset B$ of a. By the definition of a trivially covered neighborhood, we have $p^{-1}(U) = \bigcup U_\alpha$. Let $x \in p^{-1}(U)$, consider an open sets U_α containing x and take x to the pair $(p(x), c)$, where $\{c\} = p^{-1}(a) \cap U_\alpha$. It is clear that the correspondence $x \mapsto (p(x), c)$ determines a homeomorphism $h : p^{-1}(U) \to U \times p^{-1}(a)$.
$\boxed{\Leftarrow}$ By assertion *33.1*, U is a trivially covered neighborhood, hence, $p : X \to B$ is a covering.

33.C For each point $z \in S^1$, the set $U_z = S^1 \smallsetminus \{-z\}$ is a trivially covered neighborhood of z. Indeed, let $z = e^{2\pi i x}$. Then the preimage of U_z is the union $\bigcup_{k \in \mathbb{Z}}(x + k - \frac{1}{2}, x + k + \frac{1}{2})$, and the restriction of the covering to each of the above intervals is a homeomorphism.

33.D The product $(S^1 \smallsetminus \{-z\}) \times \mathbb{R}$ is a trivially covered neighborhood of a point $(z, y) \in S^1 \times \mathbb{R}$; cf. *33.E*.

33.E Verify that the product of trivially covered neighborhoods of points $b \in B$ and $b' \in B'$ is a trivially covered neighborhood of the point $(b, b') \in B \times B'$.

33.F Consider the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{h} & \mathbb{C} \\ {\scriptstyle q}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ S^1 \times \mathbb{R} & \xrightarrow{g} & \mathbb{C} \smallsetminus 0, \end{array}$$

where $g(z, x) = ze^x$, $h(x, y) = y + 2\pi i x$, and $q(x, y) = (e^{2\pi i x}, y)$. The equality $g(q(x, y)) = e^{2\pi i x} \cdot e^y = e^{y + 2\pi i x} = p(h(x, y))$ implies that the diagram is commutative. Clearly, g and h are homeomorphisms. Since q is a covering by *33.D*, p is also a covering.

33.G By *33.E*, this assertion follows from *33.C*. Certainly, it is not difficult to prove it directly. The product $(S^1 \smallsetminus \{-z\}) \times (S^1 \smallsetminus \{-z'\})$ is a trivially covered neighborhood of the point $(z, z') \in S^1 \times S^1$.

33.H Let $z \in S^1$. The preimage $-z$ under the projection consists of n points, which partition the covering space into n arcs, and the restriction of the projection to each of them determines a homeomorphism of this arc onto the neighborhood $S^1 \smallsetminus \{-z\}$ of z.

33.I By *33.E*, this assertion follows from *33.H*.

33.J The preimage of a point $y \in \mathbb{R}P^n$ is a pair $\{x, -x\} \subset S^n$ of antipodal points. The plane passing through the center of the sphere and orthogonal to the vector x splits the sphere into two open hemispheres, each of which is homeomorphially projected to a neighborhood (homeomorphi to $\mathbb{R}^n$) of the point $y \in \mathbb{R}P^n$.

33.K No, it is not, because the point $1 \in S^1$ has no trivially covered neighborhood.

33.L The open intervals mentioned in the statement are not open subsets of the plane. Furthermore, since the preimage of any interval is a connected set, it cannot be split into disjoint open subsets at all.

33.M Prove that the definition of a covering implies that the set of the points in the base with preimage of prescribed cardinality is open and use the fact that the base of the covering is connected.

33.N Those coverings where the covering space is $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^n \smallsetminus 0$ with $n \geq 3$, and S^n with $n \geq 2$, i.e., a simply connected space.

34.A Assume that there exists a lifting g of the identity map $S^1 \to S^1$; this is a continuous injection $S^1 \to \mathbb{R}$. We show that there are no such injections. Let $g(S^1) = [a, b]$. The Intermediate Value Theorem implies that each point $x \in (a, b)$ is the image of at least two points of the circle. Consequently, g is not an injection.

34.B Cover the base by trivially covered neighborhoods and partition the segment $[0, 1]$ by points $0 = a_0 < a_1 < \dots < a_n = 1$, such that the image $s([a_i, a_{i+1}])$ is entirely contained in one of the trivially covered neighborhoods; $s([a_i, a_{i+1}]) \subset U_i$, $i = 0, 1, \dots, n-1$. Since the restriction of the covering to $p^{-1}(U_0)$ is a trivial covering and $f([a_0, a_1]) \subset U_0$, there exists a lifting of $s|_{[a_0,a_1]}$ such that $\widetilde{s}(a_0) = x_0$, let $x_1 = \widetilde{s}(a_1)$. Similarly, there exists a unique lifting $\widetilde{s}|_{[a_1,a_2]}$ such that $\widetilde{s}(a_1) = x_1$; let $x_2 = \widetilde{s}(a_2)$, and so on. Thus, there exists a lifting $\widetilde{s} : I \to X$. Its uniqueness is obvious. If you do not agree, use induction.

34.C Let $h : I \times I \to B$ be a homotopy between the paths u and v, thus, $h(\tau, 0) = u(\tau)$, $h(\tau, 1) = v(\tau)$, $h(0, t) = b_0$, and $h(1, t) = b_1 \in B$. We show that there exists a map $\tilde{h} : I \times I \to X$ covering h and such that $h(0, 0) = x_0$. The proof of the existence of the covering homotopy is similar to that of the Path Lifting Theorem. We subdivide the square $I \times I$ into smaller squares such that the h-image of each of them is contained in a certain trivially covered neighborhood in B. The restriction $h_{k,l}$ of the homotopy h to each of the "little" squares $I_{k,l}$ is covered by the corresponding map $\widetilde{h}_{k,l}$. In order to obtain a homotopy covering h, we must only ensure that these maps coincide on the intersections of these squares. By *34.3*, it suffices to require that these maps coincide at least at one point. Let us make the first step: let $h(I_{0,0}) \subset U_{b_0}$ and let $\widetilde{h}_{0,0} : I_{0,0} \to X$ be a covering map such that $\widetilde{h}_{0,0}(a_0, c_0) = x_0$. Now we put $b_1 = h(a_1, c_0)$ and $x_1 = \widetilde{h}(a_1, c_0)$. There is a map $\widetilde{h}_{1,0} : I_{1,0} \to X$ covering $h|_{I_{1,0}}$ such that $\widetilde{h}_{1,0}(a_1, c_0) = x_1$. Proceeding in this way, we obtain a map $\widetilde{h}$ defined on the entire square. It remains to verify that $\widetilde{h}$ is a homotopy of paths. Consider the covering path $\widetilde{u} : t \mapsto \widetilde{h}(0, t)$. Since $p \circ \widetilde{u}$ is a constant path, the path $\widetilde{u}$ must also be constant, whence $\widetilde{h}(0, t) = x_0$. Similarly, $\widetilde{h}(1, t) = x_1$ is a marked point of the covering space. Therefore, $\widetilde{h}$ is a homotopy of paths. In conclusion, we observe that the uniqueness of this homotopy follows, once more, from Lemma *34.3*.

34.D Formally speaking, this is indeed a corollary, but actually we already proved this when proving Theorem *34.C*.

34.E A constant path is covered by a constant path. By *34.D*, each null-homotopic loop is covered by a loop.

35.A Consider the paths $\tilde{s}_n : I \to \mathbb{R} \ : t \mapsto nt$, $\tilde{s}_{n-1} : I \to \mathbb{R} \ : t \mapsto (n-1)t$, and $\tilde{s}_1 : I \to \mathbb{R} \ : t \mapsto n-1+t$ covering the paths s_n, s_{n-1}, and s_1, respectively. Since the product $\widetilde{s}_{n-1}\widetilde{s}_1$ is defined and has the same starting and ending points as the path $\widetilde{s}_n$, we have $\widetilde{s}_n \sim \widetilde{s}_{n-1}\widetilde{s}_1$, whence $s_n \sim s_{n-1}s_1$. Therefore, $[s_n] = [s_{n-1}]\alpha$. Reasoning by induction, we obtain the required equality $[s_n] = \alpha^n$.

35.B See the proof of assertion *35.A*: this is the path defined by the formula $\widetilde{s}_n(t) = nt$.

35.C By *35.C.1*, the map in question is indeed a well-defined homomorphism. By *35.C.2*, it is an epimorphism, and by *35.C.3* it is a monomorphism. Therefore, it is an isomorphism.

35.C.1 If $n \mapsto \alpha^n$ and $k \mapsto \alpha^k$, then $n + k \mapsto \alpha^{n+k} = \alpha^n \cdot \alpha^k$.

35.C.2 Since $\mathbb{R}$ is simply connected, the paths $\widetilde{s}$ and $\widetilde{s}_n$ are homotopic, therefore, the paths s and s_n are also homotopic, whence $[s] = [s_n] = \alpha^n$.

35.C.3 If $n \neq 0$, then the path $\widetilde{s}_n$ ends at the point n, hence, it is not a loop. Consequently, the loop s_n is not null-homotopic.

35.D This follows from the above computation of the fundamental group of the circle and assertion *31.H*:

$$\pi_1(\underbrace{S^1 \times \ldots \times S^1}_{n \text{ factors}}, (1, 1, \ldots, 1)) \cong \underbrace{\pi_1(S^1, 1) \times \ldots \times \pi_1(S^1, 1)}_{n \text{ factors}} \cong \mathbb{Z}^n.$$

35.E Let $S^1 \times S^1 = \{(z, w) : |z| = 1, |w| = 1\} \subset \mathbb{C} \times \mathbb{C}$. The generators of $\pi_1(S^1 \times S^1, (1, 1))$ are the loops $s_1 : t \mapsto (e^{2\pi it}, 1)$ and $s_2 : t \mapsto (1, e^{2\pi it})$.

35.F Since $\mathbb{R}^2 \smallsetminus 0 \cong S^1 \times \mathbb{R}$, we have $\pi_1(\mathbb{R}^2 \smallsetminus 0, (1, 0)) \cong \pi_1(S^1, 1) \times \pi_1(\mathbb{R}, 1) \cong \mathbb{Z}$.

35.G.1 Let u be a loop in $\mathbb{R}P^n$, and let $\tilde{u}$ be the covering u the path in S^n. For $n \geq 2$, the sphere S^n is simply connected, and if $\widetilde{u}$ is a loop, then $\widetilde{u}$ and hence also u are null-homotopic. Now if $\widetilde{u}$ is not a loop, then, once more since S^n is simply connected, we have $\widetilde{u} \sim \widetilde{l}$, whence $u \sim l$.

35.G By *35.G.1*, the fundamental group consists of two elements, therefore, it is a cyclic group of order two.

35.H See 35°5.

35.M See the paragraph following the present assertion.

35.N This obviously follows from the definition of P.

35.O This obviously follows from the definition of p.

35.P Use induction.

35.Q Use the fact that the image of any loop, as a compact set, intersects only a finite number of the segments constituting the covering space X, and use induction on the number of such segments.