Chapter VIII

Fundamental Group and Maps

36. Induced Homomorphisms and Their First Applications

36°1. Homomorphisms Induced by a Continuous Map

Let $f : X \to Y$ be a continuous map of a topological space X to a topological space Y. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $f(x_0) = y_0$. The latter property of f is expressed by saying that f *maps pair* (X, x_0) *to pair* (Y, y_0) and writing $f : (X, x_0) \to (Y, y_0)$.

Consider the map $f_\# : \Omega(X, x_0) \to \Omega(Y, y_0) : s \mapsto f \circ s$. This map assigns to a loop its composition with f.

36.A. $f_\#$ maps homotopic loops to homotopic loops.

Therefore, $f_\#$ induces a map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

36.B. *$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism for any continuous map $f : (X, x_0) \to (Y, y_0)$.*

$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$ is the *homomorphism induced* by f.

36.C. *Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be (continuous) maps. Then*

$$(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(Z, z_0).$$

36.D. *Let $f, g : (X, x_0) \to (Y, y_0)$ be continuous maps homotopic via a homotopy fixed at x_0. Then $f_* = g_*$.*

36.E. Riddle. How can we generalize Theorem *36.D* to the case of freely homotopic f and g?

36.F. *Let $f : X \to Y$ be a continuous map, x_0 and x_1 points of X connected by a path $s : I \to X$. Denote $f(x_0)$ by y_0 and $f(x_1)$ by y_1. Then the diagram*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ {\scriptstyle T_s}\downarrow & & \downarrow{\scriptstyle T_{f\circ s}} \\ \pi_1(X, x_1) & \xrightarrow{f_*} & \pi_1(Y, y_1) \end{array}$$

is commutative, i.e., $T_{f\circ s} \circ f_* = f_* \circ T_s$.

36.1. Prove that the map $\mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0 : z \mapsto z^3$ is not homotopic to the identity map $\mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0 : z \mapsto z$.

36.2. Let X be a subset of $\mathbb{R}^n$. Prove that if a continuous map $f : X \to Y$ extends to a continuous map $\mathbb{R}^n \to Y$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a trivial homomorphism (i.e., maps everything to unit) for any $x_0 \in X$.

36.3. Prove that if a Hausdorff space X contains an open set homeomorphic to $S^1 \times S^1 \smallsetminus (1, 1)$, then X has infinite noncyclic fundamental group.

36.3.1. Prove that a space X satisfying the conditions of *36.3* can be continuously mapped to a space with infinite noncyclic fundamental group in such a way that the map would induce an epimorphism of $\pi_1(X)$ onto this infinite group.

36.4. Prove that the fundamental group of the space $GL(n, \mathbb{C})$ of complex $n \times n$-matrices with nonzero determinant is infinite.

36°2. Fundamental Theorem of Algebra

Our goal here is to prove the following theorem, which at first glance has no relation to fundamental group.

36.G Fundamental Theorem of Algebra. *Every polynomial of positive degree in one variable with complex coefficients has a complex root.*

In more detail:

Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a polynomial of degree $n > 0$ in z with complex coefficients. Then there exists a complex number w such that $p(w) = 0$.

Although it is formulated in an algebraic way and called “The Fundamental Theorem of Algebra,” it has no simple algebraic proof. Its proofs usually involve topological arguments or use complex analysis. This is so because the field $\mathbb{C}$ of complex numbers as well as the field $\mathbb{R}$ of reals is extremely difficult to describe in purely algebraic terms: all customary constructive descriptions involve a sort of completion construction, cf. Section 17.

36.G.1 Reduction to Problem on a Map. Deduce Theorem *36.G* from the following statement:

For any complex polynomial $p(z)$ of a positive degree, the zero belongs to the image of the map $\mathbb{C} \to \mathbb{C} : z \mapsto p(z)$. In other words, the formula $z \mapsto p(z)$ does not determine a map $\mathbb{C} \to \mathbb{C} \setminus 0$.

36.G.2 Estimate of Remainder. Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a complex polynomial, $q(z) = z^n$, and $r(z) = p(z) - q(z)$. Then there exists a positive real R such that $|r(z)| < |q(z)| = R^n$ for any z with $|z| = R$

36.G.3 Lemma on Lady with Doggy. (Cf. *29.11*.) A lady $q(z)$ and her dog $p(z)$ walk on the punctured plane $\mathbb{C} \setminus 0$ periodically (i.e., say, with $z \in S^1$). Prove that if the lady does not let the dog to run further than by $|q(z)|$ from her, then the doggy's loop $S^1 \to \mathbb{C} \setminus 0 : z \mapsto p(z)$ is homotopic to the lady's loop $S^1 \to \mathbb{C} \setminus 0 : z \mapsto q(z)$.

36.G.4 Lemma for Dummies. (Cf. *29.12*.) If $f : X \to Y$ is a continuous map and $s : S^1 \to X$ is a null-homotopic loop, then $f \circ s : S^1 \to Y$ is also null-homotopic.

36°3x. Generalization of Intermediate Value Theorem

36.Ax. Riddle. How to generalize Intermediate Value Theorem *12.A* to the case of maps $f : D^n \to \mathbb{R}^n$?

36.Bx. Find out whether Intermediate Value Theorem *12.A* is equivalent to the following statement:
Let $f : D^1 \to \mathbb{R}^1$ be a continuous map. If $0 \notin f(S^0)$ and the submap $f|_{S^0} : S^0 \to \mathbb{R}^1 \setminus 0$ of f induces a nonconstant map $\pi_0(S^0) \to \pi_0(\mathbb{R}^1 \setminus 0)$, then there exists a point $x \in D^1$ such that $f(x) = 0$.

36.Cx. Riddle. Suggest a generalization of Intermediate Value Theorem to maps $D^n \to \mathbb{R}^n$ which would generalize its reformulation *36.Bx*. To do it, you must give a definition of the induced homomorphism for homotopy groups.

36.Dx. *Let $f : D^n \to \mathbb{R}^n$ be a continuous map. If $f(S^{n-1})$ does not contain $0 \in \mathbb{R}^n$ and the submap $f|_{S^{n-1}} : S^{n-1} \to \mathbb{R}^n \setminus 0$ of f induces a nonconstant map*

$$\pi_{n-1}(S^{n-1}) \to \pi_{n-1}(\mathbb{R}^n \setminus 0),$$

then there exists a point $x \in D^1$ such that $f(x) = 0$.

Usability of Theorem *36.Dx* is impeded by a condition which is difficult to check if $n > 0$. For $n = 1$, this is still possible in the frameworks of the theory developed above.

36.1x. Let $f : D^2 \to \mathbb{R}^2$ be a continuous map. If $f(S^1)$ does not contain $a \in \mathbb{R}^2$ and the circular loop $f|_{S^1} : S^1 \to \mathbb{R}^2 \setminus a$ determines a nontrivial element of $\pi_1(\mathbb{R}^2 \setminus a)$, then there exists $x \in D^2$ such that $f(x) = a$.

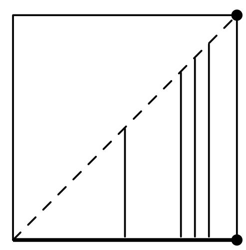
36.2x. Let $f : D^2 \to \mathbb{R}^2$ be a continuous map that leaves fixed each point of the boundary circle S^1. Then $f(D^2) \supset D^2$.

36.3x. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map and there exists a real number m such that $|f(x) - x| \le m$ for any $x \in \mathbb{R}^2$. Prove that f is a surjection.

36.4x. Let $u, v : I \to I \times I$ be two paths such that $u(0) = (0, 0)$, $u(1) = (1, 1)$ and $v(0) = (0, 1)$, $v(1) = (1, 0)$. Prove that $u(I) \cap v(I) \ne \varnothing$.

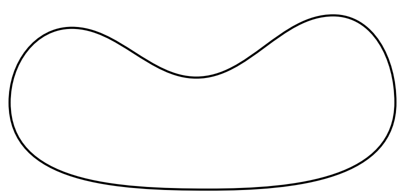
> ***36.4x.1.*** Let u, v be as in *36.4x*. Prove that $0 \in \mathbb{R}^2$ is a value of the map $w : I^2 \to \mathbb{R}^2 : (x, y) \mapsto u(x) - v(y)$.

36.5x. Prove that there exist connected disjoint sets $F, G \subset I^2$ such that $(0, 0), (1, 1) \in F$ and $(0, 1), (1, 0) \in G$.



36.6x. Can we require in addition that the sets F and G satisfying the assumptions of Problem *36.5x* be closed?

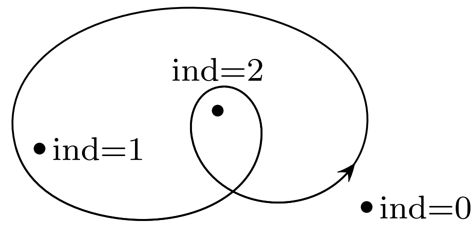
36.7x. Let C be a smooth simple closed curve on the plane with two inflection points. Prove that there is a line intersecting C in four points a, b, c, and d with segments $[a, b]$, $[b, c]$ and $[c, d]$ of the same length.



36°4x. Winding Number

As we know (see *35.F*), the fundamental group of the punctured plane $\mathbb{R}^2 \smallsetminus 0$ is isomorphic to $\mathbb{Z}$. There are two isomorphisms, which differ by multiplication by -1. We choose that taking the homotopy class of the loop $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ to $1 \in \mathbb{Z}$. In terms of circular loops, the isomorphism means that to any loop $f : S^1 \to \mathbb{R}^2 \smallsetminus 0$ we assign an integer. Roughly speaking, it is the number of times the loop goes around 0 (with account of direction).

Now we change the viewpoint in this consideration, and fix the loop, but vary the point. Let $f : S^1 \to \mathbb{R}^2$ be a circular loop and let $x \in \mathbb{R}^2 \smallsetminus f(S^1)$. Then f determines an element in $\pi_1(\mathbb{R}^2 \smallsetminus x) = \mathbb{Z}$ (here we choose basically the same identification of $\pi_1(\mathbb{R}^2 \smallsetminus x)$ with $\mathbb{Z}$ that takes 1 to the homotopy class of $t \mapsto x + (\cos 2\pi t, \sin 2\pi t))$. This number is denoted by $\operatorname{ind}(f, x)$ and called the *winding number* or *index* of x with respect to f.



It is also convenient to characterize the number $\operatorname{ind}(u, x)$ as follows. Along with the circular loop $u : S^1 \to \mathbb{R}^2 \smallsetminus x$, consider the map $\varphi_{u,x} : S^1 \to S^1 : z \mapsto \frac{u(z)-x}{|u(z)-x|}$. The homomorphism $\left(\varphi_{u,x}\right)_* : \pi_1(S^1) \to \pi_1(S^1)$ takes the generator α of the fundamental group of the circle to the element $k\alpha$, where $k = \operatorname{ind}(u, x)$.

36.Ex. The formula $x \mapsto \operatorname{ind}(u, x)$ defines a locally constant function on $\mathbb{R}^2 \smallsetminus u(S^1)$.

36.8x. Let $f : S^1 \to \mathbb{R}^2$ be a loop and $x, y \in \mathbb{R}^2 \smallsetminus f(S^1)$. Prove that if $\operatorname{ind}(f, x) \neq \operatorname{ind}(f, y)$, then any path connecting x and y in $\mathbb{R}^2$ meets $f(S^1)$.

36.9x. Prove that if $u(S^1)$ is contained in a disk, while a point x is not, then $\operatorname{ind}(u, x) = 0$.

36.10x. Find the set of values of function $\operatorname{ind} : \mathbb{R}^2 \smallsetminus u(S^1) \to \mathbb{Z}$ for the following loops u:
a) $u(z) = z$; b) $u(z) = \bar{z}$; c) $u(z) = z^2$; d) $u(z) = z + z^{-1} + z^2 - z^{-2}$
(here $z \in S^1 \subset \mathbb{C}$).

36.11x. Choose several loops $u : S^1 \to \mathbb{R}^2$ such that $u(S^1)$ is a bouquet of two circles (a "lemniscate"). Find the winding number with respect to these loops for various points.

36.12x. Find a loop $f : S^1 \to \mathbb{R}^2$ such that there exist points $x, y \in \mathbb{R}^2 \smallsetminus f(S^1)$ with $\operatorname{ind}(f, x) = \operatorname{ind}(f, y)$, but belonging to different connected components of $\mathbb{R}^2 \smallsetminus f(S^1)$.

36.13x. Prove that any ray R radiating from x meets $f(S^1)$ at least at $|\operatorname{ind}(f, x)|$ points (i.e., the number of points in $f^{-1}(R)$ is not less than $|\operatorname{ind}(f, x)|$).

36.Fx. *If $u : S^1 \to \mathbb{R}^2$ is a restriction of a continuous map $F : D^2 \to \mathbb{R}^2$ and $\operatorname{ind}(u, x) \neq 0$, then $x \in F(D^2)$.*

36.Gx. *If u and v are two circular loops in $\mathbb{R}^2$ with common base point (i. e., $u(1) = v(1)$) and uv is their product, then $\operatorname{ind}(uv, x) = \operatorname{ind}(u, x) + \operatorname{ind}(v, x)$ for each $x \in \mathbb{R}^2 \smallsetminus uv(S^1)$.*

36.Hx. *Let u and v be circular loops in $\mathbb{R}^2$, and $x \in \mathbb{R}^2 \smallsetminus (u(S^1) \cup v(S^1))$. If there exists a (free) homotopy u_t, $t \in I$ connecting u and v such that $x \in \mathbb{R}^2 \smallsetminus u_t(S^1)$ for each $t \in I$, then* $\operatorname{ind}(u, x) = \operatorname{ind}(v, x)$.

36.Ix. *Let $u : S^1 \to \mathbb{C}$ be a circular loop and $a \in \mathbb{C}^2 \smallsetminus u(S^1)$. Then*

$$\operatorname{ind}(u, a) = \frac{1}{2\pi i} \int_{S^1} \frac{|u(z) - a|}{u(z) - a}\, dz.$$

36.Jx. *Let $p(z)$ be a polynomial with complex coefficients, $R > 0$, and let $z_0 \in \mathbb{C}$. Consider the circular loop $u : S^1 \to \mathbb{C} : z \mapsto p(Rz)$. If $z_0 \in \mathbb{C} \smallsetminus u(S^1)$, then the polynomial $p(z) - z_0$ has (counting the multiplicities) precisely* $\operatorname{ind}(u, z_0)$ *roots in the open disk $B^2_R = \{z : |z| < R\}$.*

36.Kx. Riddle. By what can we replace the circular loop u, the domain B_R, and the polynomial $p(z)$ so that the assertion remain valid?

36°5x. Borsuk–Ulam Theorem

36.Lx One-Dimensional Borsuk–Ulam. For each continuous map $f : S^1 \to \mathbb{R}^1$ there exists $x \in S^1$ such that $f(x) = f(-x)$.

36.Mx Two-Dimensional Borsuk–Ulam. For each continuous map $f : S^2 \to \mathbb{R}^2$ there exists $x \in S^2$ such that $f(x) = f(-x)$.

36.Mx.1 Lemma. If there exists a continuous map $f : S^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for each $x \in S^2$, then there exists a continuous map $\varphi : \mathbb{R}P^2 \to \mathbb{R}P^1$ inducing a nonzero homomorphism $\pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^1)$.

36.14x. Prove that at each instant of time, there is a pair of antipodal points on the earth's surface where the pressures and also the temperatures are equal.

Theorems *36.Lx* and *36.Mx* are special cases of the following general theorem. We do not assume the reader to be ready to prove Theorem *36.Nx* in the full generality, but is there another easy special case?

36.Nx Borsuk–Ulam Theorem. For each continuous map $f : S^n \to \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = f(-x)$.

37. Retractions and Fixed Points

37°1. Retractions and Retracts

A continuous map of a topological space onto a subspace is a *retraction* if the restriction of the map to the subspace is the identity map. In other words, if X is a topological space and $A \subset X$, then $\rho : X \to A$ is a retraction if ρ is continuous and $\rho|_A = \mathrm{id}_A$.

37.A. Let ρ be a continuous map of a space X onto its subspace A. Then the following statements are equivalent:

(1) ρ is a retraction,

(2) $\rho(a) = a$ for any $a \in A$,

(3) $\rho \circ \mathrm{in} = \mathrm{id}_A$,

(4) $\rho : X \to A$ is an extension of the identity map $A \to A$.

A subspace A of a space X is a *retract* of X if there exists a retraction $X \to A$.

37.B. Any one-point subset is a retract.

Two-point set may be a non-retract.

37.C. Any subset of $\mathbb{R}$ consisting of two points is not a retract of $\mathbb{R}$.

37.1. If A is a retract of X and B is a retract of A, then B is a retract of X.

37.2. If A is a retract of X and B is a retract of Y, then $A \times B$ is a retract of $X \times Y$.

37.3. A closed interval $[a, b]$ is a retract of $\mathbb{R}$.

37.4. An open interval (a, b) is not a retract of $\mathbb{R}$.

37.5. What topological properties of ambient space are inherited by a retract?

37.6. Prove that a retract of a Hausdorff space is closed.

37.7. Prove that the union of Y-axis and the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y = \sin \frac{1}{x}\}$ is not a retract of $\mathbb{R}^2$ and, moreover, is not a retract of any of its neighborhoods.

37.D. S^0 is not a retract of D^1.

The role of the notion of retract is clarified by the following theorem.

37.E. *A subset A of a topological space X is a retract of X iff for each space Y each continuous map $A \to Y$ extends to a continuous map $X \to Y$.*

37°2. Fundamental Group and Retractions

37.F. *If $\rho : X \to A$ is a retraction, $i : A \to X$ is the inclusion, and $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is an epimorphism and $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a monomorphism.*

37.G. Riddle. Which of the two statements of Theorem *37.F* (about ρ_* or i_*) is easier to use for proving that a set $A \subset X$ is not a retract of X?

37.H Borsuk Theorem in Dimension 2. *S^1 is not a retract of D^2.*

37.8. Is the projective line a retract of the projective plane?

The following problem is more difficult than *37.H* in the sense that its solution is not a straightforward consequence of Theorem *37.F*, but rather demands to reexamine the arguments used in proof of *37.F*.

37.9. Prove that the boundary circle of Möbius band is not a retract of Möbius band.

37.10. Prove that the boundary circle of a handle is not a retract of the handle.

The Borsuk Theorem in its whole generality cannot be deduced like Theorem *37.H* from Theorem *37.F*. However, it can be proven using a generalization of *37.F* to higher homotopy groups. Although we do not assume that you can successfully prove it now relying only on the tools provided above, we formulate it here.

37.I Borsuk Theorem. *The $(n-1)$-sphere S^{n-1} is not a retract of the n-disk D^n.*

At first glance this theorem seems to be useless. Why could it be interesting to know that a map with a very special property of being a retraction does not exist in this situation? However, in mathematics nonexistence theorems are often closely related to theorems that may seem to be more attractive. For instance, the Borsuk Theorem implies the Brouwer Theorem discussed below. But prior to this we must introduce an important notion related to the Brouwer Theorem.

37°3. Fixed-Point Property

Let $f : X \to X$ be a continuous map. A point $a \in X$ is a *fixed point* of f if $f(a) = a$. A space X has the *fixed-point property* if every continuous map $X \to X$ has a fixed point. The fixed point property implies solvability of a wide class of equations.

37.11. Prove that the fixed point property is a topological property.

37.12. A closed interval $[a, b]$ has the fixed point property.

37.13. Prove that if a topological space has the fixed point property, then so does each of its retracts.

37.14. Let X and Y be two topological spaces, $x_0 \in X$ and $y_0 \in Y$. Prove that X and Y have the fixed point property iff so does their bouquet $X \vee Y = X \sqcup Y/_{[x_0 \sim y_0]}$.

37.15. Prove that any finite tree (i.e., a connected space obtained from a finite collection of closed intervals by some identifying of their endpoints such that deleting of an internal point of each of the segments makes the space disconnected, see 42°4x) has the fixed-point property. Is this statement true for infinite trees?

37.16. Prove that $\mathbb{R}^n$ with $n > 0$ does not have the fixed point property.

37.17. Prove that S^n does not have the fixed point property.

37.18. Prove that $\mathbb{R}P^n$ with odd n does not have the fixed point property.

37.19*. Prove that $\mathbb{C}P^n$ with odd n does not have the fixed point property.

Information. $\mathbb{R}P^n$ and $\mathbb{C}P^n$ with any even n have the fixed point property.

37.J Brouwer Theorem. *D^n has the fixed point property.*

37.J.1. Deduce from Borsuk Theorem in dimension n (i.e., from the statement that S^{n-1} is not a retract of D^n) Brouwer Theorem in dimension n (i.e., the statement that any continuous map $D^n \to D^n$ has a fixed point).

37.K. Derive the Borsuk Theorem from the Brouwer Theorem.

The existence of fixed points can follow not only from topological arguments.

37.20. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a periodic affine transformation (i.e., $\underbrace{f \circ \cdots \circ f}_{p \text{ times}} = \mathrm{id}_{\mathbb{R}^n}$ for a certain p), then f has a fixed point.

38. Homotopy Equivalences

38°1. Homotopy Equivalence as Map

Let X and Y be two topological spaces, $f : X \to Y$ and $g : Y \to X$ continuous maps. Consider the compositions $f \circ g : Y \to Y$ and $g \circ f : X \to X$. They would be equal to the corresponding identity maps if f and g were mutually inverse homeomorphisms. If $f \circ g$ and $g \circ f$ are only homotopic to the identity maps, then f and g are said to be *homotopy inverse* to each other. If a continuous map f possesses a homotopy inverse map, then f is a *homotopy invertible map* or a *homotopy equivalence*.

38.A. Prove the following properties of homotopy equivalences:

(1) any homeomorphism is a homotopy equivalence,
(2) a map homotopy inverse to a homotopy equivalence is a homotopy equivalence,
(3) the composition of two homotopy equivalences is a homotopy equivalence.

38.1. Find a homotopy equivalence that is not a homeomorphism.

38°2. Homotopy Equivalence as Relation

Two topological spaces X and Y are *homotopy equivalent* if there exists a homotopy equivalence $X \to Y$.

38.B. Homotopy equivalence of topological spaces is an equivalence relation.

The classes of homotopy equivalent spaces are *homotopy types*. Thus homotopy equivalent spaces are said to be of the same homotopy type.

38.2. Prove that homotopy equivalent spaces have the same number of path-connected components.

38.3. Prove that homotopy equivalent spaces have the same number of connected components.

38.4. Find an infinite series of topological spaces that belong to the same homotopy type, but are pairwise not homeomorphic.

38°3. Deformation Retraction

A retraction $\rho : X \to A$ is a *deformation retraction* if its composition $\mathrm{in} \circ \rho$ with the inclusion $\mathrm{in} : A \to X$ is homotopic to the identity id_X. If $\mathrm{in} \circ \rho$ is A-homotopic to id_X, then ρ is a *strong deformation retraction*. If X admits a (strong) deformation retraction onto A, then A is a (*strong*) *deformation retract* of X.

38.C. Each deformation retraction is a homotopy equivalence.

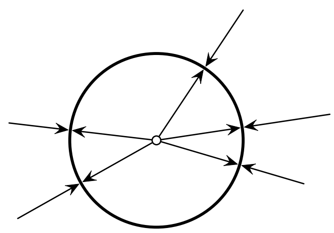
38.D. If A is a deformation retract of X, then A and X are homotopy equivalent.

38.E. Any two deformation retracts of one and the same space are homotopy equivalent.

38.F. If A is a deformation retract of X and B is a deformation retract of Y, then $A \times B$ is a deformation retract of $X \times Y$.

38°4. Examples

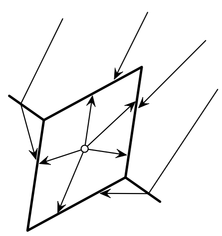
38.G. Circle S^1 is a deformation retract of $\mathbb{R}^2 \setminus 0$.



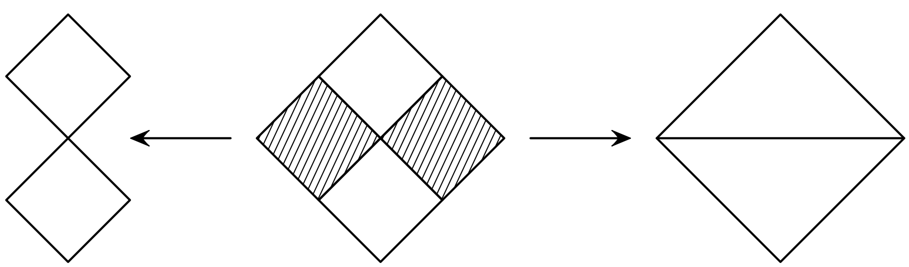
38.5. Prove that the Möbius strip is homotopy equivalent to a circle.

38.6. Classify letters of Latin alphabet up to homotopy equivalence.

38.H. Prove that a plane with s punctures is homotopy equivalent to a union of s circles intersecting in a single point.



38.I. Prove that the union of a diagonal of a square and the contour of the same square is homotopy equivalent to a union of two circles intersecting in a single point.



38.7. Prove that a handle is homotopy equivalent to a bouquet of two circles. (E.g., construct a deformation retraction of the handle to a union of two circles intersecting in a single point.)

38.8. Prove that a handle is homotopy equivalent to a union of three arcs with common endpoints (i.e., letter θ).

38.9. Prove that the space obtained from S^2 by identification of a two (distinct) points is homotopy equivalent to the union of a two-sphere and a circle intersecting in a single point.

38.10. Prove that the space $\{(p, q) \in \mathbb{C} : z^2 + pz + q$ has two distinct roots$\}$ of quadratic complex polynomials with distinct roots is homotopy equivalent to the circle.

38.11. Prove that the space $GL(n, \mathbb{R})$ of invertible $n\times n$ real matrices is homotopy equivalent to the subspace $O(n)$ consisting of orthogonal matrices.

38.12. Riddle. Is there any relation between a solution of the preceding problem and the Gram–Schmidt orthogonalization? Can the Gram–Schmidt orthogonalization algorithm be considered a deformation retraction?

38.13. Construct the following deformation retractions: (a) $\mathbb{R}^3 \smallsetminus \mathbb{R}^1 \to S^1$; (b) $\mathbb{R}^n \smallsetminus \mathbb{R}^m \to S^{n-m-1}$; (c) $S^3 \smallsetminus S^1 \to S^1$; (d) $S^n \smallsetminus S^m \to S^{n-m-1}$ (e) $\mathbb{R}P^n \smallsetminus \mathbb{R}P^m \to \mathbb{R}P^{n-m-1}$.

38°5. Deformation Retraction versus Homotopy Equivalence

38.J. Spaces of Problem *38.I* cannot be embedded one to another. On the other hand, they can be embedded as deformation retracts in the plane with two punctures.

Deformation retractions comprise a special type of homotopy equivalences. For example, they are easier to visualize. However, as follows from *38.J*, it may happen that two spaces are homotopy equivalent, but none of them can be embedded in the other one, and so none of them is homeomorphic to a deformation retract of the other one. Therefore, deformation retractions seem to be insufficient for establishing homotopy equivalences.

However, this is not the case:

38.14*. Prove that any two homotopy equivalent spaces can be embedded as deformation retracts in the same topological space.

38°6. Contractible Spaces

A topological space X is *contractible* if the identity map $\mathrm{id} : X \to X$ is null-homotopic.

38.15. Show that $\mathbb{R}$ and I are contractible.

38.16. Prove that any contractible space is path-connected.

38.17. Prove that the following three statements about a topological space X are equivalent:

(1) X is contractible,
(2) X is homotopy equivalent to a point,
(3) there exists a deformation retraction of X onto a point,
(4) any point a of X is a deformation retract of X,
(5) any continuous map of any topological space Y to X is null-homotopic,
(6) any continuous map of X to any topological space Y is null-homotopic.

38.18. Is it true that if X is a contractible space, then for any topological space Y

(1) any two continuous maps $X \to Y$ are homotopic?
(2) any two continuous maps $Y \to X$ are homotopic?

38.19. Find out if the spaces on the following list are contractible:

(1) $\mathbb{R}^n$,
(2) a convex subset of $\mathbb{R}^n$,
(3) a star-shaped subset of $\mathbb{R}^n$,
(4) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \le 1\}$,
(5) a finite tree (i.e., a connected space obtained from a finite collection of closed intervals by some identifying of their endpoints such that deleting of an internal point of each of the segments makes the space disconnected, see 42°4x.)

38.20. Prove that $X \times Y$ is contractible iff both X and Y are contractible.

38°7. Fundamental Group and Homotopy Equivalences

38.K. *Let $f : X \to Y$ and $g : Y \to X$ be homotopy inverse maps, and let $x_0 \in X$ and $y_0 \in Y$ be two points such that $f(x_0) = y_0$ and $g(y_0) = x_0$ and, moreover, the homotopies relating $f \circ g$ to id_Y and $g \circ f$ to id_X are fixed at y_0 and x_0, respectively. Then f_* and g_* are inverse to each other isomorphisms between groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.*

38.L Corollary. *If $\rho : X \to A$ is a strong deformation retraction, $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ and $\mathrm{in}_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ are mutually inverse isomorphisms.*

38.21. Calculate the fundamental group of the following spaces:

(a) $\mathbb{R}^3 \smallsetminus \mathbb{R}^1$,	(b) $\mathbb{R}^N \smallsetminus \mathbb{R}^n$,	(c) $\mathbb{R}^3 \smallsetminus S^1$,	(d) $\mathbb{R}^N \smallsetminus S^n$,
(e) $S^3 \smallsetminus S^1$,	(f) $S^N \smallsetminus S^k$,	(g) $\mathbb{R}P^3 \smallsetminus \mathbb{R}P^1$,	(h) handle,
(i) Möbius band,		(j) sphere with s holes,	
(k) Klein bottle with a point removed,		(l) Möbius band with s holes.	

38.22. Prove that the boundary circle of the Möbius band standardly embedded in $\mathbb{R}^3$ (see *21.18*) could not be the boundary of a disk embedded in $\mathbb{R}^3$ in such a way that its interior does not intersect the band.

38.23. 1) Calculate the fundamental group of the space Q of all complex polynomials $ax^2 + bx + c$ with distinct roots. 2) Calculate the fundamental group of the subspace Q_1 of Q consisting of polynomials with $a = 1$ (unital polynomials).

38.24. Riddle. Can you solve *38.23* along the lines of deriving the customary formula for the roots of a quadratic trinomial?

38.M. Suppose that the assumptions of Theorem *38.K* are weakened as follows: $g(y_0) \neq x_0$ and/or the homotopies relating $f \circ g$ to id_Y and $g \circ f$ to id_X are *not* fixed at y_0 and x_0, respectively. How would f_* and g_* be related? Would $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ be isomorphic?

39. Covering Spaces via Fundamental Groups

39°1. Homomorphisms Induced by Covering Projections

39.A. Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 = p(x_0)$. Then $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism. Cf. *34.C*.

The image of the monomorphism $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ induced by the covering projection $p : X \to B$ is the *group of the covering* p *with base point* x_0.

39.B. Riddle. Is the group of covering determined by the covering?

39.C Group of Covering versus Lifting of Loops. Describe loops in the base space of a covering, whose homotopy classes belong to the group of the covering, in terms provided by Path Lifting Theorem *34.B*.

39.D. Let $p : X \to B$ be a covering, let $x_0, x_1 \in X$ belong to the same path-component of X, and $b_0 = p(x_0) = p(x_1)$. Then $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ are conjugate subgroups of $\pi_1(B, b_0)$ (i.e., there exists an $\alpha \in \pi_1(B, b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1} p_*(\pi_1(X, x_0))\alpha$).

39.E. Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 = p(x_0)$. For each $\alpha \in \pi_1(B, b_0)$, there exists an $x_1 \in p^{-1}(b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1} p_*(\pi_1(X, x_0))\alpha$.

39.F. Let $p : X \to B$ be a covering in a narrow sense, $G \subset \pi_1(B, b_0)$ the group of this covering with a base point x_0. A subgroup $H \subset \pi_1(B, b_0)$ is a group of the same covering iff H is conjugate to G.

39°2. Number of Sheets

39.G Number of Sheets and Index of Subgroup. Let $p : X \to B$ be a covering in a narrow sense with finite number of sheets. Then the number of sheets is equal to the index of the group of this covering.

39.H Sheets and Right Cosets. Let $p : X \to B$ be a covering in a narrow sense, $b_0 \in B$, and $x_0 \in p^{-1}(b_0)$. Construct a natural bijection of $p^{-1}(b_0)$ and the set $p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0)$ of right cosets of the group of the covering in the fundamental group of the base space.

39.1 Number of Sheets in Universal Covering. The number of sheets of a universal covering equals the order of the fundamental group of the base space.

39.2 Nontrivial Covering Means Nontrivial π_1***.*** Any topological space that has a nontrivial path-connected covering space has a nontrivial fundamental group.

39.3. What numbers can appear as the number of sheets of a covering of the Möbius strip by the cylinder $S^1 \times I$?

39.4. What numbers can appear as the number of sheets of a covering of the Möbius strip by itself?

39.5. What numbers can appear as the number of sheets of a covering of the Klein bottle by torus?

39.6. What numbers can appear as the number of sheets of a covering of the Klein bottle by itself?

39.7. What numbers can appear as the numbers of sheets for a covering of the Klein bottle by plane $\mathbb{R}^2$?

39.8. What numbers can appear as the numbers of sheets for a covering of the Klein bottle by $S^1 \times \mathbb{R}$?

39°3. Hierarchy of Coverings

Let $p : X \to B$ and $q : Y \to B$ be two coverings, $x_0 \in X$, $y_0 \in Y$, and $p(x_0) = q(y_0) = b_0$. The covering q with base point y_0 is *subordinate* to p with base point x_0 if there exists a map $\varphi : X \to Y$ such that $q \circ \varphi = p$ and $\varphi(x_0) = y_0$. In this case, the map φ is a *subordination*.

39.I. A subordination is a covering map.

39.J. If a subordination exists, then it is unique. Cf. *34.B*.

Two coverings $p : X \to B$ and $q : Y \to B$ are *equivalent* if there exists a homeomorphism $h : X \to Y$ such that $p = q \circ h$. In this case, h and h^{-1} are *equivalences*.

39.K. If two coverings are mutually subordinate, then the corresponding subordinations are equivalences.

39.L. The equivalence of coverings is, indeed, an equivalence relation in the set of coverings with a given base space.

39.M. Subordination determines a nonstrict partial order in the set of equivalence classes of coverings with a given base.

39.9. What equivalence class of coverings is minimal (i.e., subordinate to all other classes)?

39.N. Let $p : X \to B$ and $q : Y \to B$ be coverings, $x_0 \in X$, $y_0 \in Y$ and $p(x_0) = q(y_0) = b_0$. If q with base point y_0 is subordinate to p with base point x_0, then the group of covering p is contained in the group of covering q, i.e., $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$.

39°4x. Existence of Subordinations

A topological space X is *locally path-connected* if for each point $a \in X$ and each neighborhood U of a the point a has a path-connected neighborhood $V \subset U$.

39.1x. Find a path connected, but not locally path connected topological space.

39.Ax. Let B be a locally path-connected space, $p : X \to B$ and $q : Y \to B$ be coverings in a narrow sense, $x_0 \in X$, $y_0 \in Y$ and $p(x_0) = q(y_0) = b_0$. If $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$, then q is subordinate to p.

39.Ax.1. Under the conditions of *39.Ax*, if two paths $u, v : I \to X$ have the same initial point x_0 and a common final point, then the paths that cover $p \circ u$ and $p \circ v$ and have the same initial point y_0 also have the same final point.

39.Ax.2. Under the conditions of *39.Ax*, the map $X \to Y$ defined by *39.Ax.1* (guess, what this map is!) is continuous.

39.2x. Construct an example proving that the hypothesis of local path connectedness in *39.Ax.2* and *39.Ax* is necessary.

39.Bx. Two coverings $p : X \to B$ and $q : Y \to B$ with a common locally path-connected base are equivalent iff for some $x_0 \in X$ and $y_0 \in Y$ with $p(x_0) = q(y_0) = b_0$ the groups $p_*(\pi_1(X, x_0))$ and $q_*(\pi_1(Y, y_0))$ are conjugate in $\pi_1(B, b_0)$.

39.3x. Construct an example proving that the assumption of local path connectedness of the base in *39.Bx* is necessary.

39°5x. Micro Simply Connected Spaces

A topological space X is *micro simply connected* if each point $a \in X$ has a neighborhood U such that the inclusion homomorphism $\pi_1(U, a) \to \pi_1(X, a)$ is trivial.

39.4x. Any simply connected space is micro simply connected.

39.5x. Find a micro simply connected, but not simply connected space.

A topological space is *locally contractible at point a* if each neighborhood U of a contains a neighborhood V of a such that the inclusion $V \to U$ is null-homotopic. A topological space is *locally contractible* if it is locally contractible at each of its points.

39.6x. Any finite topological space is locally contractible.

39.7x. Any locally contractible space is micro simply connected.

39.8x. Find a space which is not micro simply connected.

In the literature, the micro simply connectedness is also called *weak local simply connectedness*, while a strong local simply connectedness is the following property: any neighborhood U of any point x contains a neighborhood V such that any loop at x in V is null-homotopic in U.

> ***39.9*x.** Find a micro simply connected space which is not strong locally simply connected.

39°6x. Existence of Coverings

***39.C*x.** *A space having a universal covering space is micro simply connected.*

***39.D*x Existence of Covering With a Given Group.** *If a topological space B is path connected, locally path connected, and micro simply connected, then for any $b_0 \in B$ and any subgroup π of $\pi_1(B, b_0)$ there exists a covering $p : X \to B$ and a point $x_0 \in X$ such that $p(x_0) = b_0$ and $p_*(\pi_1(X, x_0)) = \pi$.*

> ***39.D*x.*1.*** Suppose that in the assumptions of Theorem *39.Dx* there exists a covering $p : X \to B$ satisfying all requirements of this theorem. For each $x \in X$, describe all paths in B that are p-images of paths connecting x_0 to x in X.
>
> ***39.D*x.*2.*** Does the solution of Problem *39.Dx.1* determine an equivalence relation in the set of all paths in B starting at b_0, so that we obtain a one-to-one correspondence between the set X and the set of equivalence classes?
>
> ***39.D*x.*3.*** Describe a topology in the set of equivalence classes from *39.Dx.2* such that the natural bijection between X and this set be a homeomorphism.
>
> ***39.D*x.*4.*** Prove that the reconstruction of X and $p : X \to B$ provided by problems *39.Dx.1–39.Dx.4* under the assumptions of Theorem *39.Dx* determine a covering whose existence is claimed by Theorem *39.Dx*.

Essentially, assertions *39.Dx.1–39.Dx.3* imply the uniqueness of the covering with a given group. More precisely, the following assertion holds true.

***39.E*x Uniqueness of the Covering With a Given Group.** *Assume that B is path-connected, locally path-connected, and micro simply connected. Let $p : X \to B$ and $q : Y \to B$ be two coverings, and let $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0))$. Then the coverings p and q are equivalent, i.e., there exists a homeomorphism $f : X \to Y$ such that $f(x_0) = y_0$ and $p \circ f = q$.*

***39.F*x Classification of Coverings Over a Good Space.** *There is a one-to-one correspondence between classes of equivalent coverings (in a narrow sense) over a path-connected, locally path-connected, and micro simply connected space B with base point b_0, on the one hand, and conjugacy classes of subgroups of $\pi_1(B, b_0)$, on the other hand. This correspondence identifies the hierarchy of coverings (ordered by subordination) with the hierarchy of subgroups (ordered by inclusion).*

Under the correspondence of Theorem *39.Fx*, the trivial subgroup corresponds to a covering with simply connected covering space. Since this covering subordinates any other covering with the same base space, it is said to be ***universal***.

> ***39.10x.*** Describe all coverings of the following spaces up to equivalence and subordination:
>
> (1) circle S^1;
> (2) punctured plane $\mathbb{R}^2 \smallsetminus 0$;
> (3) Möbius strip;
> (4) four point digital circle (the space formed by 4 points, a, b, c, d; with the base of open sets formed by $\{a\}$, $\{c\}$, $\{a, b, c\}$ and $\{c, d, a\}$)
> (5) torus $S^1 \times S^1$;

39°7x. Action of Fundamental Group on Fiber

39.Gx Action of π_1 ***on Fiber.*** Let $p : X \to B$ be a covering, $b_0 \in B$. Construct a natural right action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$.

39.Hx. When the action in *39.Gx* is transitive?

39°8x. Automorphisms of Covering

A homeomorphism $\varphi : X \to X$ is an ***automorphism*** of a covering $p : X \to B$ if $p \circ \varphi = p$.

39.Ix. Automorphisms of a covering form a group.

Denote the group of automorphisms of a covering $p : X \to B$ by $\mathrm{Aut}(p)$.

39.Jx. *An automorphism $\varphi : X \to X$ of covering $p : X \to B$ is recovered from the image $\varphi(x_0)$ of any $x_0 \in X$. Cf. 39.J.*

39.Kx. *Any two-fold covering has a nontrivial automorphism.*

> ***39.11x.*** Find a three-fold covering without nontrivial automorphisms.

Let G be a group and H its subgroup. Recall that the ***normalizer*** $Nr(H)$ of H is the subset of G consisting of $g \in G$ such that $g^{-1}Hg = H$. This is a subgroup of G, which contains H as a normal subgroup. So, $Nr(H)/_H$ is a group.

39.Lx. Let $p : X \to B$ be a covering, $x_0 \in X$ and $b_0 = p(x_0)$. Construct a map $\pi_1(B, b_0) \to p^{-1}(b_0)$ which induces a bijection of the set $p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0)$ of right cosets onto $p^{-1}(b_0)$.

39.Mx. Show that the bijection $p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0) \to p^{-1}(b_0)$ from *39.Lx* maps the set of images of a point x_0 under all automorphisms of a covering $p : X \to B$ to the group $Nr(p_*(\pi_1(X, x_0)))/_{p_*(\pi_1(X, x_0))}$.

***39.N*x.** *For any covering $p : X \to B$ in a narrow sense, there is a natural injective map $Aut(p)$ to the group $Nr(p_*(\pi_1(X, x_0)))/_{p_*(\pi_1(X,x_0))}$. This map is an antihomomorphism.*[1]

***39.O*x.** *Under assumptions of Theorem 39.N*x*, if B is locally path connected, then the antihomomorphism $Aut(p) \to Nr(p_*(\pi_1(X, x_0)))/_{p_*(\pi_1(X,x_0))}$ is bijective.*

39°9x. Regular Coverings

39.P*x *Regularity of Covering. *Let $p : X \to B$ be a covering in a narrow sense, $b_0 \in B$, $x_0 \in p^{-1}(b_0)$. The following conditions are equivalent:*

(1) *$p_*\big(\pi_1(X, x_0)\big)$ is a normal subgroup of $\pi_1(B, b_0)$;*

(2) *$p_*\big(\pi_1(X, x)\big)$ is a normal subgroup of $\pi_1(B, p(x))$ for each $x \in X$;*

(3) *all groups $p_*\pi_1(X, x)$ for $x \in p^{-1}(b)$ are the same;*

(4) *for any loop $s : I \to B$ either every path in X covering s is a loop (independent on the its initial point) or none of them is a loop;*

(5) *the automorphism group acts transitively on $p^{-1}(b_0)$.*

A covering satisfying to (any of) the equivalent conditions of Theorem *39.P*x is said to be *regular*.

***39.12*x.** The coverings $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$ and $S^1 \to S^1 : z \mapsto z^n$ for integer $n > 0$ are regular.

***39.Q*x.** *The automorphism group of a regular covering $p : X \to B$ is naturally anti-isomorphic to the quotient group $\pi_1(B, b_0)/_{p_*\pi_1(X,x_0)}$ of the group $\pi_1(B, b_0)$ by the group of the covering for any $x_0 \in p^{-1}(b_0)$.*

39.R*x *Classification of Regular Coverings Over a Good Base. *There is a one-to-one correspondence between classes of equivalent coverings (in a narrow sense) over a path connected, locally path connected, and micro simply connected space B with a base point b_0, on one hand, and anti-epimorphisms $\pi_1(B, b_0) \to G$, on the other hand.*

Algebraic properties of the automorphism group of a regular covering are often referred to as if they were properties of the covering itself. For instance, a *cyclic covering* is a regular covering with cyclic automorphism group, an *Abelian covering* is a regular covering with Abelian automorphism group, etc.

[1] Recall that a map $\varphi : G \to H$ from a group G to a group H is an *antihomomorphism* if $\varphi(ab) = \varphi(b)\varphi(a)$ for any $a, b \in G$.

39.13x. Any two-fold covering is regular.

39.14x. Which coverings considered in Problems of Section 33 are regular? Is out there any nonregular covering?

39.15x. Find a three-fold nonregular covering of a bouquet of two circles.

39.16x. Let $p : X \to B$ be a regular covering, $Y \subset X$, $C \subset B$, and let $q : Y \to C$ be a submap of p. Prove that if q is a covering, then this covering is regular.

39°10x. Lifting and Covering Maps

39.Sx. Riddle. Let $p : X \to B$ and $f : Y \to B$ be continuous maps. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $p(x_0) = f(y_0)$. Formulate in terms of homomorphisms $p_* : \pi_1(X, x_0) \to \pi_1(B, p(x_0))$ and $f_* : \pi_1(Y, y_0) \to \pi_1(B, f(y_0))$ a necessary condition for existence of a lifting $\widetilde{f} : Y \to X$ of f such that $\widetilde{f}(y_0) = x_0$. Find an example where this condition is not sufficient. What additional assumptions can make it sufficient?

39.Tx Theorem on Lifting a Map. *Let $p : X \to B$ be a covering in a narrow sense and $f : Y \to B$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $p(x_0) = f(y_0)$. If Y is a locally path-connected space and $f_*\pi(Y, y_0) \subset p_*\pi(X, x_0)$, then there exists a unique continuous map $\widetilde{f} : Y \to X$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(y_0) = x_0$.*

39.Ux. *Let $p : X \to B$ and $q : Y \to C$ be coverings in a narrow sense and $f : B \to C$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $fp(x_0) = q(y_0)$. If there exists a continuous map $F : X \to Y$ such that $fp = qF$ and $F(x_0) = y_0$, then $f_*p_*\pi_1(X, x_0) \subset q_*\pi_1(Y, y_0)$.*

39.Vx Theorem on Covering of a Map. *Let $p : X \to B$ and $q : Y \to C$ be coverings in a narrow sense and $f : B \to C$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $fp(x_0) = q(y_0)$. If Y is locally path connected and $f_*p_*\pi_1(X, x_0) \subset q_*\pi_1(Y, y_0)$, then there exists a unique continuous map $F : X \to Y$ such that $fp = qF$ and $F(x_0) = y_0$.*

39°11x. Induced Coverings

39.Wx. *Let $p : X \to B$ be a covering and $f : A \to B$ a continuous map. Denote by W a subspace of $A \times X$ consisting of points (a, x) such that $f(a) = p(x)$. Let $q : W \to A$ be a restriction of $A \times X \to A$. Then $q : W \to A$ is a covering with the same number of sheets as p.*

A covering $q : W \to A$ obtained as in Theorem *39.Wx* is said to be *induced* from $p : X \to B$ by $f : A \to B$.

39.17x. Represent coverings from problems *33.D* and *33.F* as induced from $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$.

***39.18*x.** Which of the coverings considered above can be induced from the covering of Problem *35.7*?

39°12x. High-Dimensional Homotopy Groups of Covering Space

***39.X*x.** *Let $p : X \to B$ be a covering. Then for any continuous map $s : I^n \to B$ and a lifting $u : I^{n-1} \to X$ of the restriction $s|_{I^{n-1}}$ there exists a unique lifting of s extending u.*

***39.Y*x.** *For any covering $p : X \to B$ and points $x_0 \in X$, $b_0 \in B$ such that $p(x_0) = b_0$ the homotopy groups $\pi_r(X, x_0)$ and $\pi_r(B, b_0)$ with $r > 1$ are canonically isomorphic.*

***39.Z*x.** Prove that homotopy groups of dimensions greater than 1 of circle, torus, Klein bottle and Möbius strip are trivial.

Proofs and Comments

36.A This follows from *29.I*.

36.B Let $[u], [v] \in \pi_1(X, x_0)$. Since $f \circ (uv) = (f \circ u)(f \circ v)$, we have $f_\#(uv) = f_\#(u) f_\#(v)$ and

$$f_*([u][v]) = f_*\big([uv]\big) = \big[f_\#(uv)\big] = \big[f_\#(u) f_\#(v)\big] =$$
$$= \big[f_\#(u)\big]\big[f_\#(v)\big] = f_*([u]) f_*([v]).$$

36.C Let $[u] \in \pi_1(X, x_0)$. Since $(g \circ f)_\#(u) = g \circ f \circ u = g_\#(f_\#(u))$, consequently,

$$(g \circ f)_*([u]) = \big[(g \circ f)_\#(u)\big] = \big[g_\#(f_\#(u))\big] = g_*\left([f_\#(u)]\right) = g_*(f_*(u)),$$

thus, $(g \circ f)_* = g_* \circ f_*$.

36.D Let $H : X \times I \to Y$ be a homotopy between f and g, and let $H(x_0, t) = y_0$ for all $t \in I$; u is a certain loop in X. Consider a map $h = H \circ (u \times \mathrm{id}_I)$, thus, $h : (\tau, t) \mapsto H(u(\tau), t)$. Then $h(\tau, 0) = H(u(\tau), 0) = f(u(\tau))$ and $h(\tau, 1) = H(u(\tau), 1) = g(u(\tau))$, so that h is a homotopy between the loops $f \circ u$ and $g \circ u$. Furthermore, $h(0, t) = H(u(0), t) = H(x_0, t) = y_0$, and we similarly have $h(1, t) = y_0$, therefore, h is a homotopy between the loops $f_\#(u)$ and $g_\#(v)$, whence

$$f_*\left([u]\right) = \left[f_\#\left(u\right)\right] = \left[g_\#\left(u\right)\right] = g_*\left([u]\right).$$

36.E Let H be a homotopy between the maps f and g and the loop s is defined by the formula $s(t) = H(x_0, t)$. By assertion *32.2*, $g_* = T_s \circ f_*$.

36.F This obviously follows from the equality

$$f_\#(s^{-1} u s) = (f \circ s)^{-1} f_\#(u)(f \circ s).$$

36.G.1 This is the assertion of Theorem *36.G*.

36.G.2 For example, it is sufficient to take R such that

$$R > \max\{1, |a_1| + |a_2| + \ldots + |a_n|\}.$$

36.G.3 Use the rectilinear homotopy $h(z, t) = tp(z) + (1 - t)q(z)$. It remains to verify that $h(z, t) \neq 0$ for all z and t. Indeed, since $|p(z) - q(z)| < q(z)$ by assumption, we have

$$|h(z, t)| \geq |q(z)| - t|p(z) - q(z)| \geq |q(z)| - |p(z) - q(z)| > 0.$$

36.G.4 Indeed, this is a quite obvious lemma; see *36.A*.

36.G Take a number R satisfying the assumptions of assertion *36.G.2* and consider the loop $u : u(t) = Re^{2\pi it}$. The loop u, certainly, is null-homotopic in $\mathbb{C}$. Now we assume that $p(z) \neq 0$ for all z with $|z| \leq R$. Then the loop $p \circ u$ is null-homotopic in $\mathbb{C} \smallsetminus 0$, by *36.G.3*, and the loop $q \circ u$ is null-homotopic in $\mathbb{C} \smallsetminus 0$. However, $(q \circ u)(t) = R^n e^{2\pi int}$, therefore, this loop is not null-homotopic. A contradiction.

***36.A*x** See *36.Dx*.

***36.B*x** Yes, it is.

***36.C*x** See *36.Dx*.

***36.D*x** Let $i : S^{n-1} \to D^n$ be the inclusion. Assume that $f(x) \neq 0$ for all $x \in D^n$. We preserve the designation f for the submap $D^n \to \mathbb{R}^n \smallsetminus 0$ and consider the inclusion homomorphisms $i_* : \pi_{n-1}(S^{n-1}) \to \pi_{n-1}(D^n)$ and $f_* : \pi_{n-1}(D^n) \to \pi_{n-1}(\mathbb{R}^n \smallsetminus 0)$. Since all homotopy groups of D^n are trivial, the composition $(f \circ i)_* = f_* \circ i_*$ is a zero homomorphism. However, the composition $f \circ i$ is the map f_0, which, by assumption, induces a nonzero homomorphism $\pi_{n-1}(S^{n-1}) \to \pi_{n-1}(\mathbb{R}^n \smallsetminus 0)$.

***36.E*x** Consider a circular neighborhood U of x disjoint with the image $u(S^1)$ of the circular loop under consideration and let $y \in U$. Join x and y by a rectilinear path $s : t \mapsto ty + (1-t)x$. Then

$$h(z,t) = \varphi_{u,s(t)}(z) = \frac{u(z) - s(t)}{|u(z) - s(t)|}$$

determines a homotopy between $\varphi_{u,x}$ and $\varphi_{u,y}$, whence $\left(\varphi_{u,x}\right)_* = \left(\varphi_{u,y}\right)_*$, whence it follows that $\operatorname{ind}(u,y) = \operatorname{ind}(u,x)$ for any point $y \in U$. Consequently, the function $\operatorname{ind} : x \mapsto \operatorname{ind}(u,x)$ is constant on U.

***36.F*x** If $x \notin F(D^2)$, then the circular loop u is null-homotopic in $\mathbb{R}^2 \smallsetminus x$ because $u = F \circ i$, where i is the standard embedding $S^1 \to D^2$, and i is null-homotopic in D^2.

***36.G*x** This is true because we have $[uv] = [u][v]$ and $\pi_1(\mathbb{R}^2 \smallsetminus x) \to \mathbb{Z}$ is a homomorphism.

***36.H*x** The formula

$$h(z,t) = \varphi_{u_t,x}(z) = \frac{u_t(z) - x}{|u_t(z) - x|}$$

determines a homotopy between $\varphi_{u,x}$ and $\varphi_{v,x}$, whence $\operatorname{ind}(u,x) = \operatorname{ind}(v,x)$; cf. *36.Ex*.

***36.L*x** We define a map $\varphi : S^1 \to \mathbb{R} : x \mapsto f(x) - f(-x)$. Then

$$\varphi(-x) = f(-x) - f(x) = -(f(x) - f(-x)) = -\varphi(x),$$

thus φ is an odd map. Consequently, if, for example, $\varphi(1) \neq 0$, then the image $\varphi(S^1)$ contains values with distinct signs. Since the circle is connected, there is a point $x \in S^1$ such that $f(x) - f(-x) = \varphi(x) = 0$.

36.Mx.1 Assume that $f(x) \neq f(-x)$ for all $x \in S^2$. In this case, the formula $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ determines a map $g : S^2 \to S^1$. Since $g(-x) = -g(x)$, it follows that g takes antipodal points of S^2 to antipodal points of S^1. The quotient map of g is a continuous map $\varphi : \mathbb{R}P^2 \to \mathbb{R}P^1$. We show that the induced homomorphism $\varphi_* : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^1)$ is nontrivial. The generator λ of the group $\pi_1(\mathbb{R}P^2)$ is the class of the loop l covered by the path $\tilde{l}$ joining two opposite points of S^2. The path $g \circ \tilde{l}$ also joins two opposite points lying on the circle, consequently, the loop $\varphi \circ l$ covered by $g \circ \tilde{l}$ is not null-homotopic. Thus, $\varphi_*(\lambda)$ is a nontrivial element of $\pi_1(\mathbb{R}P^1)$.

36.Mx To prove the Borsuk–Ulam Theorem, it only remains to observe that there are no nontrivial homomorphisms $\pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^1)$ because the first of these groups is isomorphic to $\mathbb{Z}_2$, while the second one is isomorphic to $\mathbb{Z}$.

37.A Prove this assertion on your own.

37.B Since any map to a singleton is continuous, the map $\rho : X \to \{x_0\}$ is a retraction.

37.C The line is connected. Therefore, its retract (being its continuous image) is connected, too. However, a pair of points in the line is not connected.

37.D See the proof of assertion *37.C*.

37.E $\Longrightarrow$ Let $\rho : X \to A$ be a retraction. and let $f : A \to Y$ be a continuous map. Then the composition $F = f \circ \rho : X \to Y$ extends f.
$\Longleftarrow$ Consider the identity map $\mathrm{id} : A \to A$. Its continuous extension to X is the required retraction $\rho : X \to A$.

37.F Since $\rho_* \circ i_* = (\rho \circ i)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A,x_0)}$, it follows that the homomorphism ρ_* is an epimorphism, and the homomorphism i_* is a monomorphism.

37.G About i_*; for example, see the proof of the following assertion.

37.H Since the group $\pi_1(D^2)$ is trivial, while $\pi_1(S^1)$ is not, it follows that $i_* : \pi_1(S^1, 1) \to \pi_1(D^2, 1)$ cannot be a monomorphism. Consequently, by assertion *37.F*, the disk D^2 cannot be retracted to its boundary S^1.

37.I The proof word by word repeats that of Theorem *37.H*, only instead of fundamental groups we must use $(n-1)$-dimensional homotopy groups. The reason for this is that the group $\pi_{n-1}(D^n)$ is trivial, while $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$ (i.e., this group is nontrivial).

37.J Assume that a map $f : D^n \to D^n$ has no fixed points. For each $x \in D^n$, consider the ray starting at $f(x) \in D^n$ and passing through x, and denote by $\rho(x)$ the point of its intersection with the boundary sphere S^{n-1}. It is clear that $\rho(x) = x$ for $x \in S^{n-1}$. Prove that the map ρ is continuous. Therefore, $\rho : D^n \to S^{n-1}$ is a retraction. However, this contradicts the Borsuk Theorem.

38.A Prove this assertion on your own.

38.B This immediately follows from assertion *38.A*.

38.C Since ρ is a retraction, it follows that one of the conditions in the definition of homotopically inverse maps is automatically fulfilled: $\rho \circ \mathrm{in} = \mathrm{id}_A$. The second requirement: $\mathrm{in} \circ \rho$ is homotopic to id_X, is fulfilled by assumption.

38.D This immediately follows from assertion *38.C*.

38.E This follows from *38.D* and *38.B*.

38.F Let $\rho_1 : X \to A$ and $\rho_2 : Y \to B$ be deformation retractions. Prove that $\rho_1 \times \rho_2$ is a deformation retraction.

38.G Let the map $\rho : \mathbb{R}^2 \smallsetminus 0 \to S^1$ be defined by the formula $\rho(x) = \frac{x}{|x|}$. The formula $h(x,t) = (1-t)x + t\frac{x}{|x|}$ determines a rectilinear homotopy between the identity map of $\mathbb{R}^2 \smallsetminus 0$ and the composition $\rho \circ i$, where i is the standard inclusion $S^1 \to \mathbb{R}^2 \smallsetminus 0$.

38.H The topological type of $\mathbb{R}^2 \smallsetminus \{x_1, x_2, \dots, x_s\}$ does not depend on the position of the points $x_1, x_2, \dots, x_s$ in the plane. We put them on the unit circle: for example, let them be roots of unity of degree s. Consider s simple closed curves on the plane each of which encloses exactly one of the points and passes through the origin, and which have no other common points except the origin. Instead of curves, maybe it is simpler to take, e.g., rhombi with centers at our points. It remains to prove that the union of the curves (or rhombi) is a deformation retract of the plane with s punctures. Clearly, it makes little sense to write down explicit formulas, although this is possible. Consider an individual rhombus R and its center c. The central projection maps $R \smallsetminus c$ to the boundary of R, and there is a rectilinear homotopy between the projection and the identical map of $R \smallsetminus c$. It remains to show that the part of the plane lying outside the union of the rhombi also admits a deformation retraction to the union of their boundaries. What can we do in order to make the argument look more like a proof? First consider the polygon P whose vertices are the vertices of the rhombi opposite to the origin. We easily see that P is a strong deformation retract of the plane (as well as the disk is). It remains to show that the union of the rhombi is a deformation retract of P, which is obvious, is not it?

38.I We subdivide the square into four parts by two midlines and consider the set K formed by the contour, the midlines, and the two quarters of the square containing one of the diagonals. Show that each of the following sets is a deformation retract of K: the union of the contour and the mentioned diagonal of the square; the union of the contours of the "empty" quarters of this square.

38.J 1) None of these spaces can be embedded in another. Prove this on your own, using the following lemma. Let J_n be the union of n segments with a common endpoint. Then J_n cannot be embedded in J_k for any $n > k \geq 2$. 2) The second question is answered in the affirmative; see the proof of assertion *38.I*.

38.K Since the composition $g \circ f$ is x_0-null-homotopic, we have $g_* \circ f_* = (g \circ f)_* = \mathrm{id}_{\pi_1(X,x_0)}$. Similarly, $f_* \circ g_* = \mathrm{id}_{\pi_1(Y,y_0)}$. Thus, f_* and g_* are mutually inverse homomorphisms.

38.L Indeed, this immediately follows from Theorem *38.K*.

38.M Let $x_1 = g(x_0)$. For any homotopy h between id_X and $g \circ f$, the formula $s(t) = h(x_0, t)$ determines a path at x_0. By the answer to Riddle *36.E*, the composition $g_* \circ f_* = T_s$ is an isomorphism. Similarly, the composition $f_* \circ g_*$ is an isomorphism. Therefore, f_* and g_* are isomorphisms.

39.A If u is a loop in X such that the loop $p \circ u$ in B is null-homotopic, then by the Path Homotopy Lifting Theorem *34.C* the loop u is also null-homotopic. Thus, if $p_*([u]) = [p \circ u] = 0$, then $[u] = 0$, which precisely means that p_* is a monomorphism.

39.B No, it is not. If $p(x_0) = p(x_1) = b_0$, $x_0 \neq x_1$, and the group $\pi_1(B, b_0)$ is non-Abelian, then the subgroups $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ can easily be distinct (see *39.D*).

39.C The group $p_*(\pi_1(X, x_0))$ of the covering consists of the homotopy classes of those loops at b_0 whose covering path starting at x_0 is a loop.

39.D Let s be a path in X joining x_0 and x_1. Denote by α the class of the loop $p \circ s$ and consider the inner automorphism $\varphi : \pi_1(B, b_0) \to \pi_1(B, b_0) : \beta \mapsto \alpha^{-1}\beta\alpha$. We prove that the following diagram is commutative:

$$\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{T_s} & \pi_1(X, x_1) \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\
\pi_1(B, b_0) & \xrightarrow{\varphi} & \pi_1(B, b_0).
\end{array}$$

Indeed, since $T_s([u]) = [s^{-1}us]$, we have

$$p_*\big(T_s([u])\big) = [p \circ (s^{-1}us)] = [(p \circ s^{-1})(p \circ u)(p \circ s)] = \alpha^{-1}p_*\big([u]\big)\alpha.$$

Since the diagram is commutative and T_s is an isomorphism, it follows that

$$p_*(\pi_1(X, x_1)) = \varphi(p_*(\pi_1(X, x_0))) = \alpha^{-1}p_*(\pi_1(X, x_0))\alpha,$$

thus, the groups $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ are conjugate.

39.E Let s be a loop in X representing the class $\alpha \in \pi_1(B, b_0)$. Let the path $\widetilde{s}$ cover s and start at x_0. If we put $x_1 = \widetilde{s}(1)$, then, as it follows from the proof of assertion *39.D*, we have $p_*(\pi_1(X, x_1)) = \alpha^{-1}p_*(\pi_1(X, x_0))\alpha$.

39.F This follows from *39.D* and *39.E*.

39.G See *39.H*.

39.H For brevity, put $H = p_*(\pi_1(X, x_0))$. Consider an arbitrary point $x_1 \in p^{-1}(b_0)$; let s be the path starting at x_0 and ending at x_1, and $\alpha = [p \circ s]$. Take x_1 to the right coset $H\alpha \subset \pi_1(B, b_0)$. Let us verify that this definition is correct. Let s_1 be another path from x_0 to x_1, $\alpha_1 = [p \circ s_1]$. The path ss_1^{-1} is a loop, so that $\alpha\alpha_1^{-1} \in H$, whence $H\alpha = H\alpha_1$. Now we prove that the described correspondence is a surjection. Let $H\alpha$ be a coset. Consider a loop u representing the class α, let $\widetilde{u}$ be the path covering u and starting at x_0, and $x_1 = \tilde{u}(1) \in p^{-1}(b_0)$. By construction, x_1 is taken to the coset $H\alpha$, therefore, the above correspondence is surjective. Finally, let us prove that it is injective. Let $x_1, x_2 \in p^{-1}(b_0)$, and let s_1 and s_2 be two paths joining x_0 with x_1 and x_2, respectively; let $\alpha_i = [p \circ s_i]$, $i = 1, 2$. Assume that $H\alpha_1 = H\alpha_2$ and show that then $x_1 = x_2$. Consider a loop $u = (p \circ s_1)(p \circ s_2^{-1})$ and the path $\widetilde{u}$ covering u, which is a loop because $\alpha_1\alpha_2^{-1} \in H$. It remains to observe that the paths s'_1 and s'_2, where $s'_1(t) = u\big(\frac{t}{2}\big)$ and $s'_2(t) = u\big(1 - \frac{t}{2}\big)$, start at x_0 and cover the paths $p \circ s_1$ and $p \circ s_2$, respectively. Therefore, $s_1 = s'_1$ and $s_2 = s'_2$, thus,

$$x_1 = s_1(1) = s'_1(1) = \widetilde{u}\big(\tfrac{1}{2}\big) = s'_2(1) = s_2(1) = x_2.$$

39.I Consider an arbitrary point $y \in Y$, let $b = q(y)$, and let U_b be a neighborhood of b that is trivially covered for both p and q. Further, let V be the sheet over U_b containing y, and let $\{W_\alpha\}$ be the collection of sheets over U_b the union of which is $\varphi^{-1}(V)$. Clearly, the map $\varphi|_{W_\alpha} = (q|_V)^{-1} \circ p|_{W_\alpha}$ is a homeomorphism.

39.J Let p and q be two coverings. Consider an arbitrary point $x \in X$ and a path s joining the marked point x_0 with x. Let $u = p \circ s$. By assertion *34.B*, there exists a unique path $\widetilde{u} : I \to Y$ covering u and starting at y_0. Therefore, $\widetilde{u} = \varphi \circ s$, consequently, the point $\varphi(x) = \varphi(s(1)) = \widetilde{u}(1)$ is uniquely determined.

39.K Let $\varphi : X \to Y$ and $\psi : Y \to X$ be subordinations, and let $\varphi(x_0) = y_0$ and $\psi(y_0) = x_0$. Clearly, the composition $\psi\circ\varphi$ is a subordination of the covering $p : X \to B$ to itself. Consequently, by the uniqueness of a subordination (see *39.J*), we have $\psi \circ \varphi = \mathrm{id}_X$. Similarly, $\varphi \circ \psi = \mathrm{id}_Y$, which precisely means that the subordinations φ and ψ are mutually inverse equivalences.

39.L This relation is obviously symmetric, reflexive, and transitive.

39.M It is clear that if two coverings p and p' are equivalent and q is subordinate to p, then q is also subordinate to p', therefore, the subordination relation is transferred from coverings to their equivalence classes. This relation is obviously reflexive and transitive, and it is proved in *39.K* that two coverings subordinate to each other are equivalent, therefore this relationb is antisymmetric.

39.N Since $p_* = (q \circ \varphi)_* = q_* \circ \varphi_*$, we have

$$p_*(\pi_1(X, x_0)) = q_*(\varphi_*(\pi_1(X, x_0))) \subset q_*(\pi_1(Y, y_0)).$$

39.Ax.1 Denote by $\widetilde{u}, \widetilde{v} : I \to Y$ the paths starting at y_0 and covering the paths $p \circ u$ and $p \circ v$, respectively. Consider the path uv^{-1}, which is a loop at x_0 by assumption, the loop $(p \circ u)(p \circ v)^{-1} = p \circ (uv^{-1})$, and its class $\alpha \in p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. Thus, $\alpha \in q_*(\pi_1(Y, y_0))$, therefore, the path starting at y_0 and covering the loop $(p \circ u)(p \circ v)^{-1}$ is also a loop. Consequently, the paths covering $p \circ u$ and $p \circ v$ and starting at y_0 end at one and the same point. It remains to observe that they are the paths $\widetilde{u}$ and $\widetilde{v}$.

39.Ax.2 We define the map $\varphi : X \to Y$ as follows. Let $x \in X$, u – a path joining x_0 and x. Then $\varphi(x) = y$, where y is the endpoint of the path $\widetilde{u} : I \to Y$ covering the path $p \circ u$. By assertion *39.Ax.1*, the map φ is well defined. We prove that $\varphi : X \to Y$ is continuous. Let $x_1 \in X$, $b_1 = p(x_1)$ and $y_1 = \varphi(x_1)$; by construction, we have $q(y_1) = b_1$. Consider an arbitrary neighborhood V of y_1. We can assume that V is a sheet over a trivially covered path-connected neighborhood U of b_1. Let W be the sheet over U containing x_1, thus, the neighborhood W is also path-connected. Consider an arbitrary point $x \in W$. Let a path $v : I \to W$ join x_1 and x. It is clear that the image of the path $\tilde{v}$ starting at y_1 and covering the path $p \circ v$ is contained in the neighborhood V, whence $\varphi(x) \in V$. Thus, $\varphi(W) \subset V$, consequently, φ is continuous at x.

39.Bx This follows from *39.E*, *39.Ax*, and *39.K*.

39.Cx Let $X \to B$ be a universal covering, U a trivially covered neighborhood of a point $a \in B$, and V one of the “sheets” over U. Then the inclusion $i : U \to B$ is the composition $p \circ j \circ (p|_V)^{-1}$, where j is the inclusion $V \to X$. Since the group $\pi_1(X)$ is trivial, the inclusion homomorphism $i_* : \pi_1(U, a) \to \pi_1(B, a)$ is also trivial.

39.Dx.1 Let two paths u_1 and u_2 join b_0 and b. The paths covering them and starting at x_0 end at one and the same point x iff the class of the loop $u_1 u_2^{-1}$ lies in the subgroup π.

39.Dx.2 Yes, it does. Consider the set of all paths in B starting at b_0, endow it with the following equivalence relation: $u_1 \sim u_2$ if $[u_1 u_2^{-1}] \in \pi$, and let $\widetilde{X}$ be the quotient set by this relation. A natural bijection between X and $\tilde{X}$ is constructed as follows. For each point $x \in X$, we consider a path u joining the marked point x_0 with of a point x. The class of the path $p \circ u$ in $\widetilde{X}$ is the image of x. The described correspondence is obviously a bijection $f : X \to \widetilde{X}$. The map $g : \widetilde{X} \to X$ inverse to f has the following structure. Let $u : I \to B$ represent a class $y \in \tilde{X}$. Consider the path $v : I \to X$ covering u and starting at x_0. Then $g(y) = v(1)$.

39.Dx.3 We define a base for the topology in $\widetilde{X}$. For each pair (U, x), where U is an open set in B and $x \in \widetilde{X}$, the set U_x consists of the classes of all possible paths uv, where u is a path in the class x, and v is a path in U starting at $u(1)$. It is not difficult to prove that for each point $y \in U_x$ we have the identity $U_y = U_x$, whence it follows that the collection of the sets of the form U_x is a base for the topology in $\widetilde{X}$. In order to prove that f and g are homeomorphisms, it is sufficient to verify that each of them maps each set in a certain base for the topology to an open set. Consider the base consisting of trivially covered neighborhoods $U \subset B$, each of which, firstly, is path-connected, and, secondly, each loop in which is null-homotopic in B.

39.Dx.4 The space $\widetilde{X}$ is defined in *39.Dx.2*. The projection $p : \widetilde{X} \to B$ is defined as follows: $p(y) = u(1)$, where u is a path in the class $y \in \widetilde{X}$. The map p is continuous without any assumptions on the properties of B. Prove that if a set U in B is open and path-connected and each loop in U is null-homotopic in B, then U is a trivially covered neighborhood.

39.Fx Consider the subgroups $\pi \subset \pi_0 \subset \pi_1(B, b_0)$ and let $p : \widetilde{Y} \to B$ and $q : \widetilde{Y} \to B$ be the coverings constructed by π and π_0, respectively. The construction of the covering implies that there exists a map $f : \widetilde{X} \to \widetilde{Y}$. Show that f is the required subordination.

39.Gx We say that the group G acts from the right on a set F if each element $\alpha \in G$ determines a map $\varphi_\alpha : F \to F$ so that: 1) $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta$; 2)

if e is the unity of the group G, then $\varphi_e = \mathrm{id}_F$. Put $F = p^{-1}(b_0)$. For each $\alpha \in \pi_1(B, b_0)$, we define a map $\varphi_\alpha : F \to F$ as follows. Let $x \in F$. Consider a loop u at b_0, such that $[u] = \alpha$. Let the path $\widetilde{u}$ cover u and start at x. Put $\varphi_\alpha(x) = \widetilde{u}(1)$.
The Path Homotopy Lifting Theorem implies that the map φ_α depends only on the homotopy class of u, therefore, the definition is correct. If $[u] = e$, i.e., the loop u is null-homotopic, then the path $\widetilde{u}$ is also a loop, whence $\widetilde{u}(1) = x$, thus, $\varphi_e = \mathrm{id}_F$. Verify that the first property in the definition of an action of a group on a set is also fulfilled.

***39.H*x** See *39.P*x.

***39.I*x** The group operation in the set of all automorphisms is their composition.

***39.J*x** This follows from *39.J*.

***39.K*x** Show that the map transposing the two points in the preimage of each point in the base, is a homeomorphism.

***39.L*x** This is assertion *39.H*.

***39.Q*x** This follows from *39.N*x and *39.P*x.