Chapter IX

Cellular Techniques

40. Cellular Spaces

40°1. Definition of Cellular Spaces

In this section, we study a class of topological spaces that play a very important role in algebraic topology. Their role in the context of this book is more restricted: this is the class of spaces for which we learn how to calculate the fundamental group. [1]

A *zero-dimensional cellular space* is just a discrete space. Points of a 0-dimensional cellular space are also called (*zero-dimensional*) *cells*, or 0-*cells*.

A *one-dimensional cellular space* is a space that can be obtained as follows. Take any 0-dimensional cellular space X_0. Take a family of maps $\varphi_\alpha : S^0 \to X_0$. Attach to X_0 via φ_α the sum of a family of copies of D^1 (indexed by the same indices α as the maps φ_α):

$$X_0 \cup_{\sqcup \varphi_\alpha} \left(\bigsqcup_\alpha D^1 \right).$$

[1]This class of spaces was introduced by J. H. C. Whitehead. He called these spaces *CW-complexes*, and they are known under this name. However, it is not a good name for plenty of reasons. With very rare exceptions (one of which is *CW*-complex, the other is simplicial complex), the word *complex* is used nowadays for various algebraic notions, but not for spaces. We have decided to use the term *cellular space* instead of *CW-complex,* following D. B. Fuchs and V. A. Rokhlin [**6**].

The images of the interior parts of copies of D^1 are called (*open*) 1-*dimensional cells*, 1-*cells*, *one-cells*, or *edges*. The subsets obtained from D^1 are *closed* 1-*cells*. The cells of X_0 (i.e., points of X_0) are also called *vertices*. Open 1-cells and 0-cells constitute a partition of a one-dimensional cellular space. This partition is included in the notion of cellular space, i.e., a one-dimensional cellular space is a topological space equipped with a partition that can be obtained in this way. [2]

A *two-dimensional cellular space* is a space that can be obtained as follows. Take any cellular space X_1 of dimension 0 or 1. Take a family of continuous[3] maps $\varphi_\alpha : S^1 \to X_1$. Attach the sum of a family of copies of D^2 to X_1 via φ_α:

$$X_1 \cup_{\sqcup\varphi_\alpha} \Big(\bigsqcup_\alpha D^2\Big).$$

The images of the interior parts of copies of D^2 are (*open*) 2-*dimensional cells*, 2-*cells*, *two-cells*, or *faces*. The cells of X_1 are also regarded as cells of the 2-dimensional cellular space. Open cells of both kinds constitute a partition of a 2-dimensional cellular space. This partition is included in the notion of cellular space, i.e., a two-dimensional cellular space is a topological space equipped with a partition that can be obtained in the way described above. The set obtained out of a copy of the whole D^2 is a *closed* 2-*cell*.

A *cellular space of dimension* n is defined in a similar way: This is a space equipped with a partition. It is obtained from a cellular space X_{n-1} of dimension less than n by attaching a family of copies of the n-disk D^n via by a family of continuous maps of their boundary spheres:

$$X_{n-1} \cup_{\sqcup\varphi_\alpha} \Big(\bigsqcup_\alpha D^n\Big).$$

[2]One-dimensional cellular spaces are also associated with the word *graph*. However, rather often this word is used for objects of other classes. For example, one can call in this way one-dimensional cellular spaces in which attaching maps of different one-cells are not allowed to coincide, or the boundary of a one-cell is prohibited to consist of a single vertex. When one-dimensional cellular spaces are to be considered anyway, despite of this terminological disregard, they are called *multigraphs* or *pseudographs*. Furthermore, sometimes one includes into the notion of graph an additional structure. Say, a choice of orientation on each edge. Certainly, all these variations contradict a general tendency in mathematical terminology to call in a simpler way decent objects of a more general nature, passing to more complicated terms along with adding structures and imposing restrictions. However, in this specific situation there is no hope to implement that tendency. Any attempt to fix a meaning for the word *graph* apparently only contributes to this chaos, and we just keep this word away from important formulations, using it as a short informal synonym for more formal term of one-dimensional cellular space. (Other overused common words, like *curve* and *surface*, also deserve this sort of caution.)

[3]In the above definition of a 1-dimensional cellular space, the attaching maps φ_α also were continuous, although their continuity was not required since any map of S^0 to any space is continuous.

The images of the interiors of the attached n-dosks are (*open*) *n-dimensional cells* or simply *n-cells*. The images of the entire n-disks are *closed n-cells*. Cells of X_{n-1} are also regarded as cells of the n-dimensional cellular space. The mappings φ_α are the *attaching maps*, and the restrictions of the factorization map to the n-disks D^n are the *characteristic maps*.

A *cellular space* is obtained as a union of increasing sequence of cellular spaces $X_0 \subset X_1 \subset \dots \subset X_n \subset \dots$ obtained in this way from each other. The sequence may be finite or infinite. In the latter case, the topological structure is introduced by saying that the cover of the union by X_n's is fundamental, i.e., a set $U \subset \bigcup_{n=0}^{\infty} X_n$ is open iff its intersection $U \cap X_n$ with each X_n is open in X_n.

The partition of a cellular space into its open cells is a *cellular decomposition*. The union of all cells of dimension less than or equal to n of a cellular space X is the *n-dimensional skeleton* of X. This term may be misleading since the n-dimensional skeleton may contain no n-cells, and so it may coincide with the $(n-1)$-dimensional skeleton. Thus, the n-dimensional skeleton may have dimension less than n. For this reason, it is better to speak about the *nth skeleton* or *n-skeleton*.

40.1. In a cellular space, skeletons are closed.

A cellular space is *finite* if it contains a finite number of cells. A cellular space is *countable* if it contains a countable number of cells. A cellular space is *locally finite* if each of its points has a neighborhood intersecting finitely many cells.

Let X be a cellular space. A subspace $A \subset X$ is a *cellular subspace* of X if A is a union of open cells and together with each cell e contains the closed cell $\bar{e}$. This definition admits various equivalent reformulations. For instance, $A \subset X$ is a *cellular subspace* of X iff A is both a union of closed cells and a union of open cells. Another option: together with each point $x \in A$ the subspace A contains the closed cell $\overline{e} \in x$. Certainly, A is equipped with a partition into the open cells of X contained in A. Obviously, the k-skeleton of a cellular space X is a cellular subspace of X.

40.2. Prove that the union and intersection of any collection of cellular subspaces are cellular subspaces.

40.A. Prove that a cellular subspace of a cellular space is a cellular space. (Probably, your proof will involve assertion *40.Gx*.)

40.A.1. Let X be a topological space, and let $X_1 \subset X_2 \subset \dots$ be an increasing sequence of subsets constituting a fundamental cover of X. Let $A \subset X$ be a subspace, put $A_i = A \cap X_i$. Let one of the following conditions be fulfilled:
1) X_i are open in X;
2) A_i are open in X;

3) A_i are closed in X.
Then $\{A_i\}$ is a fundamental cover of A.

40°2. First Examples

40.B. A cellular space consisting of two cells, one of which is a 0-cell and the other one is an n-cell, is homeomorphic to S^n.

40.C. Represent D^n with $n > 0$ as a cellular space made of three cells.

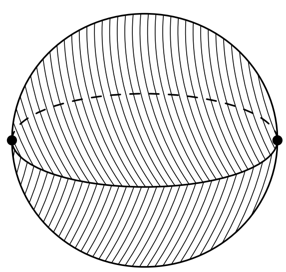
40.D. A cellular space consisting of a single 0-cell and q one-cells is a bouquet of q circles.

40.E. Represent torus $S^1 \times S^1$ as a cellular space with one 0-cell, two 1-cells, and one 2-cell.

40.F. How to obtain a presentation of torus $S^1 \times S^1$ as a cellular space with 4 cells from a presentation of S^1 as a cellular space with 2 cells?

> ***40.3.*** Prove that if X and Y are finite cellular spaces, then $X \times Y$ has a natural structure of a finite cellular space.
>
> ***40.4*.*** Does the statement of *40.3* remain true if we skip the finiteness condition in it? If yes, prove this; if no, find an example where the product is not a cellular space.

40.G. Represent sphere S^n as a cellular space such that spheres $S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^{n-1}$ are its skeletons.



40.H. Represent $\mathbb{R}P^n$ as a cellular space with $n + 1$ cells. Describe the attaching maps of the cells.

> ***40.5.*** Represent $\mathbb{C}P^n$ as a cellular space with $n+1$ cells. Describe the attaching maps of its cells.
>
> ***40.6.*** Represent the following topological spaces as cellular ones
>
> (a) handle; (b) Möbius strip; (c) $S^1 \times I$,
> (d) sphere with p handles; (e) sphere with p crosscaps.
>
> ***40.7.*** What is the minimal number of cells in a cellular space homeomorphic to
>
> (a) Möbius strip; (b) sphere with p handles; (c) sphere with p crosscaps?

40.8. Find a cellular space where the closure of a cell is not equal to a union of other cells. What is the minimal number of cells in a space containing a cell of this sort?

40.9. Consider the disjoint sum of a countable collection of copies of closed interval I and identify the copies of 0 in all of them. Represent the result (which is the bouquet of the countable family of intervals) as a countable cellular space. Prove that this space is not first countable.

40.I. Represent $\mathbb{R}^1$ as a cellular space.

40.10. Prove that for any two cellular spaces homeomorphic to $\mathbb{R}^1$ there exists a homeomorphism between them homeomorphically mapping each cell of one of them onto a cell of the other one.

40.J. Represent $\mathbb{R}^n$ as a cellular space.

Denote by $\mathbb{R}^\infty$ the union of the sequence of Euclidean spaces $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \dots \subset \mathbb{R}^n \subset$ canonically included to each other: $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Equip $\mathbb{R}^\infty$ with the topological structure for which the spaces $\mathbb{R}^n$ constitute a fundamental cover.

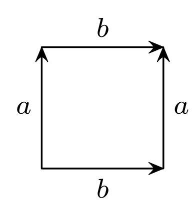
40.K. Represent $\mathbb{R}^\infty$ as a cellular space.

40.11. Show that $\mathbb{R}^\infty$ is not metrizable.

40°3. Further Two-Dimensional Examples

Let us consider a class of 2-dimensional cellular spaces that admit a simple combinatorial description. Each space in this class is a quotient space of a finite family of convex polygons by identification of sides via affine homeomorphisms. The identification of vertices is determined by the identification of the sides. The quotient space has a natural decomposition into 0-cells, which are the images of vertices, 1-cells, which are the images of sides, and faces, the images of the interior parts of the polygons.

To describe such a space, we need, first, to show, what sides are identified. Usually this is indicated by writing the same letters at the sides to be identified. There are only two affine homeomorphisms between two closed intervals. To specify one of them, it suffices to show the orientations of the intervals that are identified by the homeomorphism. Usually this is done by drawing arrows on the sides. Here is a description of this sort for the standard presentation of torus $S^1 \times S^1$ as the quotient space of square:



We can replace a picture by a combinatorial description. To do this, put letters on *all* sides of polygon, go around the polygons counterclockwise and write down the letters that stay at the sides of polygon along the contour. The letters corresponding to the sides whose orientation is opposite to the counterclockwise direction are put with exponent -1. This yields a collection of words, which contains sufficient information about the family of polygons and the partition. For instance, the presentation of the torus shown above is encoded by the word $ab^{-1}a^{-1}b$.

40.12. Prove that:

(1) the word $a^{-1}a$ describes a cellular space homeomorphic to S^2,
(2) the word aa describes a cellular space homeomorphic to $\mathbb{R}P^2$,
(3) the word $aba^{-1}b^{-1}c$ describes a handle,
(4) the word $abcb^{-1}$ describes cylinder $S^1 \times I$,
(5) each of the words aab and $abac$ describe Möbius strip,
(6) the word $abab$ describes a cellular space homeomorphic to $\mathbb{R}P^2$,
(7) each of the words $aabb$ and $ab^{-1}ab$ describe Klein bottle,
(8) the word

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}.$$

describes sphere with g handles,
(9) the word $a_1a_1a_2a_2\dots a_ga_g$ describes sphere with g crosscaps.

40°4. Embedding to Euclidean Space

40.L. Any countable 0-dimensional cellular space can be embedded into $\mathbb{R}$.

40.M. Any countable locally finite 1-dimensional cellular space can be embedded into $\mathbb{R}^3$.

40.13. Find a 1-dimensional cellular space which you cannot embed into $\mathbb{R}^2$. (We do not ask you to prove rigorously that no embedding is possible.)

40.N. *Any finite dimensional countable locally finite cellular space can be embedded into Euclidean space of sufficiently high dimension.*

40.N.1. Let X and Y be topological spaces such that X can be embedded into $\mathbb{R}^p$ and Y can be embedded into $\mathbb{R}^q$, and both embeddings are proper maps (see 18°3x; in particular, their images are closed in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively). Let A be a closed subset of Y. Assume that A has a neighborhood U in Y such that there exists a homeomorphism $h : \operatorname{Cl} U \to A \times I$ mapping A to $A \times 0$. Let $\varphi : A \to X$ be a proper continuous map. Then the initial embedding $X \to \mathbb{R}^p$ extends to an embedding $X \cup_\varphi Y \to \mathbb{R}^{p+q+1}$.

40.N.2. Let X be a locally finite countable k-dimensional cellular space and A be the $(k-1)$-skeleton of X. Prove that if A can be embedded to $\mathbb{R}^p$, then X can be embedded into $\mathbb{R}^{p+k+1}$.

40.O. Any countable locally finite cellular space can be embedded into $\mathbb{R}^\infty$.

40.P. Any finite cellular space is metrizable.

40.Q. Any finite cellular space is normal.

40.R. Any countable cellular space can be embedded into $\mathbb{R}^\infty$.

40.S. Any cellular space is normal.

40.T. Any locally finite cellular space is metrizable.

40°5x. Simplicial Spaces

Recall that in 23°3x we introduced a class of topological spaces: simplicial spaces. Each simplicial space is equipped with a partition into subsets, called open simplices, which are indeed homeomorphic to open simplices of Euclidean space.

40.Ax. Any simplicial space is cellular, and its partition into open simplices is the corresponding partition into open cells.

40°6x. Topological Properties of Cellular Spaces

The present section contains assertions of mixed character. For example, we study conditions ensuring that a cellular space is compact (*40.Kx*) or separable (*40.Ox*). We also prove that a cellular space X is connected, iff X is path-connected (*40.Sx*), iff the 1-skeleton of X is path-connected (*40.Vx*). On the other hand, we study the cellular topological structure as such. For example, any cellular space is Hausdorff (*40.Bx*). Further, is not obvious at all from the definition of a cellular space that a closed cell is the closure of the corresponding open cell (or that closed cells are closed at all). In this connection, the present section includes assertions of technical character. (We do not formulate them as lemmas to individual theorems because often they are lemmas for several assertions.) For example: closed cells constitute a fundamental cover of a cellular space (*40.Dx*).

We notice that, say, in the textbook [FR], a cellular space is defined as a Hausdorff topological space equipped by a cellular partition with two properties:
(C) each closed cell intersects only a finite number of (open) cells;
(W) closed cells constitute a fundamental cover of the space. The results of assertions *40.Bx*, *40.Cx*, and *40.Fx* imply that cellular spaces in the sense of the above definition are cellular spaces in the sense of Rokhlin–Fuchs' textbook (i.e., in the standard sense), the possibility of inductive construction for which is proved in [RF]. Thus, both definitions of a cellular space are equivalent.

An advice to the reader: first try to prove the above assertions for finite cellular spaces.

40.Bx. Each cellular space is a Hausdorff topological space.

40.Cx. In a cellular space, the closure of any cell e is the closed cell $\overline{e}$.

40.Dx. Closed cells constitute a fundamental cover of a cellular space.

40.Ex. *Each cover of a cellular space by cellular subspaces is fundamental.*

40.Fx. In a cellular space, any closed cell intersects only a finite number of open cells.

40.Gx. If A is cellular subspace of a cellular space X, then A is closed in X.

40.Hx. The space obtained as a result of pasting two cellular subspaces together along their common subspace, is cellular.

40.Ix. If a subset A of a cellular space X intersects each open cell along a finite set, then A is closed. Furthermore, the induced topology on A is discrete.

40.Jx. Prove that any compact subset of a cellular space intersects a finite number of cells.

40.Kx Corollary. *A cellular space is compact iff it is finite.*

40.Lx. Any cell of a cellular space is contained in a finite cellular subspace of this space.

40.Mx. Any compact subset of a cellular space is contained in a finite cellular subspace.

40.Nx. *A subset of a cellular space is compact iff it is closed and intersects only a finite number of open cells.*

40.Ox. A cellular space is separable iff it is countable.

40.Px. Any path-connected component of a cellular space is a cellular subspace.

40.Qx. A cellular space is locally path-connected.

40.Rx. Any path-connected component of a cellular space is both open and closed. It is a connected component.

40.Sx. A cellular space is connected iff it is path connected.

40.Tx. A locally finite cellular space is countable iff it has countable 0-skeleton.

40.Ux. Any connected locally finite cellular space is countable.

40.Vx. *A cellular space is connected iff its* 1*-skeleton is connected.*

41. Cellular Constructions

41°1. Euler Characteristic

Let X be a finite cellular space. Let $c_i(X)$ denote the number of its cells of dimension i. The *Euler characteristic* of X is the alternating sum of $c_i(X)$:

$$\chi(X) = c_0(X) - c_1(X) + c_2(X) - \dots + (-1)^i c_i(X) + \dots$$

41.A. Prove that Euler characteristic is additive in the following sense: for any cellular space X and its finite cellular subspaces A and B we have

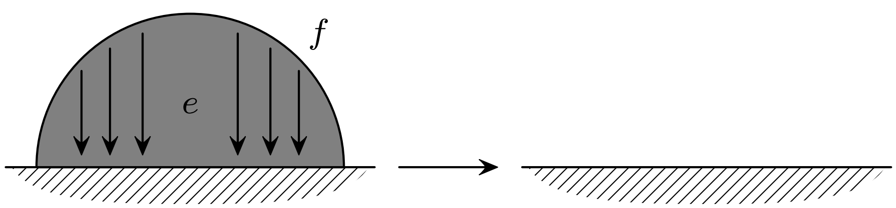
$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

41.B. Prove that Euler characteristic is multiplicative in the following sense: for any finite cellular spaces X and Y the Euler characteristic of their product $X \times Y$ is $\chi(X)\chi(Y)$.

41°2. Collapse and Generalized Collapse

Let X be a cellular space, e and f its open cells of dimensions n and $n-1$, respectively. Suppose:

- the attaching map $\varphi_e : S^{n-1} \to X_{n-1}$ of e determines a homeomorphism of the open upper hemisphere S^{n-1}_+ onto f,
- f does not meet images of attaching maps of cells, distinct from e,
- the cell e is disjoint from the image of attaching map of any cell.



41.C. $X \smallsetminus (e \cup f)$ is a cellular subspace of X.

41.D. *$X \smallsetminus (e \cup f)$ is a deformation retract of X.*

We say that $X \smallsetminus (e \cup f)$ is obtained from X by an *elementary collapse*, and we write $X \searrow X \smallsetminus (e \cup f)$.

If a cellular subspace A of a cellular space X is obtained from X by a sequence of elementary collapses, then we say that X *is collapsed* onto A and also write $X \searrow A$.

41.E. *Collapsing does not change the Euler characteristic: if X is a finite cellular space and $X \searrow A$, then $\chi(A) = \chi(X)$.*

As above, let X be a cellular space, let e and f be its open cells of dimensions n and $n-1$, respectively, and let the attaching map $\varphi_e : S^n \to X_{n-1}$ of e determine a homeomorphism S^{n-1}_+ on f. Unlike the preceding situation, here we assume neither that f is disjoint from the images of attaching maps of cells different from e, nor that e is disjoint from the images of attaching maps of whatever cells. Let $\chi_e : D^n \to X$ be a characteristic map of e. Furthermore, let $\psi : D^n \to S^{n-1} \smallsetminus \varphi_e^{-1}(f) = S^{n-1} \smallsetminus S^{n-1}_+$ be a deformation retraction.

41.F. Under these conditions, the quotient space $X/[\chi_e(x) \sim \varphi_e(\psi(x))]$ of X is a cellular space where the cells are the images under the natural projections of all cells of X except e and f.

Cellular space $X/[\chi_e(x) \sim \varphi_e(\psi(x))]$ is said to be obtained by *cancellation of cells* e and f.

41.G. *The projection* $X \to X/[\chi_e(x) \sim \varphi_e(\psi(x))]$ *is a homotopy equivalence.*

41.G.1. Find a cellular subspace Y of a cellular space X such that the projection $Y \to Y/[\chi_e(x) \sim \varphi_e(\psi(x))]$ would be a homotopy equivalence by Theorem *41.D*.

41.G.2. Extend the map $Y \to Y \smallsetminus (e \cup f)$ to a map $X \to X'$, which is a homotopy equivalence by *41.6*x.

41°3x. Homotopy Equivalences of Cellular Spaces

***41.1*x.** Let $X = A \cup_\varphi D^n$ be the space obtained by attaching an n-disk to a topological space A via a continuous map $\varphi : S^{n-1} \to A$. Prove that the complement $X \smallsetminus x$ of any point $x \in X \smallsetminus A$ admits a (strong) deformation retraction to A.

***41.2*x.** Let X be an n-dimensional cellular space, and let K be a set intersecting each of the open n-cells of X at a single point. Prove that the $(n-1)$-skeleton X_{n-1} of X is a deformation retract of $X \smallsetminus K$.

***41.3*x.** Prove that the complement $\mathbb{R}P^n \smallsetminus$point is homotopy equivalent to $\mathbb{R}P^{n-1}$; the complement $\mathbb{C}P^n \smallsetminus$ point is homotopy equivalent to $\mathbb{C}P^{n-1}$.

***41.4*x.** Prove that the punctured solid torus $D^2 \times S^1 \smallsetminus$ point, where point is an arbitrary interior point, is homotopy equivalent to a torus with a disk attached along the meridian $S^1 \times 1$.

***41.5*x.** Let A be cellular space of dimension n, let $\varphi : S^n \to A$ and $\psi : S^n \to A$ be continuous maps. Prove that if φ and ψ are homotopic, then the spaces $X_\varphi = A \cup_\varphi D^{n+1}$ and $X_\psi = A \cup_\psi D^{n+1}$ are homotopy equivalent.

Below we need a more general fact.

***41.6*x.** Let $f : X \to Y$ be a homotopy equivalence, $\varphi : S^{n-1} \to X$ and $\varphi' : S^{n-1} \to Y$ continuous maps. Prove that if $f \circ \varphi \sim \varphi'$, then $X \cup_\varphi D^n \simeq Y \cup_{\varphi'} D^n$.

41.7x. Let X be a space obtained from a circle by attaching of two copies of disk by maps $S^1 \to S^1 : z \mapsto z^2$ and $S^1 \to S^1 : z \mapsto z^3$, respectively. Find a cellular space homotopy equivalent to X with smallest possible number of cells.

41.8x. Riddle. Generalize the result of Problem *41.7x*.

41.9x. Prove that if we attach a disk to the torus $S^1 \times S^1$ along the parallel $S^1 \times 1$, then the space K obtained is homotopy equivalent to the bouquet $S^2 \vee S^1$.

41.10x. Prove that the torus $S^1 \times S^1$ with two disks attached along the meridian $\{1\} \times S^1$ and parallel $S^1 \times 1$, respectively, is homotopy equivalent to S^2.

41.11x. Consider three circles in $\mathbb{R}^3$: $S_1 = \{x^2 + y^2 = 1,\ z = 0\}$, $S_2 = \{x^2 + y^2 = 1,\ z = 1\}$, and $S_3 = \{z^2 + (y-1)^2 = 1,\ x = 0\}$. Since $\mathbb{R}^3 \cong S^3 \smallsetminus$ point, we can assume that S_1, S_2, and S_3 lie in S^3. Prove that the space $X = S^3 \smallsetminus (S_1 \cup S_2)$ is not homotopy equivalent to the space $Y = S^3 \smallsetminus (S_1 \cup S_3)$.

41.Ax. Let X be a cellular space, $A \subset X$ a cellular subspace. Then the union $(X \times 0) \cup (A \times I)$ is a retract of the cylinder $X \times I$.

41.Bx. Let X be a cellular space, $A \subset X$ a cellular subspace. Assume that we are given a map $F : X \to Y$ and a homotopy $h : A \times I \to Y$ of the restriction $f = F|_A$. Then the homotopy h extends to a homotopy $H : X \times I \to Y$ of F.

41.Cx. Let X be a cellular space, $A \subset X$ a contractible cellular subspace. Then the projection $\mathrm{pr} : X \to X/_A$ is a homotopy equivalence.

Problem *41.Cx* implies the following assertions.

41.Dx. If a cellular space X contains a closed 1-cell e homeomorphic to I, then X is homotopy equivalent to the cellular space $X/_e$ obtained by contraction of e.

41.Ex. *Any connected cellular space is homotopy equivalent to a cellular space with one-point* 0*-skeleton.*

41.Fx. *A simply connected finite* 2*-dimensional cellular space is homotopy equivalent to a cellular space with one-point* 1*-skeleton.*

41.12x. Solve Problem *41.9x* with the help of Theorem *41.Cx*.

41.13x. Prove that the quotient space

$$\mathbb{C}P^2/_{[(z_0 : z_1 : z_2) \sim (\overline{z_0} : \overline{z_1} : \overline{z_2})]}$$

of the complex projective plane $\mathbb{C}P^2$ is homotopy equivalent to S^4.

Information. We have $\mathbb{C}P^2/_{[z \sim \tau(z)]} \cong S^4$.

41.Gx. Let X be a cellular space, and let A be a cellular subspace of X such that the inclusion $\mathrm{in} : A \to X$ is a homotopy equivalence. Then A is a deformation retract of X.

42. One-Dimensional Cellular Spaces

42°1. Homotopy Classification

42.A. *Any connected finite* 1*-dimensional cellular space is homotopy equivalent to a bouquet of circles.*

> ***42.A.1 Lemma.*** Let X be a 1-dimensional cellular space, e a 1-cell of X attached by an injective map $S^0 \to X_0$ (i.e., e has two distinct endpoints). Prove that the projection $X \to X/_e$ is a homotopy equivalence. Describe the homotopy inverse map explicitly.

42.B. *A finite connected cellular space X of dimension one is homotopy equivalent to the bouquet of* $1 - \chi(X)$ *circles, and its fundamental group is a free group of rank* $1 - \chi(X)$.

42.C Corollary. *The Euler characteristic of a finite connected one-dimensional cellular space is invariant under homotopy equivalence. It is not greater than one. It equals one iff the space is homotopy equivalent to point.*

42.D Corollary. *The Euler characteristic of a finite one-dimensional cellular space is not greater than the number of its connected components. It is equal to this number iff each of its connected components is homotopy equivalent to a point.*

42.E Homotopy Classification of Finite 1-Dimensional Cellular Spaces. *Finite connected one-dimensional cellular spaces are homotopy equivalent, iff their fundamental groups are isomorphic, iff their Euler characteristics are equal.*

> *42.1.* The fundamental group of a 2-sphere punctured at n points is a free group of rank $n - 1$.
>
> *42.2.* Prove that the Euler characteristic of a cellular space homeomorphic to S^2 is equal to 2.
>
> ***42.3 The Euler Theorem.*** For any convex polyhedron in $\mathbb{R}^3$, the sum of the number of its vertices and the number of its faces equals the number of its edges plus two.
>
> *42.4.* Prove the Euler Theorem without using fundamental groups.
>
> *42.5.* Prove that the Euler characteristic of any cellular space homeomorphi to the torus is equal to 0.

Information. The Euler characteristic is homotopy invariant, but the usual proof of this fact involves the machinery of singular homology theory, which lies far beyond the scope of our book.

42°2. Spanning Trees

A one-dimensional cellular space is a *tree* if it is connected, while the complement of each of its (open) 1-cells is disconnected. A cellular subspace A of a cellular space X is a *spanning tree* of X if A is a tree and is not contained in any other cellular subspace $B \subset X$ which is a tree.

42.F. Any finite connected one-dimensional cellular space contains a spanning tree.

42.G. Prove that a cellular subspace A of a cellular space X is a spanning tree iff A is a tree and contains all vertices of X.

Theorem *42.G* explains the term *spanning tree*.

42.H. Prove that a cellular subspace A of a cellular space X is a spanning tree iff it is a tree and the quotient space $X/_A$ is a bouquet of circles.

42.I. Let X be a one-dimensional cellular space and A its cellular subspace. Prove that if A is a tree, then the projection $X \to X/_A$ is a homotopy equivalence.

Problems *42.F*, *42.I*, and *42.H* provide one more proof of Theorem *42.A*.

42°3x. Dividing Cells

42.Ax. *In a one-dimensional connected cellular space each connected component of the complement of an edge meets the closure of the edge. The complement has at most two connected component.*

A complete local characterization of a vertex in a one-dimensional cellular space is its *valency*. This is the total number of points in the preimages of the vertex under attaching maps of all one-cells of the space. It is more traditional to define the degree of a vertex v as the number of edges incident to v, counting with multiplicity 2 the edges that are incident only to v.

42.Bx. *1) Each connected component of the complement of a vertex in a connected one-dimensional cellular space contains an edge with boundary containing the vertex. 2) The complement of a vertex of valency m has at most m connected components.*

42°4x. Trees and Forests

A one-dimensional cellular space is a *tree* if it is connected, while the complement of each of its (open) 1-cells is disconnected. A one-dimensional cellular space is a *forest* if each of its connected components is a tree.

42.Cx. Any cellular subspace of a forest is a forest. In particular, any connected cellular subspace of a tree is a tree.

***42.D*x.** In a tree the complement of an edge consists of two connected components.

***42.E*x.** In a tree, the complement of a vertex of valency m has consists of m connected components.

***42.F*x.** A finite tree has there exists a vertex of valency one.

***42.G*x.** Any finite tree collapses to a point and has Euler characteristic one.

***42.H*x.** Prove that any point of a tree is its deformation retract.

***42.I*x.** Any finite one-dimensional cellular space that can be collapsed to a point is a tree.

***42.J*x.** In any finite one-dimensional cellular space the sum of valencies of all vertices is equal to the number of edges multiplied by two.

***42.K*x.** A finite connected one-dimensional cellular space with Euler characteristic one has a vertex of valency one.

***42.L*x.** A finite connected one-dimensional cellular space with Euler characteristic one collapses to a point.

42°5x. Simple Paths

Let X be a one-dimensional cellular space. A *simple path of length* n in X is a finite sequence $(v_1, e_1, v_2, e_2, \dots, e_n, v_{n+1})$, formed by vertices v_i and edges e_i of X such that each term appears in it only once and the boundary of every edge e_i consists of the preceding and subsequent vertices v_i and v_{i+1}. The vertex v_1 is the *initial* vertex, and v_{n+1} is the *final* one. The simple path *connects* these vertices. They are connected by a path $I \to X$, which is a topological embedding with image contained in the union of all cells involved in the simple path. The union of these cells is a cellular subspace of X. It is called a *simple broken line*.

***42.M*x.** *In a connected one-dimensional cellular space, any two vertices are connected by a simple path.*

***42.N*x Corollary.** *In a connected one-dimensional cellular space X, any two points are connected by a path $I \to X$ which is a topological embedding.*

> *42.1*x. Can a path-connected space contain two distinct points that cannot be connected by a path which is a topological embedding?
>
> *42.2*x. Can you find a Hausdorff space with this property?

***42.O*x.** *A connected one-dimensional cellular space X is a tree iff there exists no topological embedding $S^1 \to X$.*

42.Px. *In a one-dimensional cellular space X there exists a loop $S^1 \to X$ that is not null-homotopic iff there exists a topological embedding $S^1 \to X$.*

42.Qx. *A one-dimensional cellular space is a tree iff any two distinct vertices are connected in it by a unique simple path.*

42.3x. Prove that any finite tree has fixed point property.

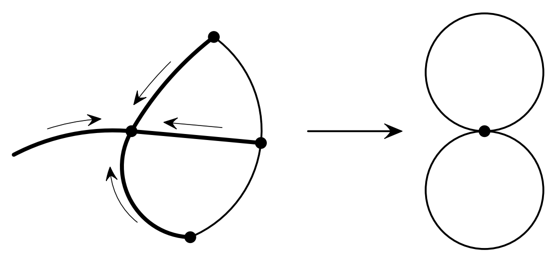
Cf. *37.12*, *37.13*, and *37.14*.

42.4x. Is this true for any tree; for any finite connected one-dimensional cellular space?

43. Fundamental Group of a Cellular Space

43°1. One-Dimensional Cellular Spaces

43.A. *The fundamental group of a connected finite one-dimensional cellular space X is a free group of rank $1-\chi(X)$.*



43.B. Let X be a finite connected one-dimensional cellular space, T a spanning tree of X, and $x_0 \in T$. For each 1-cell $e \subset X \smallsetminus T$, choose a loop s_e that starts at x_0, goes inside T to e, then goes once along e, and then returns to x_0 in T. Prove that $\pi_1(X, x_0)$ is freely generated by the homotopy classes of s_e.

43°2. Generators

43.C. Let A be a topological space, $x_0 \in A$. Let $\varphi : S^{k-1} \to A$ be a continuous map, $X = A \cup_\varphi D^k$. If $k > 1$, then the inclusion homomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. Cf. *43.G.4* and *43.G.5*.

43.D. *Let X be a cellular space, x_0 its 0-cell and X_1 the 1-skeleton of X. Then the inclusion homomorphism*

$$\pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is surjective.

43.E. Let X be a finite cellular space, T a spanning tree of X_1, and $x_0 \in T$. For each cell $e \subset X_1 \smallsetminus T$, choose a loop s_e that starts at x_0, goes inside T to e, then goes once along e, and finally returns to x_0 in T. Prove that $\pi_1(X, x_0)$ is generated by the homotopy classes of s_e.

> ***43.1.*** Deduce Theorem *31.G* from Theorem *43.D*.
>
> ***43.2.*** Find $\pi_1(\mathbb{C}P^n)$.

43°3. Relations

Let X be a cellular space, x_0 its 0-cell. Denote by X_n the n-skeleton of X. Recall that X_2 is obtained from X_1 by attaching copies of the disk

D^2 via continuous maps $\varphi_\alpha : S^1 \to X_1$. The attaching maps are circular loops in X_1. For each α, choose a path $s_\alpha : I \to X_1$ connecting $\varphi_\alpha(1)$ with x_0. Denote by N the normal subgroup of $\pi_1(X, x_0)$ generated (as a normal subgroup[4]) by the elements

$$T_{s_\alpha}[\varphi_\alpha] \in \pi_1(X_1, x_0).$$

43.F. N does not depend on the choice of the paths s_α.

43.G. *The normal subgroup N is the kernel of the inclusion homomorphism* $\text{in}_* : \pi_1(X_1, x_0) \to \pi_1(X, x_0)$.

Theorem *43.G* can be proved in various ways. For example, we can derive it from the Seifert–van Kampen Theorem (see *43.4x*). Here we prove Theorem *43.G* by constructing a "rightful" covering space. The inclusion $N \subset \operatorname{Ker} \text{in}_*$ is rather obvious (see *43.G.1*). The proof of the converse inclusion involves the existence of a covering $p : Y \to X$, whose submap over the 1-skeleton of X is a covering $p_1 : Y_1 \to X_1$ with group N, and the fact that $\operatorname{Ker} \text{in}_*$ is contained in the group of each covering over X_1 that extends to a covering over the entire X. The scheme of argument suggested in Lemmas 1–7 can also be modified. The thing is that the inclusion $X_2 \to X$ induces an isomorphism of fundamental groups. It is not difficult to prove this, but the techniques involved, though quite general and natural, nevertheless lie beyond the scope of our book. Here we just want to emphasize that this result replaces Lemmas 4 and 5.

43.G.1 Lemma 1. $N \subset \operatorname{Ker} i_*$, cf. *31.J* (3).

43.G.2 Lemma 2. Let $p_1 : Y_1 \to X_1$ be a covering with covering group N. Then for any α and a point $y \in p_1^{-1}(\varphi_\alpha(1))$ there exists a lifting $\widetilde{\varphi}_\alpha : S^1 \to Y_1$ of φ_α with $\widetilde{\varphi}_\alpha(1) = y$.

43.G.3 Lemma 3. Let Y_2 be a cellular space obtained by attaching copies of disk to Y_1 by all liftings of attaching maps φ_α. Then there exists a map $p_2 : Y_2 \to X_2$ extending p_1 which is a covering.

43.G.4 Lemma 4. Attaching maps of n-cells with $n \geq 3$ are lift to any covering space. Cf. *39.Xx* and *39.Yx*.

43.G.5 Lemma 5. Covering $p_2 : Y_2 \to X_2$ extends to a covering of the whole X.

43.G.6 Lemma 6. Any loop $s : I \to X_1$ realizing an element of $\operatorname{Ker} i_*$ (i.e., null-homotopic in X) is covered by a loop of Y. The covering loop is contained in Y_1.

43.G.7 Lemma 7. $N = \operatorname{Ker} \text{in}_*$.

---

[4]Recall that a subgroup N is *normal* if N coincides with all conjugate subgroups of N. The normal subgroup N generated by a set A is the minimal normal subgroup containing A. As a subgroup, N is generated by elements of A and elements conjugate to them. This means that each element of N is a product of elements conjugate to elements of A.

43.H. *The inclusion* $\mathrm{in}_2 : X_2 \to X$ *induces an isomorphism between the fundamental groups of a cellular space and its 2-skeleton.*

> *43.3.* Check that the covering over the cellular space X constructed in the proof of Theorem *43.G* is universal.

43°4. Writing Down Generators and Relations

Theorems *43.E* and *43.G* imply the following recipe for writing down a presentation for the fundamental group of a finite dimensional cellular space by generators and relations:

Let X be a finite cellular space, x_0 a 0-cell of X. Let T a spanning tree of the 1-skeleton of X. For each 1-cell $e \not\subset T$ of X, choose a loop s_e that starts at x_0, goes inside T to e, goes once along e, and then returns to x_0 in T. Let $g_1, \dots, g_m$ be the homotopy classes of these loops. Let $\varphi_1, \dots, \varphi_n : S^1 \to X_1$ be the attaching maps of 2-cells of X. For each φ_i choose a path s_i connecting $\varphi_i(1)$ with x_0 in the 1-skeleton of X. Express the homotopy class of the loop $s_i^{-1}\varphi_i s_i$ as a product of powers of generators g_j. Let $r_1, \dots, r_n$ are the words in letters $g_1, \dots, g_m$ obtained in this way. The fundamental group of X is generated by $g_1, \dots, g_m$, which satisfy the defining relations $r_1 = 1, \dots, r_n = 1$.

43.I. Check that this rule gives correct answers in the cases of $\mathbb{R}P^n$ and $S^1 \times S^1$ for the cellular presentations of these spaces provided in Problems *40.H* and *40.E*.

In assertion *41.Fx* proved above we assumed that the cellular space is 2-dimensional. The reason for this was that at that moment we did not know that the inclusion $X_2 \to X$ induces an isomorphism of fundamental groups.

43.J. Each finite simply connected cellular space is homotopy equivalent to a cellular space with one-point 1-skeleton.

43°5. Fundamental Groups of Basic Surfaces

43.K. *The fundamental group of a sphere with g handles admits presentation*

$$\langle a_1, b_1, a_2, b_2, \dots a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$

43.L. *The fundamental group of a sphere with g crosscaps admits the following presentation*

$$\langle a_1, a_2, \dots a_g \mid a_1^2 a_2^2 \dots a_g^2 = 1 \rangle.$$

43.M. *Fundamental groups of spheres with different numbers of handles are not isomorphic.*

When we want to prove that two finitely presented groups are not isomorphic, one of the first natural moves is to abelianize the groups. (Recall that to *abelianize* a group G means to quotient it out by the commutator subgroup. The commutator subgroup $[G, G]$ is the normal subgroup generated by the commutators $a^{-1}b^{-1}ab$ for all $a, b \in G$. Abelianization means adding relations that $ab = ba$ for any $a, b \in G$.)

Abelian finitely generated groups are well known. Any finitely generated Abelian group is isomorphic to a product of a finite number of cyclic groups. If the abelianized groups are not isomorphic, then the original groups are not isomorphic as well.

43.M.1. *The abelianized fundamental group of a sphere with g handles is a free Abelian group of rank $2g$ (i.e., is isomorphic to $\mathbb{Z}^{2g}$).*

43.N. *Fundamental groups of spheres with different numbers of crosscaps are not isomorphic.*

43.N.1. *The abelianized fundamental group of a sphere with g crosscaps is isomorphic to $\mathbb{Z}^{g-1} \times \mathbb{Z}_2$.*

43.O. *Spheres with different numbers of handles are not homotopy equivalent.*

43.P. *Spheres with different numbers of crosscaps are not homotopy equivalent.*

43.Q. *A sphere with handles is not homotopy equivalent to a sphere with crosscaps.*

If X is a path-connected space, then the abelianized fundamental group of X is the 1-*dimensional* (or *first*) *homology group* of X and denoted by $H_1(X)$. If X is not path-connected, then $H_1(X)$ is the direct sum of the first homology groups of all path-connected components of X. Thus *43.M.1* can be rephrased as follows: if F_g is a sphere with g handles, then $H_1(F_g) = \mathbb{Z}^{2g}$.

43°6x. Seifert–van Kampen Theorem

To calculate fundamental group, one often uses the Seifert–van Kampen Theorem, instead of the cellular techniques presented above.

43.Ax Seifert–van Kampen Theorem. *Let X be a path-connected topological space, A and B be its open path-connected subspaces covering X, and let $C = A \cap B$ be also path-connected. Then $\pi_1(X)$ can be presented as amalgamated product of $\pi_1(A)$ and $\pi_1(B)$ with identified subgroup $\pi_1(C)$. In other words, if $x_0 \in C$,*

$$\pi_1(A, x_0) = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$

$$\pi_1(B, x_0) = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1\rangle,$$

$\pi_1(C, x_0)$ is generated by its elements $\gamma_1, \dots, \gamma_t$, and $in_A : C \to A$ and $in_B : C \to B$ are inclusions, then $\pi_1(X, x_0)$ can be presented as

$$\langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \operatorname{in}_{A*}(\gamma_1) = \operatorname{in}_{B*}(\gamma_1), \dots, \operatorname{in}_{A*}(\gamma_t) = \operatorname{in}_{B*}(\gamma_t)\rangle.$$

Now we consider the situation where the space X and its subsets A and B are cellular.

***43.B*x.** Assume that X is a connected finite cellular space, and A and B are two cellular subspaces of X covering X. Denote $A \cap B$ by C. How are the fundamental groups of X, A, B, and C related to each other?

***43.C*x Seifert–van Kampen Theorem.** *Let X be a connected finite cellular space, A and B – connected cellular subspaces covering X, $C = A \cap B$. Assume that C is also connected. Let $x_0 \in C$ be a 0-cell,*

$$\pi_1(A, x_0) = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1\rangle,$$

$$\pi_1(B, x_0) = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1\rangle,$$

and let the group $\pi_1(C, x_0)$ be generated by the elements $\gamma_1, \dots, \gamma_t$. Denote by $\xi_i(\alpha_1, \dots, \alpha_p)$ and $\eta_i(\beta_1, \dots, \beta_q)$ the images of the elements γ_i (more precisely, their expression via the generators) under the inclusion homomorphisms

$$\pi_1(C, x_0) \to \pi_1(A, x_0) \text{ and, respectively, } \pi_1(C, x_0) \to \pi_1(B, x_0).$$

Then

$$\pi_1(X, x_0) = \langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \xi_1 = \eta_1, \dots, \xi_t = \eta_t\rangle.$$

***43.1*x.** Let X, A, B, and C be as above. Assume that A and B are simply connected and C consists of two connected components. Prove that $\pi_1(X)$ is isomorphic to $\mathbb{Z}$.

***43.2*x.** Is Theorem *43.C*x a special case of Theorem *43.A*x?

***43.3*x.** May the assumption of openness of A and B in *43.A*x be omitted?

***43.4*x.** Deduce Theorem *43.G* from the Seifert–van Kampen Theorem *43.A*x.

***43.5*x.** Compute the fundamental group of the *lens space*, which is obtained by pasting together two solid tori via the homeomorphism $S^1 \times S^1 \to S^1 \times S^1 : (u, v) \mapsto (u^k v^l, u^m v^n)$, where $kn - lm = 1$.

43.6x. Determine the homotopy and the topological type of the lens space for $m = 0, 1$.

43.7x. Find a presentation for the fundamental group of the complement in $\mathbb{R}^3$ of a torus knot K of type (p, q), where p and q are relatively prime positive integers. This knot lies on the revolution torus T, which is described by parametric equations

$$\begin{cases} x = & (2 + \cos 2\pi u) \cos 2\pi v \\ y = & (2 + \cos 2\pi u) \sin 2\pi v \\ z = & \sin 2\pi u, \end{cases}$$

and K is described on T by equation $pu = qv$.

43.8x. Let (X, x_0) and (Y, y_0) be two simply connected topological spaces with marked points, and let $Z = X \vee Y$ be their bouquet.

(1) Prove that if X and Y are cellular spaces, then Z is simply connected.
(2) Prove that if x_0 and y_0 have neighborhoods $U_{x_0} \subset X$ and $V_{y_0} \subset Y$ that admit strong deformation retractions to x_0 and y_0, respectively, then Z is simply connected.
(3) Construct two simply connected topological spaces X and Y with a non-simply connected bouquet.

43°7x. Group-Theoretic Digression: Amalgamated Product of Groups

At first glance, description of the fundamental group of X given above in the statement of Seifert - van Kampen Theorem is far from being invariant: it depends on the choice of generators and relations of other groups involved. However, this is actually a detailed description of a group - theoretic construction in terms of generators and relations. By solving the next problem, you will get a more complete picture of the subject.

43.Dx. Let A and B be groups,

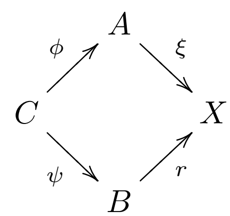
$$A = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$

$$B = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1 \rangle,$$

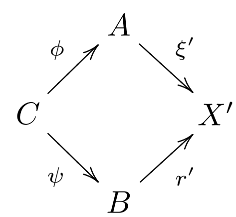
and C be a group generated by $\gamma_1, \dots \gamma_t$. Let $\xi : C \to A$ and $\eta : C \to B$ be arbitrary homomorphisms. Then

$$X = \langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \xi(\gamma_1) = \eta(\gamma_1), \dots, \xi(\gamma_t) = \eta(\gamma_t) \rangle.$$

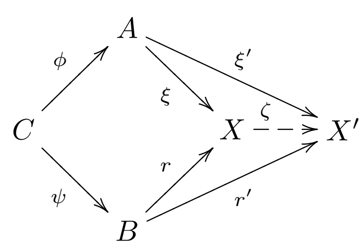
and homomorphisms $\phi : A \to X : \alpha_i \mapsto \alpha_i,\ i = 1, \dots, p$ and $\psi : B \to X : \beta_j \mapsto \beta_j,\ j = 1, \dots, q$ take part in commutative diagram



and for each group X' and homomorphisms $\varphi' : A \to X'$ and $\psi' : B \to X'$ involved in commutative diagram



there exists a unique homomorphism $\zeta : X \to X'$ such that diagram



is commutative. The latter determines the group X up to isomorphism.

The group X described in *43.Dx* is a *free product of A and B with amalgamated subgroup C*, it is denoted by $A *_C B$. Notice that the name is not quite precise, as it ignores the role of the homomorphisms ϕ and ψ and the possibility that they may be not injective.

If the group C is trivial, then $A *_C B$ is denoted by $A * B$ and called the *free product* of A and B.

***43.9*x.** Is a free group of rank n a free product of n copies of $\mathbb{Z}$?

***43.10*x.** Represent the fundamental group of Klein bottle as $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$. Does this decomposition correspond to a decomposition of Klein bottle?

43.11*x. *Riddle. Define a free product as a set of equivalence classes of words in which the letters are elements of the factors.

***43.12*x.** Investigate algebraic properties of free multiplication of groups: is it associative, commutative and, if it is, then in what sense? Do homomorphisms of the factors determine a homomorphism of the product?

43.13x*. Find decomposition of modular group $Mod = SL(2,\mathbb{Z})/\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ as free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

43°8x. Addendum to Seifert–van Kampen Theorem

Seifert-van Kampen Theorem appeared and used mainly as a tool for calculation of fundamental groups. However, it helps not in any situation. For example, it does not work under assumptions of the following theorem.

43.Ex. *Let X be a topological space, A and B open sets covering X and $C = A \cap B$. Assume that A and B are simply connected and C consists of two connected components. Then $\pi_1(X)$ is isomorphic to $\mathbb{Z}$.*

Theorem *43.Ex* also holds true if we assume that C consists of two path-connected components. The difference seems to be immaterial, but the proof becomes incomparably more technical.

Seifert and van Kampen needed more universal tool for calculation of fundamental group, and theorems published by them were much more general than *43.Ax*. Theorem *43.Ax* is all that could penetrate from there original papers to textbooks. Theorem *43.1x* is another special case of their results. The most general formulation is cumbersome, and we restrict ourselves to one more special case, which was distinguished by van Kampen. Together with *43.Ax*, it allows one to calculate fundamental groups in all situations that are available with the most general formulations by van Kampen, although not that fast. We formulate the original version of this theorem, but recommend, first, to restrict to a cellular version, in which the results presented in the beginning of this section allow one to obtain a complete answer about calculation of fundamental groups, and only after that to consider the general situation.

First, let us describe the situation common for both formulations. Let A be a topological space, B its closed subset and U a neighborhood of B in A such that $U \smallsetminus B$ is a union of two disjoint sets, M_1 and M_2, open in A. Put $N_i = B \cup M_i$. Let C be a topological space that can be represented as $(A \smallsetminus U) \cup (N_1 \sqcup N_2)$ and in which the sets $(A \smallsetminus U) \cup N_1$ and $(A \smallsetminus U) \cup N_2$ with the topology induced from A form a fundamental cover. There are two copies of B in C, which come from N_1 and N_2. The space A can be identified with the quotient space of C obtained by identification of the two copies of B via the natural homeomorphism. However, our description begins with A, since this is the space whose fundamental group we want to calculate, while the space B is auxiliary constructed out of A (see Figure 1).

In the cellular version of the statement formulated below, space A is supposed to be cellular, and B its cellular subspace. Then C is also equipped with a natural cellular structure such that the natural map $C \to A$ is cellular.

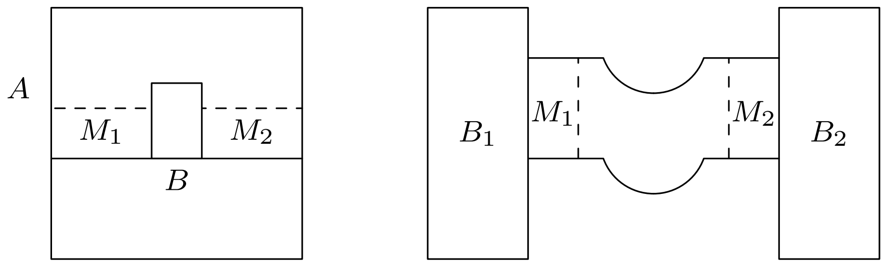


Figure 1

***43.F*x.** *Let in the situation described above C is path-connected and $x_0 \in C \smallsetminus (B_1 \cup B_2)$. Let $\pi_1(C, x_0)$ is presented by generators $\alpha_1, \dots, \alpha_n$ and relations $\psi_1 = 1, \dots, \psi_m = 1$. Assume that base points $y_i \in B_i$ are mapped to the same point y under the map $C \to A$, and σ_i is a homotopy class of a path connecting x_0 with y_i in C. Let $\beta_1, \dots, \beta_p$ be generators of $\pi_1(B, y)$, and $\beta_{1i}, \dots, \beta_{pi}$ the corresponding elements of $\pi_1(B_i, y_i)$. Denote by φ_{li} a word representing $\sigma_i \beta_{li} \sigma_i^{-1}$ in terms of $\alpha_1, \dots, \alpha_n$. Then $\pi_1(A, x_0)$ has the following presentation:*

$$\langle \alpha_1, \dots, \alpha_n, \gamma \mid \psi_1 = \dots = \psi_m = 1, \gamma\varphi_{11} = \varphi_{12}\gamma, \dots, \gamma\varphi_{p1} = \varphi_{p2}\gamma \rangle.$$

***43.14*x.** Using *43.F*x, calculate fundamental groups of torus and Klein bottle.

***43.15*x.** Using *43.F*x, calculate the fundamental groups of basic surfaces.

***43.16*x.** Deduce Theorem *43.1*x from *43.A*x and *43.F*x.

43.17*x.** ***Riddle. Develop an algebraic theory of group-theoretic construction contained in Theorem *43.F*x.

Proofs and Comments

40.A Let A be a cellular subspace of a cellular space X. For $n = 0, 1, \dots,$ we see that $A \cap X_{n+1}$ is obtained from $A \cap X_n$ by attaching the $(n+1)$-cells contained in A. Therefore, if A is contained in a certain skeleton, then A certainly is a cellular space and the intersections $A_n = A \cap X_n$, $n = 0, 1, \dots$, are the skeletons of A. In the general case, we must verify that the cover of A by the sets A_n is fundamental, which follows from assertion 3 of Lemma *40.A.1* below, Problem *40.1*, and assertion *40.G*x.

40.A.1 We prove only assertion 3 because it is needed for the proof of the theorem. Assume that a subset $F \subset A$ intersects each of the sets A_i along a set closed in A_i. Since $F \cap X_i = F \cap A_i$ is closed in A_i, it follows that this set is closed in X_i. Therefore, F is closed in X since the cover $\{X_i\}$ is fundamental. Consequently, F is also closed in A, which proves that the cover $\{A_i\}$ is fundamental.

40.B This is true because attaching D^n to a point along the boundary sphere we obtain the quotient space $D^n/_{S^{n-1}} \cong S^n$.

40.C These (open) cells are: a point, the $(n-1)$-sphere S^{n-1} without this point, the n-ball B^n bounded by S^{n-1}: $e^0 = x \in S^{n-1} \subset D^n$, $e^{n-1} = S^n \smallsetminus x$, $e^n = B^n$.

40.D Indeed, factorizing the disjoint union of segments by the set of all of their endpoints, we obtain a bouquet of circles.

40.E We present the product $I \times I$ as a cellular space consisting of 9 cells: four 0-cells – the vertices of the square, four 1-cells – the sides of the square, and a 2-cell – the interior of the square. After the standard factorization under which the square becomes a torus, from the four 0-cells we obtain one 0-cell, and from the four 1-cells we obtain two 1-cells.

40.F Each open cell of the product is a product of open cells of the factors, see Problem *40.3*.

40.G Let $S^k = S^n \cap \mathbb{R}^{k+1}$, where

$$\mathbb{R}^{k+1} = \{(x_1, x_2, \dots, x_{k+1}, 0, \dots, 0)\} \subset \mathbb{R}^{n+1}.$$

If we present S^n as the union of the constructed spheres of smaller dimensions: $S^n = \bigcup_{k=0}^{n} S^k$, then for each $k \in \{1, \dots, n\}$ the difference $S^k \smallsetminus S^{k-1}$ consists of exactly two k-cells: open hemispheres.

40.H Consider the cellular partition of S^n described in the solution of Problem *40.G*. Then the factorization $S^n \to \mathbb{R}P^n$ identifies both cells in each dimension into one. Each of the attaching maps is the projection $D^k \to \mathbb{R}P^k$ mapping the boundary sphere S^{k-1} onto $\mathbb{R}P^{k-1}$.

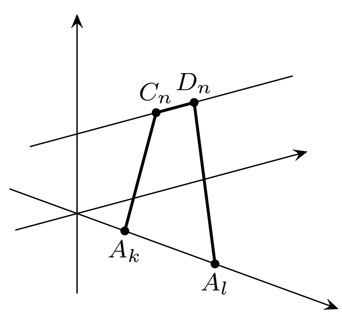
40.I 0-cells are all integer points, and 1-cells are the open intervals $(k, k+1)$, $k \in \mathbb{Z}$.

40.J Since $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ (n factors), the cellular structure of $\mathbb{R}^n$ can be determined by those of the factors (see *40.3*). Thus, the 0-cells are the points with integer coordinates. The 1-cells are open intervals with endpoints $(k_1, \ldots, k_i, \ldots, k_n)$ and $(k_1, \ldots, k_i + 1, \ldots, k_n)$, i.e., segments parallel to the coordinate axes. The 2-cells are squares parallel to the coordinate 2-planes, etc.

40.K See the solution of Problem *40.J*.

40.L This is obvious: each infinite countable 0-dimensional space is homeomorphi to $\mathbb{N} \subset \mathbb{R}$.

40.M We map 0-cells to integer points $A_k(k, 0, 0)$ on the x axis. The embeddings of 1-cells will be piecewise linear and performed as follows. Take the nth 1-cell of X to the pair of points with coordinates $C_n(0, 2n-1, 1)$ and $D_n(0, 2n, 1)$, $n \in \mathbb{N}$. If the endpoints of the 1-cell are mapped to A_k and A_l, then the image of the 1-cell is the three-link polyline $A_k C_n D_n A_l$ (possibly, closed). We easily see that the images of distinct open cells are disjoint (because their outer third parts lie on two skew lines). We have thus constructed an injection $f : X \to \mathbb{R}^3$, which is obviously continuous. The inverse map is continuous because it is continuous on each of the constructed polylines, which in addition constitute a closed locally-finite cover of $f(X)$, which is fundamental by *9.U*.



40.N Use induction on skeletons and *40.N.2*. The argument is simplified a great deal in the case where the cellular space is finite.

40.N.1 We assume that $X \subset \mathbb{R}^p \subset \mathbb{R}^{p+q+1}$, where $\mathbb{R}^p$ is the coordinate space of the first p coordinate lines in $\mathbb{R}^{p+q+1}$, and $Y \subset \mathbb{R}^q \subset \mathbb{R}^{p+q+1}$, where $\mathbb{R}^q$ is the coordinate space of the last q coordinate lines in $\mathbb{R}^{p+q+1}$. Now we define a map $f : X \sqcup Y \to \mathbb{R}^{p+q+1}$. Put $f(x) = x$ if $x \in X$, and $f(y) = (0, \ldots, 0, 1, y)$ if $y \notin V = h^{-1}\big(A \times \big[0, \frac{1}{2}\big)\big)$. Finally, if $y \in U$,

$h(y) = (a, t)$, and $t \in \left[0, \frac{1}{2}\right]$, then we put

$$f(y) = \big((1 - 2t)\varphi(a), 2t, 2ty\big).$$

We easily see that f is a proper map. The quotient map $\widehat{f} : X \cup_\varphi Y \to \mathbb{R}^{p+q+1}$ is a proper injection, therefore, $\widehat{f}$ is an embedding by *18.Ox* (cf. *18.Px*).

40.N.2 By the definition of a cellular space, X is obtained by attaching a disjoint union of closed k-disks to the $(k-1)$-skeleton of X. Let Y be a countable union of k-balls, A the union of their boundary spheres. (The assumptions of Lemma *40.N.1* is obviously fulfilled: let the neighborhood U be the complement of the union of concentric disks with radius $\frac{1}{2}$.) Thus, Lemma *40.N.2* follows from *40.N.1*.

40.O This follows from *40.N.2* by the definition of the cellular topology.

40.P This follows from *40.O* and *40.N*.

40.Q This follows from *40.P*.

40.R Try to prove this assertion at least for 1-dimensional spaces.

40.S This can be proved by somewhat complicating the argument used in the proof of *40.Bx*.

40.T See, [FR, p. 93].

40.Ax We easily see that the closure of any open simplex is canonically homeomorphi to the closed n-simplex. and, since any simplicial space Σ is Hausdorff, Σ is homeomorphi to the quotient space obtained from a disjoint union of several closed simplices by pasting them together along entire faces via affine homeomorphisms. Since each simplex Δ is a cellular space and the faces of Δ are cellular subspaces of Δ, it remains to use Problem *40.Hx*.

40.Bx Let X be a cellular space, $x, y \in X$. Let n be the smallest number such that $x, y \in X_n$. We construct their disjoint neighborhoods U_n and V_n in X_n. Let, for example, $x \in e$, where e is an open n-cell. Then let U_n be a small ball centered at x, and let V_n be the complement (in X_n) of the closure of U_n. Now let a be the center of an $(n+1)$-cell, $\varphi : S^n \to X_n$ the attaching map. Consider the open cones over $\varphi^{-1}(U_n)$ and $\varphi^{-1}(V_n)$ with vertex a. Let U_{n+1} and V_{n+1} be the unions of the images of such cones over all $(n+1)$-cells of X. Clearly, they are disjoint neighborhoods of x and y in X_{n+1}. The sets $U = \cup_{k=n}^{\infty} U_k$ and $V = \cup_{k=n}^{\infty} V_k$ are disjoint neighborhoods of x and y in X.

40.Cx Let X be a cellular space, $e \subset X$ a cell of X, $\psi : D^n \to X$ the characteristic map of e, $B = B^n \subset D^n$ the open unit ball. Since the map ψ is continuous, we have $\overline{e} = \psi(D^n) = \psi(\operatorname{Cl} B) \subset \operatorname{Cl}(\psi(B)) = \operatorname{Cl}(e)$. On the other hand, $\psi(D^n)$ is a compact set, which is closed by *40.Bx*, whence $\overline{e} = \psi(D^n) \supset \operatorname{Cl}(e)$.

***40.D*x** Let X be a cellular space, X_n the n-skeleton of X, $n \in \mathbb{N}$. The definition of the quotient topology easily implies that X_{n-1} and closed n-cells of X form a fundamental cover of X_n. Starting with $n = 0$ and reasoning by induction, we prove that the cover of X_n by closed k-cells with $k \leq n$ is fundamental. And since the cover of X by the skeletons X_n is fundamental by the definition of the cellular topology, so is the cover of X by closed cells (see *9.31*).

***40.E*x** This follows from assertion *40.D*x, the fact that, by the definition of a cellular subspace, each closed cell is contained in an element of the cover, and assertion *9.31*.

***40.F*x** Let X be a cellular space, X_k the k-skeleton of X. First, we prove that each compact set $K \subset X_k$ intersects only a finite number of open cells in X_k. We use induction on the dimension of the skeleton. Since the topology on the 0-skeleton is discrete, each compact set can contain only a finite number of 0-cells of X. Let us perform the step of induction. Consider a compact set $K \subset X_n$. For each n-cell e_α meeting K, take an open ball $U_\alpha \subset e_\alpha$ such that $K \cap U_\alpha \neq \varnothing$. Consider the cover $\Gamma = \{e_\alpha, X_n \smallsetminus \cup \operatorname{Cl}(U_\alpha)\}$. It is clear that Γ is an open cover of K. Since K is compact, Γ contains a finite subcovering. Therefore, K intersects finitely many n-cells. The intersection of K with the $(n-1)$-skeleton is closed, therefore, it is compact. By the inductive hypothesis, this set (i.e., $K \cap X_{n-1}$) intersects finitely many open cells. Therefore, the set K also intersects finitely many open cells.
Now let $\varphi : S^{n-1} \to X_{n-1}$ be the attaching map for the n-cell, $F = \varphi(S^{n-1}) \subset X_{n-1}$. Since F is compact, F can intersect only a finite number of open cells. Thus we see that each closed cell intersects only a finite number of open cells.

***40.G*x** Let A be a cellular subspace of X. By *40.D*x, it is sufficient to verify that $A \cap \overline{e}$ is closed for each cell e of X. Since a cellular subspace is a union of open (as well as of closed) cells, i.e., $A = \cup e_\alpha = \cup \overline{e}_\alpha$, it follows from *40.F*x that we have

$$A \cap \overline{e} = \big(\cup e_\alpha\big) \cap \overline{e} = (\cup_{i=1}^{n} e_{\alpha_i}) \cap \overline{e} \subset (\cup_{i=1}^{n} \overline{e}_{\alpha_i}) \cap \overline{e} \subset A \cap \overline{e}$$

and, consequently, the inclusions in this chain are equalities. Consequently, by *40.C*x, the set $A \cap \overline{e} = \cup_{i=1}^{n} (\overline{e}_{\alpha_i} \cap \overline{e})$ is closed as a union of a finite number of closed sets.

***40.I*x** Since, by *40.F*x, each closed cell intersects only a finite number of open cells, it follows that the intersection of any closed cell $\overline{e}$ with A is finite and consequently (since cellular spaces are Hausdorff) closed, both in X, and *a fortiori* in $\overline{e}$. Since, by *40.D*x, closed cells constitute a fundamental cover, the set A itself is also closed. Similarly, each subset of A is also closed in X and *a fortiori* in A. Thus, indeed, the induced topology in A is discrete.

***40.J*x** Let $K \subset X$ be a compact subset. In each of the cells e_α meeting K, we take a point $x_\alpha \in e_\alpha \cap K$ and consider the set $A = \{x_\alpha\}$. By *40.I*x, the set A is closed, and the topology on A is discrete. Since A is compact as a closed subset of a compact set, therefore, A is finite. Consequently, K intersects only a finite number of open cells.

***40.K*x** $\boxed{\Longrightarrow}$ Use *40.J*x. $\boxed{\Longleftarrow}$ A finite cellular space is compact as a union of a finite number of compact sets – closed cells.

***40.L*x** We can use induction on the dimension of the cell because the closure of any cell intersects finitely many cells of smaller dimension. Notice that the closure itself is not necessarily a cellular subspace.

***40.M*x** This follows from *40.J*x, *40.L*x, and *40.2*.

***40.N*x** $\boxed{\Longrightarrow}$ Let K be a compact subset of a cellular space. Then K is closed because each cellular space is Hausdorff. Assertion *40.J*x implies that K meets only a finite number of open cells.
$\boxed{\Longleftarrow}$ If K intersects finitely many open cells, then by *40.L*x K lies in a finite cellular subspace Y, which is compact by *40.K*x, and K is a closed subset of Y.

***40.O*x** Let X be a cellular space. $\boxed{\Longrightarrow}$ We argue by contradiction. Let X contain an uncountable set of n-cells e^n_α. Put $U^n_\alpha = e^n_\alpha$. Each of the sets U^n_α is open in the n-skeleton X_n of X. Now we construct an uncountable collection of disjoint open sets in X. Let a be the center of a certain $(n+1)$-cell, $\varphi : S^n \to X_n$ the attaching map of the cell. We construct the cone over $\varphi^{-1}(U^n_\alpha)$ with vertex at a and denote by U^{n+1}_α the union of such cones over all $(n+1)$-cells of X. It is clear that $\{U^{n+1}_\alpha\}$ is an uncountable collection of sets open in X_{n+1}. Then the sets $U_\alpha = \bigcup_{k=n}^{\infty} U^k_\alpha$ constitute an uncountable collection of disjoint sets that are open in the entire X. Therefore, X is not second countable and, therefore, nonseparable.
$\boxed{\Longleftarrow}$ If X has a countable set of cells, then, taking in each cell a countable everywhere dense set and uniting them, we obtain a countable set dense in the entire X (check this!). Thus, X is separable.

***40.P*x** Indeed, any path-connected component Y of a cellular space together with each point $x \in Y$ entirely contains each closed cell containing x and, in particular, it contains the closure of the open cell containing x.

***40.R*x** Cf. the argument used in the solution of Problem *40.O*x.

***40.R*x** This is so because a cellular space is locally path-connected, see *40.Q*x.

***40.S*x** This follows from *40.R*x.

***40.T*x** $\boxed{\Longrightarrow}$ Obvious. $\boxed{\Longleftarrow}$ We show by induction that the number of cells in each dimension is countable. For this purpose, it is sufficient to prove that each cell intersects finitely many closed cells. It is more convenient to prove a stronger assertion: any closed cell $\overline{e}$ intersects finitely many closed cells. It is clear that any neighborhood meeting the closed cell also meets the cell itself. Consider the cover of $\overline{e}$ by neighborhoods each of which intersects finitely many closed cells. It remains to use the fact that $\overline{e}$ is compact.

***40.U*x** By Problem *40.T*x, the 1-skeleton of X is connected. The result of Problem *40.T*x implies that it is sufficient to prove that the 0-skeleton of X is countable. Fix a 0-cell x_0. Denote by A_1 the union of all closed 1-cells containing x_0. Now we consider the set A_2 – the union of all closed 1-cells meeting A_1. Since X is locally finite, each of the sets A_1 and A_2 contains a finite number of cells. Proceeding in a similar way, we obtain an increasing sequence of 1-dimensional cellular subspaces $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$, each of which is finite. Put $A = \bigcup_{k=1}^{\infty} A_k$. The set A contains countably many cells. The definition of the cellular topology implies that A is both open and closed in X_1. Since X_1 is connected, we have $A = X_1$.

***40.V*x** ⟹ Assume the contrary: let the 1-skeleton X_1 be disconnected. Then X_1 is the union of two closed sets: $X_1 = X_1' \cup X_1''$. Each 2-cell is attached to one of these sets, whence $X_2 = X_2' \cup X_2''$. A similar argument shows that for each positive integer n the n-skeleton is a union of its closed subsets. Put $X' = \bigcup_{n=0}^{\infty} X_n'$ and $X'' = \bigcup_{n=0}^{\infty} X_n''$. By the definition of the cellular topology, X' and X'' are closed, consequently, X is disconnected. ⟸ This is obvious.

41.A This immediately follows from the obvious equality $c_i(A \cup B) = c_i(A) + c_i(B) - c_i(A \cap B)$.

41.B Here we use the following artificial trick. We introduce the polynomial $\chi_A(t) = c_0(A) + c_1(A)t + \dots + c_i(A)t^i + \dots$. This is the *Poincaré polynomial*, and its most important property for us here is that $\chi(X) = \chi_X(-1)$. Since $c_k(X \times Y) = \sum_{i=0}^{k} c_i(X)c_{k-i}(Y)$, we have

$$\chi_{X\times Y}(t) = \chi_X(t) \cdot \chi_Y(t),$$

whence $\chi(X \times Y) = \chi_{X\times Y}(-1) = \chi_X(-1) \cdot \chi_Y(-1) = \chi(X) \cdot \chi(Y)$.

41.C Set $X' = X \smallsetminus (e \cup f)$. It follows from the definition that the union of all open cells in X' coincides with the union of all closed cells in X', consequently, X' is a cellular subspace of X.

41.D The deformation retraction of D^n to the lower closed hemisphere S^{n-1}_- determines a deformation retraction $X \to X \smallsetminus (e \cup f)$.

41.E The assertion is obvious because each elementary combinatorial collapse decreases by one the number of cells in each of two neighboring dimensions.

41.F Let $p : X \to X'$ be the factorization map. The space X' has the same open cells as X except e and f. The attaching map for each of them is the composition of the initial attaching map and p.

41.G.1 Put $Y = X_{n-1} \cup_{\varphi_e} D^n$. Clearly, $Y' \cong Y \smallsetminus (e \cup f)$, and so we identify these spaces. Then the projection $p' : Y \to Y'$ is a homotopy equivalence by *41.D*.

41.G.2 Let $\{e_\alpha\}$ be a collection of n-cells of X distinct from the cell e, φ_α – the corresponding attaching maps. Consider the map $p' : Y \to Y'$. Since

$$X_n = Y \cup_{(\bigsqcup_\alpha \varphi_\alpha)} \left(\bigsqcup_\alpha D^n_\alpha \right),$$

we have

$$X'_n = Y' \cup_{(\bigsqcup_\alpha p' \circ \varphi_\alpha)} \left(\bigsqcup_\alpha D^n_\alpha \right).$$

Since p' is a homotopy equivalence by *41.G.1*, the result of *41.6x* implies that p' extends to a homotopy equivalence $p_n : X_n \to X'_n$. Using induction on skeletons, we obtain the required assertion.

41.Ax We use induction on the dimension. Clearly, we should consider only those cells which do not lie in A. If there is a retraction

$$\rho_{n-1} : (X_{n-1} \cup A) \times I \to (X_{n-1} \times 0) \cup (A \times I),$$

and we construct a retraction

$$\widetilde{\rho}_n : (X_n \cup A) \times I \to (X_n \times 0) \cup ((X_{n-1} \cup A) \times I),$$

then it is obvious how, using their "composition", we can obtain a retraction

$$\rho_n : (X_n \cup A) \times I \to (X_n \times 0) \cup (A \times I).$$

We need the standard retraction $\rho : D^n \times I \to (D^n \times 0) \cup (S^{n-1} \times I)$. (It is most easy to define ρ geometrically. Place the cylinder in a standard way in $\mathbb{R}^{n+1}$ and consider a point p lying over the center of the upper base. For $z \in D^n \times I$, let $\rho(z)$ be the point of intersection of the ray starting at p and passing through z with the union of the base $D^n \times 0$ and the lateral area $S^{n-1} \times I$ of the cylinder.) The quotient map ρ is a map $\overline{e} \times I \to (X_n \times 0) \cup (X_{n-1} \times I)$. Extending it identically to $X_{n-1} \times I$, we obtain a map

$$\rho_e : (\overline{e} \times I) \cup (X_{n-1} \times I) \to (X_n \times 0) \cup (X_{n-1} \times I).$$

Since the closed cells constitute a fundamental cover of a cellular space, the retraction $\widetilde{\rho}_n$ is thus defined.

***41.B*x** The formulas $\widetilde{H}(x,0) = F(x)$ for $x \in X$ and $\widetilde{H}(x,t) = h(x,t)$ for $(x,t) \in A \times I$ determine a map $\widetilde{H} : (X \times 0) \cup (A \times I) \to Y$. By *41.Ax*, there is a retraction $\rho : X \times I \to (X \times 0) \cup (A \times I)$. The composition $H = \widetilde{H} \circ \rho$ is the required homotopy.

***41.C*x** Denote by $h : A \times I \to A$ a homotopy between the identity map of A and the constant map $A \to A : \ a \mapsto x_0$. Consider the homotopy $\widetilde{h} = i \circ h : A \times I \to X$. By Theorem *41.Bx*, $\widetilde{h}$ extends to a homotopy $H : X \times I \to X$ of the identity map of the entire X. Consider the map $f : X \to X$, $f(x) = H(x,1)$. By the construction of the homotopy $\widetilde{h}$, we have $f(A) = \{x_0\}$, consequently, the quotient map of f is a continuous map $g : X/_A \to X$. We prove that pr and g are mutually inverse homotopy equivalences. To do this we must verify that $g \circ \mathrm{pr} \sim \mathrm{id}_X$ and $\mathrm{pr} \circ g \sim \mathrm{id}_{X/A}$.
1) We observe that $H(x,1) = g(\mathrm{pr}(x))$ by the definition of g. Since $H(x,0) = x$ for all $x \in X$, it follows that H is a homotopy between id_X and the composition $g \circ \mathrm{pr}$.
2) If we factorize each fiber $X \times t$ by $A \times t$, then, since $H(x,t) \in A$ for all $x \in A$ and $t \in I$, the homotopy H determines a homotopy $\widetilde{H} : X/_A \to X/_A$ between $\mathrm{id}_{X/A}$ and the composition $p \circ g$.

***41.F*x** Let X be the space. By *41.Ex*, we can assume that X has one 0-cell, and therefore the 1-skeleton X_1 is a bouquet of circles. Consider the characteristic map $\psi : I \to X_1$ of a certain 1-cell. Instead of the loop ψ, it is more convenient to consider the circular loop $S^1 \to X_1$, which we denote by the same letter. Since X is simply connected, the loop ψ extends to a map $f : D^2 \to X$. Now consider the disk D^3. To simplify the notation, we assume that f is defined on the lower hemisphere $S^2_- \subset D^3$. Put $Y = X \cup_f D^3 \simeq X$. The space Y is cellular and is obtained by adding two cells to X: a 2- and a 3-cell. The new 2-cell e, i.e., the image of the upper hemisphere in D^3, is a contractible cellular space. Therefore, we have $Y/_e \simeq Y$, and $Y/_e$ contains one 1-cell less than the initial space X. Proceeding in this way, we obtain a space with one-point 2-skeleton. Notice that our construction yielded a 3-dimensional cellular space. Actually, in our assumptions the space is homotopy equivalent to: a point, a 2-sphere, or a bouquet of 2-spheres, but the proof of this fact involves more sophisticated techniques (the homology).

***41.G*x** Let the map $f : X \to A$ be homotopically inverse to the inclusion in_A. By assumption, the restriction of f to the subspace A, i.e., the composition $f \circ \mathrm{in}$, is homotopic to the identity map id_A. By Theorem *41.Bx*, this homotopy extends to a homotopy $H : X \times I \to A$ of f. Put $\rho(x) = H(x,1)$; then $\rho(x,1) = x$ for all $x \in A$. Consequently, ρ is a retraction. It remains to observe that, since ρ is homotopic to f, it follows that $\text{in} \circ \rho$ is homotopic to the composition $\text{in}_A \circ f$, which is homotopic to id_X because f and in are homotopically inverse by assumption.

42.A Prove this by induction, using Lemma *42.A.1*.

42.A.1 Certainly, the fact that the projection is a homotopy equivalence is a special case of assertions *41.Dx* and *41.G*. However, here we present an independent argument, which is more visual in the 1-dimensional case. All homotopies will be fixed outside a neighborhood of the 1-cell e of the initial cellular space X and outside a neighborhood of the 0-cell x_0, which is the image of e in the quotient space $Y = X/_e$. For this reason, we consider only the closures of such neighborhoods. Furthermore, to simplify the notation, we assume that the spaces under consideration coincide with these neighborhoods. In this case, X is the 1-cell e with the segments $I_1, I_2, \dots, I_k$ (respectively, $J_1, J_2, \dots, J_n$) attached to the left endpoint, (respectively, to the right endpoint). The space Y is simply a bouquet of all these segments with a common point x_0. The map $f : X \to Y$ has the following structure: each of the segments I_i and J_j is mapped onto itself identically, and the cell e is mapped to x_0. The map $g : Y \to X$ takes x_0 to the midpoint of e and maps a half of each of the segments I_s and J_t to the left and to the right half of e, respectively. Finally, the remaining half of each of these segments is mapped (with double stretching) onto the entire segment. We prove that the described maps are homotopically inverse. Here it is important that the homotopies be fixed on the free endpoints of I_s and J_t. The composition $f \circ g : Y \to Y$ has the following structure. The restriction of $f \circ g$ to each of the segments in the bouquet is, strictly speaking, the product of the identical path and the constant path, which is known to be homotopic to the identical path. Furthermore, the homotopy is fixed both on the free endpoints of the segments and on x_0. The composition $g \circ f$ maps the entire cell e to the midpoint of e, while the halves of each of the segments I_s and J_t adjacent to e are mapped a half of e, and their remaining parts are doubly stretched and mapped onto the entire corresponding segment. Certainly, the map under consideration is homotopic to the identity.

42.B By *42.A.1*, each connected 1-dimensional finite cellular space X is homotopy equivalent to a space X', where the number of 0- and 1-cells is one less than in X, whence $\chi(X) = \chi(X')$. Reasoning by induction, we obtain as a result a space with a single 0-cell and with Euler characteristic equal to $\chi(X)$ (cf. *41.E*). Let k be the number of 1-cells in this space. Then $\chi(X) = 1-k$, whence $k = 1-\chi(X)$. It remains to observe that k is precisely the rang of $\pi_1(X)$.

42.C This follows from *42.B* because the fundamental group of a space is invariant with respect to homotopy equivalences.

42.D This follows from *42.C*.

42.E By *42.B*, if two finite connected 1-dimensional cellular spaces have isomorphic fundamental groups (or equal Euler characteristics), then each of them is homotopy equivalent to a bouquet consisting of one and the same number of circles, therefore, the spaces are homotopy equivalent. If the spaces are homotopy equivalent, then, certainly, their fundamental groups are isomorphic, and, by *42.C*, their Euler characteristics are also equal.

***42.A*x** Let e be an open cell. If the image $\varphi_e(S^0)$ of the attaching map of e is one-point, then $X \smallsetminus e$ is obviously connected. Assume that $\varphi_e(S^0) = \{x_0, x_1\}$. Prove that each connected component of $X \smallsetminus e$ contains at least one of the points x_0 and x_1.

***42.B*x** 1) Let X be a connected 1-dimensional cellular space, $x \in X$ a vertex. If a connected component of $X \smallsetminus x$ contains no edges whose closure contains x, then, since cellular spaces are locally connected, the component is both open and closed in the entire X, contrary to the connectedness of X. 2) This follows from the fact that a vertex of degree m lies in the closure of at most m distinct edges.

43.A See *42.B*.

43.B This follows from *42.I* (or *41.C*x) because of *35.L*.

43.C It is sufficient to prove that each loop $u : I \to X$ is homotopic to a loop v with $v(I) \subset A$. Let $U \subset D^k$ be the open ball with radius $\frac{2}{3}$, and let V be the complement in X of a closed disk with radius $\frac{1}{3}$. By the Lebesgue Lemma *16.W*, the segment I can be subdivided segments $I_1, \dots, I_N$ the image of each of which is entirely contained in one of the sets U or V. Assume that $u(I_l) \subset U$. Since in D^k any two paths with the same starting and ending points are homotopic, it follows that the restriction $u|_{I_l}$ is homotopic to a path that does not meet the center $a \in D^k$. Therefore, the loop u is homotopic to a loop u' whose image does not contain a. It remains to observe that the space A is a deformation retract of $X \smallsetminus a$, therefore, u' is homotopic to a loop v with image lying in A.

43.D Let s be a loop at x_0. Since the set $s(I)$ is compact, $s(I)$ is contained in a finite cellular subspace Y of X. It remains to apply assertion *43.C* and use induction on the number of cells in Y.

43.E This follows from *43.D* and *43.B*.

43.F If we take another collection of paths s'_α, then the elements $T_{s_\alpha}[\varphi_\alpha]$ and $T_{s'_\alpha}[\varphi_\alpha]$ will be conjugate in $\pi_1(X_1, x_0)$, and since the subgroup N is normal, N contains the collection of elements $\{T_{s_\alpha}[\varphi_\alpha]\}$ iff N contains the collection $\{T_{s'_\alpha}[\varphi_\alpha]\}$.

43.G We can assume that the 0-skeleton of X is the singleton $\{x_0\}$, so that the 1-skeleton X_1 is a bouquet of circles. Consider a covering

$p_1 : Y_1 \to X_1$ with group N. Its existence follows from the more general Theorem *39.Dx* on the existence of a covering with given group. In the case considered, the covering space is a 1-dimensional cellular space. Now the proof of the theorem consists of several steps, each of which is the proof of one of the following seven lemmas. It will also be convenient to assume that $\varphi_\alpha(1) = x_0$, so that $T_{s_\alpha}[\varphi_\alpha] = [\varphi_\alpha]$.

43.G.1 Since, clearly, $\text{in}_*([\varphi_\alpha]) = 1$ in $\pi_1(X, x_0)$, we have $\text{in}_*([\varphi_\alpha]) = 1$ in $\pi_1(X, x_0)$, therefore, each of the elements $[\varphi_\alpha] \in \operatorname{Ker} i_*$. Since the subgroup $\operatorname{Ker} i_*$ is normal, it contains N, which is the smallest subgroup generated by these elements.

43.G.2 This follows from *39.Px*.

43.G.3 Let $F = p_1^{-1}(x_0)$ be the fiber over x_0. The map p_2 is a quotient map

$$Y_1 \sqcup \left(\bigsqcup_{\alpha} \bigsqcup_{y \in F_\alpha} D^2_{\alpha,y} \right) \to X_1 \sqcup \left(\bigsqcup_{\alpha} D^2_\alpha \right),$$

whose submap $Y_1 \to X_1$ is p_1, and the maps $\bigsqcup_{y \in F_\alpha} D^2_\alpha \to D^2_\alpha$ are identities on each of the disks D^2_α. It is clear that for each point $x \in \operatorname{Int} D^2_\alpha \subset X_2$ the entire interior of the disk is a trivially covered neighborhood. Now assume that for point $x \in X_1$ the set U_1 is a trivially covered neighborhood of x with respect to the covering p_1. Put $U = U_1 \cup (\bigcup_{\alpha'} \psi_{\alpha'}(B_{\alpha'}))$, where $B_{\alpha'}$ is the open cone with vertex at the center of $D^2_{\alpha'}$ and base $\varphi^{-1}_{\alpha'}(U)$. The set U is a trivially covered neighborhood of x with respect to p_2.

43.G.4 First, we prove this for $n = 3$. So, let $p : X \to B$ be an arbitrary covering, $\varphi : S^2 \to B$ an arbitrary map. Consider the subset $A = S^1 \times 0 \cup 1 \times I \cup S^1 \times 1$ of the cylinder $S^1 \times I$, and let $q : S^1 \times I \to S^1 \times I/_A$ be the factorization map. We easily see that $S^1 \times I/_A \cong S^2$. Therefore, we assume that $q : S^1 \times I \to S^2$. The composition $h = \varphi \circ q : S^1 \times I \to B$ is a homotopy between one and the same constant loop in the base of the covering. By the Path Homotopy Lifting Theorem *34.C*, the homotopy h is covered by the map $\widetilde{h}$, which also is a homotopy between two constant paths, therefore, the quotient map of $\widetilde{h}$ is the map $\widetilde{\varphi} : S^2 \to X$ covering φ. For $n > 3$, use *39.Yx*.

43.G.5 The proof is similar to that of Lemma 3.

43.G.6 Since the loop $\text{in} \circ s : I \to X$ is null-homotopic, it is covered by a loop, the image of which automatically lies in Y_1.

43.G.7 Let s be a loop in X_1 such that $[s] \in \operatorname{Ker}(i_1)_*$. Lemma 6 implies that s is covered by a loop $\widetilde{s} : I \to Y_1$, whence $[s] = (p_1)_*([\widetilde{s}]) \in N$. Therefore, $\operatorname{Ker} \operatorname{in}_* \subset N$, whence $N = \operatorname{Ker} \operatorname{in}_*$ by Lemma 1.

43.I For example, $\mathbb{R}P^2$ is obtained by attaching D^2 to S^1 via the map $\varphi : S^1 \to S^1 : z \mapsto z^2$. The class of the loop φ in $\pi_1(S^1) = \mathbb{Z}$ is the doubled generator, whence $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, as it should have been expected. The torus $S^1 \times S^1$ is obtained by attaching D^2 to the bouquet $S^1 \vee S^1$ via a map φ representing the commutator of the generators of $\pi_1(S^1 \vee S^1)$. Therefore, as it should have been expected, the fundamental group of the torus is $\mathbb{Z}^2$.

43.K See *40.12* (h).

43.L See *40.12* (i).

43.M.1 Indeed, the single relation in the fundamental group of the sphere with g handles means that the product of g commutators of the generators a_i and b_i equals 1, and so it “vanishes” after the abelianization.

43.N.1 Taking the elements $a_1, \dots, a_{g-1}$, and $b_n = a_1 a_2 \dots a_g$ as generators in the commuted group, we obtain an Abelian group with a single relation $b_n^2 = 1$.

43.O This follows from *43.M.1*.

43.O This follows from *43.N.1*.

43.Q This follows from *43.M.1* and *43.N.1*.

***43.A*x** We do not assume that you can prove this theorem on your own. The proof can be found, for example, in [Massey].

***43.B*x** Draw a commutative diagram comprising all inclusion homomorphisms induced by all inclusions occurring in this situation.

***43.C*x** Since, as we will see in Section 43°7x, the group presented as above, actually, up to canonical isomorphism does not depend on the choice of generators and relations in $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ and the choice of generators in $\pi_1(C, x_0)$, we can use the presentation which is most convenient for us. We derive the theorem from Theorems *43.D* and *43.G*. First of all, it is convenient to replace X, A, B, and C by homotopy equivalent spaces with one-point 0-skeletons. We do this with the help of the following construction. Let T_C be a spanning tree in the 1-skeleton of C. We complete T_C to a spanning tree $T_A \supset T_C$ in A, and also complete T_C to a spanning tree $T_B \supset T_C$. The union $T = T_A \cup T_B$ is a spanning tree in X. It remains to replace each of the spaces under consideration with its quotient space by a spanning tree. Thus, the 1-skeleton of each of the spaces X, A, B, and C either coincides with the 0-cell x_0, or is a bouquet of circles. Each of the circles of the bouquets determines a generator of the fundamental group of the corresponding space. The image of $\gamma_i \in \pi_1(C, x_0)$ under the inclusion homomorphism is one of the generators, let it be α_i (β_i) in $\pi_1(A, x_0)$

(respectively, in $\pi_1(B, x_0)$). Thus, $\xi_i = \alpha_i$ and $\eta_i = \beta_i$. The relations $\xi_i = \eta_i$, and, in this case, $\alpha_i = \beta_i$, $i = 1, \dots, t$ arise because each of the circles lying in C determines a generator of $\pi_1(X, x_0)$. All the remaining relations, as it follows from assertion *43.G*, are determined by the attaching maps of the 2-cells of X, each of which lies in at least one of the sets A or B, and hence is a relation between the generators of the fundamental groups of these spaces.

***43.D*x** Let $\mathcal{F}$ be a free group with generators $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$. By definition, the group X is the quotient group of F by the normal hull N of the elements

$$\{\rho_1, \dots, \rho_r,\ \sigma_1, \dots, \sigma_s,\ \xi(\gamma_1)\eta(\gamma_1)^{-1},\ \dots,\ \xi(\gamma_t)\eta(\gamma_t)^{-1}\}.$$

Since the first diagram is commutative, it follows that the subgroup N lies in the kernel of the homomorphism $F \to X' :\ \alpha_i \mapsto \varphi'(\alpha_i), \beta_i \mapsto \psi'(\alpha_i)$, consequently, there is a homomorphism $\zeta : X \to X'$. Its uniqueness is obvious. Prove the last assertion of the theorem on your own.

***43.E*x** Construct a universal covering of X.