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*Dedicated to the memory of Vladimir Abramovich Rokhlin (1919–1984)  
– our teacher*

## Introduction

### The subject of the book, Elementary Topology

Elementary means close to elements, basics. It is impossible to determine precisely, once and for all, which topology is elementary, and which is not. The elementary part of a subject is the part with which an expert starts to teach a novice.

We suppose that our student is ready to study topology. So, we do not try to win her or his attention and benevolence by hasty and obscure stories about mysterious and attractive things such as the Klein bottle.<sup>1</sup> All in good time: the Klein bottle will appear in its turn. However, we start with what a topological space is. That is, we start with general topology.

General topology became a part of the general mathematical language long ago. It teaches one to speak clearly and precisely about things related to the idea of continuity. It is needed not only in order to explain what, finally, the Klein bottle is. This is also a way to introduce geometrical images into any area of mathematics, no matter how far from geometry the area may be at first glance.

As an active research area, general topology is practically completed. A permanent usage in the capacity of a general mathematical language has polished its system of definitions and theorems. Nowadays, study of general topology indeed resembles rather a study of a language than a study of mathematics: one has to learn many new words, while the proofs of the majority of theorems are extremely simple. But the quantity of the theorems is huge. This comes as no surprise because they play the role of rules that regulate usage of words.

The book consists of two parts. General topology is the subject of the first one. The second part is an introduction to algebraic topology via its most classical and elementary segment which emerges from the notions of fundamental group and covering space.

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<sup>1</sup>A person who is looking for such elementary topology will easily find it in one of the numerous books with beautiful pictures on visual topology.

In our opinion, elementary topology also includes basic topology of manifolds, i.e., spaces that look locally as the Euclidean space. One- and two-dimensional manifolds, i.e., curves and surfaces, are especially elementary. But a book should not be too thick, and so we had to stop.

Chapter 5 keeps somewhat aloof. Its material is intimately related to a number of different areas of Mathematics. Although it plays a profound role in these areas, it is not that important in the initial study of general topology. Therefore mastering of this material may be postponed until it appears in a substantial way in other mathematical courses (which will concern the Lie groups, functional analysis, etc.). The main reason why we included this material is that it provides a great variety of examples and exercises.

### Organization of the text

Even a cursory overview detects unusual features in organization of this book. We dared to come up with several innovations and hope that the reader will quickly get used to them and even find them useful.

We know that needs and interests of our readers vary, and realize that it is very difficult to make a book interesting and useful for *each* reader. To solve this problem, we decorated the text in such a way that the reader could easily determine what (s)he can expect from each piece of the text. We hope that this will allow the reader to organize studying the material of the book in accordance with his or her tastes and abilities. To achieve this goal, we use several tricks.

First of all, we distinguished the basic, so to speak, lecture line. This is the material which we consider basic. It constitutes a minor part of the text.

The basic material is often interrupted by specific examples, illustrative and training problems, and discussion of the notions that are related to these examples and problems, but are not used in what follows. Some of the notions play a fundamental role in other areas of mathematics, but here they are of minor importance.

In a word, the basic line is interrupted by *variations* wherever possible. The variations are clearly separated from the *basic theme* by graphical means.

The second feature distinguishing the present book from the majority of other textbooks is that proofs are separated from formulations. The book looks nearly as a problem book. It would be easy to make the book looking like hundreds of other mathematical textbooks. For this purpose, it suffices to move all variations to the ends of their sections so that they would look exercises to the basic text, and put the proofs of theorems immediately after their formulations.

**For whom is this book?**

A reader who has safely reached the university level in her/his education may bravely approach this book. Super brave daredevils may try it even earlier. However, we cannot say that no preliminary knowledge is required. We suppose that the reader is familiar with real numbers. And, surely, with natural, integer, and rational numbers, too. Complex numbers will also do no harm, although one can manage without them in the first part of the book.

We assume that the reader is acquainted with naive set theory, but admit that this acquaintance may be superficial. For this reason, we make special set-theoretical digressions where the possession of set theory is particularly desirable.

We do not seriously rely on calculus, but since the majority of our readers are already familiar with it, at least slightly, anyway, we do not hesitate to resort to notation and notions from calculus.

In the second part, an experience in the group theory will be useful, although we give all necessary information about groups.

One of the most valuable acquisitions that the reader can make by mastering of the present book is new elements of mathematical culture and an ability to understand and appreciate an abstract axiomatic theory. The higher the degree in which the reader already possesses this ability, the easier it will be for her or him to master the material of the book.

If you want to study topology on your own, do try to work with the book. It may turn out to be precisely what you need. However, you should attentively read this Introduction again in order to understand how the material is organized and how you can use it.

**The basic theme**

The core of the book is the material of the course of topology for students major in Mathematics at the Saint Petersburg (Leningrad) State University. The material is relatively small and involves nearly no complicated arguments.

The reader should not think that by selecting the basic theme the authors just try to impose their tastes on her or him. We do not hesitate to do this occasionally, but here our prime goal is to organize study of the subject.

The basic theme forms a complete entity. The reader who has mastered the basic theme has mastered the subject. Whether the reader had looked in the variations or not is her or his business. However, the variations have been included in order to help the reader with mastering the basic material. They are not exiled to final pages of sections in order to have them at hand

precisely when they are most needed. By the way, the variations can tell you about many interesting things. However, following the variations too literally and carefully may take far too long.

We believe that the material presented in the basic theme is the minimal amount of topology that must be mastered by every student who decided to become a professional mathematician.

Certainly, a student whose interests will be related to topology and other geometrical disciplines will have to learn by far more than the basic theme includes. In this case the material can serve a good starting point.

For a student who is not going to become a professional mathematician, even a selective acquaintance with the basic theme might be useful. It may be useful for preparation to an exam or just for catching of a glimpse and feeling of abstract mathematics, with its emphasized value of definitions and precise formulations.

### **Where are the proofs?**

The book is tailored for a reader who is determined to work actively.

*The proofs of theorems are separated from their formulations and placed in the end of the current chapter.*

We believe that the first reaction to the formulation of any assertion (coming immediately after the feeling that the formulation has been understood) must be an attempt to prove the assertion. Or to disprove it, if you do not manage to prove. An attempt to disprove an assertion may be useful both for achieving a better understanding of the formulation, and for looking for a proof.

By keeping the proofs away from the formulations, we want to encourage the reader to think through each formulation, and, on the other hand, to make the book inconvenient for careless skimming. However, a reader who prefers a more traditional style and does not wish to work too actively for some reasons can either find the proof in the end of the current chapter, or skip it at all. (Certainly, in the latter case there is a high danger of misunderstanding.)

This style can also please an expert, who prefers formulations not gloomed by proofs. Most of the proofs are simple. They are easy and pleasant to invent.

### **Structure of the book**

Basic structural units of the book are sections. They are divided into numbered and titled subsections. Each subsection is devoted to a single

topic and consists of definitions, comments, theorems, exercises, problems, and riddles.

By a *riddle* we mean a problem whose solution (and often also the meaning) should be rather guessed than calculated or deduced from the formulation.

Theorems, exercises, problems, and riddles belonging to the basic material are numbered by pairs consisting of the number of the current section and a letter, separated by a dot.

**2.B. Riddle.** Taking into account the number of the riddle, determine in which section it must be contained. By the way, is this really a riddle?

The letters are assigned in the alphabetical order. They number the assertions inside a section.

Often a difficult problem (or theorem) is followed by a sequence of assertions that are lemmas to the problem. Such a chain often ends with a problem in which we suggest the reader armed with the lemmas just proven to return to the initial problem (respectively, theorem).

## Variations

The basic material is surrounded by numerous training problems and additional definitions, theorems, and assertions. In spite of their relations to the basic material, they usually are left outside of the standard lecture course.

Such additional material is easy to recognize in the book by the smaller print and wide margins, as here. Exercises, problems, and riddles that are not included into the basic material, but are closely related to it are numbered by pairs consisting of the number of a section and the number of the assertion in the limits of the section.

**2.5.** Find a problem with the same number *2.5* in the main body of the book.

*All solutions of problems are put in the end of the book.*

As is common, the problems that have seemed to be most difficult to the authors are marked by an asterisk. They are included with different purposes: to outline relations to other areas of mathematics, to indicate possible directions of development of the subject, or just to please an ambitious reader.

## Additional themes

We decided to make accessible for interested students certain theoretical topics complementing the basic material. It would be natural to include them into lecture courses designed for senior (or graduate) students. However, this does not happen usually, because the topics do not fit well into

traditional graduate courses. Furthermore, studying them seems to be more natural during the very first contacts with topology.

In the book, such topics are separated into individual subsections, whose numbers contain the symbol  $x$ , which means *extra*. (Sometimes, a whole section is marked in this way, and, in one case, even a whole chapter.)

Certainly, regarding this material as additional is a matter of taste and viewpoint. Qualifying a topic as additional, we follow our own ideas about what must be contained in the initial study of topology. We realize that some (if not most) of our colleagues may disagree with our choice, but we hope that our decorations will not hinder them from using the book.

### **Advices to the reader**

You can use the present book when preparing to an exam in topology (especially so if the exam consists in solving problems). However, if you attend lectures in topology, then it is reasonable to read the book before lectures, and try to prove the assertions in it on your own before the lecturer will prove them.

The reader who can prove assertions of the basic theme on his or her own, needn't solve *all* problems suggested in the variations, and can resort to a brief acquaintance with their formulations and solving the most difficult of them. On the other hand, the more difficult it is for you to prove assertions of the basic theme, the more attention you must pay to illustrative problems, and the less attention to problems with an asterisk.

Many of our illustrative problems are easy to invent. Moreover, when seriously studying a subject, one should permanently invent questions of this kind.

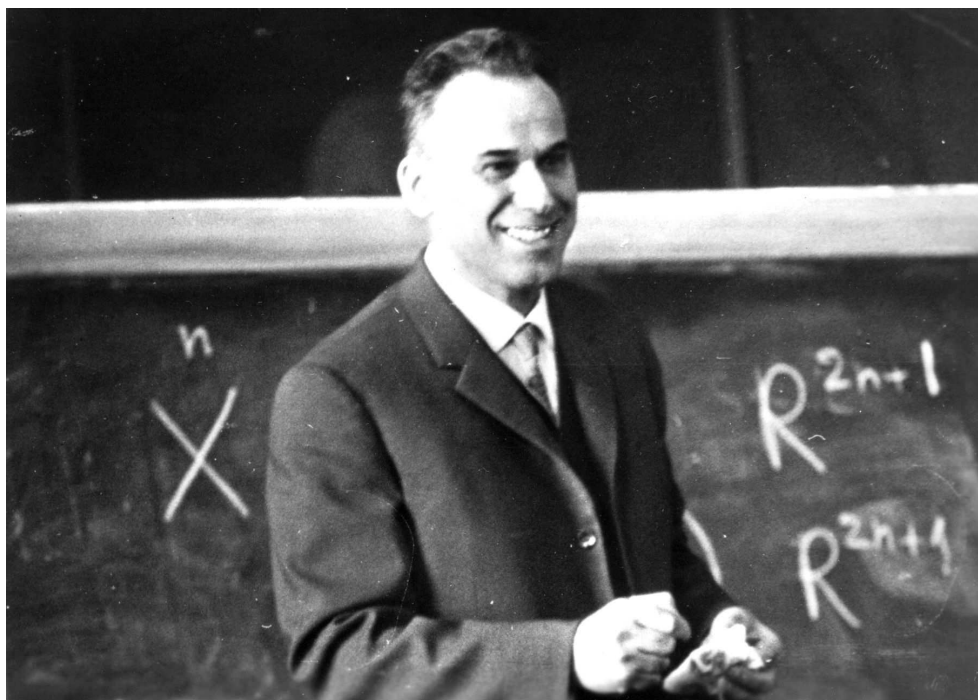
On the other hand, some problems presented in the book are not easy to invent at all. We have widely used all kinds of sources, including both literature and teachers' folklore.

### **How this book was created**

The basic theme follows the course of lectures composed by Vladimir Abramovich Rokhlin at the Faculty of Mathematics and Mechanics of the Leningrad State University in the 1960s. It seems appropriate to begin with circumstances of creating the course, although we started to write this book after Vladimir Abramovich's death (he died in 1984).

In the 1960s, mathematics was one of the most attractive areas of science for young people in the Soviet Union, being second maybe only to physics among the natural sciences. Every year more than a hundred students were enrolled in the mathematical subdivision of the Faculty.

Several dozens of them were alumnae and alumni of mathematical schools. The system and contents of lecture courses at the Faculty were seriously updated.



Vladimir Abramovich Rokhlin gives a lecture, 1960s.

Until Rokhlin created his course, topology was taught in the Faculty only in the framework of special courses. Rokhlin succeeded in including a one-semester course of topology into the system of general mandatory courses. The course consisted of three chapters devoted to general topology, fundamental group and coverings, and manifolds, respectively. The contents of the first two chapters only slightly differed from the basic material of the book. The last chapter started with a general definition of a topological manifold, included a topological classification of one-dimensional manifolds, and ended either with a topological classification of triangulated two-dimensional manifolds, or with elements of differential topology, up to embedding of a smooth manifold in the Euclidean space.

Three of the four authors belong to the first generations of students who attended Rokhlin's lecture course. This was a one-semester course, three hours a week in the first semester of the second year. At most two two-hour lessons during the whole semester were devoted to solving problems. It was not Rokhlin, but his graduate students who conducted these lessons. For instance, in 1966–68 they were conducted by Misha Gromov – an outstanding geometer, at present time a professor of the Paris Institute des Hautes Etudes Scientifiques and the New York Courant Institute. Rokhlin regarded the course as a theoretical one and did not wish to spend the lecture time to solving problems. And, indeed, in the framework of the course one did not have to teach students how to solve series of routine problems,

like problems in techniques of differentiation and integration, that are traditional for calculus.

In spite of the fact that we built our book by starting from Rokhlin's lectures, the book will give you no idea about Rokhlin's style. The lectures were brilliant. Rokhlin wrote very little on the blackboard. Nevertheless, it was very easy to take notes after him. He spoke without haste, with maximally simple and ideally correct sentences.

For the last time, Rokhlin gave his mandatory topology course in 1973. In August of 1974, because of his serious illness, the administration of the Faculty had to look for a person who would substitute Rokhlin as a lecturer. The problem was complicated by the fact that the results of the exams in the preceding year were terrible. In 1973, the time allotted for the course was increased up to four hours a week, while the number of students had grown, and, respectively, the level of their training had decreased. As a result, the grades for exams "crashed down".

It was decided that the whole class, which consisted of about 175 students, should be split into two classes. Professor Viktor A. Zalgaller was appointed to give lectures to the students who were going to specialize in applied mathematics, while Assistant Professor Oleg Ya. Viro would give the lectures to students-mathematicians. Zalgaller suggested to introduce exercise lessons – one hour a week. As a result, the time allotted for the lectures decreased, and de facto the volume of the material was also reduced along with the time.

It remained to understand *what* to do in the exercise lessons. One had to develop a system of problems and exercises that would give an opportunity to revise the definitions given in the lectures, and would allow one to develop skills in proving easy theorems from general topology in the framework of a simple axiomatic theory.

Problems of the first part of the book are a result of our efforts in this direction. Gradually, exercise lessons and problems were becoming more and more useful as long as we had to teach students with lower level of preliminary training. In 1988, the Publishing House of the Leningrad State University published the problems in a small book "Problems in Topology".

Students found the book useful. One of them, Alekseĭ Solov'ev, even translated it into English on his own initiative, when he became a graduate student at the University of California. The translation opened a new stage of the work on the book. We started developing the Russian and English versions in parallel and practically covered the entire material of Rokhlin's course. In 2000, the Publishing House of the Saint Petersburg State University published the second Russian edition of the book, which already included a chapter on the fundamental group and coverings.

The English version was used by O. Viro for his lecture course in the USA (University of California) and Sweden (Uppsala University). The Russian version was used by V. Kharlamov for his lecture courses in France (Strasbourg University). The lectures have been given for quite different audiences: both for undergraduate and graduate students. Furthermore, few professors (some of whom the authors have known personally, while the others have not) asked authors' permission to use the English version in their lectures, both in the countries mentioned above and



other ones. New demands upon the text had arisen. For instance, we were asked to include solutions of problems and proofs of theorems in the book, in order to make it meet the Western standards and transform it from a problem book into a self-sufficient textbook. After some hesitations, we fulfilled those requests, the more so that they were joined by the Publishing House of the American Mathematical Society.

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Each of us has been lucky to be a student of Vladimir Abramovich Rokhlin, to whose memory we dedicate this book.



The authors, from the left to the right:

Oleg Yanovich Viro,  
Viatcheslav Mikhaĭlovich Kharlamov,  
Nikita Yur'evich Netsvetaev,  
Oleg Aleksandrovich Ivanov.