Part 1

General Topology

The goal of this part of the book is to teach the language of mathematics. More specifically, one of its most important components: the language of set-theoretic topology, which treats the basic notions related to continuity. The term *general topology* means: this is the topology that is needed and used by most mathematicians. A permanent usage in the capacity of a common mathematical language has polished its system of definitions and theorems. Nowadays, studying general topology really more resembles studying a language rather than mathematics: one needs to learn a lot of new words, while proofs of most theorems are extremely simple. On the other hand, the theorems are numerous because they play the role of rules regulating usage of words.

We have to warn the students for whom this is one of the first mathematical subjects. Do not hurry to fall in love with it, do not let an imprinting happen. This field may seem to be charming, but it is not very active. It hardly provides as much room for exciting new research as many other fields.

Chapter I

Structures and Spaces

1. Digression on Sets

We begin with a digression, which we would like to consider unnecessary. Its subject is the first basic notions of the naive set theory. This is a part of the common mathematical language, too, but even more profound than general topology. We would not be able to say anything about topology without this part (look through the next section to see that this is not an exaggeration). Naturally, it may be expected that the naive set theory becomes familiar to a student when she or he studies Calculus or Algebra, two subjects usually preceding topology. If this is what really happened to you, then, please, glance through this section and move to the next one.

1°1. Sets and Elements

In any intellectual activity, one of the most profound actions is gathering objects into groups. The gathering is performed in mind and is not accompanied with any action in the physical world. As soon as the group has been created and assigned a name, it can be a subject of thoughts and arguments and, in particular, can be included into other groups. Mathematics has an elaborated system of notions, which organizes and regulates creating those groups and manipulating them. This system is the *naive set theory*, which is a slightly misleading name because this is rather a language than a theory.

The first words in this language are *set* and *element*. By a set we understand an arbitrary collection of various objects. An object included into the collection is an *element* of the set. A set *consists* of its elements. It is also *formed* by them. To diversify wording, the word *set* is replaced by the word *collection*. Sometimes other words, such as *class*, *family*, and *group*, are used in the same sense, but this is not quite safe because each of these words is associated in modern mathematics with a more special meaning, and hence should be used instead of the word *set* with caution.

If x is an element of a set A, then we write $x \in A$ and say that x *belongs to* A and A *contains* x. The sign $\in$ is a variant of the Greek letter epsilon, which is the first letter of the Latin word *element*. To make notation more flexible, the formula $x \in A$ is also allowed to be written in the form $A \ni x$. So, the origin of notation is sort of ignored, but a more meaningful similarity to the inequality symbols $<$ and $>$ is emphasized. To state that x is not an element of A, we write $x \notin A$ or $A \not\ni x$.

1°2. Equality of Sets

A set is determined by its elements. It is nothing but a collection of its elements. This manifests most sharply in the following principle: *two sets are considered equal if and only if they have the same elements*. In this sense, the word *set* has slightly disparaging meaning. When something is called a set, this shows, maybe unintentionally, a lack of interest to whatever organization of the elements of this set.

For example, when we say that a line is a set of points, we assume that two lines coincide if and only if they consist of the same points. On the other hand, we commit ourselves to consider all relations between points on a line (e.g., the distance between points, the order of points on the line, etc.) separately from the notion of line.

We may think of sets as boxes that can be built effortlessly around elements, just to distinguish them from the rest of the world. The cost of this lightness is that such a box is not more than the collection of elements placed inside. It is a little more than just a name: it is a declaration of our wish to think about this collection of things as of entity and not to go into details about the nature of its members-elements. Elements, in turn, may also be sets, but as long as we consider them elements, they play the role of atoms, with their own original nature ignored.

In modern Mathematics, the words *set* and *element* are very common and appear in most texts. They are even overused. There are instances when it is not appropriate to use them. For example, it is not good to use the word *element* as a replacement for other, more meaningful words. When you call something an *element*, then the *set* whose element is this one should be clear. The word *element* makes sense only in combination with the word *set*, unless we deal with a nonmathematical term (like *chemical element*), or a rare old-fashioned exception from the common mathematical terminology (sometimes the expression under the sign of integral is called an *infinitesimal element*; in old texts lines, planes, and other geometric images are also called elements). Euclid's famous book on Geometry is called *Elements*, too.

1°3. The Empty Set

Thus, an element may not be without a set. However, a set may have no elements. Actually, there is a such set. This set is unique because a set is completely determined by its elements. It is the ***empty set*** denoted[1] by $\varnothing$.

1°4. Basic Sets of Numbers

Besides $\varnothing$, there are few other sets so important that they have their own unique names and notation. The set of all positive integers, i.e., 1, 2, 3, 4, 5, ..., etc., is denoted by $\mathbb{N}$. The set of all integers, both positive, negative, and the zero, is denoted by $\mathbb{Z}$. The set of all rational numbers (add to the integers those numbers which can be presented by fractions, like $\frac{2}{3}$ and $\frac{-7}{5}$) is denoted by $\mathbb{Q}$. The set of all real numbers (obtained by adjoining to rational numbers the numbers like $\sqrt{2}$ and $\pi = 3.14\dots$) is denoted by $\mathbb{R}$. The set of complex numbers is denoted by $\mathbb{C}$.

1°5. Describing a Set by Listing Its Elements

A set presented by a list a, b, ..., x of its elements is denoted by the symbol $\{a, b, \dots, x\}$. In other words, the list of objects enclosed in curly brackets denotes the set whose elements are listed. For example, $\{1, 2, 123\}$ denotes the set consisting of the numbers 1, 2, and 123. The symbol $\{a, x, A\}$ denotes the set consisting of three elements: a, x, and A, whatever objects these three letters are.

1.1. What is $\{\varnothing\}$? How many elements does it contain?

1.2. Which of the following formulas are correct:

1) $\varnothing \in \{\varnothing, \{\varnothing\}\}$; 2) $\{\varnothing\} \in \{\{\varnothing\}\}$; 3) $\varnothing \in \{\{\varnothing\}\}$?

A set consisting of a single element is a ***singleton***. This is any set which can be presented as $\{a\}$ for some a.

1.3. Is $\{\{\varnothing\}\}$ a singleton?

Notice that sets $\{1, 2, 3\}$ and $\{3, 2, 1, 2\}$ are equal since they consist of the same elements. At first glance, lists with repetitions of elements are never needed. There arises even a temptation to prohibit usage of lists with repetitions in such a notation. However, as it often happens to temptations to prohibit something, this would not be wise. In fact, quite often one cannot say a priori whether there are repetitions or not. For example, the

[1] Other notation, like Λ, is also in use, but $\varnothing$ has become common one.

elements in the list may depend on a parameter, and under certain values of the parameter some entries of the list coincide, while for other values they don't.

1.4. How many elements do the following sets contain?

1) $\{1, 2, 1\}$;	2) $\{1, 2, \{1, 2\}\}$;	3) $\{\{2\}\}$;
4) $\{\{1\}, 1\}$;	5) $\{1, \varnothing\}$;	6) $\{\{\varnothing\}, \varnothing\}$;
7) $\{\{\varnothing\}, \{\varnothing\}\}$;	8) $\{x, 3x-1\}$ for $x \in \mathbb{R}$.	

1°6. Subsets

If A and B are sets and every element of A also belongs to B, then we say that A is a *subset* of B, or B *includes* A, and write $A \subset B$ or $B \supset A$.

The inclusion signs $\subset$ and $\supset$ resemble the inequality signs $<$ and $>$ for a good reason: in the world of sets, the inclusion signs are obvious counterparts for the signs of inequalities.

1.A. Let a set A consist of a elements, and a set B of b elements. Prove that if $A \subset B$, then $a \leq b$.

1°7. Properties of Inclusion

1.B Reflexivity of Inclusion. Any set includes itself: $A \subset A$ holds true for any A.

Thus, the inclusion signs are not completely true counterparts of the inequality signs $<$ and $>$. They are closer to $\leq$ and $\geq$. Notice that no number a satisfies the inequality $a < a$.

1.C The Empty Set Is Everywhere. $\varnothing \subset A$ for any set A. In other words, the empty set is present in each set as a subset.

Thus, each set A has two obvious subsets: the empty set $\varnothing$ and A itself. A subset of A different from $\varnothing$ and A is a *proper* subset of A. This word is used when we do not want to consider the obvious subsets (which are *improper*).

1.D Transitivity of Inclusion. If A, B, and C are sets, $A \subset B$, and $B \subset C$, then $A \subset C$.

1°8. To Prove Equality of Sets, Prove Two Inclusions

Working with sets, we need from time to time to prove that two sets, say A and B, which may have emerged in quite different ways, are equal. The most common way to do this is provided by the following theorem.

1.E Criterion of Equality for Sets.

$$A = B \textit{ if and only if } A \subset B \textit{ and } B \subset A.$$

1°9. Inclusion Versus Belonging

1.F. *$x \in A$ if and only if $\{x\} \subset A$.*

Despite this obvious relation between the notions of belonging $\in$ and inclusion $\subset$ and similarity of the symbols $\in$ and $\subset$, the concepts are quite different. Indeed, $A \in B$ means that A is an element in B (i.e., one of the indivisible pieces comprising B), while $A \subset B$ means that A is made of some of the elements of B.

In particular, $A \subset A$, while $A \notin A$ for any reasonable A. Thus, *belonging is not reflexive*. One more difference: *belonging is not transitive*, while inclusion is.

1.G Nonreflexivity of Belonging. Construct a set A such that $A \notin A$. Cf. *1.B*.

1.H Non-Transitivity of Belonging. Construct sets A, B, and C such that $A \in B$ and $B \in C$, but $A \notin C$. Cf. *1.D*.

1°10. Defining a Set by a Condition

As we know (see 1°5), a set can be described by presenting a list of its elements. This simplest way may be not available or, at least, be not the easiest one. For example, it is easy to say: "the set of all solutions of the following equation" and write down the equation. This is a reasonable description of the set. At least, it is unambiguous. Having accepted it, we may start speaking on the set, studying its properties, and eventually may be lucky to solve the equation and obtain the list of its solutions. However, the latter may be difficult and should not prevent us from discussing the set.

Thus, we see another way for description of a set: to formulate properties that distinguish the elements of the set among elements of some wider and already known set. Here is the corresponding notation: the subset of a set A consisting of the elements x that satisfy a condition $P(x)$ is denoted by $\{x \in A \mid P(x)\}$.

1.5. Present the following sets by lists of their elements (i.e., in the form $\{a, b, \dots\}$)
(a) $\{x \in \mathbb{N} \mid x < 5\}$, (b) $\{x \in \mathbb{N} \mid x < 0\}$, (c) $\{x \in \mathbb{Z} \mid x < 0\}$.

1°11. Intersection and Union

The *intersection* of sets A and B is the set consisting of their common elements, i.e., elements belonging both to A and B. It is denoted by $A \cap B$ and can be described by the formula

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Two sets A and B are *disjoint* if their intersection is empty, i.e., $A \cap B = \varnothing$.

The *union* of two sets A and B is the set consisting of all elements that belong to at least one of these sets. The union of A and B is denoted by $A \cup B$. It can be described by the formula

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Here the conjunction *or* should be understood in the inclusive way: the statement "$x \in A$ or $x \in B$" means that x belongs to *at least one* of the sets A and B, but, maybe, to both of them.

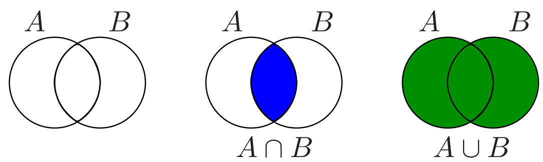


Figure 1. The sets A and B, their intersection $A \cap B$, and their union $A \cup B$.

1.I Commutativity of $\cap$ and $\cup$. For any two sets A and B, we have

$$A \cap B = B \cap A \quad \text{and} \quad A \cup B = B \cup A.$$

1.6. Prove that for any set A we have

$$A \cap A = A, \qquad A \cup A = A, \qquad A \cup \varnothing = A, \quad \text{and} \quad A \cap \varnothing = \varnothing.$$

1.7. Prove that for any sets A and B we have

$$A \subset B, \quad \text{iff} \quad A \cap B = A, \quad \text{iff} \quad A \cup B = B.$$

1.J Associativity of $\cap$ and $\cup$. For any sets A, B, and C, we have

$$(A \cap B) \cap C = A \cap (B \cap C) \quad \text{and} \quad (A \cup B) \cup C = A \cup (B \cup C).$$

Associativity allows us not to care about brackets and sometimes even omit them. We define $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$ and $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$. However, intersection and union of an arbitrarily large (in particular, infinite) collection of sets can be defined directly, without reference to intersection or union of two sets. Indeed, let Γ be a collection of sets. The *intersection* of the sets in Γ is the set formed by the elements that belong to *every* set in Γ. This set is denoted by $\bigcap_{A \in \Gamma} A$. Similarly, the *union* of the sets in Γ is the set formed by elements that belong to *at least one* of the sets in Γ. This set is denoted by $\bigcup_{A \in \Gamma} A$.

1.K. The notions of intersection and union of an arbitrary collection of sets generalize the notions of intersection and union of two sets: for $\Gamma = \{A, B\}$, we have

$$\bigcap_{C \in \Gamma} C = A \cap B \quad \text{and} \quad \bigcup_{C \in \Gamma} C = A \cup B.$$

1.8. Riddle. How do the notions of system of equations and intersection of sets related to each other?

1.L Two Distributivities. *For any sets A, B, and C, we have*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \tag{1}$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C). \tag{2}$$

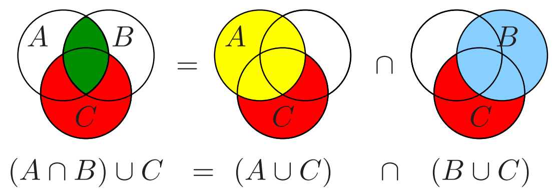


Figure 2. The left-hand side $(A \cap B) \cup C$ of equality (1) and the sets $A \cup C$ and $B \cup C$, whose intersection is the right-hand side of the equality (1).

In Figure 2, the first equality of Theorem *1.L* is illustrated by a sort of comics. Such comics are called *Venn diagrams* or *Euler circles*. They are quite useful and we strongly recommend to try to draw them for each formula about sets (at least, for formulas involving at most three sets).

1.M. Draw a Venn diagram illustrating (2). Prove (1) and (2) by tracing all details of the proofs in the Venn diagrams. Draw Venn diagrams illustrating all formulas below in this section.

1.9. Riddle. Generalize Theorem *1.L* to the case of arbitrary collections of sets.

1.N Yet Another Pair of Distributivities. *Let A be a set and Γ be a set consisting of sets. Then we have*

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \quad \text{and} \quad A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B).$$

1°12. Different Differences

The *difference* $A \smallsetminus B$ of two sets A and B is the set of those elements of A which do not belong to B. Here we do not assume that $A \supset B$.

If $A \supset B$, then the set $A \smallsetminus B$ is also called the *complement* of B in A.

1.10. Prove that for any sets A and B their union $A \cup B$ is the union of the following three sets: $A \smallsetminus B$, $B \smallsetminus A$, and $A \cap B$, which are pairwise disjoint.

1.11. Prove that $A \smallsetminus (A \smallsetminus B) = A \cap B$ for any sets A and B.

1.12. Prove that $A \subset B$ if and only if $A \smallsetminus B = \varnothing$.

1.13. Prove that $A \cap (B \smallsetminus C) = (A \cap B) \smallsetminus (A \cap C)$ for any sets A, B, and C.

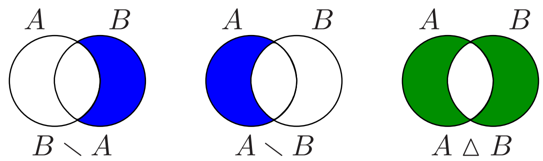


Figure 3. Differences of the sets A and B.

The set $(A \smallsetminus B) \cup (B \smallsetminus A)$ is the *symmetric difference* of the sets A and B. It is denoted by $A \bigtriangleup B$.

1.14. Prove that for any sets A and B

$$A \bigtriangleup B = (A \cup B) \smallsetminus (A \cap B)$$

1.15 Associativity of Symmetric Difference. Prove that for any sets A, B, and C we have

$$(A \bigtriangleup B) \bigtriangleup C = A \bigtriangleup (B \bigtriangleup C).$$

1.16. Riddle. Find a symmetric definition of the symmetric difference $(A \bigtriangleup B) \bigtriangleup C$ of three sets and generalize it to arbitrary finite collections of sets.

1.17 Distributivity. Prove that $(A \bigtriangleup B) \cap C = (A \cap C) \bigtriangleup (B \cap C)$ for any sets A, B, and C.

1.18. Does the following equality hold true for any sets A, B, and C:

$$(A \bigtriangleup B) \cup C = (A \cup C) \bigtriangleup (B \cup C)?$$

2. Topology in a Set

2°1. Definition of Topological Space

Let X be a set. Let Ω be a collection of its subsets such that:

(1) the union of any collection of sets that are elements of Ω belongs to Ω;

(2) the intersection of any finite collection of sets that are elements of Ω belongs to Ω;

(3) the empty set $\varnothing$ and the whole X belong to Ω.

Then

- Ω is a *topological structure* or just a *topology*[2] in X;
- the pair (X, Ω) is a *topological space*;
- elements of X are *points* of this topological space;
- elements of Ω are *open sets* of the topological space (X, Ω).

The conditions in the definition above are the *axioms of topological structure*.

2°2. Simplest Examples

A *discrete topological space* is a set with the topological structure consisting of all subsets.

2.A. Check that this is a topological space, i.e., all axioms of topological structure hold true.

An *indiscrete topological space* is the opposite example, in which the topological structure is the most meager. It consists only of X and $\varnothing$.

2.B. This is a topological structure, is it not?

Here are slightly less trivial examples.

2.1. Let X be the ray $[0, +\infty)$, and let Ω consist of $\varnothing$, X, and all rays $(a, +\infty)$ with $a \geq 0$. Prove that Ω is a topological structure.

2.2. Let X be a plane. Let Σ consist of $\varnothing$, X, and all open disks with center at the origin. Is this a topological structure?

2.3. Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topological structures in X, i.e., satisfy the axioms of topological structure:

[2] Thus Ω is important: it is called by the same word as the whole branch of mathematics. Certainly, this does not mean that Ω coincides with the subject of topology, but nearly everything in this subject is related to Ω.

(1) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a, c\}$, $\{a, b, c\}$, $\{a, b\}$;
(2) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{b, d\}$;
(3) $\varnothing$, X, $\{a, c, d\}$, $\{b, c, d\}$?

The space of *2.1* is the ***arrow***. We denote the space of *2.3* (1) by . It is a sort of toy space made of 4 points. Both spaces, as well as the space of *2.2*, are not too important, but they provide good simple examples.

2°3. The Most Important Example: Real Line

Let X be the set $\mathbb{R}$ of all real numbers, Ω the set of unions of all intervals (a, b) with $a, b \in \mathbb{R}$.

2.C. Check whether Ω satisfies the axioms of topological structure.

This is the topological structure which is always meant when $\mathbb{R}$ is considered as a topological space (unless another topological structure is explicitly specified). This space is usually called the ***real line***, and the structure is referred to as the ***canonical*** or ***standard*** topology in $\mathbb{R}$.

2°4. Additional Examples

2.4. Let X be $\mathbb{R}$, and let Ω consist of the empty set and all infinite subsets of $\mathbb{R}$. Is Ω a topological structure?

2.5. Let X be $\mathbb{R}$, and let Ω consists of the empty set and complements of all finite subsets of $\mathbb{R}$. Is Ω a topological structure?

The space of *2.5* is denoted by $\mathbb{R}_{T_1}$ and called the ***line with T_1-topology***.

2.6. Let (X, Ω) be a topological space, Y the set obtained from X by adding a single element a. Is

$$\{\{a\} \cup U \mid U \in \Omega\} \cup \{\varnothing\}$$

a topological structure in Y?

2.7. Is the set $\{\varnothing, \{0\}, \{0, 1\}\}$ a topological structure in $\{0, 1\}$?

If the topology Ω in Problem *2.6* is discrete, then the topology in Y is called a ***particular point topology*** or ***topology of everywhere dense point***. The topology in Problem *2.7* is a particular point topology; it is also called the ***topology of connected pair of points*** or ***Sierpiński topology***.

2.8. List all topological structures in a two-element set, say, in $\{0, 1\}$.

2°5. Using New Words: Points, Open Sets, Closed Sets

We recall that, for a topological space (X, Ω), elements of X are ***points***, and elements of Ω are ***open sets***.[3]

2.D. Reformulate the axioms of topological structure using the words *open set* wherever possible.

[3] The letter Ω stands for the letter O which is the initial of the words with the same meaning: *Open* in English, *Otkrytyj* in Russian, *Offen* in German, *Ouvert* in French.

A set $F \subset X$ is *closed* in the space (X, Ω) if its complement $X \smallsetminus F$ is open (i.e., $X \smallsetminus F \in \Omega$).

2°6. Set-Theoretic Digression: De Morgan Formulas

2.E. Let Γ be an arbitrary collection of subsets of a set X. Then

$$X \smallsetminus \bigcup_{A \in \Gamma} A = \bigcap_{A \in \Gamma} (X \smallsetminus A), \tag{3}$$

$$X \smallsetminus \bigcap_{A \in \Gamma} A = \bigcup_{A \in \Gamma} (X \smallsetminus A). \tag{4}$$

Formula (4) is deduced from (3) in one step, is it not? These formulas are nonsymmetric cases of a single formulation, which contains in a symmetric way sets and their complements, unions, and intersections.

2.9. Riddle. Find such a formulation.

2°7. Properties of Closed Sets

2.F. Prove that:

(1) the intersection of any collection of closed sets is closed;

(2) the union of any finite number of closed sets is closed;

(3) the empty set and the whole space (i.e., the underlying set of the topological structure) are closed.

2°8. Being Open or Closed

Notice that the property of being closed is not the negation of the property of being open. (They are not exact antonyms in everyday usage, too.)

2.G. Find examples of sets that are

(1) both open and closed simultaneously (open-closed);

(2) neither open, nor closed.

2.10. Give an explicit description of closed sets in
(a) a discrete space; (b) an indiscrete space;
(c) the arrow; (d) ;
(e) $\mathbb{R}_{T_1}$.

2.H. Is a closed segment $[a, b]$ closed in $\mathbb{R}$?

The concepts of closed and open sets are similar in a number of ways. The main difference is that the intersection of an infinite collection of open sets is not necessarily open, while the intersection of any collection of closed sets is closed. Along the same lines, the union of an infinite collection of closed sets is not necessarily closed, while the union of any collection of open sets is open.

2.11. Prove that the half-open interval $[0, 1)$ is neither open nor closed in $\mathbb{R}$, but is both a union of closed sets and an intersection of open sets.

2.12. Prove that the set $A = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is closed in $\mathbb{R}$.

2°9. Characterization of Topology in Terms of Closed Sets

2.13. Suppose a collection $\mathcal{F}$ of subsets of X satisfies the following conditions:

(1) the intersection of any family of sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(2) the union of any finite number sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(3) $\varnothing$ and X belong to $\mathcal{F}$.

Prove that then $\mathcal{F}$ is the set of all closed sets of a topological structure (which one?).

2.14. List all collections of subsets of a three-element set such that there exist topologies where these collections are complete sets of closed sets.

2°10. Neighborhoods

A *neighborhood* of a point is any open set containing this point. Analysts and French mathematicians (following N. Bourbaki) prefer a wider notion of neighborhood: they use this word for any set containing a neighborhood in the above sense.

2.15. Give an explicit description of all neighborhoods of a point in
(a) a discrete space;
(b) an indiscrete space;
(c) the arrow;
(d) ;
(e) connected pair of points;
(f) particular point topology.

2°11x. Open Sets on Line

2.Ax. *Prove that every open subset of the real line is a union of disjoint open intervals.*

At first glance, Theorem *2.Ax* suggests that open sets on the line are simple. However, an open set may lie on the line in a quite complicated manner. Its complement can be not that simple. The complement of an open set is a closed set. One can naively expect that a closed set on $\mathbb{R}$ is a union of closed intervals. The next important example shows that this is far from being true.

2°12x. Cantor Set

Let K be the set of real numbers that are sums of series of the form $\sum_{k=1}^{\infty} \dfrac{a_k}{3^k}$ with $a_k = 0$ or 2. In other words, K is the set of real numbers that are presented as $0.a_1a_2 \dots a_k \dots$ without the digit 1 in the positional system with base 3.

2.Bx. Find a geometric description of K.

2.Bx.1. Prove that

(1) K is contained in $[0, 1]$,
(2) K does not intersect $\left(\frac{1}{3}, \frac{2}{3}\right)$,
(3) K does not intersect $\left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$ for any integers k and s.

2.Bx.2. Present K as $[0, 1]$ with an infinite family of open intervals removed.

2.Bx.3. Try to sketch K.

The set K is the ***Cantor set***. It has a lot of remarkable properties and is involved in numerous problems below.

2.Cx. Prove that K is a closed set in the real line.

2°13x. Topology and Arithmetic Progressions

2.Dx*. Consider the following property of a subset F of the set $\mathbb{N}$ of positive integers: there exists $N \in \mathbb{N}$ such that F contains no arithmetic progressions of length greater than N. Prove that subsets with this property together with the whole $\mathbb{N}$ form a collection of closed subsets in some topology in $\mathbb{N}$.

When solving this problem, you probably will need the following combinatorial theorem.

2.Ex Van der Waerden's Theorem*. For every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for any subset $A \subset \{1, 2, \dots, N\}$, either A or $\{1, 2, \dots, N\} \smallsetminus A$ contains an arithmetic progression of length n.

See [**2**].

3. Bases

3°1. Definition of Base

The topological structure is usually presented by describing its part which is sufficient to recover the whole structure. A collection Σ of open sets is a ***base*** for a topology if each nonempty open set is a union of sets belonging to Σ. For instance, all intervals form a base for the real line.

3.1. Can two distinct topological structures have the same base?

3.2. Find some bases of topology of
(a) a discrete space; (b) ;
(c) an indiscrete space; (d) the arrow.
Try to choose the smallest possible bases.

3.3. Prove that any base of the canonical topology in $\mathbb{R}$ can be decreased.

3.4. Riddle. What topological structures have exactly one base?

3°2. When a Collection of Sets is a Base

3.A. *A collection Σ of open sets is a base for the topology iff for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$.*

3.B. *A collection Σ of subsets of a set X is a base for a certain topology in X iff X is a union of sets in Σ and the intersection of any two sets in Σ is a union of sets in Σ.*

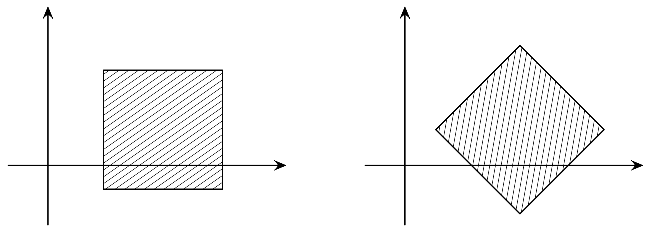
3.C. Show that the second condition in *3.B* (on the intersection) is equivalent to the following: the intersection of any two sets in Σ contains, together with any of its points, some set in Σ containing this point (cf. *3.A*).

3°3. Bases for Plane

Consider the following three collections of subsets of $\mathbb{R}^2$:

- Σ^2, which consists of all possible open disks (i.e., disks without their boundary circles);
- Σ^∞, which consists of all possible open squares (i.e., squares without their sides and vertices) with sides parallel to the coordinate axis;
- Σ^1, which consists of all possible open squares with sides parallel to the bisectors of the coordinate angles.

(The squares in Σ^∞ and Σ^1 are determined by the inequalities $\max\{|x-a|, |y-b|\} < \rho$ and $|x-a| + |y-b| < \rho$, respectively.)



3.5. Prove that every element of Σ^2 is a union of elements of Σ^∞.

3.6. Prove that the intersection of any two elements of Σ^1 is a union of elements of Σ^1.

3.7. Prove that each of the collections Σ^2, Σ^∞, and Σ^1 is a base for some topological structure in $\mathbb{R}^2$, and that the structures determined by these collections coincide.

3°4. Subbases

Let (X, Ω) be a topological space. A collection Δ of its open subsets is a *subbase* for Ω provided that the collection

$$\Sigma = \{V \mid V = \cap_{i=1}^k W_i,\ k \in \mathbb{N},\ W_i \in \Delta\}$$

of all finite intersections of sets in Δ is a base for Ω.

3.8. Let for any set X Δ be a collection of its subsets. Prove that Δ is a subbase for a topology in X iff $X = \cup_{W \in \Delta} W$.

3°5. Infiniteness of the Set of Prime Numbers

3.9. Prove that all infinite arithmetic progressions consisting of positive integers form a base for some topology in $\mathbb{N}$.

3.10. Using this topology, prove that the set of all prime numbers is infinite.

3°6. Hierarchy of Topologies

If Ω_1 and Ω_2 are topological structures in a set X such that $\Omega_1 \subset \Omega_2$, then Ω_2 is *finer* than Ω_1, and Ω_1 is *coarser* than Ω_2. For instance, the indiscrete topology is the coarsest topology among all topological structures in the same set, while the discrete topology is the finest one, is it not?

3.11. Show that the T_1-topology in the real line (see 2°4) is coarser than the canonical topology.

Two bases determining the same topological structure are *equivalent*.

3.D. Riddle. Formulate a necessary and sufficient condition for two bases to be equivalent without explicitly mentioning the topological structures determined by the bases. (Cf. *3.7*: the bases Σ^2, Σ^∞, and Σ^1 must satisfy the condition you are looking for.)

4. Metric Spaces

4°1. Definition and First Examples

A function $\rho : X \times X \to \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$ is a *metric* (or *distance function*) in X if

(1) $\rho(x, y) = 0$ iff $x = y$;
(2) $\rho(x, y) = \rho(y, x)$ for any $x, y \in X$;
(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$.

The pair (X, ρ), where ρ is a metric in X, is a *metric space*. Condition (3) is the *triangle inequality*.

4.A. Prove that the function

$$\rho : X \times X \to \mathbb{R}_+ : \ (x, y) \mapsto \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric for any set X.

4.B. Prove that $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$ is a metric.

4.C. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ is a metric.

The metrics of*4.B* and *4.C* are always meant when $\mathbb{R}$ and $\mathbb{R}^n$ are considered as metric spaces unless another metric is specified explicitly. The metric of *4.B* is a special case of the metric of *4.C*. All these metrics are called *Euclidean*.

4°2. Further Examples

4.1. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \max_{i=1,\dots,n} |x_i - y_i|$ is a metric.

4.2. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sum_{i=1}^n |x_i - y_i|$ is a metric.

The metrics in $\mathbb{R}^n$ introduced in *4.C*–*4.2* are members of an infinite series of the metrics:

$$\rho^{(p)} : \ (x, y) \mapsto \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

4.3. Prove that $\rho^{(p)}$ is a metric for any $p \geq 1$.

4.3.1 Hölder Inequality. Prove that

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p \right)^{1/p} \left(\sum_{i=1}^n y_i^q \right)^{1/q}$$

if $x_i, y_i \geq 0$, $p, q > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$.

The metric of *4.C* is $\rho^{(2)}$, that of *4.2* is $\rho^{(1)}$, and that of *4.1* can be denoted by $\rho^{(\infty)}$ and appended to the series since

$$\lim_{p \to +\infty} \left(\sum_{i=1}^{n} a_i^p \right)^{1/p} = \max a_i,$$

for any positive $a_1, a_2, \dots, a_n$.

4.4. Riddle. How is this related to Σ^2, Σ^∞, and Σ^1 from Section 3?

For a number $p \geq 1$ denote by $l^{(p)}$ the set of sequences $x = \{x_i\}_{i=1,2,\dots}$ such that the series $\sum_{i=1}^{\infty} |x|^p$ converges.

4.5. Prove that for any two sequences $x, y \in l^{(p)}$ the series $\sum_{i=1}^{\infty} |x_i - y_i|^p$ converges and that

$$(x, y) \mapsto \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}, \quad p \geq 1$$

is a metric in $l^{(p)}$.

4°3. Balls and Spheres

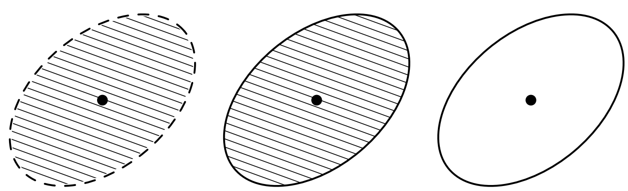
Let (X, ρ) be a metric space, $a \in X$ a point, r a positive real number. Then the sets

$$B_r(a) = \{ x \in X \mid \rho(a, x) < r \}, \tag{5}$$

$$D_r(a) = \{ x \in X \mid \rho(a, x) \leq r \}, \tag{6}$$

$$S_r(a) = \{ x \in X \mid \rho(a, x) = r \} \tag{7}$$

are, respectively, the *open ball*, *closed ball*, and *sphere* of the space (X, ρ) with center a and radius r.



4°4. Subspaces of a Metric Space

If (X, ρ) is a metric space and $A \subset X$, then the restriction of the metric ρ to $A \times A$ is a metric in A, and so $(A, \rho|_{A \times A})$ is a metric space. It is called a *subspace* of (X, ρ).

The disk $D_1(0)$ and the sphere $S_1(0)$ in $\mathbb{R}^n$ (with Euclidean metric, see *4.C*) are denoted by D^n and S^{n-1} and called the (*unit*) n-*disk* and $(n-1)$-*sphere*. They are regarded as metric spaces (with the metric induced from $\mathbb{R}^n$).

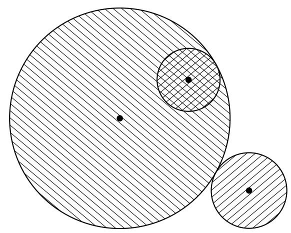
4.D. Check that D^1 is the segment $[-1, 1]$, D^2 is a plane disk, S^0 is the pair of points $\{-1, 1\}$, S^1 is a circle, S^2 is a sphere, and D^3 is a ball.

The last two assertions clarify the origin of the terms *sphere* and *ball* (in the context of metric spaces).

Some properties of balls and spheres in an arbitrary metric space resemble familiar properties of planar disks and circles and spatial balls and spheres.

4.E. Prove that for any points x and a of any metric space and any $r > \rho(a, x)$ we have

$$B_{r-\rho(a,x)}(x) \subset B_r(a) \text{ and } D_{r-\rho(a,x)}(x) \subset D_r(a).$$



4.6. Riddle. What if $r < \rho(x, a)$? What is an analog for the statement of Problem *4.E* in this case?

4°5. Surprising Balls

However, balls and spheres in other metric spaces may have rather surprising properties.

4.7. What are balls and spheres in $\mathbb{R}^2$ equipped with the metrics of *4.1* and *4.2*? (Cf. *4.4*.)

4.8. Find $D_1(a)$, $D_{\frac{1}{2}}(a)$, and $S_{\frac{1}{2}}(a)$ in the space of *4.A*.

4.9. Find a metric space and two balls in it such that the ball with the smaller radius contains the ball with the bigger one and does not coincide with it.

4.10. What is the minimal number of points in the space which is required to be constructed in *4.9*?

4.11. Prove that in *4.9* the largest radius does not exceed double the smaller radius.

4°6. Segments (What Is Between)

4.12. Prove that the segment with endpoints $a, b \in \mathbb{R}^n$ can be described as

$$\{\, x \in \mathbb{R}^n \mid \rho(a, x) + \rho(x, b) = \rho(a, b) \,\},$$

where ρ is the Euclidean metric.

4.13. How does the set defined as in *4.12* look like if ρ is the metric defined in *4.1* or *4.2*? (Consider the case, where $n = 2$ if it seems to be easier.)

4°7. Bounded Sets and Balls

A subset A of a metric space (X, ρ) is *bounded* if there is a number $d > 0$ such that $\rho(x, y) < d$ for any $x, y \in A$. The greatest lower bound for such d is the *diameter* of A, it is denoted by $\operatorname{diam}(A)$.

4.F. Prove that a set A is bounded iff A is contained in a ball.

4.14. What is the relation between the minimal radius of such a ball and $\operatorname{diam}(A)$?

4°8. Norms and Normed Spaces

Let X be a vector space (over $\mathbb{R}$). A function $X \to \mathbb{R}_+ : x \mapsto \|x\|$ is a *norm* if

(1) $\|x\| = 0$ iff $x = 0$;
(2) $\|\lambda x\| = |\lambda| \|x\|$ for any $\lambda \in \mathbb{R}$ and $x \in X$;
(3) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

4.15. Prove that if $x \mapsto \|x\|$ is a norm, then

$$\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto \|x - y\|$$

is a metric.

A vector space equipped with a norm is a *normed space*. The metric determined by the norm as in *4.15* transforms the normed space into a metric space in a canonical way.

4.16. Look through the problems of this section and figure out which of the metric spaces involved are, in fact, normed vector spaces.

4.17. Prove that every ball in a normed space is a convex[4] set symmetric with respect to the center of the ball.

4.18.* Prove that every convex closed bounded set in $\mathbb{R}^n$ that has a center of symmetry and is not contained in any affine space except $\mathbb{R}^n$ itself is a unit ball with respect to a certain norm, which is uniquely determined by this ball.

4°9. Metric Topology

4.G. *The collection of all open balls in the metric space is a base for some topology*

This topology is the *metric topology*. This topological structure is always meant whenever the metric space is regarded as a topological space (for instance, when we speak about open and closed sets, neighborhoods, etc. in this space).

4.H. Prove that the standard topological structure in $\mathbb{R}$ introduced in Section 2 is generated by the metric $(x, y) \mapsto |x - y|$.

---

[4] Recall that a set A is *convex* if for any $x, y \in A$ the segment connecting x and y is contained in A. Certainly, this definition involves the notion of *segment*, so it makes sense only for subsets of those spaces where the notion of segment connecting two points makes sense. This is the case in vector and affine spaces over $\mathbb{R}$.

4.19. What topological structure is generated by the metric of *4.A*?

4.I. *A set A is open in a metric space iff, together with each of its points, A contains a ball centered at this point.*

4°10. Openness and Closedness of Balls and Spheres

4.20. Prove that a closed ball is closed (with respect to the metric topology).

4.21. Find a closed ball that is open (with respect to the metric topology).

4.22. Find an open ball that is closed (with respect to the metric topology).

4.23. Prove that a sphere is closed.

4.24. Find a sphere that is open.

4°11. Metrizable Topological Spaces

A topological space is *metrizable* if its topological structure is generated by a certain metric.

4.J. An indiscrete space is not metrizable unless it is one-point (it has too few open sets).

4.K. A finite space X is metrizable iff it is discrete.

4.25. Which of the topological spaces described in Section 2 are metrizable?

4°12. Equivalent Metrics

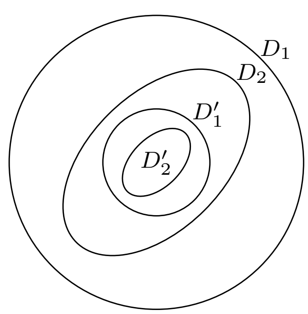
Two metrics in the same set are *equivalent* if they generate the same topology.

4.26. Are the metrics of *4.C*, *4.1*, and *4.2* equivalent?

4.27. Prove that two metrics ρ_1 and ρ_2 in X are equivalent if there are numbers $c, C > 0$ such that

$$c\rho_1(x,y) \leq \rho_2(x,y) \leq C\rho_1(x,y)$$

for any $x, y \in X$.



4.28. Generally speaking, the converse is not true.

4.29. Riddle. Hence, the condition of equivalence of metrics formulated in *4.27* can be weakened. How?

4.30. The metrics $\rho^{(p)}$ in $\mathbb{R}^n$ defined right before Problem *4.3* are equivalent.

4.31*. Prove that the following two metrics ρ_1 and ρ_C in the set of all continuous functions $[0,1] \to \mathbb{R}$ are not equivalent:

$$\rho_1(f,g) = \int_0^1 \big|f(x) - g(x)\big| dx, \qquad \rho_C(f,g) = \max_{x\in[0,1]} \big|f(x) - g(x)\big|.$$

Is it true that one of the topological structures generated by them is finer than another?

4°13. Operations With Metrics

4.32. 1) Prove that if ρ_1 and ρ_2 are two metrics in X, then $\rho_1 + \rho_2$ and $\max\{\rho_1, \rho_2\}$ also are metrics. 2) Are the functions $\min\{\rho_1, \rho_2\}$, $\dfrac{\rho_1}{\rho_2}$, and $\rho_1\rho_2$ metrics? By definition, for $\rho = \dfrac{\rho_1}{\rho_2}$ we put $\rho(x,x) = 0$.

4.33. Prove that if $\rho : X \times X \to \mathbb{R}_+$ is a metric, then

(1) the function

$$(x,y) \mapsto \frac{\rho(x,y)}{1 + \rho(x,y)}$$

is a metric;

(2) the function

$$(x,y) \mapsto \min\{\rho(x,y), 1\}$$

is a metric;

(3) the function

$$(x,y) \mapsto f\big(\rho(x,y)\big)$$

is a metric if f satisfies the following conditions:

(a) $f(0) = 0$,
(b) f is a monotone increasing function, and
(c) $f(x+y) \le f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

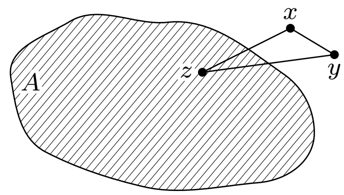
4.34. Prove that the metrics ρ and $\dfrac{\rho}{1+\rho}$ are equivalent.

4°14. Distances Between Points and Sets

Let (X, ρ) be a metric space, $A \subset X$, $b \in X$. The number $\rho(b, A) = \inf\{\, \rho(b,a) \mid a \in A \,\}$ is the *distance from the point b to the set A*.

4.L. Let A be a closed set. Prove that $\rho(b, A) = 0$ iff $b \in A$.

4.35. Prove that $|\rho(x,A) - \rho(y,A)| \le \rho(x,y)$ for any set A and any points x and y in a metric space.



$\rho(x, A) \leqslant \rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$

4°15x. Distance Between Sets

Let A and B be two bounded subsets in a metric space (X, ρ). Put

$$d_\rho(A, B) = \max\Big\{\sup_{a \in A} \rho(a, B),\ \sup_{b \in B} \rho(b, A)\Big\}.$$

This number is the ***Hausdorff distance*** between A and B.

4.Ax. Prove that the Hausdorff distance between bounded subsets of a metric space satisfies conditions (2) and (3) in the definition of a metric.

4.Bx. Prove that for every metric space the Hausdorff distance is a metric in the set of its closed bounded subsets.

Let A and B be two bounded polygons in the plane.[5] We define

$$d_\Delta(A, B) = S(A) + S(B) - 2S(A \cap B),$$

where $S(C)$ is the area of the polygon C.

4.Cx. Prove that d_Δ is a metric in the set of all bounded plane polygons.

We will call d_Δ the ***area metric***.

4.Dx. Prove that the area metric is *not* equivalent to the Hausdorff metric in the set of all bounded plane polygons.

4.Ex. Prove that the area metric is equivalent to the Hausdorff metric in the set of convex bounded plane polygons.

4°16x. Ultrametrics and p-Adic Numbers

A metric ρ is an ***ultrametric*** if it satisfies the ***ultrametric triangle inequality***:

$$\rho(x, y) \leq \max\{\rho(x, z),\ \rho(z, y)\}$$

for any x, y, and z.

A metric space (X, ρ), where ρ is an ultrametric, is an ***ultrametric space***.

[5] Although we assume that the notion of bounded polygon is well known from elementary geometry, nevertheless, we recall the definition. A ***bounded plane polygon*** is the set of the points of a simple closed polygonal line γ and the points surrounded by γ. A simple closed polygonal line is a cyclic sequence of segments each of which starts at the point where the previous one ends and these are the only pairwise intersections of the segments.

4.Fx. Check that only one metric in *4.A–4.2* is an ultrametric. Which one?

4.Gx. Prove that all triangles in an ultrametric space are isosceles (i.e., for any three points a, b, and c two of the three distances $\rho(a,b)$, $\rho(b,c)$, and $\rho(a,c)$ are equal).

4.Hx. Prove that spheres in an ultrametric space are not only closed (see *4.23*), but also open.

The most important example of an ultrametric is the *p-adic metric* in the set $\mathbb{Q}$ of rational numbers. Let p be a prime number. For $x, y \in \mathbb{Q}$, present the difference $x - y$ as $\frac{r}{s}p^{\alpha}$, where r, s, and α are integers, and r and s are co-prime with p. Put $\rho(x,y) = p^{-\alpha}$.

4.Ix. Prove that this is an ultrametric.

4°17x. Asymmetrics

A function $\rho : X \times X \to \mathbb{R}_+$ is an *asymmetric* in a set X if

(1) $\rho(x,y) = 0$ and $\rho(y,x) = 0$, iff $x = y$;
(2) $\rho(x,y) \le \rho(x,z) + \rho(z,y)$ for any $x, y, z \in X$.

Thus, an asymmetric satisfies conditions 1 and 3 of the definition of a metric, but, maybe, does not satisfy condition 2.

Here is example of an asymmetric taken from "the real life": the shortest length of path from one point to another by car in a city where there exist one-way streets.

4.Jx. Prove that if $\rho : X \times X \to \mathbb{R}_+$ is an asymmetric, then the function

$$(x,y) \mapsto \rho(x,y) + \rho(y,x)$$

is a metric in X.

Let A and B be two bounded subsets of a metric space (X,ρ). The number $a_\rho(A,B) = \sup_{b\in B} \rho(b,A)$ is the *asymmetric distance from A to B*.

4.Kx. The function a_ρ on the set of bounded subsets of a metric space satisfies the triangle inequality in the definition of an asymmetric.

4.Lx. Let (X,ρ) be a metric space. A set $B \subset X$ is contained in all closed sets containing $A \subset X$ iff $a_\rho(A,B) = 0$.

4.Mx. Prove that a_ρ is an asymmetric in the set of all bounded closed subsets of a metric space (X,ρ).

Let A and B be two polygons on the plane. Put

$$a_\Delta(A,B) = S(B) - S(A \cap B) = S(B \smallsetminus A),$$

where $S(C)$ is the area of polygon C.

4.1x. Prove that a_Δ is an asymmetric in the set of all planar polygons.

A pair (X, ρ), where ρ is an asymmetric in X, is an *asymmetric space*. Of course, any metric space is an asymmetric space, too. In an asymmetric space, balls (open and closed) and spheres are defined like in a metric space, see $4^\circ 3$.

4.Nx. *The set of all open balls of an asymmetric space is a base of a certain topology.*

This topology is *generated* by the asymmetric.

4.2x. Prove that the formula $a(x, y) = \max\{x - y, 0\}$ determines an asymmetric in $[0, \infty)$, and the topology generated by this asymmetric is the arrow topology, see $2^\circ 2$.

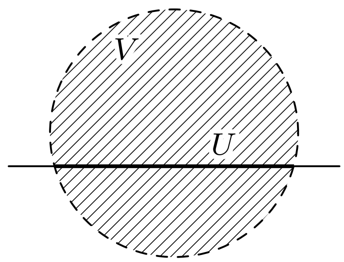
5. Subspaces

5°1. Topology for a Subset of a Space

Let (X, Ω) be a topological space, $A \subset X$. Denote by Ω_A the collection of sets $A \cap V$, where $V \in \Omega$: $\Omega_A = \{A \cap V \mid V \in \Omega\}$.

5.A. *Ω_A is a topological structure in A.*

The pair (A, Ω_A) is a *subspace* of the space (X, Ω). The collection Ω_A is the *subspace topology*, the *relative topology*, or the topology *induced* on A by Ω, and its elements are said to be sets *open* in A.



5.B. The canonical topology in $\mathbb{R}^1$ coincides with the topology induced on $\mathbb{R}^1$ as on a subspace of $\mathbb{R}^2$.

5.1. Riddle. How to construct a base for the topology induced on A by using a base for the topology in X?

5.2. Describe the topological structures induced

(1) on the set $\mathbb{N}$ of positive integers by the topology of the real line;
(2) on $\mathbb{N}$ by the topology of the arrow;
(3) on the two-point set $\{1, 2\}$ by the topology of $\mathbb{R}_{T_1}$;
(4) on the same set by the topology of the arrow.

5.3. Is the half-open interval $[0, 1)$ open in the segment $[0, 2]$ regarded as a subspace of the real line?

5.C. *A set F is closed in a subspace $A \subset X$ iff F is the intersection of A and a closed subset of X.*

5.4. If a subset of a subspace is open (respectively, closed) in the ambient space, then it is also open (respectively, closed) in the subspace.

5°2. Relativity of Openness and Closedness

Sets that are open in a subspace are not necessarily open in the ambient space.

5.D. The unique open set in $\mathbb{R}^1$ which is also open in $\mathbb{R}^2$ is $\varnothing$.

However, the following is true.

5.E. An open set of an open subspace is open in the ambient space, i.e., if $A \in \Omega$, then $\Omega_A \subset \Omega$.

The same relation holds true for closed sets. Sets that are closed in the subspace are not necessarily closed in the ambient space. However, the following is true.

5.F. Closed sets of a closed subspace are closed in the ambient space.

> ***5.5.*** Prove that a set U is open in X iff each point in U has a neighborhood V in X such that $U \cap V$ is open in V.
>
> This allows us to say that the property of being open is local. Indeed, we can reformulate *5.5* as follows: a set is open iff it is open in a neighborhood of each of its points.
>
> ***5.6.*** Show that the property of being closed is not local.

5.G Transitivity of Induced Topology. *Let (X, Ω) be a topological space, $X \supset A \supset B$. Then $(\Omega_A)_B = \Omega_B$, i.e., the topology induced on B by the relative topology of A coincides with the topology induced on B directly from X.*

> ***5.7.*** Let (X, ρ) be a metric space, $A \subset X$. Then the topology in A generated by the metric $\rho|_{A \times A}$ coincides with the relative topology on A by the topology in X generated by the metric ρ.
>
> ***5.8. Riddle.*** The statement *5.7* is equivalent to a pair of inclusions. Which of them is less obvious?

5°3. Agreement on Notation of Topological Spaces

Different topological structures in the same set are not considered simultaneously very often. That is why a topological space is usually denoted by the same symbol as the set of its points, i.e., instead of (X, Ω) we write just X. The same applies to metric spaces: instead of (X, ρ) we write just X.

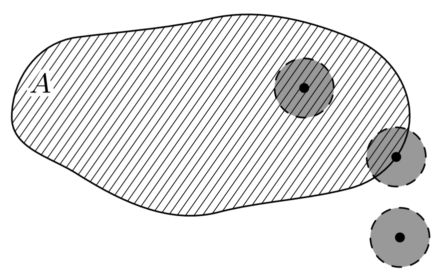
6. Position of a Point with Respect to a Set

This section is devoted to further expanding the vocabulary needed when we speak about phenomena in a topological space.

6°1. Interior, Exterior, and Boundary Points

Let X be a topological space, $A \subset X$ a subset, and $b \in X$ a point. The point b is

- an *interior* point of A if b has a neighborhood contained in A;
- an *exterior* point of A if b has a neighborhood disjoint with A;
- a *boundary* point of A if each neighborhood of b intersects both A and the complement of A.



6°2. Interior and Exterior

The *interior* of a set A in a topological space X is the greatest (with respect to inclusion) open set in X contained in A, i.e., an open set that contains any other open subset of A. It is denoted by $\operatorname{Int} A$ or, in more detail, by $\operatorname{Int}_X A$.

6.A. *Every subset of a topological space has interior. It is the union of all open sets contained in this set.*

6.B. *The interior of a set A is the set of interior points of A.*

6.C. *A set is open iff it coincides with its interior.*

6.D. Prove that in $\mathbb{R}$:

(1) $\operatorname{Int}[0,1) = (0,1)$,

(2) $\operatorname{Int} \mathbb{Q} = \varnothing$ and

(3) $\operatorname{Int}(\mathbb{R} \smallsetminus \mathbb{Q}) = \varnothing$.

6.1. Find the interior of $\{a, b, d\}$ in the space .

6.2. Find the interior of the interval $(0, 1)$ on the line with the Zariski topology.

The *exterior* of a set is the greatest open set disjoint with A. It is obvious that the exterior of A is $\operatorname{Int}(X \smallsetminus A)$.

6°3. Closure

The *closure* of a set A is the smallest closed set containing A. It is denoted $\operatorname{Cl} A$ or, more specifically, $\operatorname{Cl}_X A$.

6.E. *Every subset of topological space has closure. It is the intersection of all closed sets containing this set.*

6.3. Prove that if A is a subspace of X and $B \subset A$, then $\operatorname{Cl}_A B = (\operatorname{Cl}_X B) \cap A$. Is it true that $\operatorname{Int}_A B = (\operatorname{Int}_X B) \cap A$?

A point b is an *adherent* point for a set A if all neighborhoods of b intersect A.

6.F. *The closure of a set A is the set of the adherent points of A.*

6.G. *A set A is closed iff $A = \operatorname{Cl} A$.*

6.H. *The closure of a set A is the complement of the exterior of A. In formulas:* $\operatorname{Cl} A = X \smallsetminus \operatorname{Int}(X \smallsetminus A)$, *where X is the space and $A \subset X$.*

6.I. Prove that in $\mathbb{R}$ we have:

(1) $\operatorname{Cl}[0, 1) = [0, 1]$,
(2) $\operatorname{Cl} \mathbb{Q} = \mathbb{R}$,
(3) $\operatorname{Cl}(\mathbb{R} \smallsetminus \mathbb{Q}) = \mathbb{R}$.

6.4. Find the closure of $\{a\}$ in .

6°4. Closure in Metric Space

Let A be a subset and b a point of a metric space (X, ρ). Recall that the distance $\rho(b, A)$ from b to A is $\inf\{\, \rho(b, a) \mid a \in A \,\}$ (see 4°14).

6.J. Prove that $b \in \operatorname{Cl} A$ iff $\rho(b, A) = 0$.

6°5. Boundary

The *boundary* of a set A is the set $\operatorname{Cl} A \smallsetminus \operatorname{Int} A$. It is denoted by $\operatorname{Fr} A$ or, in more detail, $\operatorname{Fr}_X A$.

6.5. Find the boundary of $\{a\}$ in .

6.K. The boundary of a set is the set of its boundary points.

6.L. Prove that a set A is closed iff $\operatorname{Fr} A \subset A$.

6.6. 1) Prove that $\operatorname{Fr} A = \operatorname{Fr}(X \smallsetminus A)$. 2) Find a formula for $\operatorname{Fr} A$ which is symmetric with respect to A and $X \smallsetminus A$.

6.7. The boundary of a set A equals the intersection of the closure of A and the closure of the complement of A:

$$\operatorname{Fr} A = \operatorname{Cl} A \cap \operatorname{Cl}(X \smallsetminus A).$$

6°6. Closure and Interior with Respect to a Finer Topology

6.8. Let Ω_1 and Ω_2 be two topological structures in X, and $\Omega_1 \subset \Omega_2$. Let Cl_i denote the closure with respect to Ω_i. Prove that $\operatorname{Cl}_1 A \supset \operatorname{Cl}_2 A$ for any $A \subset X$.

6.9. Formulate and prove an analogous statement about interior.

6°7. Properties of Interior and Closure

6.10. Prove that if $A \subset B$, then $\operatorname{Int} A \subset \operatorname{Int} B$.

6.11. Prove that $\operatorname{Int} \operatorname{Int} A = \operatorname{Int} A$.

6.12. Do the following equalities hold true that for any sets A and B:

$$\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B, \tag{8}$$

$$\operatorname{Int}(A \cup B) = \operatorname{Int} A \cup \operatorname{Int} B? \tag{9}$$

6.13. Give an example in where one of equalities (8) and (9) is wrong.

6.14. In the example that you found when solving Problem *6.12*, an inclusion of one side into another one holds true. Does this inclusion hold true for any A and B?

6.15. Study the operator Cl in a way suggested by the investigation of Int undertaken in *6.10–6.14*.

6.16. Find $\operatorname{Cl}\{1\}$, $\operatorname{Int}[0,1]$, and $\operatorname{Fr}(2,+\infty)$ in the arrow.

6.17. Find $\operatorname{Int}\big((0,1] \cup \{2\}\big)$, $\operatorname{Cl}\{\,\frac{1}{n} \mid n \in \mathbb{N}\,\}$, and $\operatorname{Fr} \mathbb{Q}$ in $\mathbb{R}$.

6.18. Find $\operatorname{Cl} \mathbb{N}$, $\operatorname{Int}(0,1)$, and $\operatorname{Fr}[0,1]$ in $\mathbb{R}_{T_1}$. How to find the closure and interior of a set in this space?

6.19. Does a sphere contain the boundary of the open ball with the same center and radius?

6.20. Does a sphere contain the boundary of the closed ball with the same center and radius?

6.21. Find an example in which a sphere is disjoint with the closure of the open ball with the same center and radius.

6°8. Compositions of Closure and Interior

6.22 The Kuratowski Problem. How many pairwise distinct sets can one obtain from of a single set by using the operators Cl and Int?

The following problems will help you to solve problem *6.22*.

6.22.1. Find a set $A \subset \mathbb{R}$ such that the sets A, $\operatorname{Cl} A$, and $\operatorname{Int} A$ would be pairwise distinct.

6.22.2. Is there a set $A \subset \mathbb{R}$ such that

(1) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, and $\operatorname{Cl}\operatorname{Int} A$ are pairwise distinct;
(2) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, and $\operatorname{Int}\operatorname{Cl} A$ are pairwise distinct;
(3) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, $\operatorname{Cl}\operatorname{Int} A$, and $\operatorname{Int}\operatorname{Cl} A$ are pairwise distinct?

If you find such sets, keep on going in the same way, and when you fail to proceed, try to formulate a theorem explaining the failure.

6.22.3. Prove that $\operatorname{Cl}\operatorname{Int}\operatorname{Cl}\operatorname{Int} A = \operatorname{Cl}\operatorname{Int} A$.

6°9. Sets with Common Boundary

6.23*. Find three open sets in the real line that have the same boundary. Is it possible to increase the number of such sets?

6°10. Convexity and Int, Cl, Fr

Recall that a set $A \subset \mathbb{R}^n$ is *convex* if together with any two points it contains the entire segment connecting them (i.e., for any $x, y \in A$ every point z belonging to the segment $[x, y]$ belongs to A).

Let A be a convex set in $\mathbb{R}^n$.

6.24. Prove that $\operatorname{Cl} A$ and $\operatorname{Int} A$ are convex.

6.25. Prove that A contains a ball, unless A is contained in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$.

6.26. When is $\operatorname{Fr} A$ convex?

6°11. Characterization of Topology by Closure and Interior Operations

6.27*. Suppose that Cl_* is an operator in the set of all subsets of a set X, which has the following properties:

(1) $\operatorname{Cl}_* \varnothing = \varnothing$,
(2) $\operatorname{Cl}_* A \supset A$,
(3) $\operatorname{Cl}_*(A \cup B) = \operatorname{Cl}_* A \cup \operatorname{Cl}_* B$,
(4) $\operatorname{Cl}_* \operatorname{Cl}_* A = \operatorname{Cl}_* A$.

Prove that $\Omega = \{U \subset X \mid \operatorname{Cl}_*(X \smallsetminus U) = X \smallsetminus U\}$ is a topological structure and $\operatorname{Cl}_* A$ is the closure of a set A in the space (X, Ω).

6.28. Find an analogous system of axioms for Int.

6°12. Dense Sets

Let A and B be two sets in a topological space X. A is *dense in* B if $\operatorname{Cl} A \supset B$, and A is *everywhere dense* if $\operatorname{Cl} A = X$.

6.M. *A set is everywhere dense iff it intersects any nonempty open set.*

6.N. The set $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$.

6.29. Give a characterization of everywhere dense sets 1) in an indiscrete space, 2) in the arrow, and 3) in $\mathbb{R}_{T_1}$.

6.30. Prove that a topological space is discrete iff it has a unique everywhere dense set. (By the way, which one?)

6.31. Formulate a necessary and sufficient condition on the topology of a space which has an everywhere-dense point. Find spaces satisfying this condition in 2.

6.32. 1) Is it true that the union of everywhere dense sets is everywhere dense? 2) Is it true that the intersection of two everywhere-dense sets is everywhere dense?

6.33. Prove that the intersection of two open everywhere-dense sets is everywhere dense.

6.34. Which condition in the Problem *6.33* is redundant?

6.35*. 1) Prove that a countable intersection of open everywhere-dense sets in $\mathbb{R}$ is everywhere dense. 2) Is it possible to replace $\mathbb{R}$ here by an arbitrary topological space?

6.36*. Prove that $\mathbb{Q}$ is not an intersection of a countable collection of open sets in $\mathbb{R}$.

6°13. Nowhere Dense Sets

A set is *nowhere dense* if its exterior is everywhere dense.

6.37. Can a set be everywhere dense and nowhere dense simultaneously?

6.O. A set A is nowhere dense in X iff each neighborhood of each point $x \in X$ contains a point y such that the complement of A contains y together with a neighborhood of y.

6.38. Riddle. What can you say about the interior of a nowhere dense set?

6.39. Is $\mathbb{R}$ nowhere dense in $\mathbb{R}^2$?

6.40. Prove that if A is nowhere dense, then $\operatorname{Int}\operatorname{Cl} A = \varnothing$.

6.41. 1) Prove that the boundary of a closed set is nowhere dense. 2) Is this true for the boundary of an open set? 3) Is this true for the boundary of an arbitrary set?

6.42. Prove that a finite union of nowhere dense sets is nowhere dense.

6.43. Prove that for every set A there exists a greatest open set B in which A is dense. The extreme cases $B = X$ and $B = \varnothing$ mean that A is either everywhere dense or nowhere dense respectively.

6.44*. Prove that $\mathbb{R}$ is not a union of a countable collection of nowhere-dense sets in $\mathbb{R}$.

6°14. Limit Points and Isolated Points

A point b is a *limit point* of a set A, if each neighborhood of b intersects $A \smallsetminus b$.

6.P. Every limit point of a set is its adherent point.

6.45. Give an example where an adherent point is not a limit one.

A point b is an *isolated point* of a set A if $b \in A$ and b has a neighborhood disjoint with $A \setminus b$.

6.Q. *A set A is closed iff A contains all of its limit points.*

6.46. Find limit and isolated points of the sets $(0,1] \cup \{2\}$, $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ in $\mathbb{Q}$ and in $\mathbb{R}$.

6.47. Find limit and isolated points of the set $\mathbb{N}$ in $\mathbb{R}_{T_1}$.

6°15. Locally Closed Sets

A subset A of a topological space X is *locally closed* if each point of A has a neighborhood U such that $A \cap U$ is closed in U (cf. *5.5–5.6*).

6.48. Prove that the following conditions are equivalent:

(1) A is locally closed in X;
(2) A is an open subset of its closure $\operatorname{Cl} A$;
(3) A is the intersection of open and closed subsets of X.

7. Ordered Sets

This section is devoted to orders. They are structures in sets and occupy in Mathematics a position almost as profound as topological structures. After a short general introduction, we will focus on relations between structures of these two types. Like metric spaces, partially ordered sets possess natural topological structures. This is a source of interesting and important examples of topological spaces. As we will see later (in Section 20), practically all finite topological spaces appear in this way.

7°1. Strict Orders

A binary relation in a set X is a set of ordered pairs of elements of X, i.e., a subset $R \subset X \times X$. Many relations are denoted by special symbols, like $\prec$, $\vdash$, $\equiv$, or $\sim$. In the case where such a notation is used, there is a tradition to write xRy instead of writing $(x, y) \in R$. So, we write $x \vdash y$, or $x \sim y$, or $x \prec y$, etc. This generalizes the usual notation for the classical binary relations $=$, $<$, $>$, $\leq$, $\subset$, etc.

A binary relation $\prec$ in a set X is a *strict partial order*, or just a *strict order* if it satisfies the following two conditions:

- *Irreflexivity*: There is no $a \in X$ such that $a \prec a$.
- *Transitivity*: $a \prec b$ and $b \prec c$ imply $a \prec c$ for any $a, b, c \in X$.

7.A Antisymmetry. Let $\prec$ be a strict partial order in a set X. There are no $x, y \in X$ such that $x \prec y$ and $y \prec x$ simulteneously.

7.B. Relation $<$ in the set $\mathbb{R}$ of real numbers is a strict order.

Formula $a \prec b$ is read sometimes as "a is less than b" or "b is greater than a", but it is often read as "b follows a" or "a precedes b". The advantage of the latter two ways of reading is that then the relation $\prec$ is not associated too closely with the inequality between real numbers.

7°2. Nonstrict Orders

A binary relation $\preceq$ in a set X is a *nonstrict partial order*, or just *nonstrict order*, if it satisfies the following three conditions:

- *Transitivity*: If $a \preceq b$ and $b \preceq c$, then $a \preceq c$ for any $a, b, c \in X$.
- *Antisymmetry*: If $a \preceq b$ and $b \preceq a$, then $a = b$ for any $a, b \in X$.
- *Reflexivity*: $a \preceq a$ for any $a \in X$.

7.C. Relation $\leq$ in $\mathbb{R}$ is a nonstrict order.

7.D. In the set $\mathbb{N}$ of positive integers, the relation $a|b$ (a *divides* b) is a nonstrict partial order.

7.1. Is the relation $a|b$ a nonstrict partial order in the set $\mathbb{Z}$ of integers?

7.E. In the set of subsets of a set X, inclusion is a nonstrict partial order.

7°3. Relation between Strict and Nonstrict Orders

7.F. For each strict order $\prec$, there is a relation $\preceq$ defined in the same set as follows: $a \preceq b$ if either $a \prec b$, or $a = b$. This relation is a nonstrict order.

The nonstrict order $\preceq$ of *7.F* is *associated* with the original strict order $\prec$.

7.G. For each nonstrict order $\preceq$, there is a relation $\prec$ defined in the same set as follows: $a \prec b$ if $a \preceq b$ and $a \neq b$. This relation is a strict order.

The strict order $\prec$ of *7.G* is *associated* with the original nonstrict order $\preceq$.

7.H. The constructions of Problems *7.F* and *7.G* are mutually inverse: applied one after another in any order, they give the initial relation.

Thus, strict and nonstrict orders determine each other. They are just different incarnations of the same structure of order. We have already met a similar phenomenon in topology: open and closed sets in a topological space determine each other and provide different ways for describing a topological structure.

A set equipped with a partial order (either strict or nonstrict) is a *partially ordered set* or *poset*. More formally speaking, a partially ordered set is a pair $(X, \prec)$ formed by a set X and a strict partial order $\prec$ in X. Certainly, instead of a strict partial order $\prec$ we can use the corresponding nonstrict order $\preceq$.

Which of the orders, strict or nonstrict, prevails in each specific case is a matter of convenience, taste, and tradition. Although it would be handy to keep both of them available, nonstrict orders conquer situation by situation. For instance, nobody introduces notation for strict divisibility. Another example: the symbol $\subseteq$, which is used to denote nonstrict inclusion, is replaced by the symbol $\subset$, which is almost never understood as notation solely for strict inclusion.

In abstract considerations, we will use both kinds of orders: strict partial order are denoted by symbol $\prec$, nonstrict ones by symbol $\preceq$.

7°4. Cones

Let $(X, \prec)$ be a poset and let $a \in X$. The set $\{x \in X \mid a \prec x\}$ is the *upper cone* of a, and the set $\{x \in X \mid x \prec a\}$ the *lower cone* of a.

The element a does not belong to its cones. Adding a to them, we obtain *completed* cones: the *upper completed cone* or *star* $C_X^+(a) = \{x \in X \mid a \preceq x\}$ and the *lower completed cone* $C_X^-(a) = \{x \in X \mid x \preceq a\}$.

7.I Properties of Cones. *Let $(X, \prec)$ be a poset.*

(1) *$C_X^+(b) \subset C_X^+(a)$, provided that $b \in C_X^+(a)$;*

(2) *$a \in C_X^+(a)$ for each $a \in X$.*

(3) *$C_X^+(a) = C_X^+(b)$ implies $a = b$;*

7.J Cones Determine an Order. *Let X be an arbitrary set. Suppose for each $a \in X$ we fix a subset $C_a \subset X$ so that*

(1) *$b \in C_a$ implies $C_b \subset C_a$,*

(2) *$a \in C_a$ for each $a \in X$, and*

(3) *$C_a = C_b$ implies $a = b$.*

We write $a \prec b$ if $b \in C_a$. Then the relation $\prec$ is a nonstrict order in X, and for this order we have $C_X^+(a) = C_a$.

7.2. Let $C \subset \mathbb{R}^3$ be a set. Consider the relation $\lhd_C$ in $\mathbb{R}^3$ defined as follows: $a \lhd_C b$ if $b - a \in C$. What properties of C imply that $\lhd_C$ is a partial order in $\mathbb{R}^3$? What are the upper and lower cones in the poset $(\mathbb{R}^3, \lhd_C)$?

7.3. Prove that any convex cone C in $\mathbb{R}^3$ with vertex $(0,0,0)$ such that $P \cap C = \{(0,0,0)\}$ for some plane P satisfies the conditions found in the solution of Problem *7.2*.

7.4. The space-time $\mathbb{R}^4$ of special relativity theory (where points represent moment point events, the first three coordinates x_1, x_2, x_3 are the spatial coordinates, while the fourth one, t, is the time) carries a relation the *event* (x_1, x_2, x_3, t) *precedes* (and may influence) the event $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{t})$. This relation is defined by the inequality

$$c(\tilde{t} - t) \geq \sqrt{(\widetilde{x}_1 - x_1)^2 + (\widetilde{x}_2 - x_2)^2 + (\widetilde{x}_3 - x_3)^2}.$$

Is this a partial order? If yes, then what are the upper and lower cones of an event?

7.5. Answer the versions of questions of the preceding problem in the case two-dimensional and three-dimensional analogues of this space, where the number of spatial coordinates is 1 and 2, respectively.

7°5. Position of an Element with Respect to a Set

Let $(X, \prec)$ be a poset, $A \subset X$ a subset. Then b is the *greatest element* of A if $b \in A$ and $c \preceq b$ for every $c \in A$. Similarly, b is the *smallest element* of A if $b \in A$ and $b \preceq c$ for every $c \in A$.

7.K. An element $b \in A$ is the smallest element of A iff $A \subset C_X^+(b)$; an element $b \in A$ is the greatest element of A iff $A \subset C_X^-(b)$.

7.L. Each set has at most one greatest and at most one smallest element.

An element b of a set A is a ***maximal*** element of A if A contains no element c such that $b \prec c$. An element b is a ***minimal*** element of A if A contains no element c such that $c \prec b$.

7.M. An element b of A is maximal iff $A \cap C_X^-(b) = b$; an element b of A is minimal iff $A \cap C_X^+(b) = b$.

7.6. Riddle. 1) How are the notions of maximal and greatest elements related? 2) What can you say about a poset in which these notions coincide for each subset?

7°6. Linear Orders

Please, notice: the definition of a strict order does not require that for any $a, b \in X$ we have either $a \prec b$, or $b \prec a$, or $a = b$. This condition is called a ***trichotomy***. In terms of the corresponding nonstrict order, it can be reformulated as follows: any two elements $a, b \in X$ are ***comparable***: either $a \preceq b$, or $b \preceq a$.

A strict order satisfying trichotomy is ***linear***. The corresponding poset is ***linearly*** ordered. It is also called just an ***ordered set***.[6] Some orders do satisfy trichotomy.

7.N. The order $<$ in the set $\mathbb{R}$ of real numbers is linear.

This is the most important example of a linearly ordered set. The words and images rooted in it are often extended to all linearly ordered sets. For example, cones are called ***rays***, upper cones become ***right rays***, while lower cones become ***left rays***.

7.7. A poset $(X, \prec)$ is linearly ordered iff $X = C_X^+(a) \cup C_X^-(a)$ for each $a \in X$.

7.8. In the set $\mathbb{N}$ of positive integers, the order $a|b$ is not linear.

7.9. For which X is the relation of inclusion in the set of all subsets of X a linear order?

7°7. Topologies Determined by Linear Order

7.O. *Let $(X, \prec)$ be a linearly ordered set. Then set of all right rays of X, i.e., sets of the form $\{x \in X \mid a \prec x\}$, where a runs through X, and the set X itself constitute a base for a topological structure in X.*

[6] Quite a bit of confusion was brought into the terminology by Bourbaki. Then total orders were called orders, non-total orders were called partial orders, and in occasions when it was not known if the order under consideration was total, the fact that this was unknown was explicitly stated. Bourbaki suggested to withdraw the word *partial*. Their motivation for this was that a partial order, as a phenomenon more general than a linear order, deserves a shorter and simpler name. In French literature, this suggestion was commonly accepted, but in English it would imply abolishing a nice short word *poset*, which seems to be an absolutely impossible thing to do.

The topological structure determined by this base is the *right ray topology* of the linearly ordered set $(X, \prec)$. The *left ray topology* is defined similarly: it is generated by the base consisting of X and sets of the form $\{x \in X \mid x \prec a\}$ with $a \in X$.

7.10. The topology of the arrow (see 2) is the right ray topology of the half-line $[0, \infty)$ equipped with the order $<$.

7.11. Riddle. To what extent is the assumption that the order is linear necessary in Theorem *7.O*? Find a weaker condition that implies the conclusion of Theorem *7.O* and allows us to speak about the topological structure described in Problem *2.2* as the right ray topology of an appropriate partial order on the plane.

7.P. *Let $(X, \prec)$ be a linearly ordered set. Then the subsets of X having the forms*

- $\{x \in X \mid a \prec x\}$, *where a runs through X,*
- $\{x \in X \mid x \prec a\}$, *where a runs through X,*
- $\{x \in X \mid a \prec x \prec b\}$, *where a and b run through X*

constitute a base for a topological structure in X.

The topological structure determined by this base is the *interval topology* of the linearly ordered set $(X, \prec)$.

7.12. Prove that the interval topology is the smallest topological structure containing the right ray and left ray topological structures.

7.Q. *The canonical topology of the line is the interval topology of $(\mathbb{R}, <)$.*

7°8. Poset Topology

7.R. *Let $(X, \preceq)$ be a poset. Then the subsets of X having the form $\{x \in X \mid a \preceq x\}$, where a runs through the entire X, constitute a base of for topological structure in X.*

The topological structure generated by this base is the *poset topology*.

7.S. *In the poset topology, each point $a \in X$ has the smallest (with respect to inclusion) neighborhood. This is $\{x \in X \mid a \preceq x\}$.*

7.T. *The following properties of a topological space are equivalent:*

(1) *each point has a smallest neighborhood,*

(2) *the intersection of any collection of open sets is open,*

(3) *the union of any collection of closed sets is closed.*

A space satisfying the conditions of Theorem *7.T* is a ***smallest neighborhood space***.[7] In a smallest neighborhood space, open and closed sets satisfy the same conditions. In particular, the set of all closed sets of a smallest neighborhood space also is a topological structure, which is ***dual*** to the original one. It corresponds to the opposite partial order.

7.13. How to characterize points open in the poset topology in terms of the partial order? The same question about closed points.

7.14. Directly describe open sets in the poset topology of $\mathbb{R}$ with order $<$.

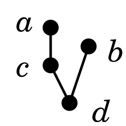
7.15. Consider a partial order in the set $\{a, b, c, d\}$ where the strict inequalities are: $c \prec a$, $d \prec c$, $d \prec a$, and $d \prec b$. Check that this is a partial order and the corresponding poset topology is the topology of described in Problem *2.3* (1).

7.16. Describe the closure of a point in a poset topology.

7.17. Which singletons are dense in a poset topology?

7°9. How to Draw a Poset

Now we can explain the pictogram, which we use to denote the space introduced in Problem *2.3* (1). It describes the partial order in $\{a, b, c, d\}$ that determines the topology of this space by *7.15*. Indeed, if we place a, b, c, and d the elements of the set under consideration at vertices of the graph of the pictogram, as shown in the picture, then the vertices corresponding to comparable elements are connected by a segment or ascending broken line, and the greater element corresponds to the higher vertex.

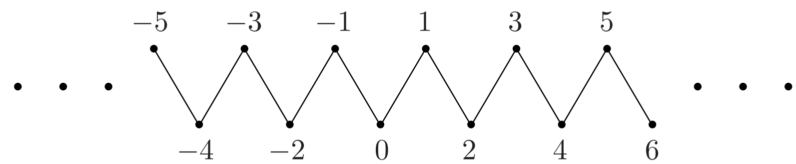


In this way, we can represent any finite poset by a diagram. Elements of the poset are represented by points. We have $a \prec b$ if and only if the following two conditions are fulfilled: 1) the point representing b lies above the point representing a and 2) those points are connected either by a segment or by a broken line consisting of segments which connect points representing intermediate elements of a chain $a \prec c_1 \prec c_2 \prec \dots \prec c_n \prec b$. We could have connected by a segment any two points corresponding to comparable elements, but this would make the diagram excessively cumbersome. This is why the segments that can be recovered from the others by transitivity are not drawn. Such a diagram representing a poset is its ***Hasse diagram***.

7.U. Prove that any finite poset can be determined by a Hasse diagram.

7.V. Describe the poset topology in the set $\mathbb{Z}$ of integers defined by the following Hasse diagram:

[7] This class of topological spaces was introduced and studied by P. S. Alexandrov in 1935. Alexandrov called them ***discrete***. Nowadays, the term discrete space is used for a much narrower class of topological spaces (see Section 2). The term ***smallest neighborhood space*** was introduced by Christer Kiselman.



The space of Problem *7.V* is the *digital line*, or *Khalimsky line*. In this space, each even number is closed and each odd one is open.

7.18. Associate with each even integer $2k$ the interval $(2k-1, 2k+1)$ of length 2 centered at this point, and with each odd integer $2k-1$, the singleton $\{2k-1\}$. Prove that a set of integers is open in the Khalimsky topology iff the union of sets associated to its elements is open in $\mathbb{R}$ with the standard topology.

7.19. Among the topological spaces described in Section 2, find all thhose can be obtained as posets with the poset topology. In the cases of finite sets, draw Hasse diagrams describing the corresponding partial orders.

7°10. Cyclic Orders in Finite Sets

Recall that a *cyclic order* in a finite set X is a linear order considered up to cyclic permutation. The linear order allows us to enumerate elements of the set X by positive integers, so that $X = \{x_1, x_2, \dots, x_n\}$. A cyclic permutation transposes the first k elements with the last $n-k$ elements without changing the order inside each of the two parts of the set:

$$(x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2}, \dots, x_n) \mapsto (x_{k+1}, x_{k+2}, \dots, x_n, x_1, x_2, \dots, x_k).$$

When we consider a cyclic order, it makes no sense to say that one of its elements is greater than another one, since an appropriate cyclic permutation put the two elements in the opposite order. However, it makes sense to say that an element is *immediately* followed by another one. Certainly, the very last element is immediately followed by the very first: indeed, any non-identity cyclic permutation puts the first element immediately after the last one.

In a cyclicly ordered finite set, each element a has a unique element b next to a, i.e., which follows a immediately. This determines a map of the set onto itself, namely the simplest cyclic permutation

$$x_i \mapsto \begin{cases} x_{i+1} & \text{if } i < n, \\ x_1 & \text{if } i = n. \end{cases}$$

This permutation acts transitively (i.e., any element is mapped to any other one by an appropriate iteration of it).

7.W. *Any map $T : X \to X$ that acts transitively in X determines a cyclic order in X such that each $a \in X$ is followed by $T(a)$.*

7.X. *A set consisting of n elements possesses exactly $(n-1)!$ pairwise distinct cyclic orders.*

In particular, a two-element set has only one cyclic order (which is so uninteresting that sometimes it is said to make no sense), while any three-element set possesses two cyclic orders.

7°11x. Cyclic Orders in Infinite Sets

One can consider cyclic orders in an infinite set. However, most of what was said above does not apply to cyclic orders in infinite sets without an adjustment. In particular, most of them cannot be described by showing pairs of elements that are next to each other. For example, points of a circle can be cyclically ordered clockwise (or counter-clockwise), but no point immediately follows another point with respect to this cyclic order.

Such "continuous" cyclic orders can be defined almost in the same way as cyclic orders in finite sets were defined above. The difference is that sometimes it is impossible to define *cyclic permutations of the set* in necessary quantity, and they have to be replaced by *cyclic transformations of the linear orders*. Namely, a cyclic order is defined as a linear order considered up to cyclic transformations, where by a *cyclic transformation* of a linear order $\prec$ in a set X we mean a passage from $\prec$ to a linear order $\prec'$ such that X splits into subsets A and B such that the restrictions of $\prec$ and $\prec'$ to each of them coincide, while $a \prec b$ and $b \prec' a$ for any $a \in A$ and $b \in B$.

7.Ax. *Existence of a cyclic transformation transforming linear orders to each other determines an equivalence relation on the set of all linear orders in a set.*

A *cyclic order* in a set is an equivalence class of linear orders under the relation of existence of a cyclic transformation.

7.Bx. Prove that for a finite set this definition is equivalent to the definition in the preceding Section.

7.Cx. Prove that the cyclic "counter-clockwise" order on a circle can be defined along the definition of this Section, but cannot be defined as a linear order modulo cyclic transformations of the set for whatever definition of cyclic transformations of circle. Describe the linear orders on the circle that determine this cyclic order up to cyclic transformations of orders.

7.Dx. Let A be a subset of a set X. If two linear orders $\prec'$ and $\prec$ on X are obtained from each other by a cyclic transformation, then their restrictions to A are also obtained from each other by a cyclic transformation.

7.Ex Corollary. A cyclic order in a set induces a well-defined cyclic order in every subset of this set.

7.Fx. *A cyclic order in a set X can be recovered from the cyclic orders induced by it in all three-element subsets of X.*

> ***7.Fx.1.*** A cyclic order in a set X can be recovered from the cyclic orders induced by it in all three-element subsets of X containing a fixed element $a \in X$.

Theorem *7.Fx* allows us to describe a cyclic order as a ternary relation. Namely, let a, b, c be elements of a cyclically ordered set. Then we write $[a \prec b \prec c]$ if the induced cyclic order on $\{a, b, c\}$ is determined by the linear order in which the inequalities in the brackets hold true (i.e., b follows a and c follows b).

7.Gx. *Let X be a cyclically ordered set. Then the ternary relation $[a \prec b \prec c]$ on X has the following properties:*

(1) *for any pairwise distinct $a, b, c \in X$, we have either $[a \prec b \prec c]$, or $[b \prec a \prec c]$ is true, but not both;*
(2) *$[a \prec b \prec c]$, iff $[b \prec c \prec a]$, iff $[c \prec a \prec b]$, for any $a, b, c \in X$;*
(3) *if $[a \prec b \prec c]$ and $[a \prec c \prec d]$, then $[a \prec b \prec d]$.*

Vice versa, a ternary relation having these four properties in a set X determines a cyclic order in X.

7°12x. Topology of Cyclic Order

7.Hx. *Let X be a cyclically ordered set. Then the sets that belong to the interval topology of every linear order determining the cyclic order on X constitute a topological structure in X.*

The topology defined in *7.Hx* is the *cyclic order topology*.

7.Ix. The cyclic order topology determined by the cyclic counterclockwise order of S^1 is the topology generated by the metric $\rho(x, y) = |x - y|$ on $S^1 \subset \mathbb{C}$.

Proofs and Comments

1.A The question is so elementary that it is difficult to find more elementary facts which we could use in the proof. What does it mean that A consists of a elements? This means, say, that we can count elements of A one by one assigning to them numbers 1, 2, 3, and the last element will receive number a. It is known that the result does not depend on the order in which we count. (In fact, one can develop a set theory which would include a theory of counting, and in which this is a theorem. However, since we have no doubts in this fact, let us use it without proof.) Therefore we can start counting of elements of B with counting the elements of A. The counting of elements of A will be done first, and then, if there are some elements of B that are not in A, counting will be continued. Thus, the number of elements in A is less than or equal to the number of elements in B.

1.B Recall that, by the definition of an inclusion, $A \subset B$ means that each element of A is an element of B. Therefore, the statement that we must prove can be rephrased as follows: each element of A is an element of A. This is a tautology.

1.C Recall that, by the definition of an inclusion, $A \subset B$ means that each element of A is an element of B. Thus we need to prove that any element of $\varnothing$ belongs to A. This is correct because there are no elements in $\varnothing$. If you are not satisfied with this argument (since it sounds too crazy), then let us resort to the question whether this can be wrong. How can it happen that $\varnothing$ is not a subset of A? This is possible only if there is an element of $\varnothing$ which is not an element of A. However, there is no such elements in $\varnothing$ because $\varnothing$ has no elements at all.

1.D We must prove that each element of A is an element of C. Let $x \in A$. Since $A \subset B$, it follows that $x \in B$. Since $B \subset C$, the latter belonging (i.e., $x \in B$) implies $x \in C$. This is what we had to prove.

1.E We have already seen that $A \subset A$. Hence if $A = B$, then, indeed, $A \subset B$ and $B \subset A$. On the other hand, $A \subset B$ means that each element of A belongs to B, while $B \subset A$ means that each element of B belongs to A. Hence A and B have the same elements, i.e., they are equal.

1.G It is easy to construct a set A with $A \notin A$. Take $A = \varnothing$, or $A = \mathbb{N}$, or $A = \{1\}, \dots$

1.H Take $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{\{\{1\}\}\}$. It is more difficult to construct sets A, B, and C such that $A \in B$, $B \in C$, and $A \in C$. Take $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{\{1\}, \{\{1\}\}\}$.

2.A What should we check? The first axiom reads here that the union of any collection of subsets of X is a subset of X. Well, this is true. If $A \subset X$ for each $A \in \Gamma$, then, obviously, $\bigcup_{A\in\Gamma} A \subset X$. Exactly in the same way we check the second axiom. Finally, of course, $\varnothing \subset X$ and $X \subset X$.

2.B Yes, it is. If one of the united sets is X, then the union is X, otherwise the union in empty. If one of the sets to intersect is $\varnothing$, then the intersection is $\varnothing$. Otherwise, the intersection equals X.

2.C First, show that $\bigcup\limits_{A\in\Gamma} A \cap \bigcup\limits_{B\in\Sigma} B = \bigcup\limits_{A\in\Gamma, B\in\Sigma} (A \cap B)$. Therefore, if A and B are intervals, then the right-hand side is a union of intervals.

If you think that a set which is a union of intervals is too simple, then, please, try to answer the following question (which has nothing to do with the problem under consideration, though). Let $\{r_n\}_{n=1}^{\infty} = \mathbb{Q}$ (i.e., we numbered all rational numbers). Prove that $\bigcup (r - 2^{-n}, r + 2^{-n}) \neq \mathbb{R}$, although this is a union of some intervals, that contains all (!) rational numbers.

2.D The union of any collection of open sets is open. The intersection of any finite collection of open sets is open. The empty set and the whole space are open.

2.E
(a)

$$x \in \bigcap_{A\in\Gamma} (X \smallsetminus A) \iff \forall\, A \in \Gamma : \ x \in X \smallsetminus A$$

$$\iff \ \forall\, A \in \Gamma : \ x \notin A \ \iff x \notin \bigcup_{A\in\Gamma} A \iff x \in X \smallsetminus \bigcup_{A\in\Gamma} A.$$

(b) Replace both sides of the formula by their complements in X and put $B = X \smallsetminus A$.

2.F (a) Let $\Gamma = \{F_\alpha\}$ be a collection of closed sets. We must verify that $\bigcap F_\alpha$ is closed, i.e. $X \smallsetminus \bigcap F_\alpha$ is open. Indeed, by the second De Morgan formula we have

$$X \smallsetminus \bigcap F_\alpha = \bigcup (X \smallsetminus F_\alpha),$$

which is open by the first axiom of topological structure.
(b) Similar to (a); use the first De Morgan formula and the second axiom of topological structure.
(c) Obvious.

2.G In any topological space, the empty set and the whole space are both open and closed. Any set in a discrete space is both open and closed.

Half-open intervals on the line are neither open nor closed. Cf. the next problem.

2.H Yes, it is, because its complement $\mathbb{R} \smallsetminus [a,b] = (-\infty, a) \cup (b, +\infty)$ is open.

***2.A*x** Let $U \subset \mathbb{R}$ be an open set. For each $x \in U$, let $(m_x, M_x) \subset U$ be the largest open interval containing x (take the union of all open intervals in U that contain x). Since U is open, such intervals exist. Any two such intervals either coincide or are disjoint.

***2.D*x** Conditions (a) and (c) from *2.13* are obviously fulfilled. To prove (b), let us use *2.Ex* and argue by contradiction. Suppose that sets A and B contain no arithmetic progressions of length at least n. If $A \cup B$ contains a sufficiently long progression, then A or B contains a progression of length more than n, a contradiction.

3.A $\boxed{\Longrightarrow}$ Present U as a union of elements of Σ. Each point $x \in U$ is contained in at least one of these sets. Such a set can be chosen as V. It is contained in U since it participates in a union equal to U.
$\boxed{\Longleftarrow}$ We must prove that each $U \in \Omega$ is a union of elements of Σ. For each point $x \in U$, choose according to the assumption a set $V_x \in \Sigma$ such that $x \in V_x \subset U$ and consider $\cup_{x \in U} V_x$. Notice that $\cup_{x \in U} V_x \subset U$ because $V_x \subset U$ for each $x \in U$. On the other hand, each point $x \in U$ is contained in its own V_x and hence in $\cup_{x \in U} V_x$. Therefore, $U \subset \cup_{x \in U} V_x$. Thus, $U = \cup_{x \in U} V_x$.

3.B $\boxed{\Longrightarrow}$ X, being an open set in any topology, is a union of some sets in Σ. The intersection of any two sets in to Σ is open, therefore it also is a union of base sets. $\boxed{\Longleftarrow}$ Let us prove that the set of unions of all collections of elements of Σ satisfies the axioms of topological structure. The first axiom is obviously fulfilled since the union of unions is a union. Let us prove the second axiom (the intersection of two open sets is open). Let $U = \cup_\alpha A_\alpha$ and $V = \cup_\beta B_\beta$, where $A_\alpha, B_\beta \in \Sigma$. Then

$$U \cap V = (\cup_\alpha A_\alpha) \cap (\cup_\beta B_\beta) = \cup_{\alpha,\beta}(A_\alpha \cap B_\beta),$$

and since, by assumption, $A_\alpha \cap B_\beta$ is a union of elements of Σ, so is the intersection $U \cap V$. In the third axiom, we need to check only the part concerning the entire X. By assumption, X is a union of sets belonging to Σ.

3.D Let Σ_1 and Σ_2 be bases of topological structures Ω_1 and Ω_2 in a set X. Obviously, $\Omega_1 \subset \Omega_2$ iff $\forall U \in \Sigma_1 \ \forall x \in U \ \exists V \in \Sigma_2 : \ x \in V \subset U$. Now recall that $\Omega_1 = \Omega_2$ iff $\Omega_1 \subset \Omega_2$ and $\Omega_2 \subset \Omega_1$.

4.A Indeed, it makes sense to check that all conditions in the definition of a metric are fulfilled for *each* triple of points x, y, and z.

4.B The triangle inequality in this case takes the form $|x-y| \le |x-z| + |z-y|$. Putting $a = x - z$ and $b = z - y$, we transform the triangle inequality into the well-known inequality $|a+b| \le |a| + |b|$.

4.C As in the solution of Problem *4.B*, the triangle inequality takes the form: $\sqrt{\sum_{i=1}^n (a_i+b_i)^2} \le \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}$. Two squarings followed by an obvious simplification reduce this inequality to the well-known Cauchy inequality $(\sum a_i b_i)^2 \le \sum a_i^2 \sum b_i^2$.

4.E We must prove that every point $y \in B_{r-\rho(a,x)}(x)$ belongs to $B_r(a)$. In terms of distances, this means that $\rho(y,a) < r$ if $\rho(y,x) < r - \rho(a,x)$ and $\rho(a,x) < r$. By the triangle inequality, $\rho(y,a) \le \rho(y,x) + \rho(x,a)$. Replacing the first summand on the right-hand side of the latter inequality by a greater number $r - \rho(a,x)$, we obtain the required inequality. The second inclusion is proved similarly.

4.F $\boxed{\Rightarrow}$ Show that if $d = \operatorname{diam} A$ and $a \in A$, then $A \subset D_d(a)$. $\boxed{\Leftarrow}$ Use the fact that $\operatorname{diam} D_d(a) \le 2d$. (Cf. *4.11*.)

4.G This follows from Problem *4.E*, Theorem *3.B* and Assertion *3.C*.

4.H For this metric, the balls are open intervals. Each open interval in $\mathbb{R}$ is as a ball. The standard topology in $\mathbb{R}$ is determined by the base consisting of all open intervals.

4.I $\boxed{\Rightarrow}$ If $a \in A$, then $a \in B_r(x) \subset A$ and $B_{r-\rho(a,x)}(a) \in B_r(x) \subset U$, see *4.E*. $\boxed{\Leftarrow}$ A is a union of balls, therefore, A is open in the metric topology.

4.J An indiscrete space does not have sufficiently many open sets. For $x, y \in X$ and $r = \rho(x,y) > 0$, the ball $D_r(x)$ is nonempty and does not coincide with the whole space (it does not contain y).

4.K $\boxed{\Rightarrow}$ For $x \in X$, put $r = \min\{\rho(x,y) \mid y \in X \smallsetminus x\}$. Which points are in $B_r(x)$? $\boxed{\Leftarrow}$ Obvious. (Cf. *4.19*.)

4.L $\boxed{\Rightarrow}$ The condition $\rho(b,A) = 0$ means that each ball centered at b meets A, i.e., b does not belong to the complement of A (since A is closed, the complement of A is open). $\boxed{\Leftarrow}$ Obvious.

***4.A*x** Condition (2) is obviously fulfilled. Put $r(A,B) = \sup_{a\in A} \rho(a,B)$, so that $d_\rho(A,B) = \max\{r(A,B), r(B,A)\}$. To prove that (3) is also fulfilled, it suffices to prove that $r(A,C) \le r(A,B) + r(B,C)$ for any $A, B, C \subset X$. We easily see that $\rho(a,C) \le \rho(a,b) + \rho(b,C)$ for all $a \in A$ and $b \in B$. Hence, $\rho(a,C) \le \rho(a,b) + r(B,C)$, whence

$$\rho(a,C) \le \inf_{b\in B} \rho(a,b) + r(B,C) = \rho(a,B) + r(B,C) \le r(A,B) + r(B,C),$$

which implies the required inequality.

***4.B*x** By *4.A*x, d_ρ satisfies conditions (2) and (3) from the definition of a metric. From *4.L* it follows that if the Hausdorff distance between two closed sets A and B equals zero, then $A \subset B$ and $B \subset A$, i.e., $A = B$. Thus, d_ρ satisfies the condition (1).

***4.C*x** $d_\Delta(A, B)$ is the area of the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ of A and B. The first two axioms of metric are obviously fulfilled. Prove the triangle inequality by using the inclusion $A \setminus B \subset (C \setminus B) \cup (A \setminus C)$.

***4.F*x** Clearly, the metric in *4.A* is an ultrametric. The other metrics are not: for each of them you can find points x, y, and z such that $\rho(x, y) = \rho(x, z) + \rho(z, y)$.

***4.G*x** The definition of an ultrametric implies that none of the pairwise distances between the points a, b, and c is greater than each of the other two.

***4.H*x** By *4.G*x, if $y \in S_r(x)$ and $r > s > 0$, then $B_s(y) \subset S_r(x)$.

***4.I*x** Let $x - z = \frac{r_1}{s_1} p^{\alpha_1}$ and $z - y = \frac{r_2}{s_2} p^{\alpha_2}$, where $\alpha_1 \leq \alpha_2$. Then we have

$$x - y = p^{\alpha_1}\left(\tfrac{r_1}{s_1} + \tfrac{r_2}{s_2} p^{\alpha_2 - \alpha_1}\right) = p^{\alpha_1} \frac{r_1 s_2 + r_2 s_1 p^{\alpha_2 - \alpha_1}}{s_1 s_2},$$

whence $p(x, y) \leq p^{-\alpha_1} = \max\{\rho(x, z), \rho(z, y)\}$.

5.A We must check that Ω_A satisfies the axioms of topological structure. Consider the first axiom. Let $\Gamma \subset \Omega_A$ be a collection of sets in Ω_A. We must prove that $\bigcup_{U \in \Gamma} U \in \Omega_A$. For each $U \in \Gamma$, find $U_X \in \Omega$ such that $U = A \cap U_X$. This is possible due to the definition of Ω_A. Transform the union under consideration: $\bigcup_{U \in \Gamma} U = \bigcup_{U \in \Gamma}(A \cap U_X) = A \cap \bigcup_{U \in \Gamma} U_X$. The union $\bigcup_{U \in \Gamma} U_X$ belongs to Ω (i.e., is open in X) as the union of sets open in X. (Here we use the fact that Ω, being a topology in X, satisfies the first axiom of topological structure.) Therefore, $A \cap \bigcup_{U \in \Gamma} U_X$ belongs to Ω_A. Similarly we can check the second axiom. The third axiom: $A = A \cap X$, and $\varnothing = A \cap \varnothing$.

5.B Let us prove that a subset of $\mathbb{R}^1$ is open in the relative topology iff it is open in the canonical topology. $\boxed{\Rightarrow}$ The intersection of an open disk with $\mathbb{R}^1$ is either an open interval or the empty set. Any open set in the plane is a union of open disks. Therefore the intersection of any open set of the plane with $\mathbb{R}^1$ is a union of open intervals. Thus, it is open in $\mathbb{R}^1$. $\boxed{\Leftarrow}$ Prove this part on your own.

5.C $\boxed{\Rightarrow}$ The complement $A \setminus F$ is open in A, i.e., $A \setminus F = A \cap U$, where U is open in X. What closed set cuts F on A? It is cut by $X \setminus U$.

Indeed, $A \cap (X \smallsetminus U) = A \smallsetminus (A \cap U) = A \smallsetminus (A \smallsetminus F) = F$. $\boxed{\Leftarrow}$ This is proved in a similar way.

5.D No disk of $\mathbb{R}^2$ is contained in $\mathbb{R}$.

5.E If $A \in \Omega$ and $B \in \Omega_A$, then $B = A \cap U$, where $U \in \Omega$. Therefore, $B \in \Omega$ as the intersection of two sets, A and U, belonging to Ω.

5.F Act as in the solution of the preceding problem *5.E*, but use *5.C* instead of the definition of the relative topology.

5.G The core of the proof is the equality $(U \cap A) \cap B = U \cap B$. It holds true because $B \subset A$, and we apply it to $U \in \Omega$. As U runs through Ω, the right-hand side of the equality $(U \cap A) \cap B = U \cap B$ runs through Ω_B, while the left-hand side runs through $(\Omega_A)_B$. Indeed, elements of Ω_B are intersections $U \cap B$ with $U \in \Omega$, and elements of $(\Omega_A)_B$ are intersections $V \cap B$ with $V \in \Omega_A$, but V, in turn, being an element of Ω_A, is the intersection $U \cap A$ with $U \in \Omega$.

6.A The union of all open sets contained in A, firstly, is open (as a union of open sets), and, secondly, contains every open set that is contained in A (i.e., it is the greatest one among those sets).

6.B Let x be an interior point of A (i.e., there exists an open set U_x such that $x \in U_x \subset A$). Then $U_x \subset \operatorname{Int} A$ (because $\operatorname{Int} A$ is *the greatest* open set contained in A), whence $x \in \operatorname{Int} A$. Vice versa, if $x \in \operatorname{Int} A$, then the set $\operatorname{Int} A$ itself is a neighborhood of x contained in A.

6.C $\boxed{\Rightarrow}$ If U is open, then U is the greatest open subset of U, and hence coincides with the interior ofU. $\boxed{\Leftarrow}$ A set coinciding with its interior is open since the interior is open.

6.D

(1) $[0, 1)$ is not open in the line, while $(0, 1)$ is. Therefore $\operatorname{Int}[0, 1) = (0, 1)$.

(2) Since any interval contains an irrational point, $\mathbb{Q}$ does not contain a nonempty sets open in the classical topology of $\mathbb{R}$. Therefore, $\operatorname{Int} \mathbb{Q} = \varnothing$.

(3) Since any interval contains rational points, $\mathbb{R} \smallsetminus \mathbb{Q}$ does not contain a nonempty set open in the classical topology of $\mathbb{R}$. Therefore, $\operatorname{Int}(\mathbb{R} \smallsetminus \mathbb{Q}) = \varnothing$.

6.E The intersection of all closed sets containing A, firstly, is closed (as an intersection of closed sets), and, secondly, is contained in every closed set that contains A (i.e., it is the smallest one among those sets). Cf. the proof of Theorem *6.A*. In general, properties of closure can be obtained from properties of interior by replacing unions with intersections and vice versa.

6.F If $x \notin \operatorname{Cl} A$, then there exists a closed set F such that $F \supset A$ and $x \notin F$, whence $x \in U = X \smallsetminus F$. Thus, x is not an adherent point for A. Prove the inverse implication on your own, cf. *6.H.*

6.G Cf. the proof of Theorem *6.C.*

6.H The intersection of all closed sets containing A is the complement of the union of all open sets contained in $X \smallsetminus A$.

6.I (a) The half-open interval $[0, 1)$ is not closed, and $[0, 1]$ is closed; (b)–(c) The exterior of each of the sets $\mathbb{Q}$ and $\mathbb{R} \smallsetminus \mathbb{Q}$ is empty since each interval contains both rational and irrational numbers.

6.J $\boxed{\Longrightarrow}$ If b is an adherent point for A, then $\forall\, \varepsilon > 0\ \exists\, a \in A \cap D_\varepsilon(b)$, whence $\forall\, \varepsilon > 0\ \exists\, a \in A : \rho(a, b) < \varepsilon$. Thus, $\rho(b, A) = 0$. $\boxed{\Longleftarrow}$ This is an easy exercise.

6.K If $x \in \operatorname{Fr} A$, then $x \in \operatorname{Cl} A$ and $x \notin \operatorname{Int} A$. Hence, firstly, each neighborhood of x meets A, secondly, no neighborhood of x is contained in A, and therefore each neighborhood of x meets $X \smallsetminus A$. Thus, x is a boundary point of A. Prove the converse on your own.

6.L Since $\operatorname{Int} A \subset A$, it follows that $\operatorname{Cl} A = A$ iff $\operatorname{Fr} A \subset A$.

6.M $\boxed{\Longrightarrow}$ Argue by contradiction. A set A disjoint with an open set U is contained in the closed set $X \smallsetminus U$. Therefore, if U is nonempty, then A is not everywhere dense. $\boxed{\Longleftarrow}$ A set meeting each nonempty open set is contained in only one closed set: the entire space. Hence, its closure is the whole space, and this set is everywhere dense.

6.N This is *6.I*(b).

6.O The condition means that each neighborhood of each point contains an exterior point of A. This, in turn, means that the exterior of A is everywhere dense.

6.Q $\boxed{\Longrightarrow}$ This is *6.P.* $\boxed{\Longleftarrow}$ Hint: any point of $\operatorname{Cl} A \smallsetminus A$ is a limit point of A.

7.F We need to check that the relation "$a \prec b$ or $a = b$" satisfies the three conditions from the definition of a nonstrict order. Doing this, we can use only the fact that $\prec$ satisfies the conditions from the definition of a strict order. Let us check the transitivity. Suppose that $a \preceq b$ and $b \preceq c$. This means that either 1) $a \prec b \prec c$, or 2) $a = b \prec c$, or 3) $a \prec b = c$, or 4) $a = b = c$.
1) In this case, $a \prec c$ by transitivity of $\prec$, and so $a \preceq c$. 2) We have $a \prec c$, whence $a \preceq c$. 3) We have $a \prec c$, whence $a \preceq c$. 4) Finally, $a = c$, whence $a \preceq c$. Other conditions are checked similarly.

7.I Assertion (a) follows from transitivity of the order. Indeed, consider an arbitrary an $c \in C^+_X(b)$. By the definition of a cone, we have $b \preceq c$, while

the condition $b \in C^+_X(a)$ means that $a \preceq b$. By transitivity, this implies that $a \preceq c$, i.e., $c \in C^+_X(a)$. We have thus proved that each element of $C^+_X(b)$ belongs to $C^+_X(a)$. Hence, $C^+_X(b) \subset C^+_X(a)$, as required.
Assertion (b) follows from the definition of a cone and the reflexivity of order. Indeed, by definition, $C^+_X(a)$ consists of all b such that $a \preceq b$, and, by reflexivity of order, $a \preceq a$.
Assertion (c) follows similarly from antisymmetry: the assumption $C^+_X(a) = C^+_X(b)$ together with assertion (b) implies that $a \preceq b$ and $b \preceq a$, which together with antisymmetry implies that $a = b$.

7.J By Theorem *7.I*, cones in a poset have the properties that form the hypothesis of the theorem to be proved. When proving Theorem *7.I*, we showed that these properties follow from the corresponding conditions in the definition of a partial nonstrict order. In fact, they are equivalent to these conditions. Permuting words in the proof of Theorem *7.I*, we to obtain a proof of Theorem *7.J*.

7.O By Theorem *3.B*, it suffices to prove that the intersection of any two right rays is a union of right rays. Let $a, b \in X$. Since the order is linear, either $a \prec b$, or $b \prec a$. Let $a \prec b$. Then

$$\{x \in X \mid a \prec x\} \cap \{x \in X \mid b \prec x\} = \{x \in X \mid b \prec x\}.$$

7.R By Theorem *3.C*, it suffices to prove that each element of the intersection of two cones, say, $C^+_X(a)$ and $C^+_X(b)$, is contained in the intersection together with a whole cone of the same kind. Assume that $c \in C^+_X(a) \cap C^+_X(b)$ and $d \in C^+_X(c)$. Then $a \preceq c \preceq d$ and $b \preceq c \preceq d$, whence $a \preceq d$ and $b \preceq d$. Therefore $d \in C^+_X(a) \cap C^+_X(b)$. Hence, $C^+_X(c) \subset C^+_X(a) \cap C^+_X(b)$.

7.T Equivalence of the second and third properties follows from the De Morgan formulas, as in *2.F*. Let us prove that the first property implies the second one. Consider the intersection of an arbitrary collection of open sets. For each of its points, every set of this collection is a neighborhood. Therefore, its smallest neighborhood is contained in each of the sets to be intersected. Hence, the smallest neighborhood of the point is contained in the intersection, which we denote by U. Thus, each point of U lies in U together with its neighborhood. Since U is the union of these neighborhoods, it is open.
Now let us prove that if the intersection of any collection of open sets is open, then any point has a smallest neighborhood. Where can one get such a neighborhood from? How to construct it? Take all neighborhoods of a point x and consider their intersection U. By assumption, U is open. It contains

x. Therefore, U is a neighborhood of x. This neighborhood, being the intersection of all neighborhoods, is contained in each of the neighborhoods. Thus, U is the smallest neighborhood.

7.V The minimal base of this topology consists of singletons of the form $\{2k-1\}$ with $k \in \mathbb{Z}$ and three-point sets of the form $\{2k-1, 2k, 2k+1\}$, where again $k \in \mathbb{Z}$.

Chapter II

Continuity

8. Set-Theoretic Digression: Maps

8°1. Maps and Main Classes of Maps

A *map* f of a set X to a set Y is a triple consisting of X, Y, and a rule,[1] which assigns to every element of X exactly one element of Y. There are other words with the same meaning: *mapping*, *function*, etc.

If f is a map of X to Y, then we write $f : X \to Y$, or $X \xrightarrow{f} Y$. The element b of Y assigned by f to an element a of X is denoted by $f(a)$ and called the *image* of a under f, or the f-image of a. We write $b = f(a)$, or $a \overset{f}{\mapsto} b$, or $f : a \mapsto b$.

A map $f : X \to Y$ is a *surjective map*, or just a *surjection* if every element of Y is the image of at least one element of X. A map $f : X \to Y$ is an *injective map*, *injection*, or *one-to-one map* if every element of Y is the image of at most one element of X. A map is a *bijective map*, *bijection*, or *invertible map* if it is both surjective and injective.

[1] Certainly, the rule (as everything in set theory) may be thought of as a set. Namely, we consider the set of the ordered pairs (x, y) with $x \in X$ and $y \in Y$ such that the rule assigns y to x. This is the *graph* of f. It is a subset of $X \times Y$. (Recall that $X \times Y$ is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.)

8°2. Image and Preimage

The *image* of a set $A \subset X$ under a map $f : X \to Y$ is the set of images of all points of A. It is denoted by $f(A)$. Thus

$$f(A) = \{f(x) \mid x \in A\}.$$

The image of the entire set X (i.e., the set $f(X)$) is the *image* of f, it is denoted by $\operatorname{Im} f$.

The *preimage* of a set $B \subset Y$ under a map $f : X \to Y$ is the set of elements of X whith images in to B. It is denoted by $f^{-1}(B)$. Thus

$$f^{-1}(B) = \{a \in X \mid f(a) \in B\}.$$

Be careful with these terms: their etymology can be misleading. For example, the image of the preimage of a set B can differ from B. And even if it does not differ, it may happen that the preimage is not the only set with this property. Hence, the preimage *cannot* be defined as a set whose image is the given set.

8.A. We have $f\big(f^{-1}(B)\big) \subset B$ for any map $f : X \to Y$ and any $B \subset Y$.

8.B. $f\big(f^{-1}(B)\big) = B$ iff $B \subset \operatorname{Im} f$.

8.C. Let $f : X \to Y$ be a map and let $B \subset Y$ be such that $f\big(f^{-1}(B)\big) = B$. Then the following statements are equivalent:

(1) $f^{-1}(B)$ is the unique subset of X whose image equals B;

(2) for any $a_1, a_2 \in f^{-1}(B)$ the equality $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

8.D. A map $f : X \to Y$ is an injection iff for each $B \subset Y$ such that $f\big(f^{-1}(B)\big) = B$ the preimage $f^{-1}(B)$ is the unique subset of X with image equal to B.

8.E. We have $f^{-1}\big(f(A)\big) \supset A$ for any map $f : X \to Y$ and any $A \subset X$.

8.F. $f^{-1}\big(f(A)\big) = A$ iff $f(A) \cap f(X \smallsetminus A) = \varnothing$.

8.1. Do the following equalities hold true for any $A, B \subset Y$ and $f : X \to Y$:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \tag{10}$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \tag{11}$$

$$f^{-1}(Y \smallsetminus A) = X \smallsetminus f^{-1}(A)? \tag{12}$$

8.2. Do the following equalities hold true for any $A, B \subset X$ and any $f : X \to Y$:

$$f(A \cup B) = f(A) \cup f(B), \tag{13}$$

$$f(A \cap B) = f(A) \cap f(B), \tag{14}$$

$$f(X \smallsetminus A) = Y \smallsetminus f(A)? \tag{15}$$

8.3. Give examples in which two of the above equalities (13)–(15) are false.

8.4. Replace false equalities of *8.2* by correct inclusions.

8.5. Riddle. What simple condition on $f : X \to Y$ should be imposed in order to make correct all equalities of *8.2* for any $A, B \subset X$?

8.6. Prove that for any map $f : X \to Y$ and any subsets $A \subset X$ and $B \subset Y$ we have:

$$B \cap f(A) = f\big(f^{-1}(B) \cap A\big).$$

8°3. Identity and Inclusion

The *identity map* of a set X is the map $\mathrm{id}_X : X \to X : x \mapsto x$. It is denoted just by id if there is no ambiguity. If A is a subset of X, then the map $\mathrm{in} : A \to X : \ x \mapsto x$ is the *inclusion map*, or just *inclusion*, of A into X. It is denoted just by in when A and X are clear.

8.G. The preimage of a set B under the inclusion $\mathrm{in} : A \to X$ is $B \cap A$.

8°4. Composition

The *composition* of maps $f : X \to Y$ and $g : Y \to Z$ is the map $g \circ f : X \to Z : x \mapsto g\big(f(x)\big)$.

8.H Associativity. $h \circ (g \circ f) = (h \circ g) \circ f$ for any maps $f : X \to Y$, $g : Y \to Z$, and $h : Z \to U$.

8.I. $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for any $f : X \to Y$.

8.J. A composition of injections is injective.

8.K. If the composition $g \circ f$ is injective, then so is f.

8.L. A composition of surjections is surjective.

8.M. If the composition $g \circ f$ is surjective, then so is g.

8.N. A composition of bijections is a bijection.

8.7. Let a composition $g \circ f$ be bijective. Is then f or g necessarily bijective?

8°5. Inverse and Invertible

A map $g : Y \to X$ is *inverse* to a map $f : X \to Y$ if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. A map having an inverse map is *invertible*.

8.O. A map is invertible iff it is a bijection.

8.P. If an inverse map exists, then it is unique.

8°6. Submaps

If $A \subset X$ and $B \subset Y$, then for every $f : X \to Y$ such that $f(A) \subset B$ we have a map $\mathrm{ab}(f) : A \to B : x \mapsto f(x)$, which is called the *abbreviation* of f to A and B, a *submap*, or a *submapping*. If $B = Y$, then $\mathrm{ab}(f) : A \to Y$ is denoted by $f|_A$ and called the *restriction* of f to A. If $B \neq Y$, then $\mathrm{ab}(f) : A \to B$ is denoted by $f|_{A,B}$ or even simply $f|$.

8.Q. The restriction of a map $f : X \to Y$ to $A \subset X$ is the composition of the inclusion $\mathrm{in} : A \to X$ and f. In other words, $f|_A = f \circ \mathrm{in}$.

8.R. Any submap (in particular, any restriction) of an injection is injective.

8.S. If a map possesses a surjective restriction, then it is surjective.

9. Continuous Maps

9°1. Definition and Main Properties of Continuous Maps

Let X and Y be two topological spaces. A map $f : X \to Y$ is *continuous* if the preimage of any open subset of Y is an open subset of X.

9.A. *A map is continuous iff the preimage of each closed set is closed.*

9.B. *The identity map of any topological space is continuous.*

9.1. Let Ω_1 and Ω_2 be two topological structures in a space X. Prove that the identity map

$$\mathrm{id} : (X, \Omega_1) \to (X, \Omega_2)$$

is continuous iff $\Omega_2 \subset \Omega_1$.

9.2. Let $f : X \to Y$ be a continuous map. Find out whether or not it is continuous with respect to

(1) a finer topology in X and the same topology in Y,
(2) a coarser topology in X and the same topology in Y,
(3) a finer topology in Y and the same topology in X,
(4) a coarser topology in Y and the same topology in X.

9.3. Let X be a discrete space and Y an arbitrary space. 1) Which maps $X \to Y$ are continuous? 2) Which maps $Y \to X$ are continuous?

9.4. Let X be an indiscrete space and Y an arbitrary space. 1) Which maps $X \to Y$ are continuous? 2) Which maps $Y \to X$ are continuous?

9.C. *Let A be a subspace of X. The inclusion* $\mathrm{in} : A \to X$ *is continuous.*

9.D. The topology Ω_A induced on $A \subset X$ by the topology of X is the coarsest topology in A with respect to which the inclusion $\mathrm{in} : A \to X$ is continuous.

9.5. Riddle. The statement *9.D* admits a natural generalization with the inclusion map replaced by an arbitrary map $f : A \to X$ of an arbitrary set A. Find this generalization.

9.E. *A composition of continuous maps is continuous.*

9.F. *A submap of a continuous map is continuous.*

9.G. *A map $f : X \to Y$ is continuous iff* $\mathrm{ab}\, f : X \to f(X)$ *is continuous.*

9.H. Any constant map (i.e., a map with image consisting of a single point) is continuous.

9°2. Reformulations of Definition

9.6. Prove that a map $f : X \to Y$ is continuous iff
$$\operatorname{Cl} f^{-1}(A) \subset f^{-1}(\operatorname{Cl} A)$$
for any $A \subset Y$.

9.7. Formulate and prove similar criteria of continuity in terms of $\operatorname{Int} f^{-1}(A)$ and $f^{-1}(\operatorname{Int} A)$. Do the same for $\operatorname{Cl} f(A)$ and $f(\operatorname{Cl} A)$.

9.8. Let Σ be a base for topology in Y. Prove that a map $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open for each $U \in \Sigma$.

9°3. More Examples

9.9. Consider the map
$$f : [0,2] \to [0,2] : f(x) = \begin{cases} x & \text{if } x \in [0,1), \\ 3 - x & \text{if } x \in [1,2]. \end{cases}$$
Is it continuous (with respect to the topology induced from the real line)?

9.10. Consider the map f from the segment $[0,2]$ (with the relative topology induced by the topology of the real line) into the arrow (see Section 2) defined by the formula
$$f(x) = \begin{cases} x & \text{if } x \in [0,1], \\ x+1 & \text{if } x \in (1,2]. \end{cases}$$
Is it continuous?

9.11. Give an explicit characterization of continuous maps of $\mathbb{R}_{T_1}$ (see Section 2) to $\mathbb{R}$.

9.12. Which maps $\mathbb{R}_{T_1} \to \mathbb{R}_{T_1}$ are continuous?

9.13. Give an explicit characterization of continuous maps of the arrow to itself.

9.14. Let f be a map of the set $\mathbb{Z}_+$ of nonnegative numbers onto $\mathbb{R}$ defined by formula
$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Let $g : \mathbb{Z}_+ \to f(\mathbb{Z}_+)$ be its submap. Induce a topology on $\mathbb{Z}_+$ and $f(\mathbb{Z}_+)$ from $\mathbb{R}$. Are f and the map g^{-1} inverse to g continuous?

9°4. Behavior of Dense Sets

9.15. Prove that the image of an everywhere dense set under a surjective continuous map is everywhere dense.

9.16. Is it true that the image of nowhere dense set under a continuous map is nowhere dense?

9.17*. Do there exist a nowhere dense set A of $[0,1]$ (with the topology induced from the real line) and a continuous map $f : [0,1] \to [0,1]$ such that $f(A) = [0,1]$?

9°5. Local Continuity

A map f from a topological space X to a topological space Y is said to be *continuous at a point* $a \in X$ if for every neighborhood V of $f(a)$ there exists a neighborhood U of a such that $f(U) \subset V$.

9.I. *A map $f : X \to Y$ is continuous iff it is continuous at each point of X.*

9.J. *Let X and Y be two metric spaces, $a \in X$. A map $f : X \to Y$ is continuous at a iff for every ball with center at $f(a)$ there exists a ball with center at a whose image is contained in the first ball.*

9.K. *Let X and Y be two metric spaces. A map $f : X \to Y$ is continuous at a point $a \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$.*

Theorem *9.K* means that the definition of continuity usually studied in Calculus, when applicable, is equivalent to the above definition stated in terms of topological structures.

9°6. Properties of Continuous Functions

9.18. Let $f, g : X \to \mathbb{R}$ be continuous. Prove that the maps $X \to \mathbb{R}$ defined by formulas

$$x \mapsto f(x) + g(x), \tag{16}$$

$$x \mapsto f(x)g(x), \tag{17}$$

$$x \mapsto f(x) - g(x), \tag{18}$$

$$x \mapsto \big|f(x)\big|, \tag{19}$$

$$x \mapsto \max\{f(x), g(x)\}, \tag{20}$$

$$x \mapsto \min\{f(x), g(x)\} \tag{21}$$

are continuous.

9.19. Prove that if $0 \notin g(X)$, then the map

$$X \to \mathbb{R} : x \mapsto \frac{f(x)}{g(x)}$$

is continuous.

9.20. Find a sequence of continuous functions $f_i : \mathbb{R} \to \mathbb{R}$, $(i \in \mathbb{N})$, such that the function

$$\mathbb{R} \to \mathbb{R} : x \mapsto \sup\{\, f_i(x) \mid i \in \mathbb{N}\,\}$$

is not continuous.

9.21. Let X be a topological space. Prove that a function $f : X \to \mathbb{R}^n : x \mapsto (f_1(x), \dots, f_n(x))$ is continuous iff so are all functions $f_i : X \to \mathbb{R}$ with $i = 1, \dots, n$.

Real $p \times q$-matrices form a space $Mat(p \times q, \mathbb{R})$, which differs from $\mathbb{R}^{pq}$ only in the way of numeration of its natural coordinates (they are numerated by pairs of indices).

9.22. Let $f : X \to Mat(p \times q, \mathbb{R})$ and $g : X \to Mat(q \times r, \mathbb{R})$ be continuous maps. Prove that then

$$X \to Mat(p \times r, \mathbb{R}) : x \mapsto g(x)f(x)$$

is a continuous map.

Recall that $GL(n; \mathbb{R})$ is the subspace of $Mat(n \times n, \mathbb{R})$ consisting of all invertible matrices.

9.23. Let $f : X \to GL(n; \mathbb{R})$ be a continuous map. Prove that $X \to GL(n; \mathbb{R}) : x \mapsto (f(x))^{-1}$ is continuous.

9°7. Continuity of Distances

9.L. *For every subset A of a metric space X, the function $X \to \mathbb{R} : x \mapsto \rho(x, A)$ (see Section 4) is continuous.*

9.24. Prove that a topology of a metric space is the coarsest topology with respect to which the function $X \to \mathbb{R} : x \mapsto \rho(x, A)$ is continuous for every $A \subset X$.

9°8. Isometry

A map f of a metric space X into a metric space Y is an *isometric embedding* if $\rho\big(f(a), f(b)\big) = \rho(a, b)$ for any $a, b \in X$. A bijective isometric embedding is an *isometry*.

9.M. Every isometric embedding is injective.

9.N. Every isometric embedding is continuous.

9°9. Contractive Maps

A map $f : X \to X$ of a metric space X is *contractive* if there exists $\alpha \in (0, 1)$ such that $\rho\big(f(a), f(b)\big) \le \alpha\rho(a, b)$ for any $a, b \in X$.

9.25. Prove that every contractive map is continuous.

Let X and Y be metric spaces. A map $f : X \to Y$ is a *Hölder* map if there exist $C > 0$ and $\alpha > 0$ such that $\rho\big(f(a), f(b)\big) \le C\rho(a, b)^{\alpha}$ for any $a, b \in X$.

9.26. Prove that every Hölder map is continuous.

9°10. Sets Defined by Systems of Equations and Inequalities

9.O. *Let f_i $(i = 1, \dots, n)$ be continuous maps $X \to \mathbb{R}$. Then the subset of X consisting of solutions of the system of equations*

$$f_1(x) = 0, \dots, f_n(x) = 0$$

is closed.

9.P. *Let f_i $(i = 1, \dots, n)$ be continuous maps $X \to \mathbb{R}$. Then the subset of X consisting of solutions of the system of inequalities*

$$f_1(x) \geq 0, \dots, f_n(x) \geq 0$$

is closed, while the set consisting of solutions of the system of inequalities

$$f_1(x) > 0, \dots, f_n(x) > 0$$

is open.

9.27. Where in *9.O* and *9.P* a finite system can be replaced by an infinite one?

9.28. Prove that in $\mathbb{R}^n$ $(n \geq 1)$ every proper algebraic set (i.e., a set defined by algebraic equations) is nowhere dense.

9°11. Set-Theoretic Digression: Covers

A collection Γ of subsets of a set X is a *cover* or a *covering* of X if X is the union of sets belonging to Γ, i.e., $X = \bigcup_{A \in \Gamma} A$. In this case, elements of Γ *cover* X.

There is also a more general meaning of these words. A collection Γ of subsets of a set Y is a *cover* or a *covering* of a set $X \subset Y$ if X is contained in the union of the sets in Γ, i.e., $X \subset \bigcup_{A \in \Gamma} A$. In this case, the sets belonging to Γ are also said to *cover* X.

9°12. Fundamental Covers

Consider a cover Γ of a topological space X. Each element of Γ inherits a topological structure from X. When are these structures sufficient for recovering the topology of X? In particular, under what conditions on Γ does the continuity of a map $f : X \to Y$ follow from that of its restrictions to elements of Γ? To answer these questions, solve Problems *9.29–9.30* and *9.Q–9.V*.

9.29. Find out whether or not this is true for the following covers:

(1) $X = [0, 2]$, and $\Gamma = \{[0, 1], (1, 2]\}$;
(2) $X = [0, 2]$, and $\Gamma = \{[0, 1], [1, 2]\}$;
(3) $X = \mathbb{R}$, and $\Gamma = \{\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}\}$;
(4) $X = \mathbb{R}$, and Γ is a set of all one-point subsets of $\mathbb{R}$.

A cover Γ of a space X is *fundamental* if a set $U \subset X$ is open iff for every $A \in \Gamma$ the set $U \cap A$ is open in A.

9.Q. A cover Γ of a space X is fundamental iff a set $U \subset X$ is open, provided $U \cap A$ is open in A for every $A \in \Gamma$.

9.R. A cover Γ of a space X is fundamental iff a set $F \subset X$ is closed, provided $F \cap A$ is closed A for every $A \in \Gamma$.

9.30. The cover of a topological space by singletons is fundamental iff the space is discrete.

A cover of a topological space is *open* if it consists of open sets, and it is *closed* if it consists of closed sets. A cover of a topological space is *locally finite* if every point of the space has a neighborhood intersecting only a finite number of elements of the cover.

9.S. *Every open cover is fundamental.*

9.T. *A finite closed cover is fundamental.*

9.U. Every locally finite closed cover is fundamental.

9.V. *Let Γ be a fundamental cover of a topological space X, and let $f : X \to Y$ be a map. If the restriction of f to each element of Γ is continuous, then so is f.*

A cover Γ' is a *refinement* of a cover Γ if every element of Γ' is contained in an element of Γ.

9.31. Prove that if a cover Γ' is a refinement of a cover Γ and Γ' is fundamental, then so is Γ.

9.32. Let Δ be a fundamental cover of a topological space X, and Γ be a cover of X such that $\Gamma_A = \{ U \cap A \mid U \in \Gamma \}$ is a fundamental cover for subspace $A \subset X$ for every $A \in \Delta$. Prove that Γ is a fundamental cover.

9.33. Prove that the property of being fundamental is local, i.e., if every point of a space X has a neighborhood V such that $\Gamma_V = \{ U \cap V \mid U \in \Gamma \}$ is fundamental, then Γ is fundamental.

9°13x. Monotone Maps

Let $(X, \prec)$ and $(Y, \prec)$ be posets. A map $f : X \to Y$ is

- (*non-strictly*) *monotonically increasing* or just *monotone* if $f(a) \preceq f(b)$ for any $a, b \in X$ with $a \preceq b$;
- (*non-strictly*) *monotonically decreasing* or *antimonotone* if $f(b) \preceq f(a)$ for any $a, b \in X$ with $a \preceq b$;
- *strictly monotonically increasing* or just *strictly monotone* if $f(a) \prec f(b)$ for any $a, b \in X$ with $a \prec b$;
- *strictly monotonically decreasing* or *strictly antimonotone* if $f(b) \prec f(a)$ for any $a, b \in X$ with $a \prec b$.

9.Ax. *Let X and Y be linearly ordered sets. With respect to the interval topology in X and Y any surjective strictly monotone or antimonotone map $X \to Y$ is continuous.*

9.1x. Show that the surjectivity condition in *9.Ax* is needed.

9.2x. Is it possible to remove the word *strictly* from the hypothesis of Theorem *9.Ax*?

9.3x. Under conditions of Theorem *9.Ax*, is f continuous with respect to the right-ray or left-ray topologies?

9.Bx. *A map of a poset to a poset is monotone iff it is continuous with respect to the poset topologies.*

9°14x. Gromov–Hausdorff Distance

9.Cx. For any metric spaces X and Y, there exists a metric space Z such that X and Y can be isometrically embedded into Z.

Having isometrically embedded two metric space in a single one, we can consider the Hausdorff distance between their images (see. 4°15x). The infimum of such Hausdorff distances over all pairs of isometric embeddings of metric spaces X and Y into metric spaces is the *Gromov–Hausdorff distance* between X and Y.

9.Dx. Does there exist metric spaces with infinite Gromov–Hausdorff distance?

9.Ex. Prove that the Gromov–Hausdorff distance is symmetric and satisfies the triangle inequality.

9.Fx. ***Riddle.*** In what sense the Gromov–Hausdorff distance can satisfy the first axiom of metric?

9°15x. Functions on the Cantor Set and Square-Filling Curves

Recall that the Cantor set K is the set of real numbers that can be presented as sums of series of the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n \in \{0, 2\}$.

9.Gx. Consider the map

$$\gamma_1 : K \to [0,1] : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

Prove that it is a continuous surjection. Sketch the graph of γ_1.

9.Hx. Prove that the function

$$K \to K : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \sum_{n=1}^{\infty} \frac{a_{2n}}{3^n}$$

is continuous.

Denote by K^2 the set $\{(x, y) \in \mathbb{R}^2 \mid x \in K, y \in K\}$.

9.Ix. Prove that the map

$$\gamma_2 : K \to K^2 : \sum_{n=1}^{\infty} \frac{a_n}{3^n} \mapsto \left(\sum_{n=1}^{\infty} \frac{a_{2n-1}}{3^n}, \sum_{n=1}^{\infty} \frac{a_{2n}}{3^n} \right)$$

is a continuous surjection.

The unit segment $[0,1]$ is denoted by I, the set

$$\{(x_1, \dots, x_n) \subset \mathbb{R}^n \mid 0 \le x_i \le 1 \text{ for each } i\}$$

is denoted by I^n and called the (unit) *n-cube*.

9.Jx. Prove that the map $\gamma_3 : K \to I^2$ defined as the composition of $\gamma_2 : K \to K^2$ and $K^2 \to I^2 : (x, y) \mapsto (\gamma_1(x), \gamma_1(y))$ is a continuous surjection.

9.Kx. Prove that the map $\gamma_3 : K \to I^2$ is a restriction of a continuous map. (Cf. *2.Bx.2*.)

The latter map is a continuous surjection $I \to I^2$. Thus, this is a curve filling the square. A curve with this property was first constructed by G. Peano in 1890. Though the construction sketched above involves the same ideas as the original Peano's construction, the two constructions are slightly different. Since then a lot of other similar examples have been found. You may find a nice survey of them in Hans Sagan's book Space-Filling Curves, Springer-Verlag 1994. Here is a sketch of Hilbert's construction.

9.Lx. Prove that there exists a sequence of polygonal maps $f_n : I \to I^2$ such that

(1) f_n connects all centers of the squares forming the obvious subdivision of I^2 into 4^n equal squares with side $1/2^n$;

(2) $\operatorname{dist}(f_n(x), f_{n-1}(x)) \le \sqrt{2}/2^{n+1}$ for any $x \in I$ (here dist denotes the metric induced on I^2 from the standard Euclidean metric of R^2).

9.Mx. Prove that any sequence of paths $f_n : I \to I^2$ satisfying the conditions of *9.Lx* converges to a map $f : I \to I^2$ (i.e., for any $x \in I$ there exists a limit $f(x) = \lim_{n\to\infty} f_n(x)$), this map is continuous, and its image is dense in I^2.

9.Nx.[2] Prove that any continuous map $I \to I^2$ with dense image is surjective.

9.Ox. Generalize *9.Ix – 9.Kx*, *9.Lx – 9.Nx* to obtain a continuous surjection of I onto I^n.

[2]Although this problem can be solved by using theorems that are well known from Calculus, we have to mention that it would be more appropriate to solve it after Section 16. Cf. Problems *16.P*, *16.U*, and *16.K*.

10. Homeomorphisms

10°1. Definition and Main Properties of Homeomorphisms

An invertible map is a *homeomorphism* if both this map and its inverse are continuous.

10.A. Find an example of a continuous bijection which is not a homeomorphism.

10.B. Find a continuous bijection $[0,1) \to S^1$ which is not a homeomorphism.

10.C. The identity map of a topological space is a homeomorphism.

10.D. *A composition of homeomorphisms is a homeomorphism.*

10.E. *The inverse of a homeomorphism is a homeomorphism.*

10°2. Homeomorphic Spaces

A topological space X is *homeomorphic* to a space Y if there exists a homeomorphism $X \to Y$.

10.F. *Being homeomorphic is an equivalence relation.*

10.1. Riddle. How is Theorem *10.F* related to *10.C*–*10.E*?

10°3. Role of Homeomorphisms

10.G. Let $f : X \to Y$ be a homeomorphism. Then $U \subset X$ is open (in X) iff $f(U)$ is open (in Y).

10.H. $f : X \to Y$ is a homeomorphism iff f is a bijection and determines a bijection between the topological structures of X and Y.

10.I. Let $f : X \to Y$ be a homeomorphism. Then for every $A \subset X$

(1) A is closed in X iff $f(A)$ is closed in Y;
(2) $f(\operatorname{Cl} A) = \operatorname{Cl}(f(A))$;
(3) $f(\operatorname{Int} A) = \operatorname{Int}(f(A))$;
(4) $f(\operatorname{Fr} A) = \operatorname{Fr}(f(A))$;
(5) A is a neighborhood of a point $x \in X$ iff $f(A)$ is a neighborhood of the point $f(x)$;
(6) etc.

Therefore, from the topological point of view, homeomorphic spaces are completely identical: a homeomorphism $X \to Y$ establishes a one-to-one correspondence between all phenomena in X and Y that can be expressed in terms of topological structures. [3]

10°4. More Examples of Homeomorphisms

10.J. Let $f : X \to Y$ be a homeomorphism. Prove that for every $A \subset X$ the submap $\mathrm{ab}(f) : A \to f(A)$ is also a homeomorphism.

10.K. Prove that every isometry (see Section 9) is a homeomorphism.

10.L. Prove that every nondegenerate affine transformation of $\mathbb{R}^n$ is a homeomorphism.

10.M. *Let X and Y be two linearly ordered sets. Any strictly monotone surjection $f : X \to Y$ is a homeomorphism with respect to the interval topological structures in X and Y.*

10.N Corollary. *Any strictly monotone surjection $f : [a, b] \to [c, d]$ is a homeomorphism.*

10.2. Let R be a positive real. Prove that the inversion

$$\tau : \mathbb{R}^n \smallsetminus 0 \to \mathbb{R}^n \smallsetminus 0 : x \mapsto \frac{Rx}{|x|^2}$$

is a homeomorphism.

10.3. Let $\mathcal{H} = \{\, z \in \mathbb{C} \mid \operatorname{Im} z > 0 \,\}$ be the upper half-plane, let $a, b, c, d \in \mathbb{R}$, and let $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$. Prove that

$$f : \mathcal{H} \to \mathcal{H} : z \mapsto \frac{az + b}{cz + d}$$

is a homeomorphism.

10.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a bijection. Prove that f is a homeomorphism iff f is a monotone function.

10.5. 1) Prove that every bijection of an indiscrete space onto itself is a homeomorphism. Prove the same 2) for a discrete space and 3) $\mathbb{R}_{T_1}$.

10.6. Find all homeomorphisms of the space  (see Section 2) to itself.

[3] This phenomenon was used as a basis for a definition of the subject of topology in the first stages of its development, when the notion of topological space had not been developed yet. Then mathematicians studied only subspaces of Euclidean spaces, their continuous maps, and homeomorphisms. Felix Klein in his famous Erlangen Program classified various geometries that had emerged up to that time, like Euclidean, Lobachevsky, affine, and projective geometries, and defined topology as a part of geometry that deals with properties preserved by homeomorphisms. In fact, it was not assumed to be a program in the sense of being planned, although it became a kind of program. It was a sort of dissertation presented by Klein for getting a professor position at the Erlangen University.

10.7. Prove that every continuous bijection of the arrow onto itself is a homeomorphism.

10.8. Find two homeomorphic spaces X and Y and a continuous bijection $X \to Y$ which is not a homeomorphism.

10.9. Is $\gamma_2 : K \to K^2$ considered in Problem *9.Ix* a homeomorphism? Recall that K is the Cantor set, $K^2 = \{(x, y) \in \mathbb{R}^2 \mid x \in K, y \in K\}$ and γ_2 is defined by

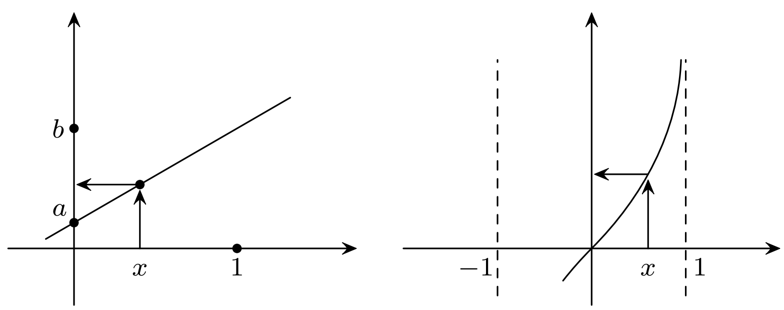
$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{3^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k} \right)$$

10°5. Examples of Homeomorphic Spaces

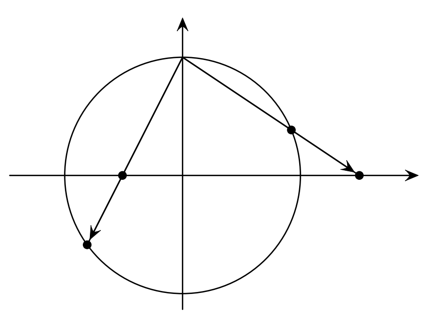
Below the homeomorphism relation is denoted by $\cong$. This notation it is not commonly accepted. In other textbooks, any sign close to, but distinct from $=$, e.g., $\sim$, $\simeq$, $\approx$, is used.

10.O. Prove that

(1) $[0,1] \cong [a,b]$ for any $a < b$;
(2) $[0,1) \cong [a,b) \cong (0,1] \cong (a,b]$ for any $a < b$;
(3) $(0,1) \cong (a,b)$ for any $a < b$;
(4) $(-1,1) \cong \mathbb{R}$;
(5) $[0,1) \cong [0,+\infty)$ and $(0,1) \cong (0,+\infty)$.



10.P. Let $N = (0,1) \in S^1$ be the North Pole of the unit circle. Prove that $S^1 \smallsetminus N \cong \mathbb{R}^1$.



10.Q. The graph of a continuous real-valued function defined on an interval is homeomorphic to the interval.

10.R. $S^n \smallsetminus \text{point} \cong \mathbb{R}^n$. (The first space is the "punctured sphere".)

> ***10.10.*** Prove that the following plane domains are homeomorphic. (Here and below, our notation is sometimes slightly incorrect: in the curly brackets, we drop the initial part "$(x, y) \in \mathbb{R}^2 \mid$".)
>
> (1) The whole plane $\mathbb{R}^2$;
> (2) open square $\operatorname{Int} I^2 = \{ x, y \in (0, 1) \}$;
> (3) open strip $\{ x \in (0, 1) \}$;
> (4) open half-plane $\mathcal{H} = \{ y > 0 \}$;
> (5) open half-strip $\{ x > 0,\ y \in (0, 1) \}$;
> (6) open disk $B^2 = \{ x^2 + y^2 < 1 \}$;
> (7) open rectangle $\{ a < x < b,\ c < y < d \}$;
> (8) open quadrant $\{ x, y > 0 \}$;
> (9) open angle $\{ x > y > 0 \}$;
> (10) $\{ y^2 + |x| > x \}$, i.e., plane without the ray $\{ y = 0 \le x \}$;
> (11) open half-disk $\{ x^2 + y^2 < 1,\ y > 0 \}$;
> (12) open sector $\{ x^2 + y^2 < 1,\ x > y > 0 \}$.

10.S. Prove that

(1) the closed disk D^2 is homeomorphic to the square $I^2 = \{ (x, y) \in \mathbb{R}^2 \mid x, y \in [0, 1] \}$;
(2) the open disk $B^2 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$ is homeomorphic to the open square $\operatorname{Int} I^2 = \{ (x, y) \in \mathbb{R}^2 \mid x, y \in (0, 1) \}$;
(3) the circle S^1 is homeomorphic to the boundary $\partial I^2 = I^2 \smallsetminus \operatorname{Int} I^2$ of the square.

10.T. Let $\Delta \subset \mathbb{R}^2$ be a planar bounded closed convex set with nonempty interior U. Prove that

(1) Δ is homeomorphic to the closed disk D^2;
(2) U is homeomorphic to the open disk B^2;
(3) $\operatorname{Fr} \Delta = \operatorname{Fr} U$ is homeomorphic to S^1.

> ***10.11.*** In which of the assertions in *10.T* can we omit the assumption that the closed convex set Δ be bounded?
>
> ***10.12.*** Classify up to homeomorphism all (nonempty) closed convex sets in the plane. (Make a list without repeats; prove that every such a set is homeomorphic to one in the list; postpone a proof of nonexistence of homeomorphisms till Section 11.)
>
> ***10.13*.*** Generalize the previous three problems to the case of sets in $\mathbb{R}^n$ with arbitrary n.

The latter four problems show that angles are not essential in topology, i.e., for a line or the boundary of a domain the property of having angles is not preserved by homeomorphism. Here are two more problems on this.

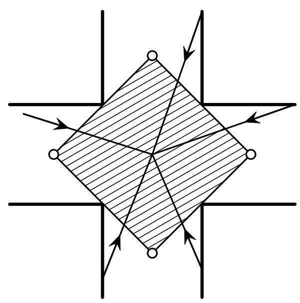
10.14. Prove that every simple (i.e., without self-intersections) closed polygon in $\mathbb{R}^2$ (as well as in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the circle S^1.

10.15. Prove that every nonclosed simple finite unit polyline in $\mathbb{R}^2$ (as well as in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the segment $[0, 1]$.

The following problem generalizes the technique used in the previous two problems and is used more often than it may seem at first glance.

10.16. Let X and Y be two topological spaces equipped with fundamental covers: $X = \bigcup_\alpha X_\alpha$ and $Y = \bigcup_\alpha Y_\alpha$. Suppose $f : X \to Y$ is a map such that $f(X_\alpha) = Y_\alpha$ for each α and the submap $\mathrm{ab}(f) : X_\alpha \to Y_\alpha$ is a homeomorphism. Then f is a homeomorphism.

10.17. Prove that $\mathbb{R}^2 \smallsetminus \{\, |x|, |y| > 1 \,\} \cong I^2 \smallsetminus \{x, y \in \{0, 1\}\}$. (An "infinite cross" is homeomorphic to a square without vertices.)



10.18*. A nonempty set $\Sigma \subset \mathbb{R}^2$ is "star-shaped with respect to a point c" if Σ is a union of segments and rays with an endpoint at c. Prove that if Σ is open, then $\Sigma \cong B^2$. (What can you say about a closed star-shaped set with nonempty interior?)

10.19. Prove that the following plane figures are homeomorphic to each other. (See *10.10* for our agreement about notation.)

(1) A half-plane: $\{\, x \geq 0 \,\}$;
(2) a quadrant: $\{\, x, y \geq 0 \,\}$;
(3) an angle: $\{\, x \geq y \geq 0 \,\}$;
(4) a semi-open strip: $\{\, y \in [0, 1) \,\}$;
(5) a square without three sides: $\{\, 0 < x < 1,\ 0 \leq y < 1 \,\}$;
(6) a square without two sides: $\{\, 0 \leq x,\ y < 1 \,\}$;
(7) a square without a side: $\{\, 0 \leq x \leq 1,\ 0 \leq y < 1 \,\}$;
(8) a square without a vertex: $\{\, 0 \leq x,\ y \leq 1 \,\} \smallsetminus (1, 1)$;
(9) a disk without a boundary point: $\{\, x^2 + y^2 \leq 1,\ y \neq 1 \,\}$;
(10) a half-disk without the diameter: $\{\, x^2 + y^2 \leq 1,\ y > 0 \,\}$;
(11) a disk without a radius: $\{\, x^2 + y^2 \leq 1 \,\} \smallsetminus [0, 1]$;
(12) a square without a half of the diagonal: $\{\, |x| + |y| \leq 1 \,\} \smallsetminus [0, 1]$.

10.20. Prove that the following plane domains are homeomorphic to each other:

(1) punctured plane $\mathbb{R}^2 \smallsetminus (0, 0)$;
(2) punctured open disk $B^2 \smallsetminus (0, 0) = \{\, 0 < x^2 + y^2 < 1 \,\}$;
(3) annulus $\{\, a < x^2 + y^2 < b \,\}$, where $0 < a < b$;
(4) plane without a disk: $\mathbb{R}^2 \smallsetminus D^2$;
(5) plane without a square: $\mathbb{R}^2 \smallsetminus I^2$;

(6) plane without a segment: $\mathbb{R}^2 \smallsetminus [0,1]$;
(7) $\mathbb{R}^2 \smallsetminus \Delta$, where Δ is a closed bounded convex set with $\operatorname{Int}\Delta \neq \varnothing$.

10.21. Let $X \subset \mathbb{R}^2$ be an union of several segments with a common endpoint. Prove that the complement $\mathbb{R}^2 \smallsetminus X$ is homeomorphic to the punctured plane.

10.22. Let $X \subset \mathbb{R}^2$ be a simple nonclosed finite polyline. Prove that its complement $\mathbb{R}^2 \smallsetminus X$ is homeomorphic to the punctured plane.

10.23. Let $K = \{a_1, \dots, a_n\} \subset \mathbb{R}^2$ be a finite set. The complement $\mathbb{R}^2 \smallsetminus K$ is a *plane with n punctures*. Prove that any two planes with n punctures are homeomorphic, i.e., the position of $a_1, \dots, a_n$ in $\mathbb{R}^2$ does not affect the topological type of $\mathbb{R}^2 \smallsetminus \{a_1, \dots, a_n\}$.

10.24. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed disks. Prove that the complement of their union is homeomorphic to a plane with n punctures.

10.25. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed disks. The complement of the union of its interiors is said to be *plane with n holes*. Prove that any two planes with n holes are homeomorphic, i.e., the location of disks $D_1, \dots, D_n$ does not affect the topological type of $\mathbb{R}^2 \smallsetminus \cup_{i=1}^n \operatorname{Int} D_i$.

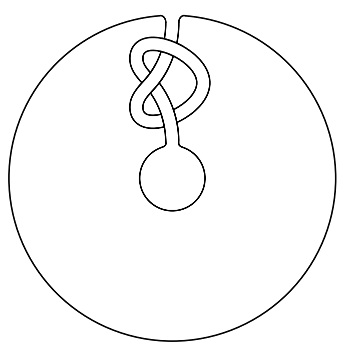
10.26. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions such that $f < g$. Prove that the "strip" $\{\, (x,y) \in \mathbb{R}^2 \mid f(x) \le y \le g(x) \,\}$ bounded by their graphs is homeomorphic to the closed strip $\{\, (x,y) \mid y \in [0,1] \,\}$.

10.27. Prove that a mug (with a handle) is homeomorphic to a doughnut.

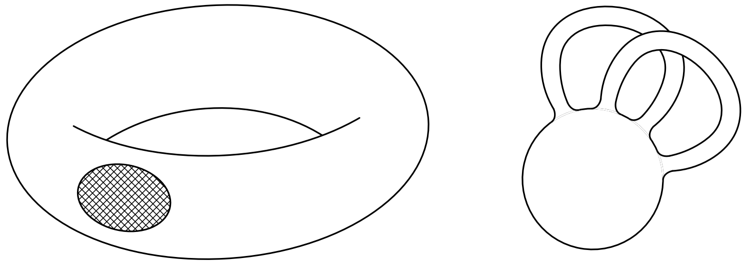
10.28. Arrange the following items to homeomorphism classes: a cup, a saucer, a glass, a spoon, a fork, a knife, a plate, a coin, a nail, a screw, a bolt, a nut, a wedding ring, a drill, a flower pot (with a hole in the bottom), a key.

10.29. In a spherical shell (the space between two concentric spheres), one drilled out a cylindrical hole connecting the boundary spheres. Prove that the rest is homeomorphic to D^3.

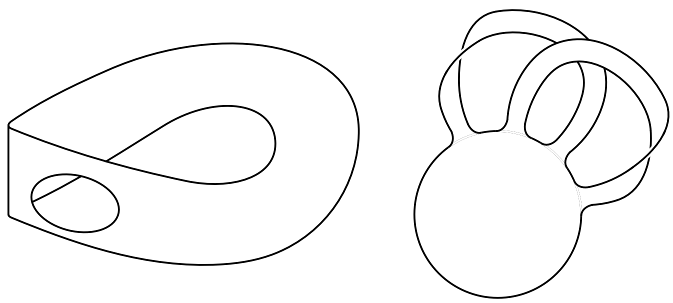
10.30. In a spherical shell, one made a hole connecting the boundary spheres and having the shape of a knotted tube (see Figure).Prove that the rest of the shell is homeomorphic to D^3.



10.31. Prove that surfaces shown in the Figure are homeomorphic (they are called *handles*).



10.32. Prove that surfaces shown in the the Figure are homeomorphic. (They are homeomorphic to a *Klein bottle with two holes*. More details about this is given in Section 21.)



10.33*. Prove that $\mathbb{R}^3 \smallsetminus S^1 \cong \mathbb{R}^3 \smallsetminus \big(\mathbb{R}^1 \cup (0,0,1)\big)$. (What can you say in the case of $\mathbb{R}^n$?)

10.34. Prove that the subset of S^n defined in the standard coordinates in $\mathbb{R}^{n+1}$ by the inequality $x_1^2 + x_2^2 + \dots + x_k^2 < x_{k+1}^2 + \dots + x_n^2$ is homeomorphic to $\mathbb{R}^n \smallsetminus \mathbb{R}^{n-k}$.

10°6. Examples of Nonhomeomorphic Spaces

10.U. Spaces consisting of different number of points are not homeomorphic.

10.V. A discrete space and an indiscrete space (having more than one point) are not homeomorphic.

10.35. Prove that the spaces $\mathbb{Z}$, $\mathbb{Q}$ (with topology induced from $\mathbb{R}$), $\mathbb{R}$, $\mathbb{R}_{T_1}$, and the arrow are pairwise not homeomorphic.

10.36. Find two spaces X and Y that are not homeomorphic, but there exist continuous bijections $X \to Y$ and $Y \to X$.

10°7. Homeomorphism Problem and Topological Properties

One of the classical problems in topology is the *homeomorphism problem*: to find out whether two given topological spaces are homeomorphic. In each special case, the character of solution depends mainly on the answer. In order to prove that two spaces are homeomorphic, it suffices to present a homeomorphism between them. Essentially this is what one usually does in this case (see the examples above). To prove that two spaces are **not** homeomorphic, it does not suffice to consider any special map, and usually it is impossible to review all the maps. Therefore, for proving the nonexistence of a homeomorphism one uses indirect arguments. In particular, we can find a property or a characteristic shared by homeomorphic spaces and such that one of the spaces has it, while the other does not. Properties and characteristics that are shared by homeomorphic spaces are called *topological properties* and *invariants*. Obvious examples are the cardinality (i.e., the number of elements) of the set of points and the set of open sets (cf. Problems *10.34* and *10.U*). Less obvious properties are the main object of the next chapter.

10°8. Information: Nonhomeomorphic Spaces

Euclidean spaces of different dimensions are not homeomorphic. The disks D^p and D^q with $p \neq q$ are not homeomorphic. The spheres S^p, S^q with $p \neq q$ are not homeomorphic. Euclidean spaces are homeomorphic neither to balls, nor to spheres (of any dimension). Letters A and B are not homeomorphic (if the lines are absolutely thin!). The punctured plane $\mathbb{R}^2 \smallsetminus$ point is not homeomorphic to the plane with a hole: $\mathbb{R}^2 \smallsetminus \{ x^2+y^2 < 1 \}$.

These statements are of different degrees of difficulty. Some of them will be considered in the next section. However, some of them can not be proved by techniques of this course. (See, e.g., [**6**].)

10°9. Embeddings

A continuous map $f : X \to Y$ is a (*topological*) *embedding* if the submap $\operatorname{ab}(f) : X \to f(X)$ is a homeomorphism.

10.W. The inclusion of a subspace into a space is an embedding.

10.X. Composition of embeddings is an embedding.

10.Y. Give an example of a continuous injection which is not a topological embedding. (Find such an example above and create a new one.)

10.37. Find topological spaces X and Y such that X can be embedded into Y, Y can be embedded into X, but $X \not\cong Y$.

10.38. Prove that $\mathbb{Q}$ cannot be embedded into $\mathbb{Z}$.

10.39. 1) Can a discrete space be embedded into an indiscrete space? 2) How about vice versa?

10.40. Prove that the spaces $\mathbb{R}$, $\mathbb{R}_{T_1}$, and the arrow cannot be embedded into each other.

10.41 Corollary of Inverse Function Theorem. Deduce from the Inverse Function Theorem (see, e.g., any course of advanced calculus) the following statement:

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map whose Jacobian $\det(\partial f_i/\partial x_j)$ does not vanish at the origin $0 \in \mathbb{R}^n$. Then there exists a neighborhood U of the origin such that the restriction $f|_U : U \to \mathbb{R}^n$ is an embedding and $f(U)$ is open.

It is of interest that if $U \subset \mathbb{R}^n$ is an open set, then any continuous injection $f : U \to \mathbb{R}^n$ is an embedding and $f(U)$ is also open in $\mathbb{R}^n$.

10°10. Equivalence of Embeddings

Two embeddings $f_1, f_2 : X \to Y$ are *equivalent* if there exist homeomorphisms $h_X : X \to X$ and $h_Y : Y \to Y$ such that $f_2 \circ h_X = h_Y \circ f_1$. (The latter equality may be stated as follows: the diagram

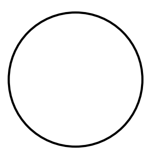
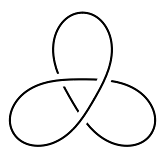
$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "h_X"'] & Y \arrow[d, "h_Y"] \\ X \arrow[r, "f_2"] & Y \end{tikzcd}$$

is commutative.)

An embedding $S^1 \to \mathbb{R}^3$ is called a *knot*.

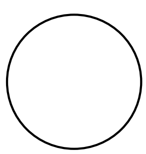
10.42. Prove that knots $f_1, f_2 : S^1 \to \mathbb{R}^3$ with $f_1(S^1) = f_2(S^1)$ are equivalent.

10.43. Prove that knots with images

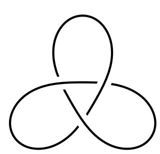



are equivalent.

Information: There are nonequivalent knots. For instance, those with images



and



Proofs and Comments

8.A If $x \in f^{-1}(B)$, then $f(x) \in B$.

8.B $\boxed{\Longrightarrow}$ Obvious. $\boxed{\Longleftarrow}$ For each $y \in B$, there exists an element x such that $f(x) = y$. By the definition of the preimage, $x \in f^{-1}(B)$, whence $y \in f(f^{-1}(B))$. Thus, $B \subset f(f^{-1}(B))$. The opposite inclusion holds true for any set, see *8.A*.

8.C (a) $\Longrightarrow$ (b) Assume that $f(C) = B$ implies $C = f^{-1}(B)$. If there exist distinct $a_1, a_2 \in f^{-1}(B)$ such that $f(a_1) = f(a_2)$, then also $f(f^{-1}(B) \smallsetminus a_2) = B$, which contradicts the assumption.
(b) $\Longrightarrow$ (a) Assume now that there exists $C \neq f^{-1}(B)$ such that $f(C) = B$. Clearly, $C \subset f^{-1}(B)$. Therefore, C can differ from $f^{-1}(B)$ only if $f^{-1}(B) \smallsetminus C \neq \varnothing$. Take $a_1 \in f^{-1}(B) \smallsetminus C$, let $b = f(a_1)$. Since $f(C) = B$, there exists $a_2 \in C$ with $f(a_2) = f(a_1)$, but $a_2 \neq a_1$ because $a_2 \in C$, while $a_1 \notin C$.

8.D This follows from *8.C*.

8.E Let $x \in A$. Then $f(x) = y \in f(A)$, whence $x \in f^{-1}(f(A))$.

8.F Both equalities are obviously equivalent to the following statement: $f(x) \notin f(A)$ for each $x \notin A$.

8.G $\mathrm{in}^{-1}(B) = \{x \in A \mid x \in B\} = A \cap B$.

8.H Let $x \in X$. Then

$$h \circ (g \circ f)(x) = h(g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = (h \circ g) \circ f(x).$$

8.J Let $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$, because f is injective, and $g(f(x_1)) \neq g(f(x_2))$, because g is injective.

8.K If f is not injective, then there exist $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. However, then $(g \circ f)(x_1) = (g \circ f)(x_2)$, which contradicts the injectivity of $g \circ f$.

8.L Let $f : X \to Y$ and $g : Y \to Z$ be surjective. Then we have $f(X) = Y$, whence $g(f(X)) = g(Y) = Z$.

8.M This follows from the obvious inclusion $\mathrm{Im}(g \circ f) \subset \mathrm{Im}\, g$.

8.N This follows from *8.J* and *8.L*.

8.O $\boxed{\Longrightarrow}$ Use *8.K* and *8.M*. $\boxed{\Longleftarrow}$ Let $f : X \to Y$ be a bijection. Then, by the surjectivity, for each $y \in Y$ there exists $x \in X$ such that $y = f(x)$, and, by the injectivity, such an element of X is unique. Putting $g(y) = x$, we obtain a map $g : Y \to X$. It is easy to check (please, do it!) that g is inverse to f.

8.P This is actually obvious. On the other hand, it is interesting to look at "mechanical" proof.Let two maps $g, h : Y \to X$ be inverse to a map $f : X \to Y$. Consider the composition $g \circ f \circ h : Y \to X$. On the one hand, $g \circ f \circ h = (g \circ f) \circ h = \mathrm{id}_X \circ h = h$. On the other hand, $g \circ f \circ h = g \circ (f \circ h) = g \circ \mathrm{id}_Y = g$.

9.A Let $f : X \to Y$ be a map. $\boxed{\Longrightarrow}$ If $f : X \to Y$ is continuous, then, for each closed set $F \subset Y$, the set $X \smallsetminus f^{-1}(F) = f^{-1}(Y \smallsetminus F)$ is open, and therefore $f^{-1}(F)$ is closed. $\boxed{\Longleftarrow}$ Exchange the words *open* and *closed* in the above argument.

9.C If a set U is open in X, then its preimage $\mathrm{in}^{-1}(U) = U \cap A$ is open in A by the definition of the relative topology.

9.D If $U \in \Omega_A$, then $U = V \cap A$ for some $V \in \Omega$. If the map $\mathrm{in} : (A, \Omega') \to (X, \Omega)$ is continuous, then the preimage $U = \mathrm{in}^{-1}(V) = V \cap A$ of a set $V \in \Omega$ belongs to Ω'. Thus, $\Omega_A \subset \Omega'$.

9.E Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. We must show that for every $U \subset Z$ which is open in Z its preimage $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X. The set $g^{-1}(U)$ is open in Y by continuity of g. In turn, its preimage $f^{-1}(g^{-1}(U))$ is open in X by the continuity of f.

9.F $(f|_{A,B})^{-1}(V) = (f|_{A,B})^{-1}(U \cap B) = A \cap f^{-1}(U)$.

9.G $\boxed{\Longrightarrow}$ Use *9.F*. $\boxed{\Longleftarrow}$ Use the fact that $f = \mathrm{in}_{f(X)} \circ \mathrm{ab}\, f$.

9.H The preimage of any set under a continuous map either is empty or coincides with the whole space.

9.I $\boxed{\Longrightarrow}$ Let $a \in X$. Then for any neighborhood U of $f(a)$ we can construct a desired neighborhood V of a just by putting $V = f^{-1}(U)$: indeed, $f(V) = f(f^{-1}(U)) \subset U$. $\boxed{\Longleftarrow}$ We must check that the preimage of each open set is open. Let $U \subset Y$ be an open set in Y. Take $a \in f^{-1}(U)$. By continuity of f at a, there exists a neighborhood V of a such that $f(V) \subset U$. Then, obviously, $V \subset f^{-1}(U)$. This proves that any point of $f^{-1}(U)$ is internal, and hence $f^{-1}(U)$ is open.

9.J Proving each of the implications, use Theorem *4.I*, according to which any neighborhood of a point in a metric space contains a ball centered at the point.

9.K The condition "for every point $x \in X$ the inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$" means that $f(B_\delta(a)) \subset B_{Ge}(f(a))$. Now, apply *9.J*.

9.L This immediately follows from the inequality of Problem *4.35*.

9.M If $f(x) = f(y)$, then $\rho(f(x), f(y)) = 0$, whence $\rho(x, y) = 0$.

9.N Use the obvious fact that the primage of any open ball under isometric embedding is an open ball of the same radius.

9.O The set of solutions of the system is the intersection of the preimages of the point $0 \in \mathbb{R}$. As the maps are continuos and the point is closed, the preimages of the point are closed, and hence the intersection of the preimages is closed.

9.P The set of solutions of a system of nonstrict inequalities is the intersection of preimages of closed ray $[0, +\infty)$, the set of solutions of a system of strict inequalities is the intersection of the preimages of open ray $(0, +\infty)$.

9.Q Indeed, it makes no sense to require the necessity: the intersection of an open set with any set A is open in A anyway.

9.R Consider the complement $X \smallsetminus F$ of F and apply *9.Q*.

9.S Let Γ be an open cover of a space X. Let $U \subset X$ be a set such that $U \cap A$ is open in A for any $A \in \Gamma$. By *5.E*, open subset of open subspace is open in the whole space. Therefore, $A \cap U$ is open in X. Then $U = \bigcup_{A \in \Gamma} A \cap U$ is open as a union of open sets.

9.T Argue as in the preceding proof, but instead of the definition of a fundamental cover use its reformulation *9.R*, and instead of Theorem *5.E* use Theorem *5.F*, according to which a closed set of a closed subspace is closed in the entire space.

9.U Denote the space by X and the cover by Γ. As Γ is locally finite, each point $a \in X$ has a neighborhood U_a meeting only a finite number of elements of Γ. Form the cover $\Sigma = \{U_a \mid a \in X\}$ of X. Let $U \subset X$ be a set such that $U \cap A$ is open for each $A \in \Gamma$. By *9.T*, $\{A \cap U_a \mid A \in \Gamma\}$ is a fundamental cover of U_a for any $a \in X$. Hence $U_a \cap U$ is open in U_a. By *9.S*, Σ is fundamental. Hence, U is open.

9.V Let U be a set open in Y. As the restrictions of f to elements of Γ are continuous, the preimage of U under restriction of f to any $A \in \Gamma$ is open. Obviously, $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$. Hence $f^{-1}(U) \cap A$ is open in A for any $A \in \Gamma$. By hypothesis, Γ is fundamental. Therefore $f^{-1}(U)$ is open in X. We have proved that the preimage of any open set under f is open. Thus f is continuous.

9.Ax It suffices to prove that the preimage of any base open set is open. The proof is quite straight-forward. For instance, the preimage of $\{x \mid a \prec x \prec b\}$ is $\{x \mid c \prec x \prec d\}$, where $f(c) = a$ and $f(d) = b$, which is a base open set.

9.Bx Let X and Y be two posets, $f : X \to Y$ a map. $\boxed{\Longrightarrow}$ Assume that $f : X \to Y$ is monotone. To prove the continuity of f it suffices to prove that the preimage of each base open set is open. Put $U = C_Y^+(b)$ and $V = f^{-1}(U)$. If $x \in V$ (i.e., $b \prec f(x)$), then for any $y \in C_X^+(x)$ (i.e., $x \prec y$)

we have $y \in V$. Therefore, $V = \bigcup\limits_{f(x)\in U} C_X^+(x)$. This set is open as a union of open base sets (in the poset topology of X).
$\boxed{\Longleftarrow}$ Let $a, b \in X$ and $a \prec b$. Then b is contained in any neighborhood of a. The set $C_Y^+(f(a))$ is a neighborhood of $f(a)$ in Y. By continuity of f, a has a neighborhood in X whose f-image is contained in $C_Y^+(f(a))$. However, then the minimal neighborhood of a in X (i.e., $C_X^+(a)$) also has this property. Therefore, $f(b) \in f(C_X^+(a)) \subset C_Y^+(f(a))$, and hence $f(a) \prec f(b)$.

***9.C*x** Construct Z as the disjoint union of X and Y. In the union, put the distance between two points in (the copy of) X (respectively, Y) to be equal to the distance between the corresponding points in X (respectively, Y). To define the distance between points of different copies, choose points $x_0 \in X$ and $y_0 \in Y$, and put $\rho(a, b) = \rho_X(a, x_0) + \rho_Y(y_0, b) + 1$ for $a \in X$ and $b \in Y$. Check (this is easy, really), that this defines a metric.

***9.D*x** Yes. For example, consider a singleton and any unbounded space.

***9.E*x** Although, as we have seen solving the previous problem, the Gromov–Hausdorff distance can be infinite, while symmetricity and the triangle inequality were formulated above only for functions with finite values, these two properties make sense if infinite values are admitted. (The triangle inequality should be considered fulfilled if two or three of the quantities involved are infinite, and not fulfilled if only one of them is infinite.) The following construction helps to prove the triangle inequality. Let metric spaces X and Y are isometrically embedded into a metric space A, and metric spaces Y and Z are isometrically embedded into a metric space B. Construct a new metric space in which A and B would be isometrically embedded meeting in Y. To do this, add to A all points of $B \smallsetminus A$. Put distances between these points to be equal to the distances between them in B. Put the distance between $x \in A \smallsetminus B$ and $z \in B \smallsetminus A$ equal to $\inf\{\rho_A(x, y) + \rho_B(y, z) \mid y \in A \cap B\}$. Compare this construction with the construction from the solution of Problem *9.Cx*. Prove that this gives a metric space and use the triangle inequality for the Hausdorff distance between X, Y, and Z in this space.

***9.F*x** Partially, the answer is obvious. Certainly, the Gromov–Hausdorff distance is nonnegative! But what if it is zero, in what sense the spaces should be equal then? First, the most optimistic idea is that then there should exist an isometric bijection between the spaces. But this is not true, as we can see looking at the spaces $\mathbb{Q}$ and $\mathbb{R}$ with standard distances in them. However, it is true for *compact* metric spaces.

10.A For example, consider the identity map of a discrete topological space X onto the same set but equipped with indiscrete topology. For another example, see *10.B*.

10.B Consider the map $x \mapsto (\cos 2\pi x, \sin 2\pi x)$.

10.C This and the next two statements directly follow from the definition of a homeomorphism.

10.F See the solution of *10.1*.

10.G Denote $f(U) \subset Y$ by V. Since f is a bijection, we have $U = f^{-1}(V)$. We also denote $f^{-1} : Y \to X$ by g. $\boxed{\Rightarrow}$ We have $V = g^{-1}(U)$, which is open by continuity of g. $\boxed{\Leftarrow}$ If $V = f(U)$ is open, then $U = g(V)$ is open as the preimage of an open set under a continuous map.

10.H See *10.G*.

10.I (a) A homeomorphism establishes a one-to-one correspondence between open sets of X and Y. Hence, it also establishes a one-to-one correspondence between closed sets of X and Y.
(b)–(f) Use the fact that the definitions of the closure, interior, boundary, etc. can be given in terms of open and closed sets.

10.J Obviously, $\mathrm{ab}(f)$ is a bijection. The continuity of $\mathrm{ab}(f)$ and $(\mathrm{ab}\, f)^{-1}$ follows from the general theorem *9.F* on the continuity of a submap of a continuous map.

10.K Any isometry is continuous, see *9.N* ; the map inverse to an isometry is an isometry.

10.L Recall that an affine transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by the formula $y = f(x) = Ax + b$, where A is a matrix and b a vector; f is nondegenerate if A is invertible, whence $x = A^{-1}(y-b) = A^{-1}(y) - A^{-1}(b)$, which means that f is a bijection and f^{-1} is also a nondegenerate affine transformation. Finally, f and f^{-1} are continuous, e.g., because they are given in coordinates by linear formulas (see *9.18* and *9.21*).

10.M Prove that f is invertible and f^{-1} is also strictly monotone. Then apply *9.Ax*.

10.O Homeomorphisms of the form $\langle 0, 1\rangle \to \langle a, b\rangle$ are defined, for example, by the formula $x \mapsto a + (b-a)x$, and homeomorphisms $(-1; 1) \to \mathbb{R}^1$ and $\langle 0, 1) \to \langle 0, +\infty)$ by the formula $x \mapsto \tan(\pi x/2)$. (In the latter case, you can easily find, e.g., a rational formula, but it is of interest that the above homeomorphism also arises quite often!)

10.P Observe that $(1/4, 5/4) \to S^1 \smallsetminus N : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a homeomorphism and use assertions (c) and (d) of the preceding problem. Here is another, more sophisticated construction, which can be of use in higher dimensions. The restriction f of the central projection $\mathbb{R}^2 \smallsetminus N \to \mathbb{R}^1$

(the x axis) to $S^1 \smallsetminus N$ is a homeomorphism. Indeed, f is obviously invertible: f^{-1} is a restriction of the central projection $\mathbb{R}^2 \smallsetminus N \to S^1 \smallsetminus N$. The map $S^1 \smallsetminus N \to \mathbb{R}$ is presented by formula $(x, y) \mapsto \frac{x}{1-y}$, and the inverse map by formula $x \mapsto (\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1})$. (Why are these maps continuous?)

10.Q Check that the vertical projection to the x axis determines a homeomorphism.

10.R As usual, we identify $\mathbb{R}^n$ and $\{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$. Then the restriction of the central projection

$$\mathbb{R}^{n+1} \smallsetminus (0, \dots, 0, 1) \to \mathbb{R}^n$$

to $S^n \smallsetminus (0, \dots, 0, 1)$ is a homeomorphism, which is called the *stereographic projection*. For $n = 2$, it is used in cartography. It is invertible: the inverse map is the restriction of the central projection $\mathbb{R}^{n+1} \smallsetminus (0, \dots, 0, 1) \to S^n \smallsetminus (0, \dots, 0, 1)$ to $\mathbb{R}^n$. The first map is defined by formula

$$x = (x_1, \dots, x_{n+1}) \mapsto \left(\frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right),$$

and the second one by

$$x = (x_1, \dots, x_n) \mapsto \left(\frac{2x_1}{|x|^2 + 1}, \dots, \frac{2x_n}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1}\right).$$

Check this. (Why are these maps continuous?) Explain how we can obtain a solution of this problem geometrically from the second solution to Problem *10.P*.

10.S After reading the proof, you may see that sometimes formulas are cumbersome, while a clearer verbal description is possible.
(a) Instead of I^2 it is convenient to consider the homeomorphic square $K = \{(x, y) \mid |x| \le 1, |y| \le 1\}$ of double size centered at the origin. (There is a linear homeomorphism $I^2 \to K : (x, y) \mapsto (2x - 1, 2y - 1)$.) We have a homeomorphism

$$K \to D^2 : (x, y) \mapsto \left(\frac{x \max\{|x|, |y|\}}{\sqrt{x^2 + y^2}}, \frac{y \max\{|x|, |y|\}}{\sqrt{x^2 + y^2}}\right).$$

Geometrically, this means that each segment joining the origin with a point on the contour of the square is linearly mapped to the part of the segment that lies within the circle.
(b), (c) Take suitable submaps of the above homeomorphism $K \to D^2$. Certainly, assertion (b) follows from the previous problem. It is also of

interest that in case (c) we can use a much simpler formula:

$$\partial K \to S^1 : (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

(This is simply a central projection!) We can also divide the circle into four arcs and map each of them to a side of K, cf. below.

10.T (a) For simplicity, assume that $D^2 \subset \Delta$. For $x \in \mathbb{R}^2 \setminus 0$, let $a(x)$ be the (unique) positive number such that $a(x)\frac{x}{|x|} \in \operatorname{Fr} \Delta$. Then we have a homeomorphism

$$\Delta \to D^2 : x \mapsto \frac{x}{a(x)} \quad \text{if } x \neq 0, \quad \text{while} \quad 0 \mapsto 0.$$

(Observe that in the case where Δ is the square K, we obtain the homeomorphism described in the preceding problem.)
(b), (c) Take suitable submaps of the above homeomorphism $\Delta \to D^2$.

10.U There is no bijection between them.

10.V These spaces have different numbers of open sets.

10.W Indeed, if $\mathrm{in} : A \to X$ is an inclusion, then the submap $\mathrm{ab}(\mathrm{in}) : A \to A$ is the identity homeomorphism.

10.X Let $f : X \to Y$ and $g : Y \to Z$ be two embeddings. Then the submap $\mathrm{ab}(g \circ f) : X \to g(f(X))$ is the composition of the homeomorphisms $\mathrm{ab}(f) : X \to f(X)$ and $\mathrm{ab}(g) : f(X) \to g(f(X))$.

10.Y The previous examples are $[0,1) \to S^1$ and $\mathbb{Z}_+ \to \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. Here is another one: Let $f : \mathbb{Z} \to \mathbb{Q}$ be a bijection and $\mathrm{in}_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$ the inclusion. Then the composition $\mathrm{in}_{\mathbb{Q}} \circ f : \mathbb{Z} \to \mathbb{R}$ is a continuous injection, but not an embedding.

Chapter III

Topological Properties

11. Connectedness

11°1. Definitions of Connectedness and First Examples

A topological space X is *connected* if X has only two subsets that are both open and closed: the empty set $\varnothing$ and the entire X. Otherwise, X is *disconnected*.

A *partition* of a set is a cover of this set with pairwise disjoint subsets. To *partition* a set means to construct such a cover.

11.A. *A topological space is connected, iff it has no partition into two nonempty open sets, iff it has no partition into two nonempty closed sets.*

11.1. 1) Is an indiscrete space connected? The same question for 2) the arrow and 3) $\mathbb{R}_{T_1}$.

11.2. Describe explicitly all connected discrete spaces.

11.3. Describe explicitly all disconnected two-point spaces.

11.4. 1) Is the set $\mathbb{Q}$ of rational numbers (with the relative topology induced from $\mathbb{R}$) connected? 2) The same question for the set of irrational numbers.

11.5. Let Ω_1 and Ω_2 be two topologies in a set X, and let Ω_2 be finer than Ω_1 (i.e., $\Omega_1 \subset \Omega_2$). 1) If (X, Ω_1) is connected, is (X, Ω_2) connected? 2) If (X, Ω_2) is connected, is (X, Ω_1) connected?

11°2. Connected Sets

When we say that a set A is connected, this means that A lies in some topological space (which should be clear from the context) and, equipped with the relative topology, A a connected space.

11.6. Characterize disconnected subsets without mentioning the relative topology.

11.7. Is the set $\{0, 1\}$ connected 1) in $\mathbb{R}$, 2) in the arrow, 3) in $\mathbb{R}_{T_1}$?

11.8. Describe explicitly all connected subsets 1) of the arrow, 2) of $\mathbb{R}_{T_1}$.

11.9. Show that the set $[0, 1] \cup (2, 3]$ is disconnected in $\mathbb{R}$.

11.10. Prove that every nonconvex subset of the real line is disconnected. (In other words, each connected subset of the real line is a singleton or an interval.)

11.11. Let A be a subset of a space X. Prove that A is disconnected iff A has two nonempty subsets B and C such that $A = B \cup C$, $B \cap \operatorname{Cl}_X C = \varnothing$, and $C \cap \operatorname{Cl}_X B = \varnothing$.

11.12. Find a space X and a disconnected subset $A \subset X$ such that if U and V are any two open sets partitioning X, then we have either $U \supset A$, or $V \supset A$.

11.13. Prove that for every disconnected set A in $\mathbb{R}^n$ there are disjoint open sets $U, V \subset \mathbb{R}^n$ such that $A \subset U \cup V$, $U \cap A \neq \varnothing$, and $V \cap A \neq \varnothing$.

Compare *11.11–11.13* with *11.6*.

11°3. Properties of Connected Sets

11.14. Let X be a space. If a set $M \subset X$ is connected and $A \subset X$ is open-closed, then either $M \subset A$, or $M \subset X \smallsetminus A$.

11.B. *The closure of a connected set is connected.*

11.15. Prove that if a set A is connected and $A \subset B \subset \operatorname{Cl} A$, then B is connected.

11.C. *Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subsets of a space X. Assume that any two sets in this family intersect. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected. (In other words: the union of pairwise intersecting connected sets is connected.)*

11.D Special case. If $A, B \subset X$ are two connected sets with $A \cap B \neq \varnothing$, then $A \cup B$ is also connected.

11.E. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a family of connected subsets of a space X. Assume that each set in this family intersects A_{λ_0} for some $\lambda_0 \in \Lambda$. Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is connected.

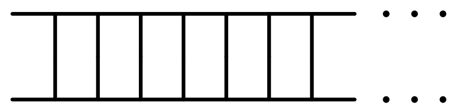
11.F. Let $\{A_k\}_{k \in \mathbb{Z}}$ be a family of connected sets such that $A_k \cap A_{k+1} \neq \varnothing$ for any $k \in \mathbb{Z}$. Prove that $\bigcup_{k \in \mathbb{Z}} A_k$ is connected.

11.16. Let A and B be two connected sets such that $A \cap \operatorname{Cl} B \neq \varnothing$. Prove that $A \cup B$ is also connected.

11.17. Let A be a connected subset of a connected space X, and let $B \subset X \smallsetminus A$ be an open-closed set in the relative topology of $X \smallsetminus A$. Prove that $A \cup B$ is connected.

11.18. Does the connectedness of $A \cup B$ and $A \cap B$ imply that of A and B?

11.19. Let A and B be two sets such that both their union and intersection are connected. Prove that A and B are connected if both of them are 1) open or 2) closed.



11.20. Let $A_1 \supset A_2 \supset \cdots$ be an infinite decreasing sequence of closed connected sets in the plane $\mathbb{R}^2$. Is $\bigcap_{k=1}^{\infty} A_k$ a connected set?

11°4. Connected Components

A *connected component* of a space X is a maximal connected subset of X, i.e., a connected subset that is not contained in any other (strictly) larger connected subset of X.

11.G. *Every point belongs to some connected component. Furthermore, this component is unique. It is the union of all connected sets containing this point.*

11.H. *Two connected components either are disjoint or coincide.*

A connected component of a space X is also called just a *component* of X. Theorems *11.G* and *11.H* mean that connected components constitute a partition of the whole space. The next theorem describes the corresponding equivalence relation.

11.I. *Prove that two points lie in the same component iff they belong to the same connected set.*

11.J Corollary. *A space is connected iff any two of its points belong to the same connected set.*

11.K. *Connected components are closed.*

11.21. If each point of a space X has a connected neighborhood, then each connected component of X is open.

11.22. Let x and y belong to the same component. Prove that any open-closed set contains either both x and y, or none of them (cf. *11.36*).

11°5. Totally Disconnected Spaces

A topological space is *totally disconnected* if all of its components are singletons.

11.L Obvious Example. Any discrete space is totally disconnected.

11.M. The space $\mathbb{Q}$ (with the topology induced from $\mathbb{R}$) is totally disconnected.

Note that $\mathbb{Q}$ is not discrete.

11.23. Give an example of an uncountable closed totally disconnected subset of the line.

11.24. Prove that Cantor set (see *2.Bx*) is totally disconnected.

11°6. Boundary and Connectedness

11.25. Prove that if A is a proper nonempty subset of a connected space, then $\operatorname{Fr} A \neq \varnothing$.

11.26. Let F be a connected subset of a space X. Prove that if $A \subset X$ and neither $F \cap A$, nor $F \cap (X \smallsetminus A)$ is empty, then $F \cap \operatorname{Fr} A \neq \varnothing$.

11.27. Let A be a subset of a connected space. Prove that if $\operatorname{Fr} A$ is connected, then so is $\operatorname{Cl} A$.

11°7. Connectedness and Continuous Maps

A *continuous image* of a space is its image under a continuous map.

11.N. *A continuous image of a connected space is connected. (In other words, if $f : X \to Y$ is a continuous map and X is connected, then $f(X)$ is also connected.)*

11.O Corollary. *Connectedness is a topological property.*

11.P Corollary. *The number of connected components is a topological invariant.*

11.Q. *A space X is disconnected iff there is a continuous surjection $X \to S^0$.*

11.28. Theorem *11.Q* often yields shorter proofs of various results concerning connected sets. Apply it for proving, e.g., Theorems *11.B–11.F* and Problems *11.D* and *11.16*.

11.29. Let X be a connected space and $f : X \to \mathbb{R}$ a continuous function. Then $f(X)$ is an interval of $\mathbb{R}$.

11.30. Suppose a space X has a group structure and the multiplication by any element of the group is a continuous map. Prove that the component of unity is a normal subgroup.

11°8. Connectedness on Line

11.R. *The segment $I = [0, 1]$ is connected.*

There are several ways to prove Theorem *11.R*. One of them is suggested by *11.Q*, but refers to a famous Intermediate Value Theorem from calculus, see *12.A*. However, when studying topology, it would be more natural to find an independent proof and deduce Intermediate Value Theorem from Theorems *11.R* and *11.Q*. Two problems below provide a sketch of basically the same proof of *11.R*. Cf. *2.Ax* below.

11.R.1 Bisection Method. Let U, V be subsets of I with $V = I \smallsetminus U$. Let $a \in U$, $b \in V$, and $a < b$. Prove that there exists a nondecreasing sequence a_n with $a_1 = a$, $a_n \in U$, and a nonincreasing sequence b_n with $b_1 = b$, $b_n \in V$, such that $b_n - a_n = \frac{b-a}{2^{n-1}}$.

11.R.2. Under assumptions of *11.R.1*, if U and V are closed in I, then which of them contains $c = \sup\{a_n\} = \inf\{b_n\}$?

11.31. Deduce *11.R* from the result of Problem *2.Ax*.

11.S. Prove that an open set in $\mathbb{R}$ has countably many connected components.

11.T. Prove that $\mathbb{R}^1$ is connected.

11.U. Each convex set in $\mathbb{R}^n$ is connected. (In particular, so are $\mathbb{R}^n$ itself, the ball B^n, and the disk D^n.)

11.V Corollary. Intervals in $\mathbb{R}^1$ are connected.

11.W. Every star-shaped set in $\mathbb{R}^n$ is connected.

11.X Connectedness on Line. *A subset of a line is connected iff it is an interval.*

11.Y. Describe explicitly all nonempty connected subsets of the real line.

11.Z. Prove that the n-sphere S^n is connected. In particular, the circle S^1 is connected.

11.32. Consider the union of spiral

$$r = \exp\left(\frac{1}{1+\varphi^2}\right), \text{ with } \varphi \geq 0$$

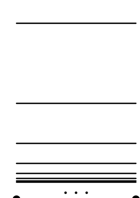
(r, φ are the polar coordinates) and circle S^1. 1) Is this set connected? 2) Will the answer change if we replace the entire circle by some of its subsets? (Cf. *11.15*.)

11.33. Are the following subsets of the plane $\mathbb{R}^2$ connected:

(1) the set of points with both coordinates rational;
(2) the set of points with at least one rational coordinate;
(3) the set of points whose coordinates are either both irrational, or both rational?

11.34. Prove that for any $\varepsilon > 0$ the ε-neighborhood of a connected subset of Euclidean space is connected.

11.35. Prove that each neighborhood U of a connected subset A of Euclidean space contains a connected neighborhood of A.



11.36. Find a space X and two points belonging to distinct components of X such that each simultaneously open and closed set contains either both points, or neither of them. (Cf. *11.22*.)

12. Application of Connectedness

12°1. Intermediate Value Theorem and Its Generalizations

The following theorem is usually included in Calculus. You can easily deduce it from the material of this section. In fact, in a sense it is equivalent to connectedness of the segment.

12.A Intermediate Value Theorem. *A continuous function*

$$f : [a, b] \to \mathbb{R}$$

takes every value between $f(a)$ and $f(b)$.

Many problems that can be solved by using Intermediate Value Theorem can be found in Calculus textbooks. Here are few of them.

12.1. Prove that any polynomial of odd degree in one variable with real coefficients has at least one real root.

12.B Generalization of 12.A. Let X be a connected space and $f : X \to \mathbb{R}$ a continuous function. Then $f(X)$ is an interval of $\mathbb{R}$.

12.C Corollary. Let $J \subset \mathbb{R}$ be an interval of the real line, $f : X \to \mathbb{R}$ a continuous function. Then $f(J)$ is also an interval of $\mathbb{R}$. (In other words, continuous functions map intervals to intervals.)

12°2. Applications to Homeomorphism Problem

Connectedness is a topological property, and the number of connected components is a topological invariant (see Section 10).

12.D. $[0, 2]$ and $[0, 1] \cup [2, 3]$ are not homeomorphic.

Simple constructions assigning homeomorphic spaces to homeomorphic ones (e.g., deleting one or several points), allow us to use connectedness for proving that some *connected* spaces are not homeomorphic.

12.E. I, $[0, \infty)$, $\mathbb{R}^1$, and S^1 are pairwise nonhomeomorphic.

12.2. Prove that a circle is not homeomorphic to a subspace of $\mathbb{R}^1$.

12.3. Give a topological classification of the letters of the alphabet: A, B, C, D, . . . , regarded as subsets of the plane (the arcs comprising the letters are assumed to have zero thickness).

12.4. Prove that square and segment are not homeomorphic.

Recall that there exist continuous surjections of the segment onto square, which are called *Peano curves*, see Section 9.

12.F. $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

Information. $\mathbb{R}^p$ and $\mathbb{R}^q$ are not homeomorphic unless $p = q$. This follows, for instance, from the Lebesgue–Brouwer Theorem on the invariance of dimension (see, e.g., W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, NJ, 1941).

12.5. The statement "$\mathbb{R}^p$ is not homeomorphic to $\mathbb{R}^q$ unless $p = q$" implies that S^p is not homeomorphic to S^q unless $p = q$.

12°3x. Induction on Connectedness

A map f is *locally constant* if each point of its source space has a neighborhood U such that the restriction of f to U is constant.

12.1x. Prove that any locally constant map is continuous.

12.2x. A locally constant map on a connected set is constant.

12.3x. Riddle. How are *11.26* and *12.2x* related?

12.4x. Let G be a group equipped with a topology such that for any $g \in G$ the map $G \to G : x \mapsto xgx^{-1}$ is continuous, and let G with this topology be connected. Prove that if the topology induced in a normal subgroup H of G is discrete, then H is contained in the center of G (i.e., $hg = gh$ for any $h \in H$ and $g \in G$).

12.5x Induction on Connectedness. Let $\mathcal{E}$ be a property of subsets of a topological space X such that the union of sets with nonempty pairwise intersections inherits this property from the sets involved. Prove that if X is connected and each point in X has a neighborhood with property $\mathcal{E}$, then X also has property $\mathcal{E}$.

12.6x. Prove *12.2x* and solve *12.4x* using *12.5x*.

For more applications of induction on connectedness, see *13.T*, *13.4x*, *13.6x*, and *13.8x*.

12°4x. Dividing Pancakes

12.7x. Any irregularly shaped pancake can be cut in half by one stroke of the knife made in any prescribed direction. In other words, if A is a bounded open set in the plane and l is a line in the plane, then there exists a line L parallel to l that divides A in half by area.

12.8x. If, under the assumptions of *12.7x*, A is connected, then L is unique.

12.9x. Suppose two irregularly shaped pancakes lie on the same platter; show that it is possible to cut both exactly in half by one stroke of the knife. In other words: if A and B are two bounded regions in the plane, then there exists a line in the plane that halves each region by area.

12.10x. Prove that a plane pancake of any shape can be divided to four pieces of equal area by two straight cuts orthogonal to each other. In other words, if A is a bounded connected open set in the plane, then there are two perpendicular lines that divide A into four parts having equal areas.

12.11x. Riddle. What if the knife is curved and makes cuts of a shape different from the straight line? For what shapes of the cuts can you formulate and solve problems similar to *12.7x*–*12.10x*?

12.12*x. *Riddle. Formulate and solve counterparts of Problems *12.7*x–*12.10*x for regions in three-space. Can you increase the number of regions in the counterpart of *12.7*x and *12.9*x?

12.13*x. *Riddle. What about pancakes in $\mathbb{R}^n$?

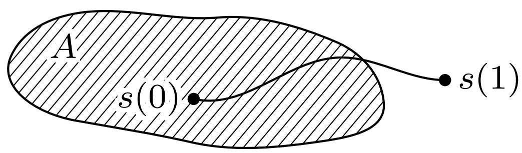
13. Path-Connectedness

13°1. Paths

A *path* in a topological space X is a continuous map of the segment $I = [0, 1]$ to X. The point $s(0)$ is the *initial* point of a path $s : I \to X$, while $s(1)$ is the *final* point of s. We say that the path s *connects* $s(0)$ with $s(1)$. This terminology is inspired by an image of a moving point: at the moment $t \in [0, 1]$, the point is at $s(t)$. To tell the truth, this is more than what is usually called a path, since besides information on the trajectory of the point it contains a complete account on the movement: the schedule saying when the point goes through each point.

13.1. If $s : I \to X$ is a path, then the image $s(I) \subset X$ is connected.

13.2. Let $s : I \to X$ be a path connecting a point in a set $A \subset X$ with a point in $X \smallsetminus A$. Prove that $s(I) \cap \operatorname{Fr}(A) \neq \varnothing$.

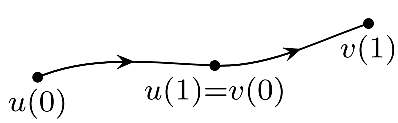


13.3. Let A be a subset of a space X, $\mathrm{in}_A : A \to X$ the inclusion. Prove that $u : I \to A$ is a path in A iff the composition $\mathrm{in}_A \circ u : I \to X$ is a path in X.

A constant map $s_a : I \to X : x \mapsto a$ is a *stationary* path. For a path s, the *inverse* path is defined by $t \mapsto s(1-t)$. It is denoted by s^{-1}. Although, strictly speaking, this notation is already used (for the inverse map), the ambiguity of notation usually leads to no confusion: as a rule, inverse maps do not appear in contexts involving paths.

Let $u : I \to X$ and $v : I \to X$ be paths such that $u(1) = v(0)$. We define

$$uv : I \to X : t \mapsto \begin{cases} u(2t) & \text{if } t \in [0, \frac{1}{2}], \\ v(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \tag{22}$$



13.A. Prove that the above map $uv : I \to X$ is continuous (i.e., it is a path). Cf. *9.T* and *9.V*.

The path uv is the *product* of u and v. Recall that it is defined only if the final point $u(1)$ of u is the initial point $v(0)$ of v.

13°2. Path-Connected Spaces

A topological space is *path-connected* (or *arcwise connected*) if any two points can be connected in it by a path.

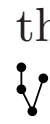
13.B. Prove that I is path-connected.

13.C. Prove that the Euclidean space of any dimension is path-connected.

13.D. Prove that the n-sphere S^n with $n > 0$ is path-connected.

13.E. Prove that the 0-sphere S^0 is not path-connected.

13.4. Which of the following spaces are path-connected:

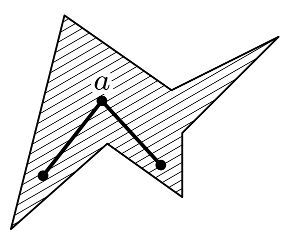
(a) a discrete space; (b) an indiscrete space;
(c) the arrow; (d) $\mathbb{R}_{T_1}$;
(e) ?

13°3. Path-Connected Sets

A *path-connected set* (or *arcwise connected set*) is a subset of a topological space (which should be clear from the context) that is path-connected as a space with the relative topology.

13.5. Prove that a subset A of a space X is path-connected iff any two points in A are connected by a path $s : I \to X$ with $s(I) \subset A$.

13.6. Prove that a convex subset of Euclidean space is path-connected.



13.7. Every star-shaped set in $\mathbb{R}^n$ is path-connected.

13.8. The image of a path is a path-connected set.

13.9. Prove that the set of plane convex polygons with topology generated by the Hausdorff metric is path-connected. (What can you say about the set of convex n-gons with fixed n?)

13.10. Riddle. What can you say about the assertion of Problem *13.9* in the case of arbitrary (not necessarily convex) polygons?

13°4. Properties of Path-Connected Sets

Path-connectedness is very similar to connectedness. Further, in some important situations it is even equivalent to connectedness. However, some properties of connectedness do not carry over to the path-connectedness (see *13.Q* and *13.R*). For the properties that do carry over, proofs are usually easier in the case of path-connectedness.

13.F. *The union of a family of pairwise intersecting path-connected sets is path-connected.*

> ***13.11.*** Prove that if two sets A and B are both closed or both open and their union and intersection are path-connected, then A and B are also path-connected.
>
> ***13.12.*** 1) Prove that the interior and boundary of a path-connected set may not be path-connected. 2) Connectedness shares this property.
>
> ***13.13.*** Let A be a subset of Euclidean space. Prove that if $\operatorname{Fr} A$ is path-connected, then so is $\operatorname{Cl} A$.
>
> ***13.14.*** Prove that the same holds true for a subset of an arbitrary path-connected space.

13°5. Path-Connected Components

A *path-connected component* or *arcwise connected component* of a space X is a path-connected subset of X that is not contained in any other path-connected subset of X.

13.G. *Every point belongs to a path-connected component.*

13.H. *Two path-connected components either coincide or are disjoint.*

Theorems *13.G* and *13.H* mean that path-connected components constitute a partition of the entire space. The next theorem describes the corresponding equivalence relation.

13.I. Prove that two points belong to the same path-connected component iff they can be connected by a path (cf. *11.I*).

Unlike to the case of connectedness, path-connected components are not necessarily closed. (See *13.Q*, cf. *13.P* and *13.R*.)

13°6. Path-Connectedness and Continuous Maps

13.J. *A continuous image of a path-connected space is path-connected.*

13.K Corollary. *Path-connectedness is a topological property.*

13.L Corollary. *The number of path-connected components is a topological invariant.*

13°7. Path-Connectedness Versus Connectedness

13.M. *Any path-connected space is connected.*

Put

$$A = \{ (x, y) \in \mathbb{R}^2 \mid x > 0,\ y = \sin(1/x) \}, \quad X = A \cup (0, 0).$$

> ***13.15.*** Sketch A.

13.N. Prove that A is path-connected and X is connected.

13.O. Prove that deleting any point from A makes A and X disconnected (and hence, not path-connected).

13.P. *X is not path-connected.*

13.Q. Find an example of a path-connected set, whose closure is not path-connected.

13.R. Find an example of a path-connected component that is not closed.

13.S. *If each point of a space has a path-connected neighborhood, then each path-connected component is open. (Cf. 11.21.)*

13.T. *Assume that each point of a space X has a path-connected neighborhood. Then X is path-connected iff X is connected.*

13.U. *For open subsets of Euclidean space connectedness is equivalent to path-connectedness.*

13.16. For subsets of the real line path-connectedness and connectedness are equivalent.

13.17. Prove that for any $\varepsilon > 0$ the ε-neighborhood of a connected subset of Euclidean space is path-connected.

13.18. Prove that any neighborhood U of a connected subset A of Euclidean space contains a path-connected neighborhood of A.

13°8x. Polygon-Connectedness

A subset A of Euclidean space is *polygon-connected* if any two points of A are connected by a finite polyline contained in A.

***13.1*x.** Each polygon-connected set in $\mathbb{R}^n$ is path-connected, and thus also connected.

***13.2*x.** Each convex set in $\mathbb{R}^n$ is polygon-connected.

***13.3*x.** Each star-shaped set in $\mathbb{R}^n$ is polygon-connected.

***13.4*x.** Prove that for open subsets of Euclidean space connectedness is equivalent to polygon-connectedness.

***13.5*x.** Construct a path-connected subset A of Euclidean space such that A consists of more than one point and no two distinct points of A can be connected by a polygon in A.

***13.6*x.** Let $X \subset \mathbb{R}^2$ be a countable set. Prove that then $\mathbb{R}^2 \smallsetminus X$ is polygon-connected.

***13.7*x.** Let $X \subset \mathbb{R}^n$ be the union of a countable collection of affine subspaces with dimensions not greater than $n-2$. Prove that then $\mathbb{R}^n \smallsetminus X$ is polygon-connected.

***13.8*x.** Let $X \subset \mathbb{C}^n$ be the union of a countable collection of algebraic subsets (i.e., subsets defined by systems of algebraic equations in the standard coordinates of $\mathbb{C}^n$). Prove that then $\mathbb{C}^n \smallsetminus X$ is polygon-connected.

13°9x. Connectedness of Some Sets of Matrices

Recall that real $n \times n$-matrices constitute a space, which differs from $\mathbb{R}^{n^2}$ only in the way of enumerating its natural coordinates (they are numerated by pairs of indices). The same relation holds true between the set of complex $n \times n$-matrix and $\mathbb{C}^{n^2}$ (homeomorphic to $\mathbb{R}^{2n^2}$).

13.9x. Find connected and path-connected components of the following subspaces of the space of real $n \times n$-matrices:

(1) $GL(n;\mathbb{R}) = \{A \mid \det A \neq 0\}$;
(2) $O(n;\mathbb{R}) = \{A \mid A \cdot ({}^tA) = \mathbb{E}\}$;
(3) $Symm(n;\mathbb{R}) = \{A \mid {}^tA = A\}$;
(4) $Symm(n;\mathbb{R}) \cap GL(n;\mathbb{R})$;
(5) $\{A \mid A^2 = \mathbb{E}\}$.

13.10x. Find connected and path-connected components of the following subspaces of the space of complex $n \times n$-matrices:

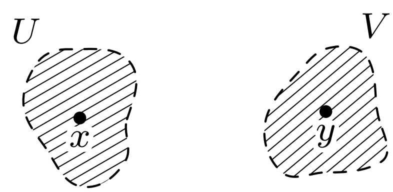
(1) $GL(n;\mathbb{C}) = \{A \mid \det A \neq 0\}$;
(2) $U(n;\mathbb{C}) = \{A \mid A \cdot ({}^t\bar{A}) = \mathbb{E}\}$;
(3) $Herm(n;\mathbb{C}) = \{A \mid {}^tA = \bar{A}\}$;
(4) $Herm(n;\mathbb{C}) \cap GL(n;\mathbb{C})$.

14. Separation Axioms

The aim of this section is to consider natural restrictions on the topological structure making the structure closer to being metrizable. A lot of separation axioms are known. We restrict ourselves to the five most important of them. They are numerated, and denoted by T_0, T_1, T_2, T_3, and T_4, respectively.[1]

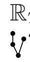
14°1. The Hausdorff Axiom

We start with the second axiom, which is most important. Besides the notation T_2, it has a name: the *Hausdorff axiom*. A topological space satisfying T_2 is a *Hausdorff space*. This axiom is stated as follows: *any two distinct points possess disjoint neighborhoods*. We can state it more formally: $\forall\, x, y \in X,\ x \neq y\ \exists U_x,\, V_y :\ U_x \cap V_y = \varnothing$.



14.A. *Any metric space is Hausdorff.*

14.1. Which of the following spaces are Hausdorff:

(1) a discrete space;
(2) an indiscrete space;
(3) the arrow;
(4) $\mathbb{R}_{T_1}$;
(5) ?

If the next problem holds you up even for a minute, we advise you to think over all definitions and solve all simple problems.

14.B. Is the segment $[0, 1]$ with the topology induced from $\mathbb{R}$ a Hausdorff space? Do the points 0 and 1 possess disjoint neighborhoods? Which if any?

14.C. A space X is Hausdorff iff for each $x \in X$ we have $\{x\} = \bigcap_{U \ni x} \operatorname{Cl} U$.

[1] Letter T in these notation originates from the German word *Trennungsaxiom*, which means *separation axiom*.

14°2. Limits of Sequence

Let $\{a_n\}$ be a sequence of points of a topological space X. A point $b \in X$ is the *limit* of the sequence if for any neighborhood U of b there exists a number N such that $a_n \in U$ for any $n \geq N$.[2] In this case, we say that the sequence *converges* or *tends* to b as n tends to infinity.

14.2. Explain the meaning of the statement "b is not a limit of sequence a_n", using as few negations (i.e., the words *no*, *not*, *none*, etc.) as you can.

14.3. The limit of a sequence does not depend on the order of the terms. More precisely, let a_n be a convergent sequence: $a_n \to b$, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the sequence $a_{\phi(n)}$ is also convergent and has the same limit: $a_{\phi(n)} \to b$. For example, if the terms in the sequence are pairwise distinct, then the convergence and the limit depend only on the set of terms, which shows that these notions actually belong to geometry.

14.D. *In a Hausdorff space any sequence has at most one limit.*

14.E. Prove that in the space $\mathbb{R}_{T_1}$ each point is a limit of the sequence $a_n = n$.

14°3. Coincidence Set and Fixed Point Set

Let $f, g : X \to Y$ be maps. Then the set $C(f, g) = \{x \in X \mid f(x) = g(x)\}$ is the *coincidence set* of f and g.

14.4. Prove that the coincidence set of two continuous maps from an arbitrary space to a Hausdorff space is closed.

14.5. Construct an example proving that the Hausdorff condition in *14.4* is essential.

A point $x \in X$ is a *fixed point* of a map $f : X \to X$ if $f(x) = x$. The set of all fixed points of a map f is the *fixed point set* of f.

14.6. Prove that the fixed-point set of a continuous map from a Hausdorff space to itself is closed.

14.7. Construct an example showing that the Hausdorff condition in *14.6* is essential.

14.8. Prove that if $f, g : X \to Y$ are two continuous maps, Y is Hausdorff, A is everywhere dense in X, and $f|_A = g|_A$, then $f = g$.

14.9. Riddle. How are problems *14.4*, *14.6*, and *14.8* related to each other?

14°4. Hereditary Properties

A topological property is *hereditary* if it carries over from a space to its subspaces, i.e., if a space X has this property, then each subspace of X also has it.

[2] You can also rephrase this as follows: each neighborhood of b contains all members of the sequence that have sufficiently large indices.

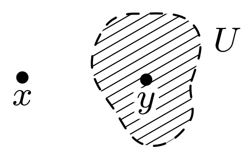
14.10. Which of the following topological properties are hereditary:

(1) finiteness of the set of points;
(2) finiteness of the topological structure;
(3) infiniteness of the set of points;
(4) connectedness;
(5) path-connectedness?

14.F. *The property of being a Hausdorff space is hereditary.*

$14^\circ 5$. The First Separation Axiom

A topological space X satisfies the *first separation axiom* T_1 if each one of any two points of X has a neighborhood that does not contain the other point.[3] More formally: $\forall\, x, y \in X,\ x \neq y\ \exists U_y:\ x \notin U_y$.



14.G. *A space X satisfies the first separation axiom,*

- *iff all one-point sets in X are closed,*
- *iff all finite sets in X are closed.*

14.11. Prove that a space X satisfies the first separation axiom iff every point of X is the intersection of all of its neighborhoods.

14.12. Any Hausdorff space satisfies the first separation axiom.

14.H. In a Hausdorff space any finite set is closed.

14.I. A metric space satisfies the first separation axiom.

14.13. Find an example showing that the first separation axiom does not imply the Hausdorff axiom.

14.J. Show that $\mathbb{R}_{T_1}$ meets the first separation axiom, but is not a Hausdorff space (cf. *14.13*).

14.K. *The first separation axiom is hereditary.*

14.14. Suppose that for any two distinct points a and b of a space X there exists a continuous map f from X to a space with the first separation axiom such that $f(a) \neq f(b)$. Prove that then X also satisfies the first separation axiom.

14.15. Prove that a continuous map of an indiscrete space to a space satisfying axiom T_1 is constant.

14.16. Prove that in every set there exists a coarsest topological structure satisfying the first separation axiom. Describe this structure.

[3] T_1 is also called the Tikhonov axiom.

14°6. The Kolmogorov Axiom

The first separation axiom emerges as a weakened Hausdorff axiom.

14.L. Riddle. How can the first separation axiom be weakened?

A topological space satisfies the *Kolmogorov axiom* or the *zeroth separation axiom* T_0 if *at least one* of any two distinct points of this space has a neighborhood that does not contain the other of these points.

14.M. An indiscrete space containing at least two points does not satisfy T_0.

14.N. *The following properties of a space X are equivalent:*

(1) *X satisfies the Kolmogorov axiom;*

(2) *any two different points of X has different closures;*

(3) *X contains no indiscrete subspace consisting of two points.*

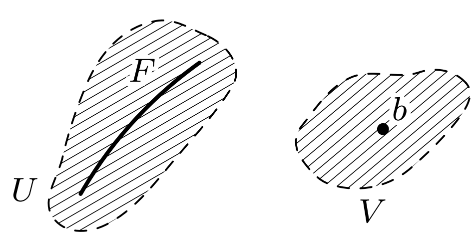
(4) *X contains no indiscrete subspace consisting of more than one point;*

14.O. *A topology is a poset topology iff it is a smallest neighborhood topology satisfying the Kolmogorov axiom.*

Thus, on the one hand, posets give rise to numerous examples of topological spaces, among which we see the most important spaces, like the line with the standard topology. On the other hand, all posets are obtained from topological spaces of a special kind, which are quite far away from the class of metric spaces.

14°7. The Third Separation Axiom

A topological space X satisfies the *third separation axiom* if every closed set in X and every point of its complement have disjoint neighborhoods, i.e., for every closed set $F \subset X$ and every point $b \in X \smallsetminus F$ there exist open sets $U, V \subset X$ such that $U \cap V = \varnothing$, $F \subset U$, and $b \in V$.



A space is *regular* if it satisfies the first and third separation axioms.

14.P. A regular space is a Hausdorff space.

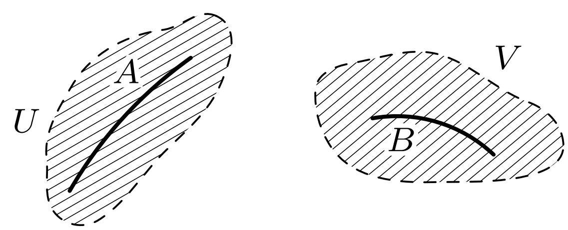
14.Q. A space is regular iff it satisfies the second and third separation axioms.

> ***14.17.*** Find a Hausdorff space which is not regular.
>
> ***14.18.*** Find a space satisfying the third, but not the second separation axiom.
>
> ***14.19.*** Prove that a space X satisfies the third separation axiom iff every neighborhood of every point $x \in X$ contains the closure of a neighborhood of x.
>
> ***14.20.*** Prove that the third separation axiom is hereditary.

14.R. *Any metric space is regular.*

14°8. The Fourth Separation Axiom

A topological space X satisfies the *fourth separation axiom* if any two disjoint closed sets in X have disjoint neighborhoods, i.e., for any two closed sets $A, B \subset X$ with $A \cap B = \varnothing$ there exist open sets $U, V \subset X$ such that $U \cap V = \varnothing$, $A \subset U$, and $B \subset V$.



A space is *normal* if it satisfies the first and fourth separation axioms.

14.S. A normal space is regular (and hence Hausdorff).

14.T. A space is normal iff it satisfies the second and fourth separation axioms.

> ***14.21.*** Find a space which satisfies the fourth, but not second separation axiom.
>
> ***14.22.*** Prove that a space X satisfies the fourth separation axiom iff every neighborhood of every closed set $F \subset X$ contains the closure of some neighborhood of F.
>
> ***14.23.*** Prove that any closed subspace of a normal space is normal.
>
> ***14.24.*** Find two closed disjoint subsets A and B of some metric space such that $\inf\{\rho(a, b) \mid a \in A,\ b \in B\} = 0$.

14.U. *Any metric space is normal.*

> ***14.25.*** Let $f : X \to Y$ be a continuous surjection such that the image of any closed set is closed. Prove that if X is normal, then so is Y.

14°9x. Niemytski's Space

Denote by $\mathcal{H}$ the open upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the topology generated by the Euclidean metric. Denote by $\mathcal{N}$ the union of $\mathcal{H}$ and the boundary line $\mathbb{R}^1$: $\mathcal{N} = \mathcal{H} \cup \mathbb{R}^1$, but equip it with the topology obtained by adjoining to the Euclidean topology the sets of the form $x \cup D$, where $x \in \mathbb{R}^1$ and D is an open disk in $\mathcal{H}$ touching $\mathbb{R}^1$ at the point x. This is the ***Niemytski space***. It can be used to clarify properties of the fourth separation axiom.

***14.1*x.** Prove that the Niemytski space is Hausdorff.

***14.2*x.** Prove that the Niemytski space is regular.

***14.3*x.** What topological structure is induced on $\mathbb{R}^1$ from $\mathcal{N}$?

***14.4*x.** Prove that the Niemytski space is not normal.

14.5*x *Corollary. There exists a regular space which is not normal.

***14.6*x.** Embed the Niemytski space into a normal space in such a way that the complement of the image would be a single point.

14.7*x *Corollary. Theorem *14.23* does not extend to nonclosed subspaces, i.e., the property of being normal is not hereditary, is it?

14°10x. Urysohn Lemma and Tietze Theorem

***14.8*x.** Let A and B be two disjoint closed subsets of a metric space X. Then there exists a continuous function $f : X \to I$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

***14.9*x.** Let F be a closed subset of a metric space X. Then any continuous function $f : X \to [-1, 1]$ can be extended over the whole X.

> ***14.9*x*.1.*** Let F be a closed subset of a metric space X. For any continuous function $f : F \to [-1, 1]$ there exists a function $g : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $|f(x) - g(x)| \le \frac{2}{3}$ for each $x \in F$.

14.A*x *Urysohn Lemma. *Let A and B be two disjoint closed subsets of a normal space X. Then there exists a continuous function $f : X \to I$ such that $f(A) = 0$ and $f(B) = 1$.*

> ***14.A*x*.1.*** Let A and B be two disjoint closed subsets of a normal space X. Consider the set $\Lambda = \left\{\frac{k}{2^n} \mid k, n \in \mathbb{Z}_+,\ k \le 2^n\right\}$. There exists a collection $\{U_p\}_{p \in \Lambda}$ of open subsets of X such that for any $p, q \in \Lambda$ we have: 1) $A \subset U_0$ and $B \subset X \smallsetminus U_1$ and 2) if $p < q$ then $\operatorname{Cl} U_p \subset U_q$.

14.B*x *Tietze Extension Theorem. *Let A be a closed subset of a normal space X. Let $f : A \to [-1, 1]$ be a continuous function. Prove that there exists a continuous function $F : X \to [-1, 1]$ such that $F|_A = f$.*

14.C*x *Corollary. Let A be a closed subset of a normal space X. Any continuous function $A \to \mathbb{R}$ can be extended to a function on the whole space.

14.10x. Will the statement of the Tietze theorem remain true if in the hypothesis we replace the segment $[-1, 1]$ by $\mathbb{R}$, $\mathbb{R}^n$, S^1, or S^2?

14.11x. Derive the Urysohn Lemma from the Tietze Extension Theorem.

15. Countability Axioms

In this section, we continue to study topological properties that are additionally imposed on a topological structure to make the abstract situation under consideration closer to special situations and hence richer in contents. The restrictions studied in this section bound a topological structure from above: they require that something be countable.

15°1. Set-Theoretic Digression: Countability

Recall that two sets have equal *cardinality* if there exists a bijection of one of them onto the other. A set of the same cardinality as a subset of the set $\mathbb{N}$ of positive integers is *countable*.

15.1. A set X is countable iff there exists an injection $X \to \mathbb{N}$ (or, more generally, an injection of X into another countable set).

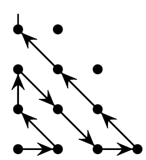
Sometimes this term is used only for infinite countable sets, i.e., for sets of the cardinality of the whole set $\mathbb{N}$ of positive integers, while sets countable in the above sense are said to be *at most countable*. This is less convenient. In particular, if we adopted this terminology, this section would be called "At Most Countability Axioms". This would also lead to other more serious inconveniences as well. Our terminology has the following advantageous properties.

15.A. Any subset of a countable set is countable.

15.B. The image of a countable set under any map is countable.

15.C. $\mathbb{Z}$ is countable.

15.D. The set $\mathbb{N}^2 = \{(k, n) \mid k, n \in \mathbb{N}\}$ is countable.



15.E. The union of a countable family of countable sets is countable.

15.F. $\mathbb{Q}$ is countable.

15.G. $\mathbb{R}$ is not countable.

15.2. Prove that any set Σ of disjoint figure eight curves in the plane is countable.

15°2. Second Countability and Separability

In this section, we study three restrictions on the topological structure. Two of them have numbers (one and two), the third one has no number. As in the previous section, we start from the restriction having number two.

A topological space X satisfies the *second axiom of countability* or is *second countable* if X has a countable base. A space is *separable* if it contains a countable dense set. (This is the countability axiom without a number that we mentioned above.)

15.H. *The second axiom of countability implies separability.*

15.I. The second axiom of countability is hereditary.

15.3. Are the arrow and $\mathbb{R}_{T_1}$ second countable?

15.4. Are the arrow and $\mathbb{R}_{T_1}$ separable?

15.5. Construct an example proving that separability is not hereditary.

15.J. *A metric separable space is second countable.*

15.K Corollary. *For metrizable spaces, separability is equivalent to the second axiom of countability.*

15.L. (Cf. *15.5.*) Prove that for metrizable spaces separability is hereditary.

15.M. Prove that Euclidean spaces and all their subspaces are separable and second countable.

15.6. Construct a metric space which is not second countable.

15.7. Prove that in a separable space any collection of pairwise disjoint open sets is countable.

15.8. Prove that the set of components of an open set $A \subset \mathbb{R}^n$ is countable.

15.N. *A continuous image of a separable space is separable.*

15.9. Construct an example proving that a continuous image of a second countable space may be not second countable.

15.O Lindelöf Theorem. *Any open cover of a second countable space contains a countable part that also covers the space.*

15.10. Prove that each base of a second countable space contains a countable part which is also a base.

15.11 Brouwer Theorem*. Let $\{K_\lambda\}$ be a family of closed sets of a second countable space and assume that for every decreasing sequence $K_1 \supset K_2 \supset \dots$ of sets belonging to this family the intersection $\cap_1^\infty K_n$ also belongs to the family. Then the family contains a minimal set A, i.e., a set such that no proper subset of A belongs to the family.

15°3. Bases at a Point

Let X be a space, a a point of X. A *neighborhood base* at a or just a *base of X at a* is a collection Σ of neighborhoods of a such that each neighborhood of a contains a neighborhood from Σ.

15.P. If Σ is a base of a space X, then $\{U \in \Sigma \mid a \in U\}$ is a base of X at a.

15.12. In a metric space the following collections of balls are neighborhood bases at a point a:

- the set of all open balls of center a;
- the set of all open balls of center a and rational radii;
- the set of all open balls of center a and radii r_n, where $\{r_n\}$ is any sequence of positive numbers converging to zero.

15.13. What are the minimal bases at a point in the discrete and indiscrete spaces?

15°4. First Countability

A topological space X satisfies the *first axiom of countability* or is a *first countable space* if X has a countable neighborhood base at each point.

15.Q. *Any metric space is first countable.*

15.R. *The second axiom of countability implies the first one.*

15.S. Find a first countable space which is not second countable. (Cf. *15.6.*)

15.14. Which of the following spaces are first countable:

(a) the arrow; (b) $\mathbb{R}_{T_1}$;
(c) a discrete space; (d) an indiscrete space?

15.15. Find a first countable separable space which is not second countable.

15.16. Prove that if X is a first countable space, then at each point it has a decreasing countable neighborhood base: $U_1 \supset U_2 \supset \dots$.

15°5. Sequential Approach to Topology

Specialists in Mathematical Analysis love sequences and their limits. Moreover, they like to talk about all topological notions relying on the notions of sequence and its limit. This tradition has almost no mathematical justification, except for a long history descending from the XIX century studies on the foundations of analysis. In fact, almost always[4] it is more convenient to avoid sequences, provided you deal with topological notions, except summing of series, where sequences are involved in the underlying

[4]The exceptions which one may find in the standard curriculum of a mathematical department can be counted on two hands.

definitions. Paying a tribute to this tradition, here we explain how and in what situations topological notions can be described in terms of sequences.

Let A be a subset of a space X. The set $\operatorname{SCl} A$ of limits of all sequences a_n with $a_n \in A$ is the *sequential closure* of A.

15.T. Prove that $\operatorname{SCl} A \subset \operatorname{Cl} A$.

15.U. *If a space X is first countable, then the for any $A \subset X$ the opposite inclusion $\operatorname{Cl} A \subset \operatorname{SCl} A$ also holds true, whence $\operatorname{SCl} A = \operatorname{Cl} A$.*

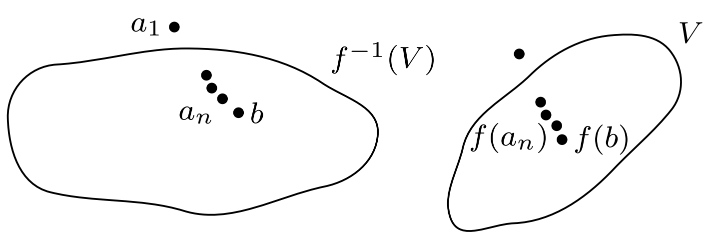
Therefore, in a first countable space (in particular, any metric spaces) we can recover (hence, define) the closure of a set provided it is known which sequences are convergent and what the limits are. In turn, the knowledge of closures allows one to determine which sets are closed. As a consequence, knowledge of closed sets allows one to recover open sets and all other topological notions.

> ***15.17.*** Let X be the set of real numbers equipped with the topology consisting of $\varnothing$ and complements of all countable subsets. (Check that this is actually a topology.) Describe convergent sequences, sequential closure and closure in X. Prove that in X there exists a set A with $\operatorname{SCl} A \neq \operatorname{Cl} A$.

15°6. Sequential Continuity

Now we consider the continuity of maps along the same lines. A map $f : X \to Y$ is *sequentially continuous* if for each $b \in X$ and each sequence $a_n \in X$ converging to b the sequence $f(a_n)$ converges to $f(b)$.

15.V. *Any continuous map is sequentially continuous.*



15.W. *The preimage of a sequentially closed set under a sequentially continuous map is sequentially closed.*

15.X. *If X is a first countable space, then any sequentially continuous map $f : X \to Y$ is continuous.*

Thus for maps of a first countable space continuity and sequential continuity are equivalent.

> ***15.18.*** Construct a sequentially continuous, but discontinuous map. (Cf. *15.17*)

15°7x. Embedding and Metrization Theorems

15.Ax. Prove that the space l_2 is separable and second countable.

15.Bx. Prove that a regular second countable space is normal.

15.Cx. Prove that a normal second countable space can be embedded into l_2. (Use the Urysohn Lemma *14.Ax*.)

15.Dx. Prove that a second countable space is metrizable iff it is regular.

16. Compactness

16°1. Definition of Compactness

This section is devoted to a topological property playing a very special role in topology and its applications. It is a sort of topological counterpart for the property of being finite in the context of set theory. (It seems though that this analogy has never been formalized.)

A topological space X is *compact* if each open cover of X contains a finite part that also covers X.

If Γ is a cover of X and $\Sigma \subset \Gamma$ is a cover of X, then Σ is a *subcover* (or *subcovering*) of Γ. Thus, a space X is compact if every open cover of X contains a finite subcovering.

16.A. Any finite space and indiscrete space are compact.

16.B. Which discrete spaces are compact?

16.1. Let $\Omega_1 \subset \Omega_2$ be two topological structures in X. 1) Does the compactness of (X, Ω_2) imply that of (X, Ω_1)? 2) And vice versa?

16.C. The line $\mathbb{R}$ is not compact.

16.D. A space X is not compact iff it has an open cover containing no finite subcovering.

16.2. Is the arrow compact? Is $\mathbb{R}_{T_1}$ compact?

16°2. Terminology Remarks

Originally the word *compactness* was used for the following weaker property: any countable open cover contains a finite subcovering.

16.E. For a second countable space, the original definition of compactness is equivalent to the modern one.

The modern notion of compactness was introduced by P. S. Alexandrov (1896–1982) and P. S. Urysohn (1898–1924). They suggested for it the term *bicompactness*. This notion appeared to be so successful that it has displaced the original one and even took its name, i.e., compactness. The term bicompactness is sometimes used (mainly by topologists of Alexandrov's school).

Another deviation from the terminology used here comes from Bourbaki: we do not include the Hausdorff property into the definition of compactness, which Bourbaki includes. According to our definition, $\mathbb{R}_{T_1}$ is compact, according to Bourbaki it is not.

16°3. Compactness in Terms of Closed Sets

A collection of subsets of a set is said to have the *finite intersection property* if the intersection of any finite subcollection is nonempty.

16.F. A collection Σ of subsets of a set X has the finite intersection property iff there exists no finite $\Sigma_1 \subset \Sigma$ such that the complements of the sets in Σ_1 cover X.

16.G. *A space is compact iff for every collection of its closed sets having the finite intersection property its intersection is nonempty.*

16°4. Compact Sets

A *compact set* is a subset A of a topological space X (the latter must be clear from the context) provided A is compact as a space with the relative topology induced from X.

16.H. *A subset A of a space X is compact iff each cover of A with sets open in X contains a finite subcovering.*

16.3. Is $[1, 2) \subset \mathbb{R}$ compact?

16.4. Is the same set $[1, 2)$ compact in the arrow?

16.5. Find a necessary and sufficient condition (formulated not in topological terms) for a subset of the arrow to be compact?

16.6. Prove that any subset of $\mathbb{R}_{T_1}$ is compact.

16.7. Let A and B be two compact subsets of a space X. 1) Does it follow that $A \cup B$ is compact? 2) Does it follow that $A \cap B$ is compact?

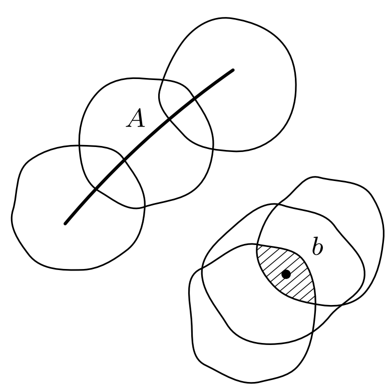
16.8. Prove that the set $A = 0 \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ in $\mathbb{R}$ is compact.

16°5. Compact Sets Versus Closed Sets

16.I. Is compactness hereditary?

16.J. *Any closed subset of a compact space is compact.*

16.K. *Any compact subset of a Hausdorff space is closed.*



16.L Lemma to 16.K, but not only .... *Let A be a compact subset of a Hausdorff space X and b a point of X that does not belong to A. Then there exist open sets $U, V \subset X$ such that $b \in V$, $A \subset U$, and $U \cap V = \varnothing$.*

> ***16.9.*** Construct a nonclosed compact subset of some topological space. What is the minimal number of points needed?

16°6. Compactness and Separation Axioms

16.M. *A compact Hausdorff space is regular.*

16.N. Prove that a compact Hausdorff space is normal.

16.O Lemma to 16.N. In a Hausdorff space, any two disjoint compact subsets possess disjoint neighborhoods.

> ***16.10.*** Prove that the intersection of any family of compact subsets of a Hausdorff space is compact. (Cf. *16.7*.)
>
> ***16.11.*** Let X be a Hausdorff space, let $\{K_\lambda\}_{\lambda\in\Lambda}$ be a family of its compact subsets, and let U be an open set containing $\bigcap_{\lambda\in\Lambda} K_\lambda$. Prove that for some finite $A \subset \Lambda$ we have $U \supset \bigcap_{\lambda\in A} K_\lambda$.
>
> ***16.12.*** Let $\{K_n\}_1^\infty$ be a decreasing sequence of nonempty compact connected sets in a Hausdorff space. Prove that the intersection $\bigcap_1^\infty K_n$ is nonempty and connected. (Cf. *11.20*)

16°7. Compactness in Euclidean Space

16.P. *The segment I is compact.*

Recall that n-dimensional cube is the set

$$I^n = \{x \in \mathbb{R}^n \mid x_i \in [0,1] \text{ for } i = 1, \dots, n\}.$$

16.Q. *The cube I^n is compact.*

16.R. *Any compact subset of a metric space is bounded.*

Therefore, any compact subset of a metric space is closed and bounded (see Theorems *14.A*, *16.K*, and *16.R*).

16.S. Construct a closed and bounded, but noncompact set in a metric space.

> ***16.13.*** Are the metric spaces of Problem *4.A* compact?

16.T. *A subset of a Euclidean space is compact iff it is closed and bounded.*

> ***16.14.*** Which of the following sets are compact:
>
> (a) $[0,1)$; (b) ray $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$; (c) S^1;
> (d) S^n; (e) one-sheeted hyperboloid; (f) ellipsoid;
> (g) $[0,1] \cap \mathbb{Q}$?

An $(n \times k)$-matrix (a_{ij}) with real entries can be regarded as a point in $\mathbb{R}^{nk}$. To do this, we only need to enumerate somehow (e.g., lexicographically) the entries of (a_{ij}) by numbers from 1 to nk. This identifies the set $L(n,k)$ of all matrices like that with $\mathbb{R}^{nk}$ and endows it with a topological structure. (Cf. Section 13.)

16.15. Which of the following subsets of $L(n,n)$ are compact:

(1) $GL(n) = \{A \in L(n,n) \mid \det A \neq 0\}$;
(2) $SL(n) = \{A \in L(n,n) \mid \det A = 1\}$;
(3) $O(n) = \{A \in L(n,n) \mid A \text{ is an orthogonal matrix}\}$;
(4) $\{A \in L(n,n) \mid A^2 = \mathbb{E}\}$, where $\mathbb{E}$ is the unit matrix?

16°8. Compactness and Continuous Maps

16.U. *A continuous image of a compact space is compact. (In other words, if X is a compact space and $f : X \to Y$ is a continuous map, then $f(X)$ is compact.)*

16.V. *A continuous numerical function on a compact space is bounded and attains its maximal and minimal values. (In other words, if X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there exist $a, b \in X$ such that $f(a) \le f(x) \le f(b)$ for every $x \in X$.) Cf. 16.U and 16.T.*

16.16. Prove that if $f : I \to \mathbb{R}$ is a continuous function, then $f(I)$ is a segment.

16.17. Let A be a subset of $\mathbb{R}^n$. Prove that A is compact iff each continuous numerical function on A is bounded.

16.18. Prove that if F and G are disjoint subsets of a metric space, F is closed, and G is compact, then $\rho(G, F) = \inf\{\rho(x,y) \mid x \in F,\, y \in G\} > 0$.

16.19. Prove that any open set U containing a compact set A of a metric space X contains an ε-*neighborhood* of A (i.e., the set $\{x \in X \mid \rho(x, A) < \varepsilon\}$) for some $\varepsilon > 0$.

16.20. Let A be a closed connected subset of $\mathbb{R}^n$ and let V be the *closed* ε-*neighborhood* of A (i.e., $V = \{x \in \mathbb{R}^n \mid \rho(x, A) \le \varepsilon\}$). Prove that V is path-connected.

16.21. Prove that if the closure of each open ball in a compact metric space is the closed ball with the same center and radius, then any ball in this space is connected.

16.22. Let X be a compact metric space, and let $f : X \to X$ be a map such that $\rho(f(x), f(y)) < \rho(x, y)$ for any $x, y \in X$ with $x \neq y$. Prove that f has a unique fixed point. (Recall that a fixed point of f is a point x such that $f(x) = x$, see *14.6*.)

16.23. Prove that for any open cover of a compact metric space there exists a (sufficiently small) number $r > 0$ such that each open ball of radius r is contained in an element of the cover.

16.W Lebesgue Lemma. *Let $f : X \to Y$ be a continuous map from a compact metric space X to a topological space Y, and let Γ be an open cover of Y. Then there exists a number $\delta > 0$ such that for any set $A \subset X$ with diameter* $\operatorname{diam}(A) < \delta$ *the image $f(A)$ is contained in an element of Γ.*

16°9. Closed Maps

A continuous map is *closed* if the image of each closed set under this map is closed.

16.24. A continuous bijection is a homeomorphism iff it is closed.

16.X. *A continuous map of a compact space to a Hausdorff space is closed.*

Here are two important corollaries of this theorem.

16.Y. *A continuous bijection of a compact space onto a Hausdorff space is a homeomorphism.*

16.Z. *A continuous injection of a compact space into a Hausdorff space is a topological embedding.*

16.25. Show that none of the assumptions in *16.Y* can be omitted without making the statement false.

16.26. Does there exist a noncompact subspace A of the Euclidian space such that any continuous map of A to a Hausdorff space is closed? (Cf. *16.V* and *16.X*.)

16.27. A restriction of a closed map to a closed subset is a also closed map.

16°10x. Norms in $\mathbb{R}^n$

16.1x. Prove that each norm $\mathbb{R}^n \to \mathbb{R}$ (see Section 4) is a continuous function (with respect to the standard topology of $\mathbb{R}^n$).

16.2x. Prove that any two norms in $\mathbb{R}^n$ are equivalent (i.e., determine the same topological structure). See *4.27*, cf. *4.31*.

16.3x. Does the same hold true for metrics in $\mathbb{R}^n$?

16°11x. Induction on Compactness

A function $f : X \to \mathbb{R}$ is *locally bounded* if for each point $a \in X$ there exist a neighborhood U and a number $M > 0$ such that $|f(x)| \le M$ for $x \in U$ (i.e., each point has a neighborhood U such that the restriction of f to U is bounded).

16.4x. Prove that if a space X is compact and a function $f : X \to \mathbb{R}$ is locally bounded, then f is bounded.

This statement is a simplest application of a general principle formulated below in *16.5x*. This principle may be called *induction on compactness* (cf. induction on connectedness, which was discussed in Section 11).

Let X be a topological space, $\mathcal{C}$ a property of subsets of X. We say that $\mathcal{C}$ is *additive* if the union of any finite family of sets having $\mathcal{C}$ also has $\mathcal{C}$. The space X *possesses $\mathcal{C}$ locally* if each point of X has a neighborhood with property $\mathcal{C}$.

16.5x. Prove that a compact space which locally possesses an additive property has this property itself.

16.6x. Using induction on compactness, deduce the statements of Problems *16.R*, *17.M*, and *17.N*.

17. Sequential Compactness

17°1. Sequential Compactness Versus Compactness

A topological space is *sequentially compact* if every sequence of its points contains a convergent subsequence.

17.A. *If a first countable space is compact, then it is sequentially compact.*

A point b is an *accumulation point* of a set A if each neighborhood of b contains infinitely many points of A.

17.A.1. Prove that in a space satisfying the first separation axiom a point is an accumulation point iff it is a limit point.

17.A.2. *In a compact space, any infinite set has an accumulation point.*

17.A.3. *A space in which each infinite set has an accumulation point is sequentially compact.*

17.B. *A sequentially compact second countable space is compact.*

17.B.1. *In a sequentially compact space a decreasing sequence of nonempty closed sets has a nonempty intersection.*

17.B.2. Prove that each nested sequence of nonempty closed sets in a space X has nonempty intersection iff each countable collection of closed sets in $X$the finite intersection property has nonempty intersection.

17.B.3. Derive Theorem *17.B* from *17.B.1* and *17.B.2*.

17.C. *For second countable spaces, compactness and sequential compactness are equivalent.*

17°2. In Metric Space

A subset A of a metric space X is an ε-*net* (where ε is a positive number) if $\rho(x, A) < \varepsilon$ for each point $x \in X$.

17.D. Prove that in any compact metric space for any $\varepsilon > 0$ there exists a finite ε-net.

17.E. Prove that in any sequentially compact metric space for any $\varepsilon > 0$ there exists a finite ε-net.

17.F. Prove that a subset A of a metric space is everywhere dense iff A is an ε-net for each $\varepsilon > 0$.

17.G. *Any sequentially compact metric space is separable.*

17.H. Any sequentially compact metric space is second countable.

17.I. *For metric spaces compactness and sequential compactness are equivalent.*

17.1. Prove that a sequentially compact metric space is bounded. (Cf. *17.E* and *17.I.*)

17.2. Prove that in any metric space for any $\varepsilon > 0$ there exists

(1) a discrete ε-net and even
(2) an ε-net such that the distance between any two of its points is greater than ε.

17°3. Completeness and Compactness

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of a metric space is a ***Cauchy sequence*** if for every $\varepsilon > 0$ there exists a number N such that $\rho(x_n, x_m) < \varepsilon$ for any $n, m \geq N$. A metric space X is ***complete*** if every Cauchy sequence in X converges.

17.J. A Cauchy sequence containing a convergent subsequence converges.

17.K. Prove that a metric space M is complete iff every nested decreasing sequence of closed balls in M with radii tending to 0 has nonempty intersection.

17.L. Prove that a compact metric space is complete.

17.M. Prove that a complete metric space is compact iff for each $\varepsilon > 0$ it contains a finite ε-net.

17.N. Prove that a complete metric space is compact iff for any $\varepsilon > 0$ it contains a compact ε-net.

17°4x. Noncompact Balls in Infinite Dimension

By l^∞ denote the set of all bounded sequences of real numbers. This is a vector space with respect to the component-wise operations. There is a natural norm in it: $\|x\| = \sup\{|x_n| \mid n \in \mathbb{N}\}$.

17.1x. Are closed balls of l^∞ compact? What about spheres?

17.2x. Is the set $\{x \in l^\infty \mid |x_n| \leq 2^{-n}, n \in \mathbb{N}\}$ compact?

17.3x. Prove that the set $\{x \in l^\infty \mid |x_n| = 2^{-n}, n \in \mathbb{N}\}$ is homeomorphic to the Cantor set K introduced in Section 2.

17.4x*. Does there exist an infinitely dimensional normed space in which closed balls are compact?

17°5x. p-Adic Numbers

Fix a prime integer p. By $\mathbb{Z}_p$ denote the set of series of the form $a_0 + a_1 p + \dots + a_n p^n + \dots$ with $0 \le a_n < p$, $a_n \in \mathbb{N}$. For $x, y \in \mathbb{Z}_p$, put $\rho(x, y) = 0$ if $x = y$, and $\rho(x, y) = p^{-m}$ if m is the smallest number such that the mth coefficients in the series x and y differ.

17.5x. Prove that ρ is a metric in $\mathbb{Z}_p$.

This metric space is the *space of integer p-adic numbers*. There is an injection $\mathbb{Z} \to \mathbb{Z}_p$ assigning to $a_0 + a_1 p + \dots + a_n p^n \in \mathbb{Z}$ with $0 \le a_k < p$ the series

$$a_0 + a_1 p + \dots + a_n p^n + 0p^{n+1} + 0p^{n+2} + \dots \in \mathbb{Z}_p$$

and to $-(a_0 + a_1 p + \dots + a_n p^n) \in \mathbb{Z}$ with $0 \le a_k < p$ the series

$$b_0 + b_1 p + \dots + b_n p^n + (p-1)p^{n+1} + (p-1)p^{n+2} + \dots,$$

where

$$b_0 + b_1 p + \dots + b_n p^n = p^{n+1} - (a_0 + a_1 p + \dots + a_n p^n).$$

Cf. *4.Ix*.

17.6x. Prove that the image of the injection $\mathbb{Z} \to \mathbb{Z}_p$ is dense in $\mathbb{Z}_p$.

17.7x. Is $\mathbb{Z}_p$ a complete metric space?

17.8x. Is $\mathbb{Z}_p$ compact?

17°6x. Spaces of Convex Figures

Let $D \subset \mathbb{R}^2$ be a closed disk of radius p. Consider the set $\mathcal{P}_n$ of all convex polygons P with the following properties:

- the perimeter of P is at most p;
- P is contained in D;
- P has at most n vertices (the cases of one and two vertices are not excluded; the perimeter of a segment is twice its length).

See *4.Ax*, cf. *4.Cx*.

17.9x. Equip $\mathcal{P}_n$ with a natural topological structure. For instance, define a natural metric on $\mathcal{P}_n$.

17.10x. Prove that $\mathcal{P}_n$ is compact.

17.11x. Prove that there exists a polygon belonging to $\mathcal{P}_n$ and having the maximal area.

17.12x. Prove that this polygon is a regular n-gon.

Consider now the set $\mathcal{P}_\infty$ of all convex polygons that have perimeter at most p and are contained in D. In other words, $\mathcal{P}_\infty = \bigcup_{n=1}^{\infty} \mathcal{P}_n$.

17.13x. Construct a topological structure in $\mathcal{P}_\infty$ inducing the structures introduced above in the spaces $\mathcal{P}_n$.

17.14x. Prove that the space $\mathcal{P}_\infty$ is not compact.

Consider now the set $\mathcal{P}$ of all convex closed subsets of the plane that have perimeter at most p and are contained in D. (Observe that all sets in $\mathcal{P}$ are compact.)

***17.15*x.** Construct a topological structure in $\mathcal{P}$ that induces the structure introduced above in the space $\mathcal{P}_\infty$.

***17.16*x.** Prove that the space $\mathcal{P}$ is compact.

***17.17*x.** Prove that there exists a convex plane set with perimeter at most p having a maximal area.

***17.18*x.** Prove that this is a disk of radius $\frac{p}{2\pi}$.

18x. Local Compactness and Paracompactness

18°1x. Local Compactness

A topological space X is *locally compact* if each point of X has a neighborhood with compact closure.

18.1x. Compact spaces are locally compact.

18.2x. Which of the following spaces are locally compact: (a) $\mathbb{R}$; (b) $\mathbb{Q}$; (c) $\mathbb{R}^n$; (d) a discrete space?

18.3x. Find two locally compact sets on the line such that their union is not locally compact.

18.Ax. Is the local compactness hereditary?

18.Bx. A closed subset of a locally compact space is locally compact.

18.Cx. Is it true that an open subset of a locally compact space is locally compact?

18.Dx. A Hausdorff locally compact space is regular.

18.Ex. An open subset of a locally compact Hausdorff space is locally compact.

18.Fx. Local compactness is a local property for a Hausdorff space, i.e., a Hausdorff space is locally compact iff each of its points has a locally compact neighborhood.

18°2x. One-Point Compactification

Let (X, Ω) be a Hausdorff topological space. Let X^* be the set obtained by adding a point x_* to X (of course, x_* does not belong to X). Let Ω^* be the collection of subsets of X^* consisting of

- sets open in X and
- sets of the form $X^* \smallsetminus C$, where $C \subset X$ is a compact set:

$$\Omega^* = \Omega \cup \{X^* \smallsetminus C \mid C \subset X \text{ is a compact set}\}.$$

18.Gx. Prove that Ω^* is a topological structure on X^*.

18.Hx. Prove that the space (X^*, Ω^*) is compact.

18.Ix. Prove that the inclusion $(X, \Omega) \hookrightarrow (X^*, \Omega^*)$ is a topological embedding.

18.Jx. Prove that if X is locally compact, then the space (X^*, Ω^*) is Hausdorff. (Recall that in the definition of X^* we assumed that X is Hausdorff.)

A topological embedding of a space X into a compact space Y is a *compactification* of X if the image of X is dense in Y. In this situation, Y is also called a *compactification* of X. (To simplify the notation, we identify X with its image in Y.)

18.Kx. Prove that if X is a locally compact Hausdorff space and Y is a compactification of X with one-point $Y \smallsetminus X$, then there exists a homeomorphism $Y \to X^*$ which is the identity on X.

Any space Y of Problem *18.Kx* is called a *one-point compactification* or *Alexandrov compactification* of X. Problem *18.Kx* says Y is essentially unique.

18.Lx. Prove that the one-point compactification of the plane is homeomorphic to S^2.

18.4x. Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.

18.5x. Give explicit descriptions of one-point compactifications of the following spaces:

(1) annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$;
(2) square without vertices $\{(x, y) \in \mathbb{R}^2 \mid x, y \in [-1, 1], |xy| < 1\}$;
(3) strip $\{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1]\}$;
(4) a compact space.

18.Mx. Prove that a locally compact Hausdorff space is regular.

18.6x. Let X be a locally compact Hausdorff space, K a compact subset of X, U a neighborhood of K. Then there exists a neighborhood V of K such that the closure $\operatorname{Cl} V$ is compact and contained in U.

18°3x. Proper Maps

A continuous map $f : X \to Y$ is *proper* if each compact subset of Y has compact preimage.

Let X, Y be Hausdorff spaces. Any map $f : X \to Y$ obviously extends to the map

$$f^* : X^* \to Y^* : x \mapsto \begin{cases} f(x) & \text{if } x \in X, \\ y^* & \text{if } x = x^*. \end{cases}$$

18.Nx. Prove that f^* is continuous iff f is a proper continuous map.

18.Ox. Prove that any proper map of a Hausdorff space to a Hausdorff locally compact space is closed.

Problem *18.Ox* is related to Theorem *16.X*.

***18.P*x.** Extend this analogy: formulate and prove statements corresponding to Theorems *16.Z* and *16.Y*.

18°4x. Locally Finite Collections of Subsets

A collection Γ of subsets of a space X is *locally finite* if each point $b \in X$ has a neighborhood U such that $A \cap U = \varnothing$ for all sets $A \in \Gamma$ except, maybe, a finite number.

***18.Q*x.** A locally finite cover of a compact space is finite.

> ***18.7*x.** If a collection Γ of subsets of a space X is locally finite, then so is $\{\operatorname{Cl} A \mid A \in \Gamma\}$.
>
> ***18.8*x.** If a collection Γ of subsets of a space X is locally finite, then each compact set $A \subset X$ intersects only a finite number of elements of Γ.
>
> ***18.9*x.** If a collection Γ of subsets of a space X is locally finite and each $A \in \Gamma$ has compact closure, then each $A \in \Gamma$ intersects only a finite number of elements of Γ.
>
> ***18.10*x.** Any locally finite cover of a sequentially compact space is finite.

***18.R*x.** Find an open cover of $\mathbb{R}^n$ that has no locally finite subcovering.

Let Γ and Δ be two covers of a set X. The cover Δ is a *refinement* of Γ if for each $A \in \Delta$ there exists $B \in \Gamma$ such that $A \subset B$.

***18.S*x.** Prove that any open cover of $\mathbb{R}^n$ has a locally finite open refinement.

***18.T*x.** Let $\{U_i\}_{i\in\mathbb{N}}$ be a (locally finite) open cover of $\mathbb{R}^n$. Prove that there exists an open cover $\{V_i\}_{i\in\mathbb{N}}$ of $\mathbb{R}^n$ such that $\operatorname{Cl} V_i \subset U_i$ for each $i \in \mathbb{N}$.

18°5x. Paracompact Spaces

A space X is *paracompact* if every open cover of X has a locally finite open refinement.

***18.U*x.** Any compact space is paracompact.

***18.V*x.** $\mathbb{R}^n$ is paracompact.

***18.W*x.** Let $X = \bigcup_{i=1}^{\infty} X_i$, where X_i are compact sets such that $X_i \subset \operatorname{Int} X_{i+1}$. Then X is paracompact.

***18.X*x.** Let X be a locally compact space. If X has a countable cover by compact sets, then X is paracompact.

> ***18.11*x.** Prove that if a locally compact space is second countable, then it is paracompact.
>
> ***18.12*x.** A closed subspace of a paracompact space is paracompact.
>
> ***18.13*x.** A disjoint union of paracompact spaces is paracompact.

18°6x. Paracompactness and Separation Axioms

18.14x. Let X be a paracompact topological space, and let F and M be two disjoint subsets of X, where F is closed. Suppose that F is covered by open sets U_α whose closures are disjoint with M: $\operatorname{Cl} U_\alpha \cap M = \varnothing$. Then F and M have disjoint neighborhoods.

18.15x. A Hausdorff paracompact space is regular.

18.16x. A Hausdorff paracompact space is normal.

18.17x. Let X be a Hausdorff locally compact and paracompact space, Γ a locally finite open cover of X. Then X has a locally finite open cover Δ such that the closures $\operatorname{Cl} V$, where $V \in \Delta$, are compact sets and $\{\operatorname{Cl} V \mid V \in \Delta\}$ is a refinement of Γ.

Here is a more general (though formally weaker) fact.

18.18x. Let X be a normal space, Γ a locally finite open cover of X. Then X has a locally finite open cover Δ such that $\{\operatorname{Cl} V \mid V \in \Delta\}$ is a refinement of Γ.

Information. *Metrizable spaces are paracompact.*

18°7x. Partitions of Unity

Let X be a topological space, $f : X \to \mathbb{R}$ a function. Then the set $\operatorname{supp} f = \operatorname{Cl}\{x \in X \mid f(x) \neq 0\}$ is the ***support*** of f.

18.19x. Let X be a topological space, and let $\{f_\alpha : X \to \mathbb{R}\}_{\alpha\in\Lambda}$ be a family of continuous functions whose supports $\operatorname{supp}(f_\alpha)$ constitute a locally finite cover of X. Prove that the formula

$$f(x) = \sum_{\alpha\in\Lambda} f_\alpha(x)$$

determines a continuous function $f : X \to \mathbb{R}$.

A family of nonnegative functions $f_\alpha : X \to \mathbb{R}_+$ is a ***partition of unity*** if the supports $\operatorname{supp}(f_\alpha)$ constitute a locally finite cover of the space X and $\sum_{\alpha\in\Lambda} f_\alpha(x) = 1$.

A partition of unity $\{f_\alpha\}$ is ***subordinate to a cover*** Γ if $\operatorname{supp}(f_\alpha)$ is contained in an element of Γ for each α. We also say that Γ ***dominates*** $\{f_\alpha\}$.

18.Yx. Let X be a normal space. Then each locally finite open cover of X dominates a certain partition of unity.

18.20x. Let X be a Hausdorff space. If each open cover of X dominates a certain partition of unity, then X is paracompact.

Information. *A Hausdorff space X is paracompact iff each open cover of X dominates a certain partition of unity.*

18°8x. Application: Making Embeddings From Pieces

***18.21*x.** Let X be a topological space, $\{U_i\}_{i=1}^k$ an open cover of X. If U_i can be embedded in $\mathbb{R}^n$ for each $i = 1, \dots, k$, then X can be embedded in $\mathbb{R}^{k(n+1)}$.

18.21*x*.1. Let $h_i : U_i \to \mathbb{R}^n$, $i = 1, \dots, k$, be embeddings, and let $f_i : X \to \mathbb{R}$ form a partition of unity subordinate to the cover $\{U_i\}_{i=1}^k$. We put $\hat{h}_i(x) = (h_i(x), 1) \in \mathbb{R}^{n+1}$. Show that the map $X \to \mathbb{R}^{k(n+1)} : x \mapsto (f_i(x)\hat{h}_i(x))_{i=1}^k$ is an embedding.

18.22*x. *Riddle. How can you generalize *18.21*x?

Proofs and Comments

11.A A set A is open and closed, iff A and $X \smallsetminus A$ are open, iff A and $X \smallsetminus A$ are closed.

11.B It suffices to prove the following apparently less general assertion: *A space having a connected everywhere dense subset is connected.* (See *6.3*.) Let $X \supset A$ be the space and the subset. To prove that X is connected, let $X = U \cup V$, where U and V are disjoint sets open in X, and prove that one of them is empty (cf. *11.A*). $U \cap A$ and $V \cap A$ are disjoint sets open in A, and

$$A = X \cap A = (U \cup V) \cap A = (U \cap A) \cup (V \cap A).$$

Since A is connected, one of these sets, say $U \cap A$, is empty. Then U is empty since A is dense, see *6.M*.

11.C To simplify the notation, we may assume that $X = \bigcup_\lambda A_\lambda$. By Theorem *11.A*, it suffices to prove that if U and V are two open sets partitioning X, then either $U = \varnothing$ or $V = \varnothing$. For each $\lambda \in \Lambda$, since A_λ is connected, we have either $A_\lambda \subset U$ or $A_\lambda \subset V$ (see *11.14*). Fix a $\lambda_0 \in \Lambda$. To be definite, let $A_{\lambda_0} \subset U$. Since each of the sets A_λ meets A_{λ_0}, all sets A_λ also lie in U, and so none of them meets V, whence

$$V = V \cap X = V \cap \bigcup_\lambda A_\lambda = \bigcup_\lambda (V \cap A_\lambda) = \varnothing.$$

11.E Apply Theorem *11.C* to the family $\{A_\lambda \cup A_{\lambda_0}\}_{\lambda \in \Lambda}$, which consists of connected sets by *11.D*. (Or just repeat the proof of Theorem *11.C*.)

11.F Using *11.D*, prove by induction that $\bigcup_{-n}^{n} A_k$ is connected, and apply Theorem *11.C*.

11.G The union of all connected sets containing a given point is connected (by *11.C*) and obviously maximal.

11.H Let A and B be two connected components with $A \cap B \neq \varnothing$. Then $A \cup B$ is connected by *11.D*. By the maximality of connected components, we have $A \supset A \cup B \subset B$, whence $A = A \cup B = B$.

11.I $\boxed{\Longrightarrow}$ This is obvious since the component is connected. $\boxed{\Longleftarrow}$ Since the components of the points are not disjoint, they coincide.

11.K If A is a connected component, then its closure $\operatorname{Cl} A$ is connected by *11.B*. Therefore, $\operatorname{Cl} A \subset A$ by the maximality of connected components. Hence, $A = \operatorname{Cl} A$, because the opposite inclusion holds true for any set A.

11.M See *11.10*.

11.N Passing to the map ab $f : X \to f(X)$, we see that it suffices to prove the following theorem:

If X is a connected space and $f : X \to Y$ is a continuous surjection, then Y is also connected.

Consider a partition of Y in two open sets U and V and prove that one of them is empty. The preimages $f^{-1}(U)$ and $f^{-1}(V)$ are open by continuity of f and constitute a partition of X. Since X is connected, one of them, say $f^{-1}(U)$, is empty. Since f is surjective, we also have $U = \varnothing$.

11.Q $\boxed{\Longrightarrow}$ Let $X = U \cup V$, where U and V are nonempty disjoint sets open in X. Set $f(x) = -1$ for $x \in U$ and $f(x) = 1$ for $x \in V$. Then f is continuous and surjective, is it not? $\boxed{\Longleftarrow}$ Assume the contrary: let X be connected. Then S^0 is also connected by *11.N*, a contradiction.

11.R By Theorem *11.Q*, this statement follows from Cauchy Intermediate Value Theorem. However, it is more natural to deduce Intermediate Value Theorem from *11.Q* and the connectedness of I.

Thus assume the contrary: let $I = [0, 1]$ be disconnected. Then $[0, 1] = U \cup V$, where U and V are disjoint and open in $[0, 1]$. Suppose $0 \in U$, consider the set $C = \{x \in [0, 1] \mid [0, x) \subset U\}$ and put $c = \sup C$. Show that each of the possibilities $c \in U$ and $c \in V$ gives rise to contradiction. A slightly different proof of Theorem *11.R* is sketched in Lemmas *11.R.1* and *11.R.2*.

11.R.1 Use induction: for $n = 1, 2, 3, \dots$, put

$$(a_{n+1}, b_{n+1}) := \begin{cases} (\frac{a_n+b_n}{2}, b_n) & \text{if } \frac{a_n+b_n}{2} \in U, \\ (a_n, \frac{a_n+b_n}{2}) & \text{if } \frac{a_n+b_n}{2} \in V. \end{cases}$$

11.R.2 On the one hand, we have $c \in U$ since $c \in \operatorname{Cl}\{a_n \mid n \in \mathbb{N}\}$, and a_n belong to U, which is closed in I. On the other hand, we have $c \in V$ since $c \in \operatorname{Cl}\{b_n \mid n \in \mathbb{N}\}$, and b_n belong to V, which is also closed in I. The contradiction means that U and V cannot be both closed, i.e., I is connected.

11.S Every open set on a line is a union of disjoint open intervals (see *2.Ax*), each of which contains a rational point. Therefore each open subset U of a line is a union of a countable collection of open intervals. Each of them is open and connected, and thus is a connected component of U (see *11.T*).

11.T Apply *11.R* and *11.J*. (Cf. *11.U* and *11.X*.)

11.U Apply *11.R* and *11.J*. (Recall that a set $K \subset \mathbb{R}^n$ is said to be convex if for any $p, q \in K$ we have $[p, q] \subset K$.)

11.V Combine *11.R* and *11.C*.

11.X $\boxed{\Longrightarrow}$ This is *11.10*. $\boxed{\Longleftarrow}$ This is *11.V*.

11.Y Singletons and all kinds of intervals (including open and closed rays and the whole line).

11.Y Use *10.R*, *11.U*, and, say Theorem *11.B* (or *11.I*).

12.A Since the segment $[a, b]$ is connected by *11.R*, its image is an interval by *11.29*. Therefore, it contains all points between $f(a)$ and $f(b)$.

12.B Combine *11.N* and *11.10*.

12.C Combine *11.V* and *11.29*.

12.D One of them is connected, while the other one is not.

12.E For each of the spaces, find the number of points with connected complement. (This is obviously a topological invariant.)

12.F Cf. *12.4*.

13.A Since the cover $\big\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\big\}$ of $[0, 1]$ is fundamental and the restriction of uv to each element of the cover is continuous, the entire map uv is also continuous.

13.B If $x, y \in I$, then $I \to I : t \mapsto (1-t)x + ty$ is a path connecting x and y.

13.C If $x, y \in \mathbb{R}^n$, then $[0, 1] \to \mathbb{R}^n : t \mapsto (1-t)x + ty$ is a path connecting x and y.

13.D Use *10.R* and *13.C*.

13.E Combine *11.R* and *11.Q*.

13.7 Use (the formula of) *13.C*, *13.A*, and *13.5*.

13.F Let x and y be two points in the union, and let A and B be the sets in the family that contain x and y. If $A = B$, there is nothing to prove. If $A \neq B$, take $z \in A \cap B$, join x with z in A by a path u, and join y with z in B by a path v. Then the path uv joins x and y in the union, and it remains to use *13.5*.

13.G Consider the union of all path-connected sets containing the point and use *13.F*. (Cf. *11.G*.)

13.H Similarly to *11.H*, only instead of *11.D* use *13.F*.

13.I $\boxed{\Longrightarrow}$ Recall the definition of a path-connected component. $\boxed{\Longleftarrow}$ This follows from (the proof of) *13.G*.

13.J Let X be path-connected, let $f : X \to Y$ be a continuous map, and let $y_1, y_2 \in f(X)$. If $y_i = f(x_i)$, $i = 1, 2$, and u is a path joining x_1 and x_2, then how can you construct a path joining y_1 and y_2?

13.M Combine *13.8* and *11.J*.

13.N By *10.Q*, A is homeomorphic to $(0, +\infty) \cong \mathbb{R}$, which is path-connected by *13.C*, and so A is also path-connected by *13.K*. Since A is connected (combine *11.T* and *11.O*, or use *13.M*) and, obviously, $A \subset X \subset \operatorname{Cl} A$ (what is $\operatorname{Cl} A$, by the way?), it follows form *11.15* that X is also connected.

13.O This is especially obvious for A since $A \cong (0, \infty)$ (you can also use *11.2*).

13.P Prove that any path in X starting at $(0, 0)$ is constant.

13.Q Let A and X be as above. Check that A is dense in X (cf. the solution to *13.N*) and plug in Problems *13.N* and *13.P*.

13.R See *13.Q*.

13.S Let C be a path-connected component of X, $x \in C$ an arbitrary point. If U_x is a path-connected neighborhood of x, then U_x lies entirely in C (by the definition of a path-connected component!), and so x is an interior point of C, which is thus open.

13.T $\boxed{\Longrightarrow}$ This is *13.M*. $\boxed{\Longleftarrow}$ Since path-connected components of X are open (see Problem *13.S*) and X is connected, there can be only one path-connected component.

13.U This follows from *13.T* because spherical neighborhoods in $\mathbb{R}^n$ (i.e., open balls) are path-connected (by *13.6* or *13.7*).

14.A If $r_1 + r_2 \le \rho(x_1, x_2)$, then the balls $B_{r_1}(x_1)$ and $B_{r_2}(x_2)$ are disjoint.

14.B Certainly, I is Hausdorff since it is metrizable. The intervals $\left[0, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right]$ are disjoint neighborhoods of 0 and 1, respectively.

14.C $\boxed{\Longrightarrow}$ If $y \ne x$, then there exist disjoint neighborhoods U_x and V_y. Therefore, $y \notin \operatorname{Cl} U_x$, whence $y \notin \bigcap\limits_{U \ni x} \operatorname{Cl} U$.

$\boxed{\Longleftarrow}$ If $y \ne x$, then $y \notin \bigcap\limits_{U \ni x} \operatorname{Cl} U$, it follows that there exists a neighborhood U_x such that $y \notin \operatorname{Cl} U_x$. Set $V_y = X \smallsetminus \operatorname{Cl} U_x$.

14.D Assume the contrary: let $x_n \to a$ and $x_n \to b$, where $a \ne b$. Let U and V be disjoint neighborhoods of a and b, respectively. Then for sufficiently large n we have $x_n \in U \cap V = \varnothing$, a contradiction.

14.E A neighborhood of a point in $\mathbb{R}_{T_1}$ has the form $U = \mathbb{R} \smallsetminus \{x_1, \dots, x_N\}$, where, say, $x_1 < x_2 < \dots < x_N$. Then, obviously, $a_n \in U$ for each $n > x_N$.

14.F Assume that X is a space, $A \subset X$ is a subspace, and $x, y \in A$ are two distinct points. If X is Hausdorff, then x and y have disjoint neighborhoods U and V in X. In this case, $U \cap A$ and $V \cap A$ are disjoint neighborhoods of x and y in A. (Recall the definition of the relative topology!)

14.G (a) $\boxed{\Longrightarrow}$ Let X satisfy T_1 and let $x \in X$. By Axiom T_1, each point $y \in X \smallsetminus x$ has a neighborhood U that does not contain x, i.e., $U \subset X \smallsetminus x$, which means that all points in $X \smallsetminus x$ are inner. Therefore, $X \smallsetminus x$ is open, and so its complement $\{x\}$ is closed. $\boxed{\Longleftarrow}$ If singletons in X are closed and $x, y \in X$ are two distinct points, then $X \smallsetminus x$ is a neighborhood of y that does not contain x, as required in T_1.

(b) $\boxed{\Longrightarrow}$ If singletons in X are closed, then so are finite subsets of X, which are finite unions of singletons. $\boxed{\Longleftarrow}$ Obvious.

14.H Combine *14.12* and *14.G*.

14.I Combine *14.A* and *14.12*.

14.J Each point in $\mathbb{R}_{T_1}$ is closed, as required by T_1, but any two nonempty sets intersect, which contradicts T_2.

14.K Combine *14.G* and *5.4*, and once more use *14.G*; or just modify the proof of *14.F*.

14.N (a) $\Rightarrow$ (b) Actually, T_0 precisely says that at least one of the points does not lie in the closure of the other (to see this, use Theorem *6.F*).
(b) $\Rightarrow$ (a) Use the above reformulation of T_0 and the fact that if $x \in \operatorname{Cl}\{y\}$ and $y \in \operatorname{Cl}\{x\}$, then $\operatorname{Cl}\{x\} = \operatorname{Cl}\{y\}$.
(a) $\Leftrightarrow$ (c) This is obvious. (Recall the definition of the relative topology!)
(c) $\Leftrightarrow$ (d) This is also obvious.

14.O $\boxed{\Longrightarrow}$ This is obvious. $\boxed{\Longleftarrow}$ Let X be a T_0 space such that each point $x \in X$ has a smallest neighborhood C_x. Then we say that $x \preceq y$ if $y \in C_x$. Let us verify the axioms of order. Reflexivity is obvious. Transitivity: assume that $x \preceq y$ and $y \preceq z$. Then C_x is a neighborhood of y, whence $C_y \subset C_x$, and so also $z \in C_x$, which means that $x \preceq z$. Antisymmetry: if $x \preceq y$ and $y \preceq x$, then $y \in C_x$ and $x \in C_y$, whence $C_x = C_y$. By T_0, this is possible only if $x = y$. Verify that this order generates the initial topology.

14.P Let X be a regular space, and let $x, y \in X$ be two distinct points. Since X satisfies T_1, the singleton $\{y\}$ is closed, and so we can apply T_3 to x and $\{y\}$.

14.Q $\boxed{\Longrightarrow}$ See Problem *14.P*. $\boxed{\Longleftarrow}$ See Problem *14.12*.

14.R Let X be a metric space, $x \in X$, and $r > 0$. Prove that, e.g., $\operatorname{Cl} B_r(x) \subset B_{2r}(x)$, and use *14.19*.

14.S Apply T_4 to a closed set and a singleton, which is also closed by T_1.

14.T $\boxed{\Longrightarrow}$ See Problem *14.S.* $\boxed{\Longleftarrow}$ See Problem *14.12*.

14.U Let A and B be two disjoint closed sets in a metric space (X, ρ). Then, obviously, $A \subset U = \{x \in X \mid \rho(x, A) < \rho(x, B)\}$ and $B \subset V = \{x \in X \mid \rho(x, A) > \rho(x, B)\}$. U and V are open (use *9.L*) and disjoint.

14.Ax.1 Put $U_1 = X \smallsetminus B$. Since X is normal, there exists an open neighborhood $U_0 \supset A$ such that $\operatorname{Cl} U_0 \subset U_1$. Let $U_{1/2}$ be an open neighborhood of $\operatorname{Cl} U_0$ such that $\operatorname{Cl} U_{1/2} \subset U_1$. Repeating the process, we obtain the required collection $\{U_p\}_{p \in \Lambda}$.

14.Ax Put $f(x) = \inf\{\lambda \in \Lambda \mid x \in \operatorname{Cl} U_\lambda\}$. We easily see that f continuous.

14.Bx Slightly modify the proof of *14.9x*, using Urysohn Lemma *14.Ax* instead of *14.9x.1*.

15.A Let $f : X \to \mathbb{N}$ be an injection and let $A \subset X$. Then the restriction $f|_A : A \to \mathbb{N}$ is also an injection. Use *15.1*.

15.B Let X be a countable set, and let $f : X \to Y$ be a map. Taking each $y \in f(X)$ to a point in $f^{-1}(y)$, we obtain an injection $f(X) \to X$. Hence, $f(X)$ is countable by *15.1*.

15.D Suggest an algorithm (or even a formula!) for enumerating elements in $\mathbb{N}^2$.

15.E Use *15.D*.

15.G Derive this from *6.44*.

15.H Construct a countable set A intersecting each base set (at least) at one point and prove that A is everywhere dense.

15.I Let X be a second countable space, $A \subset X$ a subspace. If $\{U_i\}_1^\infty$ is a countable base in X, then $\{U_i \cap A\}_1^\infty$ is a countable base in A. (See *5.1*.)

15.J Show that if the set $A = \{x_n\}_{n=1}^\infty$ is everywhere dense, then the collection $\{B_r(x) \mid x \in A,\ r \in \mathbb{Q},\ r > 0\}$ is a countable base of X. (Use Theorems *4.I* and *3.A* to show that this is a base and *15.E* to show that it is countable.)

15.L Use *15.K* and *15.I*.

15.M By *15.K* and *15.I* (or, more to the point, combine *15.J*, *15.I*, and *15.H*), it is sufficient to find a countable everywhere-dense set in $\mathbb{R}^n$. For example, take $\mathbb{Q}^n = \{x \in \mathbb{R}^n \mid x_i \in \mathbb{Q}, i = 1, \dots, n\}$. To see that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, use the metric $\rho^{(\infty)}$. To see that $\mathbb{Q}^n$ is countable, use *15.F* and *15.E*.

15.N Use *9.15*.

15.O Let X be the space, let $\{U\}$ be a countable base in X, and let $\Gamma = \{V\}$ be a cover of X. Let $\{U_i\}_{i=1}^{\infty}$ be the base sets that are contained in at least element of the cover: let $U_i \subset V_i$. Using the definition of a base, we easily see that $\{U_i\}_{i=1}^{\infty}$ is a cover of X. Then $\{V_i\}_{i=1}^{\infty}$ is the required countable subcovering of Γ.

15.P Use *3.A*.

15.Q Use *15.12*

15.R Use *15.P* and *15.A*.

15.S Consider an uncountable discrete space.

15.T If $x_n \in A$ and $x_n \to a$, then, obviously, a is an adherent point for A.

15.U Let $a \in \operatorname{Cl} A$, and let $\{U_n\}_{n\in\mathbb{N}}$ be a decreasing neighborhood base at a (see *15.16*). For each n, there is $x_n \in U_n \cap A$, and we easily see that $x_n \to a$.

15.V Indeed, let $f : X \to Y$ be a continuous map, let $b \in X$, and let $a_n \to b$ in X. We must prove that $f(a_n) \to f(b)$ in Y. Let $V \subset Y$ be a neighborhood of $f(b)$. Since f is continuous, $f^{-1}(V) \subset X$ is a neighborhood of b, and since $a_n \to b$, we have $a_n \in f^{-1}(V)$ for $n > N$. Then also $f(a_n) \in V$ for $n > N$, as required.

15.W Assume that $f : X \to Y$ is a sequentially continuous map and $A \subset Y$ is a sequentially closed set. To prove that $f^{-1}(A)$ is sequentially closed, we must prove that if $\{x_n\} \subset f^{-1}(A)$ and $x_n \to a$, then $a \in f^{-1}(A)$. Since f is sequentially continuous, we have $f(x_n) \to f(a)$, and since A is sequentially closed, we have $f(a) \in A$, whence $a \in f^{-1}(A)$, as required.

15.X It suffices to check that if $F \subset Y$ is a closed set, then so is the preimage $f^{-1}(F) \subset X$, i.e., $\operatorname{Cl}(f^{-1}(F)) \subset f^{-1}(F)$. Let $a \in \operatorname{Cl}(f^{-1}(F))$. Since X is first countable, we also have $a \in \operatorname{SCl}(f^{-1}(F))$ (see *15.U*), and so there is a sequence $\{x_n\} \subset f^{-1}(F)$ such that $x_n \to a$, whence $f(x_n) \to f(a)$ because f is sequentially continuous. Since F is closed, we have $f(a) \in F$ (by *15.T*), i.e., $a \in f^{-1}(F)$, as required.

15.Ax Since l_2 is a metric space, it is sufficient to prove that l_2 is separable (see *15.K*), i.e., to find a countable everywhere dense set $A \subset l_2$. The first idea here might be to consider the set of sequences with rational components, but this set is uncountable! Instead of this, let A be the set of all rational sequences $\{x_i\}$ such that $x_i = 0$ for all sufficiently large i. (To show that A is countable, use *15.F* and *15.E*. To show that A is everywhere dense, use the fact that if a series $\sum x_i^2$ converges, then for each $\varepsilon > 0$ there is k such that $\sum_{i=k}^{\infty} x_i^2 < \varepsilon$.)

16.A Each of the spaces has only a finite number of open sets, and so each open cover is finite.

16.B Only the finite ones. (Consider the cover consisting of all singletons.)

16.C Consider the cover of $\mathbb{R}$ by the open intervals $(-n, n)$, $n \in \mathbb{N}$.

16.D The latter condition is precisely the negation of compactness.

16.E This follows from the Lindelöf theorem *15.O*.

16.F This follows from the second De Morgan formula (see *2.E*). Indeed, $\bigcap A_\lambda \neq \varnothing$ iff $\bigcup(X \smallsetminus A_\lambda) = X \smallsetminus \bigcap A_\lambda \neq X$.

16.G $\boxed{\Rightarrow}$ Let X be a compact space and let $\Gamma = \{F_\lambda\}$ be a family of closed subsets of X with the finite intersection property. Assume the contrary: let $\bigcap F_\lambda = \varnothing$. Then by the second De Morgan formula we have $\bigcup(X \smallsetminus F_\lambda) = X \smallsetminus \bigcap F_\lambda = X$, i.e., $\{X \smallsetminus F_\lambda\}$ is an open cover of X. Since X is compact, this cover contains a finite subcovering: $\bigcup_1^n (X \smallsetminus F_i) = X$, whence $\bigcap_1^n F_i = \varnothing$, which contradicts the finite intersection property of Γ.
$\boxed{\Leftarrow}$ Prove the converse implication on your own.

16.H $\boxed{\Rightarrow}$ Let $\Gamma = \{U_\alpha\}$ be a cover of A by open subsets of X. Since A is a compact set, the cover of A with the sets $A \cap U_\alpha$ contains a finite subcovering $\{A \cap U_{\alpha_i}\}_1^n$. Hence $\{U_{\alpha_i}\}$ is a finite subcovering of Γ.
$\boxed{\Leftarrow}$ Prove the converse implication on your own.

16.I Certainly not.

16.J Let X be a compact space, $F \subset X$ a closed subset, and $\{U_\alpha\}$ an open cover of A. Then $\{X \smallsetminus F\} \cup \{U_\alpha\}$ is an open cover of X, which contains a finite subcovering $\{X \smallsetminus F\} \cup \{U_i\}_1^n$. Clearly, $\{U_i\}_1^n$ is a cover of F.

16.K This follows from *16.L*.

16.L Since X is Hausdorff, for each $x \in A$ the points x and b possess disjoint neighborhoods U_x and $V_b(x)$. Obviously, $\{U_x\}_{x \in A}$ is an open cover of A. Since A is compact, the cover contains a finite subcovering $\{U_{x_i}\}_1^n$. Put $U = \bigcup_1^n U_{x_i}$ and $V = \bigcap_1^n V_b(x_i)$. Then U and V are the required sets. (Check that they are disjoint.)

16.M Combine *16.J* and *16.L*.

16.N This follows from *16.O*.

16.O (Cf. the proof of Lemma *16.L*.) Let X be a Hausdorff space, and let $A, B \subset X$ be two compact sets. By Lemma *16.L*, each $x \in B$ has a neighborhood V_x disjoint with a certain neighborhood $U(x)$ of A. Obviously, $\{V_x\}_{x \in B}$ is an open cover of B. Since B is compact, the cover contains a finite subcovering $\{U_{x_i}\}_1^n$. Put $V = \bigcup_1^n V_{x_i}$ and $U = \bigcap_1^n U_b(x_i)$. Then U and V are the required neighborhoods. (Check that they are disjoint.)

16.P Let us argue by contradiction. If I is not compact, then I has a cover Γ_0 such that no finite part of Γ_0 covers I (see *16.D*). We bisect I

and denote by I_1 the half that also is not covered by any finite part of Γ_0. Then we bisect I_1, etc. As a result, we obtain a sequence of nested segments I_n, where the length of I_n is equal to 2^{-n}. By the completeness axiom, they have a unique point in common: $\bigcap_1^\infty I_n = \{x_0\}$. Consider an element $U_0 \in \Gamma_0$ containing x_0. Since U_0 is open, we have $I_n \subset U_0$ for sufficiently large n, in contradiction to the fact that, by construction, I_n is covered by no finite part of Γ_0.

16.Q Repeat the argument used in the proof of Theorem *16.P*, only instead of bisecting the segment each time subdivide the current cube into 2^n equal smaller cubes.

16.R Consider the cover by open balls, $\{B_n(x_0)\}_{n=1}^\infty$.

16.S Let, e.g., $X = [0,1) \cup [2,3]$. (Or just put $X = [0,1)$.) The set $[0,1)$ is bounded, it is also closed in X, but it is not compact.

16.T $\boxed{\Longrightarrow}$ Combine Theorems *14.A*, *16.K*, and *16.R*.
$\boxed{\Longleftarrow}$ If a subset $F \subset \mathbb{R}^n$ is bounded, then F lies in a certain cube, which is compact (see Theorem *16.Q*). If, in addition, F is closed, then F is also compact by *16.J*.

16.U We use Theorem *16.H*. Let $\Gamma = \{U_\lambda\}$ be a cover of $f(X)$ by open subsets of Y. Since f is continuous, $\{f^{-1}(U_\lambda)\}$ is an open cover of X. Since X is compact, this cover has a finite subcovering $\{f^{-1}(U_{\lambda_i})\}_{i=1}^n$. Then $\{U_{\lambda_i}\}_{i=1}^n$ is a finite subcovering of Γ.

16.V By *16.U* and *16.T*, the set $f(X) \subset \mathbb{R}$ is closed and bounded. Since $f(X)$ is bounded, there exist finite numbers $m = \inf f(X)$ and $M = \sup f(X)$, whence, in particular, $m \le f(x) \le M$. Since $f(X)$ is closed, we have $m, M \in f(X)$, whence it follows that there are $a, b \in X$ with $f(a) = m$ and $f(b) = M$, as required.

16.W This follows from *16.23*: consider the cover $\{f^{-1}(U) \mid U \in \Gamma\}$ of X.

16.X This immediately follows from *16.J*, *16.K*, and *16.U*.

16.Y Combine *16.X* and *16.24*.

16.Z See Problem *16.Y*.

17.A.1 $\boxed{\Longrightarrow}$ This is obvious. $\boxed{\Longleftarrow}$ Let x be a limit point. If x is not an accumulation point of A, then x has a neighborhood U_x such that the set $U_x \cap A$ is finite. Show that x has a neighborhood W_x such that $(W_x \smallsetminus x) \cap A = \varnothing$.

17.A.2 Argue by contradiction: consider the cover of the space by neighborhoods having finite intersections with the infinite set.

17.A.3 Let X be a space, and let $\{a_n\}$ be a sequence of points in X. Let A be the set of all points in the sequence. If A is finite, there is not much to prove. So, we assume that A is infinite. By Theorem *17.A.2*, A has an accumulation point x_0. Let $\{U_n\}$ be a countable neighborhood base of x_0 and $x_{n_1} \in U_1 \cap A$. Since the set $U_2 \cap A$ is infinite, there is $n_2 > n_1$ such that $x_{n_2} \in U_2 \cap A$. Prove that the subsequence $\{x_{n_k}\}$ thus constructed converges to x_0. If A is finite, then the argument simplifies a great deal.

17.B.1 Consider a sequence $\{x_n\}$, $x_n \in F_n$ and show that if $x_{n_k} \to x_0$, then $x_n \in F_n$ for all $n \in \mathbb{N}$.

17.B.2 $\boxed{\Rightarrow}$ Let $\{F_k\} \subset X$ be a sequence of closed sets the finite intersection property. Then $\left\{\bigcap_1^n F_k\right\}$ is a nested sequence of nonempty closed sets, whence $\bigcap_1^\infty F_k \neq \varnothing$. $\boxed{\Leftarrow}$ This is obvious.

17.B.3 By the Lindelöf theorem *15.O*, it is sufficient to consider countable covers $\{U_n\}$. If no finite collection of sets in this cover is not a cover, then the closed sets $F_n = X \smallsetminus U_n$ form a collection with the finite intersection property.

17.C This follows from *17.B* and *17.A*.

17.D Reformulate the definition of an ε-net: A is an ε-net if $\{B_\varepsilon(x)\}_{x\in A}$ is a cover of X. Now the proof is obvious.

17.E We argue by contradiction. If $\{x_i\}_{i=1}^{k-1}$ is not an ε-net, then there is a point x_k such that $\rho(x_i, x_k) \geq \varepsilon$, $i = 1, \dots, k-1$. As a result, we obtain a sequence in which the distance between any two points is at least ε, and so it has no convergent subsequences.

17.F $\boxed{\Rightarrow}$ This is obvious because open balls in a metric space are open sets. $\boxed{\Leftarrow}$ Use the definition of the metric topology.

17.G The union of finite $\frac{1}{n}$-nets of the space is countable and everywhere dense. (see *17.E*).

17.H Use 13.82.

17.I If X is compact, then X is sequentially compact by *17.A*. If X is sequentially compact, then X is separable, and hence X has a countable base. Then *17.C* implies that X is compact.

17.J Assume that $\{x_n\}$ is a Cauchy sequence and its subsequence x_{n_k} converges to a point a. Find a number m such that $\rho(x_l, x_k) < \frac{\varepsilon}{2}$ for $k, l \geq m$, and i such that $n_i > m$ and $\rho(x_{n_i}, a) < \frac{\varepsilon}{2}$. Then for all $l \geq m$ we have the inequality $\rho(x_l, a) \leq \rho(x_l, x_{n_i}) + \rho(x_{n_i}, a) < \varepsilon$.

17.K $\boxed{\Leftarrow}$ Obvious. $\boxed{\Rightarrow}$ Let $\{x_n\}$ be a Cauchy sequence. Let n_1 be such that $\rho(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_1$. Therefore, $x_n \in B_{1/2}(x_{n_1})$ for all $n \geq n_1$. Further, take $n_2 > n_1$ so that $\rho(x_n, x_m) < \frac{1}{4}$ for all $n, m \geq n_2$, then $B_{1/4}(x_{n_2}) \subset B_{1/2}(x_{n_1})$. Proceeding the construction, we obtain a sequence of decreasing disks such that their unique common point x_0 satisfies $x_n \to x_0$.

17.L Let $\{x_n\}$ be a Cauchy sequence of points of a compact metric space X. Since X is also sequentially compact, $\{x_n\}$ contains a convergent subsequence, and then the initial sequence also converges.

17.M $\boxed{\Rightarrow}$ Each compact space contains a finite ε-net.
$\boxed{\Leftarrow}$ Let us show that the space is sequentially compact. Consider an arbitrary sequence $\{x_n\}$. We denote by A_n a finite $\frac{1}{n}$-net in X. Since $X = \bigcup_{x \in A_1} B_1(x)$, one of the balls contains infinitely many points of the sequence; let x_{n_1} be the first of them. From the remaining members lying in the first ball, we let x_{n_2} be the first one of those lying in the ball $B_{1/2}(x)$, $x \in A_2$. Proceeding with this construction, we obtain a subsequence $\{x_{n_k}\}$. Let us show that the latter is fundamental. Since by assumption the space is complete, the constructed sequence has a limit. We have thus proved that the space is sequentially compact, hence, it is also compact.

17.N $\boxed{\Leftarrow}$ Obvious. $\boxed{\Rightarrow}$ This follows from assertion *17.M* because an $\frac{\varepsilon}{2}$-net for a $\frac{\varepsilon}{2}$-net is an ε-net for the entire space.

***18.A*x** No, it is not: consider $\mathbb{Q} \subset \mathbb{R}$.

***18.B*x** Let X be a locally compact space, $F \subset X$ a closed subset space, $x \in F$. Let $U_x \subset X$ be a neighborhood of x with compact closure. Then $U_x \cap F$ is a neighborhood of x in F. Since F is closed, the set $\mathrm{Cl}_F(U \cap F) = (\mathrm{Cl}\, U) \cap F$ (see *6.3*) is compact as a closed subset of a compact set.

***18.C*x** No, this is wrong in general. Take any space (X, Ω) that is not locally compact (e.g., let $X = \mathbb{Q}$). We put $X^* = X \cup x_*$ and $\Omega^* = \{X^*\} \cup \Omega$. The space (X^*, Ω^*) is compact for a trivial reason (which one?), hence, it is locally compact. Now, X is an open subset of X^*, but it is not locally compact by our choice of X.

***18.D*x** Let X be the space, W be a neighborhood of a point $x \in X$. Let U_0 be a neighborhood of x with compact closure. Since X is Hausdorff, it follows that $\{x\} = \bigcap_{U \ni x} \mathrm{Cl}\, U$, whence $\{x\} = \bigcap_{U \ni x} \big(\mathrm{Cl}\, U_0 \cap \mathrm{Cl}\, U\big)$. Since each of the sets $\mathrm{Cl}\, U_0 \cap \mathrm{Cl}\, U$ is compact, *16.11* implies that x has neighborhoods $U_1, \dots, U_n$ such that $\mathrm{Cl}\, U_0 \cap \mathrm{Cl}\, U_1 \cap \dots \cap \mathrm{Cl}\, U_n \subset W$. Put $V = U_0 \cap U_1 \cap \dots \cap U_n$. Then $\mathrm{Cl}\, V \subset W$. Therefore, each neighborhood of x contains the closure of a certain neighborhood (a "closed neighborhood") of x. By *14.19*, X is regular.

***18.E*x** Let X be the space, $V \subset X$ the open subset, $x \in V$ a point. Let U be a neighborhood of x such that $\mathrm{Cl}\, U$ is compact. By *18.Dx* and *14.19*, x has a neighborhood W such that $\mathrm{Cl}\, W \subset U \cap V$. Therefore, $\mathrm{Cl}_V W = \mathrm{Cl}\, W$ is compact, and so the space V is locally compact.

***18.F*x** $\boxed{\Rightarrow}$ Obvious. $\boxed{\Leftarrow}$ See the idea used in *18.Ex*.

***18.G*x** Since $\varnothing$ is both open and compact in X, we have $\varnothing, X^* \in \Omega^*$. Let us verify that unions and finite intersections of subsets in Ω^* lie in Ω^*. This is obvious for subsets in Ω. Let $X^* \smallsetminus K_\lambda \in \Omega^*$, where $K_\lambda \subset X$ are compact sets, $\lambda \in \Lambda$. Then we have $\bigcup (X^* \smallsetminus K_\lambda) = X^* \smallsetminus \bigcap K_\lambda \in \Omega^*$ because X is Hausdorff and so $\bigcap K_\lambda$ is compact. Similarly, if Λ is finite, then we also have $\bigcap (X^* \smallsetminus K_\lambda) = X^* \smallsetminus \bigcup K_\lambda \in \Omega^*$. Therefore, it suffices to consider the case where a set in Ω^* and a set in Ω are united (intersected). We leave this as an exercise.

***18.H*x** Let $U = X^* \smallsetminus K_0$ be an element of the cover that contains the added point. Then the remaining elements of the cover provide an open cover of the compact set K_0.

***18.I*x** In other words, the topology of X^* induced on X the initial topology of X (i.e., $\Omega^* \cap 2^X = \Omega$). We must check that there arise no new open sets in X. This is true because compact sets in the Hausdorff space X are closed.

***18.J*x** If $x, y \in X$, this is obvious. If, say, $y = x_*$ and U_x is a neighborhood of x with compact closure, then U_x and $X \smallsetminus \operatorname{Cl} U_x$ are neighborhoods separating x and x_*.

***18.K*x** Let $X^* \smallsetminus X = \{x_*\}$ and $Y \smallsetminus X = \{y\}$. We have an obvious bijection

$$f : Y \to X^* : x \mapsto \begin{cases} x & \text{if } x \in X, \\ x_* & \text{if } x = y. \end{cases}$$

If $U \subset X^*$ and $U = X^* \smallsetminus K$, where K is a compact set in X, then the set $f^{-1}(U) = Y \smallsetminus K$ is open in Y. Therefore, f is continuous. It remains to apply *16.Y*.

***18.L*x** Verify that if an open set $U \subset S^2$ contains the "North Pole" $(0, 0, 1)$ of S^2, then the complement of the image of U under the stereographic projection is compact in $\mathbb{R}^2$.

***18.M*x** X^* is compact and Hausdorff by *18.H*x and *18.J*x, therefore, X^* is regular by *16.M*. Since X is a subspace of X^* by *18.I*x, it remains to use the fact that regularity is hereditary by *14.20*. (Also try to prove the required assertion without using the one-point compactification.)

***18.N*x** $\boxed{\Longrightarrow}$ If1 f^* is continuous, then, obviously, so is f (by *18.I*x). Let $K \subset Y$ be a compact set, and let $U = Y \smallsetminus K$. Since f^* is continuous, the set $(f^*)^{-1}(U) = X^* \smallsetminus f^{-1}(K)$ is open in X^*, i.e., $f^{-1}(K)$ is compact in X. Therefore, f is proper. $\boxed{\Longleftarrow}$ Use a similar argument.

***18.O*x** Let $f^* : X^* \to Y^*$ be the canonical extension of a map $f : X \to Y$. Prove that if F is closed in X, then $F \cup \{x^*\}$ is closed in X^*, and hence compact. After that, use *18.N*x, *16.X*, and *18.I*x.

***18.P*x** A proper injection of a Hausdorff space into a locally compact Hausdorff space is a topological embedding. A proper bijection of a Hausdorff space onto a locally compact Hausdorff space is a homeomorphism.

***18.Q*x** Let Γ be a locally finite cover, and let Δ be a cover of X by neighborhoods each of which meets only a finite number of sets in Γ. Since X is compact, we can assume that Δ is finite. In this case, obviously, Γ is also finite.

***18.R*x** Cover $\mathbb{R}^n$ by the balls $B_n(0)$, $n \in \mathbb{N}$.

***18.S*x** Use a locally finite covering of $\mathbb{R}^n$ by equal open cubes.

***18.T*x** Cf. *18.17x*.

***18.U*x** This is obvious.

***18.V*x** This is *18.Sx*.

***18.W*x** Let Γ be an open cover of X. Since each of the sets $K_i = X_i \smallsetminus \operatorname{Int} X_{i-1}$ is compact, Γ contains a finite subcovering Γ_i of K_i. Observe that the sets $W_i = \operatorname{Int} X_{i+1} \smallsetminus X_{i-2} \supset K_i$ form a locally finite open cover of X. Intersecting for each i elements of Γ_i with W_i, we obtain a locally finite refinement of Γ.

***18.X*x** Using assertion *18.6x*, construct a sequence of open sets U_i such that for each i the closure $X_i := \operatorname{Cl} U_i$ is compact and lies in $U_{i+1} \subset \operatorname{Int} X_{i+1}$. After that, apply *18.Wx*.

***18.Y*x** Let $\Gamma = \{U_\alpha\}$ be the cover. By *18.18x*, there exists an open cover $\Delta = \{V_\alpha\}$ such that $\operatorname{Cl} V_\alpha \subset U_\alpha$ for each α. Let $\varphi_\alpha : X \to I$ be an Urysohn function with $\operatorname{supp} \varphi_\alpha = X \smallsetminus U_\alpha$ and $\varphi_\alpha^{-1}(1) = \operatorname{Cl} V_\alpha$ (see *14.Ax*). Put $\varphi(x) = \sum_\alpha \varphi_\alpha(x)$. Then the collection $\{\varphi_\alpha(x)/\varphi(x)\}$ is the required partition of unity.

Chapter IV

Topological Constructions

19. Multiplication

19°1. Set-Theoretic Digression: Product of Sets

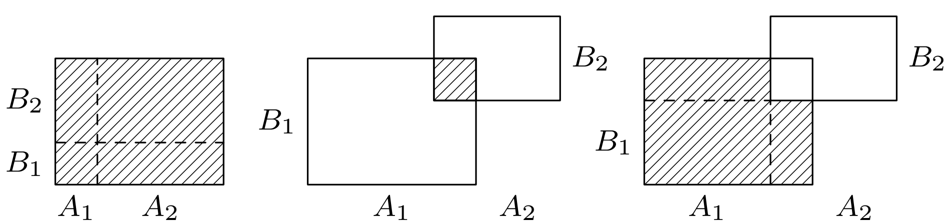
Let X and Y be sets. The set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is called the *direct product* or *Cartesian product* or just *product* of X and Y and denoted by $X \times Y$. If $A \subset X$ and $B \subset Y$, then $A \times B \subset X \times Y$. Sets $X \times b$ with $b \in Y$ and $a \times Y$ with $a \in X$ are *fibers* of the product $X \times Y$.

19.A. Prove that for any $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$ we have

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2),$$

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

$$(A_1 \times B_1) \smallsetminus (A_2 \times B_2) = \big((A_1 \smallsetminus A_2) \times B_1\big) \cap \big(A_1 \times (B_1 \smallsetminus B_2)\big).$$



The natural maps

$$\mathrm{pr}_X : X \times Y \to X : (x, y) \mapsto x \quad \text{and} \quad \mathrm{pr}_Y : X \times Y \to Y : (x, y) \mapsto y$$

are (*natural*) *projections*.

19.B. Prove that $\mathrm{pr}_X^{-1}(A) = A \times Y$ for any $A \subset X$.

> ***19.1.*** Find the corresponding formula for $B \subset Y$.

19°2. Graphs

A map $f : X \to Y$ determines a subset Γ_f of $X \times Y$ defined by $\Gamma_f = \{(x, f(x)) \mid x \in X\}$, it is called the *graph* of f.

19.C. A set $\Gamma \subset X \times Y$ is the graph of a map $X \to Y$ iff for each $a \in X$ the intersection $\Gamma \cap (a \times Y)$ is one-point.

> ***19.2.*** Prove that for any map $f : X \to Y$ and any set $A \subset X$, we have
> $$f(A) = \mathrm{pr}_Y(\Gamma_f \cap (A \times Y)) = \mathrm{pr}_Y(\Gamma_f \cap \mathrm{pr}_X^{-1}(A))$$
> and $f^{-1}(B) = \mathrm{pr}_X(\Gamma \cap (X \times B))$ for any $B \subset Y$.
>
> The set $\Delta = \{(x, x) \mid x \in X\} = \{(x, y) \in X \times X \mid x = y\}$ is the *diagonal* of $X \times X$.
>
> ***19.3.*** Let A and B be two subsets of X. Prove that $(A \times B) \cap \Delta = \varnothing$ iff $A \cap B = \varnothing$.
>
> ***19.4.*** Prove that the map $\mathrm{pr}_X \big|_{\Gamma_f}$ is bijective.
>
> ***19.5.*** Prove that f is injective iff $\mathrm{pr}_Y \big|_{\Gamma_f}$ is injective.
>
> ***19.6.*** Consider the map $T : X \times Y \to Y \times X : (x, y) \mapsto (y, x)$. Prove that $\Gamma_{f^{-1}} = T(\Gamma_f)$ for any invertible map $f : X \to Y$.

19°3. Product of Topologies

Let X and Y be two topological spaces. If U is an open set of X and B is an open set of Y, then we say that $U \times V$ is an *elementary* set of $X \times Y$.

19.D. *The set of elementary sets of $X \times Y$ is a base of a topological structure in $X \times Y$.*

The *product* of two spaces X and Y is the set $X \times Y$ with the topological structure determined by the base consisting of elementary sets.

> ***19.7.*** Prove that for any subspaces A and B of spaces X and Y the product topology on $A \times B$ coincides with the topology induced from $X \times Y$ via the natural inclusion $A \times B \subset X \times Y$.

19.E. $Y \times X$ is canonically homeomorphic to $X \times Y$.

The word *canonically* means here that a homeomorphism between $X \times Y$ and $Y \times X$, which exists according to the statement, can be chosen in a nice special (or even obvious?) way, so that we may expect that it has additional pleasant properties.

19.F. The canonical bijection $X \times (Y \times Z) \to (X \times Y) \times Z$ is a homeomorphism.

19.8. Prove that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

19.9. Prove that $\mathrm{Cl}(A \times B) = \mathrm{Cl}\, A \times \mathrm{Cl}\, B$ for any $A \subset X$ and $B \subset Y$.

19.10. Is it true that $\mathrm{Int}(A \times B) = \mathrm{Int}\, A \times \mathrm{Int}\, B$?

19.11. Is it true that $\mathrm{Fr}(A \times B) = \mathrm{Fr}\, A \times \mathrm{Fr}\, B$?

19.12. Is it true that $\mathrm{Fr}(A \times B) = (\mathrm{Fr}\, A \times B) \cup (A \times \mathrm{Fr}\, B)$?

19.13. Prove that $\mathrm{Fr}(A \times B) = (\mathrm{Fr}\, A \times B) \cup (A \times \mathrm{Fr}\, B)$ for closed A and B.

19.14. Find a formula for $\mathrm{Fr}(A \times B)$ in terms of A, $\mathrm{Fr}\, A$, B, and $\mathrm{Fr}\, B$.

19°4. Topological Properties of Projections and Fibers

19.G. *The natural projections* $\mathrm{pr}_X : X \times Y \to X$ *and* $\mathrm{pr}_Y : X \times Y \to Y$ *are continuous for any topological spaces* X *and* Y*.*

19.H. The topology of product is the coarsest topology with respect to which pr_X and pr_Y are continuous.

19.I. *A fiber of a product is canonically homeomorphic to the corresponding factor. The canonical homeomorphism is the restriction to the fiber of the natural projection of the product onto the factor.*

19.J. Prove that $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$, $(\mathbb{R}^1)^n = \mathbb{R}^n$, and $(I)^n = I^n$. (We remind the reader that I^n is the n-dimensional unit cube in $\mathbb{R}^n$.)

19.15. Let Σ_X and Σ_Y be bases of spaces X and Y. Prove that the sets $U \times V$ with $U \in \Sigma_X$ and $V \in \Sigma_Y$ constitute a base for $X \times Y$.

19.16. Prove that a map $f : X \to Y$ is continuous iff $\mathrm{pr}_X|_{\Gamma_f} : \Gamma_f \to X$ is a homeomorphism.

19.17. Prove that if W is open in $X \times Y$, then $\mathrm{pr}_X(W)$ is open in X.

A map from a space X to a space Y is *open* (*closed*) if the image of any open set under this map is open (respectively, closed). Therefore, *19.17* states that $\mathrm{pr}_X : X \times Y \to X$ is an open map.

19.18. Is pr_X a closed map?

19.19. Prove that for each space X and each compact space Y the map $\mathrm{pr}_X : X \times Y \to X$ is closed.

19°5. Cartesian Products of Maps

Let X, Y, and Z be three sets. A map $f : Z \to X \times Y$ determines the compositions $f_1 = \mathrm{pr}_X \circ f : Z \to X$ and $f_2 = \mathrm{pr}_Y \circ f : Z \to Y$, which are called the *factors* (or *components*) of f. Indeed, f can be recovered from them as a sort of product.

19.K. Prove that for any maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ there exists a unique map $f : Z \to X \times Y$ with $\mathrm{pr}_X \circ f = f_1$ and $\mathrm{pr}_Y \circ f = f_2$.

19.20. Prove that $f^{-1}(A \times B) = f_1^{-1}(A) \cap f_2^{-1}(B)$ for any $A \subset X$ and $B \subset Y$.

19.L. Let X, Y, and Z be three spaces. Prove that $f : Z \to X \times Y$ is continuous iff so are f_1 and f_2.

Any two maps $g_1 : X_1 \to Y_1$ and $g_2 : X_2 \to Y_2$ determine a map

$$g_1 \times g_2 : X_1 \times X_2 \to Y_1 \times Y_2 : (x_1, x_2) \mapsto (g_1(x_1), g_2(x_2)),$$

which is their (*Cartesian*) *product*.

19.21. Prove that $(g_1 \times g_2)(A_1 \times A_2) = g_1(A_1) \times g_2(A_2)$ for any $A_1 \subset X_1$ and $A_2 \subset X_2$.

19.22. Prove that $(g_1 \times g_2)^{-1}(B_1 \times B_2) = g_1^{-1}(B_1) \times g_2^{-1}(B_2)$ for any $B_1 \subset Y_1$ and $B_2 \subset Y_2$.

19.M. Prove that the Cartesian product of continuous maps is continuous.

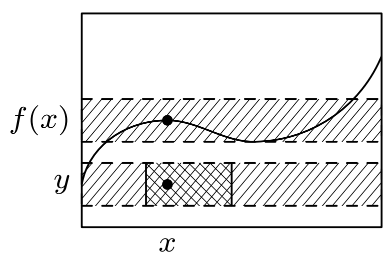
19.23. Prove that the Cartesian product of open maps is open.

19.24. Prove that a metric $\rho : X \times X \to \mathbb{R}$ is continuous with respect to the topology generated by the metric.

19.25. Let $f : X \to Y$ be a map. Prove that the graph Γ_f is the preimage of the diagonal $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$ under the map $f \times \mathrm{id}_Y : X \times Y \to Y \times Y$.

19°6. Properties of Diagonal and Other Graphs

19.26. Prove that a space X is Hausdorff iff the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.



19.27. Prove that if Y is a Hausdorff space and $f : X \to Y$ is a continuous map, then the graph Γ_f is closed in $X \times Y$.

19.28. Let Y be a compact space. Prove that if a map $f : X \to Y$ has closed graph Γ_f, then f is continuous.

19.29. Prove that the hypothesis on compactness in *19.28* is necessary.

19.30. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that its graph is:

(1) closed;
(2) connected;
(3) path connected;
(4) locally connected;
(5) locally compact.

19.31. Consider the following functions

1) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x}, & \text{otherwise.} \end{cases}$; 2) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ \sin \frac{1}{x}, & \text{otherwise.} \end{cases}$ Do their graphs possess the properties listed in *19.30*?

19.32. Does any of the properties of the graph of a function f that are mentioned in *19.30* imply that f is continuous?

19.33. Let Γ_f be closed. Then the following assertions are equivalent:

(1) f is continuous;
(2) f is locally bounded;
(3) the graph Γ_f of f is connected;
(4) the graph Γ_f of f is path-connected.

19.34. Prove that if Γ_f is connected and locally connected, then f is continuous.

19.35. Prove that if Γ_f is connected and locally compact, then f is continuous.

19.36. Are some of the assertions in Problems *19.33–19.35* true for maps $f : \mathbb{R}^2 \to \mathbb{R}$?

19°7. Topological Properties of Products

19.N. *The product of Hausdorff spaces is Hausdorff.*

19.37. Prove that the product of regular spaces is regular.

19.38. The product of normal spaces is not necessarily normal.

19.38.1*. Prove that the space $\mathcal{R}$ formed by real numbers with the topology determined by the base consisting of all semi-open intervals $[a, b)$ is normal.

19.38.2. Prove that in the Cartesian square of the space introduced in *19.38.1* the subspace $\{(x, y) \mid x = -y\}$ is closed and discrete.

19.38.3. Find two disjoint subsets of $\{(x, y) \mid x = -y\}$ that have no disjoint neighborhoods in the Cartesian square of the space of *19.38.1*.

19.O. *The product of separable spaces is separable.*

19.P. *First countability of factors implies first countability of the product.*

19.Q. *The product of second countable spaces is second countable.*

19.R. *The product of metrizable spaces is metrizable.*

19.S. *The product of connected spaces is connected.*

19.39. Prove that for connected spaces X and Y and any proper subsets $A \subset X$, $B \subset Y$ the set $X \times Y \smallsetminus A \times B$ is connected.

19.T. *The product of path-connected spaces is path-connected.*

19.U. *The product of compact spaces is compact.*

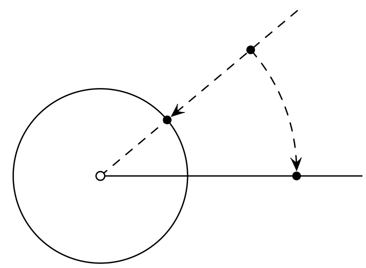
19.40. Prove that the product of locally compact spaces is locally compact.

19.41. If X is a paracompact space and Y is compact, then $X \times Y$ is paracompact.

19.42. For which of the topological properties studied above is it true that if $X \times Y$ possesses the property, then so does X?

19°8. Representation of Special Spaces as Products

19.V. Prove that $\mathbb{R}^2 \smallsetminus 0$ is homeomorphic to $S^1 \times \mathbb{R}$.



19.43. Prove that $\mathbb{R}^n \smallsetminus \mathbb{R}^k$ is homeomorphic to $S^{n-k-1} \times \mathbb{R}^{k+1}$.

19.44. Prove that $S^n \cap \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_k^2 \le x_{k+1}^2 + \dots + x_{n+1}^2\}$ is homeomorphic to $S^{k-1} \times D^{n-k+1}$.

19.45. Prove that $O(n)$ is homeomorphic to $SO(n) \times O(1)$.

19.46. Prove that $GL(n)$ is homeomorphic to $SL(n) \times GL(1)$.

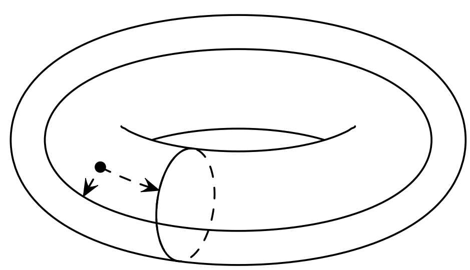
19.47. Prove that $GL_+(n)$ is homeomorphic to $SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$, where

$$GL_+(n) = \{A \in L(n,n) \mid \det A > 0\}.$$

19.48. Prove that $SO(4)$ is homeomorphic to $S^3 \times SO(3)$.

The space $S^1 \times S^1$ is a *torus*.

19.W. Construct a topological embedding of the torus to $\mathbb{R}^3$.



The product $S^1 \times \dots \times S^1$ of k factors is the *k-dimensional torus*.

19.X. Prove that the k-dimensional torus can be topologically embedded into $\mathbb{R}^{k+1}$.

19.Y. Find topological embeddings of $S^1 \times D^2$, $S^1 \times S^1 \times I$, and $S^2 \times I$ into $\mathbb{R}^3$.

20. Quotient Spaces

20°1. Set-Theoretic Digression: Partitions and Equivalence Relations

Recall that a *partition* of a set A is a cover of A consisting of pairwise disjoint sets.

Each partition of a set X determines an *equivalence relation* (i.e., a relation, which is reflexive, symmetric, and transitive): two elements of X are said to be equivalent if they belong to the same element of the partition. Vice versa, each equivalence relation in X determines the partition of X into classes of equivalent elements. Thus, partitions of a set into nonempty subsets and equivalence relations in the set are essentially the same. More precisely, they are two ways of describing the same phenomenon.

Let X be a set, S a partition. The set whose elements are members of the partition S (which are subsets of X) is the *quotient set* or *factor set* of X by S, it is denoted by $X/_S$. [1]

20.1. Riddle. How does this operation relate to division of numbers? Why is there a similarity in terminology and notation?

The set $X/_S$ is also called the *set of equivalence classes* for the equivalence relation corresponding to the partition S.

The map $\mathrm{pr} : X \to X/_S$ that maps $x \in X$ to the element of S containing x is the (*canonical*) *projection* or *factorization map*. A subset of X which is a union of elements of a partition is *saturated*. The smallest saturated set containing a subset A of X is the *saturation* of A.

20.2. Prove that $A \subset X$ is an element of a partition S of X iff $A = \mathrm{pr}^{-1}(\text{point})$, where $\mathrm{pr} : X \to X/_S$ is the natural projection.

20.A. Prove that the saturation of a set A equals $\mathrm{pr}^{-1}\big(\mathrm{pr}(A)\big)$.

20.B. Prove that a set is saturated iff it is equal to its saturation.

---

[1] At first glance, the definition of a quotient set contradicts one of the very profound principles of the set theory, which states that a set is determined by its elements. Indeed, according to this principle, we have $X/_S = S$ since S and $X/_S$ have the same elements. Hence, there seems to be no need to introduce $X/_S$. The real sense of the notion of quotient set is not in its literal set-theoretic meaning, but in our way of thinking of elements of partitions. If we remember that they are subsets of the original set and want to keep track of their internal structure (at least, of their elements), then we speak of a partition. If we think of them as atoms, getting rid of their possible internal structure, then we speak about the quotient set.

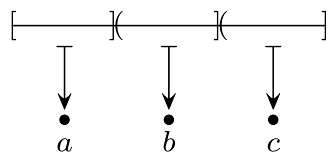
20°2. Quotient Topology

A quotient set $X/_S$ of a topological space X with respect to a partition S into nonempty subsets is provided with a natural topology: a set $U \subset X/_S$ is said to be open in $X/_S$ if its preimage $\mathrm{pr}^{-1}(U)$ under the canonical projection $\mathrm{pr} : X \to X/_S$ is open.

20.C. *The collection of these sets is a topological structure in the quotient set $X/_S$.*

This topological structure is the ***quotient topology***. The set $X/_S$ with this topology is the ***quotient space*** of X by partition S.

> ***20.3.*** Give an explicit description of the quotient space of the segment $[0, 1]$ by the partition consisting of $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3}]$, $(\frac{2}{3}, 1]$.



> ***20.4.*** What can you say about a partition S of a space X if the quotient space $X/_S$ is known to be discrete?

20.D. *A subset of a quotient space $X/_S$ is open iff it is the image of an open saturated set under the canonical projection* pr.

20.E. A subset of a quotient space $X/_S$ is closed, iff its preimage under pr is closed in X, iff it is the image of a closed saturated set.

20.F. *The canonical projection* $\mathrm{pr} : X \to X/_S$ *is continuous.*

20.G. Prove that the quotient topology is the finest topology in $X/_S$ such that the canonical projection pr is continuous with respect to it.

20°3. Topological Properties of Quotient Spaces

20.H. *A quotient space of a connected space is connected.*

20.I. *A quotient space of a path-connected space is path-connected.*

20.J. *A quotient space of a separable space is separable.*

20.K. *A quotient space of a compact space is compact.*

20.L. The quotient space of the real line by partition $\mathbb{R}_+$, $\mathbb{R} \smallsetminus \mathbb{R}_+$ is not Hausdorff.

20.M. The quotient space of a space X by a partition S is Hausdorff iff any two elements of S have disjoint saturated neighborhoods.

20.5. Formulate similar necessary and sufficient conditions for a quotient space to satisfy other separation axioms and countability axioms.

20.6. Give an example showing that the second countability can may get lost when we pass to a quotient space.

20°4. Set-Theoretic Digression: Quotients and Maps

Let S be a partition of a set X into nonempty subsets. Let $f : X \to Y$ be a map which is constant on each element of S. Then there is a map $X/_S \to Y$ which sends each element A of S to the element $f(a)$, where $a \in A$. This map is denoted by $f/_S$ and called the *quotient map* or *factor map* of f (by the partition S).

20.N. 1) Prove that a map $f : X \to Y$ is constant on each element of a partition S of X iff there exists a map $g : X/_S \to Y$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{pr}"'] & Y \\ X/_S \arrow[ur, "g"'] & \end{tikzcd}$$

2) Prove that such a map g coincides with $f/_S$.

More generally, if S and T are partitions of sets X and Y, then every map $f : X \to Y$ that maps each element of S to an element of T determines a map $X/_S \to Y/_T$ which sends an element A of partition S to the element of partition T containing $f(A)$. This map is denoted by $f/_{S,T}$ and called the *quotient map* or *factor map* of f (*with respect to* S and T).

20.O. Formulate and prove for $f/_{S,T}$ a statement generalizing *20.N*.

A map $f : X \to Y$ determines a partition of the set X into nonempty preimages of the elements of Y. This partition is denoted by $S(f)$.

20.P. The map $f/_{S(f)} : X/_{S(f)} \to Y$ is injective.

This map is the *injective factor* (or *injective quotient*) of f.

20°5. Continuity of Quotient Maps

20.Q. *Let X and Y be two spaces, S a partition of X into nonempty sets, and $f : X \to Y$ a continuous map constant on each element of S. Then the factor $f/_S$ of f is continuous.*

20.7. If the map f is open, then so is the quotient map $f/_S$.

20.8. Let X and Y be two spaces, S a partition of X into nonempty sets. Prove that the formula $f \mapsto f/_S$ determines a bijection from the set of all continuous maps $X \to Y$ that are constant on each element of S onto the set of all continuous maps $X/_S \to Y$.

20.R. Let X and Y be two spaces, S and T partitions of X and Y, respectively, and $f : X \to Y$ a continuous map which maps each element of S into an element of T. Then the map $f/_{S,T} : X/_S \to Y/_T$ is continuous.

20°6x. Closed Partitions

A partition S of a space X is *closed* if the saturation of each closed set is closed.

***20.1*x.** Prove that a partition is closed iff the canonical projection $X \to X/_S$ is a closed map.

***20.2*x.** Prove that if a partition S contains only one element consisting of more than one point, then S is closed if this element is a closed set.

***20.A*x.** Let X be a space satisfying the first separation axiom, S a closed partition of X. Then the quotient space $X/_S$ also satisfies the first separation axiom.

***20.B*x.** *The quotient space of a normal space with respect to a closed partition is normal.*

20°7x. Open Partitions

A partition S of a space X is *open* if the saturation of each open set is open.

***20.3*x.** Prove that a partition S is open iff the canonical projection $X \to X/_S$ is an open map.

***20.4*x.** Prove that if a set A is saturated with respect to an open partition, then $\operatorname{Int} A$ and $\operatorname{Cl} A$ are also saturated.

***20.C*x.** *The quotient space of a second countable space with respect to an open partition is second countable.*

***20.D*x.** *The quotient space of a first countable space with respect to an open partition is first countable.*

***20.E*x.** *Let X and Y be two spaces, and let S and T be their open partitions. Denote by $S \times T$ the partition of $X \times Y$ consisting of $A \times B$ with $A \in S$ and $B \in T$. Then the injective factor $X \times Y/_{S \times T} \to X/_S \times Y/_T$ of* $\operatorname{pr} \times \operatorname{pr} X \times Y \to X/_S \times Y/_T$ *is a homeomorphism.*

21. Zoo of Quotient Spaces

21°1. Tool for Identifying a Quotient Space with a Known Space

21.A. *If X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is a continuous map, then the injective factor $f/_{S(f)} : X/_{S(f)} \to Y$ is a homeomorphism.*

21.B. *The injective factor of a continuous map from a compact space to a Hausdorff one is a topological embedding.*

> ***21.1.*** Describe explicitly partitions of a segment such that the corresponding quotient spaces are all letters of the alphabet.
>
> ***21.2.*** Prove that there exists a partition of a segment I with the quotient space homeomorphic to square $I \times I$.

21°2. Tools for Describing Partitions

An accurate literal description of a partition can often be somewhat cumbersome, but usually it can be shortened and made more understandable. Certainly, this requires a more flexible vocabulary with lots of words having almost the same meanings. For instance, such words as *factorize* and *pass to a quotient* can be replaced by *attach*, *glue together*, *identify*, *contract*, *paste*, and other words accompanying these ones in everyday life.

Some elements of this language are easy to formalize. For instance, factorization of a space X with respect to a partition consisting of a set A and one-point subsets of the complement of A is the *contraction* (of the subset A to a point), and the result is denoted by $X/_A$.

> ***21.3.*** Let $A, B \subset X$ form a fundamental cover of a space X. Prove that the quotient map $A/_{A \cap B} \to X/_B$ of the inclusion $A \hookrightarrow X$ is a homeomorphism.

If A and B are two disjoint subspaces of a space X and $f : A \to B$ is a homeomorphism, then passing to the quotient of X by the partition into singletons in $X \smallsetminus (A \cup B)$ and two-point sets $\{x, f(x)\}$, where $x \in A$, we *glue* or *identify* the sets A and B via the homeomorphism f.

A rather convenient and flexible way for describing partitions is to describe the corresponding equivalence relations. The main advantage of this approach is that, by transitivity, it suffices to specify only some pairs of equivalent elements: if one states that $x \sim y$ and $y \sim z$, then it is not necessary to state that $x \sim z$ since this already follows.

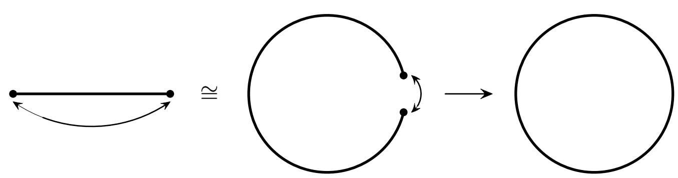
Hence, a partition is represented by a list of statements of the form $x \sim y$ that are sufficient for recovering the equivalence relation. We denote the corresponding partition by such a list enclosed into square brackets. For example, the quotient of a space X obtained by identifying subsets A and B by a homeomorphism $f : A \to B$ is denoted by $X/_{[a \sim f(a) \text{ for any } a \in A]}$ or just $X/_{[a \sim f(a)]}$.

Some partitions are easily described by a picture, especially if the original space can be embedded in the plane. In such a case, as in the pictures below, we draw arrows on the segments to be identified to show the directions to be identified.

Below we introduce all these kinds of descriptions for partitions and give examples of their usage, simultaneously providing literal descriptions. The latter are not that nice, but they may help the reader to remain confident about the meaning of the new words. On the other hand, the reader will appreciate the improvement the new words bring in.

21°3. Welcome to the Zoo

21.C. Prove that $I/_{[0 \sim 1]}$ is homeomorphic to S^1.



In other words, the quotient space of segment I by the partition consisting of $\{0, 1\}$ and $\{a\}$ with $a \in (0, 1)$ is homeomorphic to a circle.

> ***21.C.1.*** Find a surjective continuous map $I \to S^1$ such that the corresponding partition into preimages of points consists of one-point subsets of the interior of the segment and the pair of boundary points of the segment.

21.D. Prove that $D^n/_{S^{n-1}}$ is homeomorphic to S^n.

In *21.D*, we deal with the quotient space of the n-disk D^n by the partition $\{S^{n-1}\} \cup \{\{x\} \mid x \in B^n\}$.

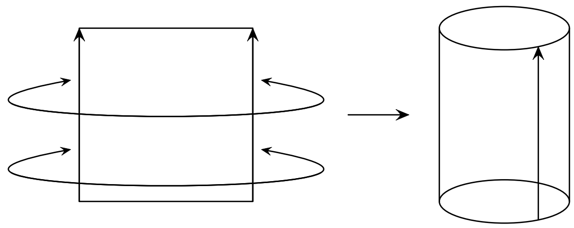
Here is a reformulation of *21.D*: *Contracting* the boundary of an n-dimensional ball to a point, we obtain gives rise an n-dimensional sphere.

> ***21.D.1.*** Find a continuous map of the n-disk D^n to the n-sphere S^n that maps the boundary of the disk to a single point and bijectively maps the interior of the disk onto the complement of this point.

21.E. Prove that $I^2/_{[(0,t) \sim (1,t) \text{ for } t \in \mathrm{I}]}$ is homeomorphic to $S^1 \times I$.

Here the partition consists of pairs of points $\{(0, t), (1, t)\}$ where $t \in I$, and one-point subsets of $(0, 1) \times I$.

Reformulation of *21.E*: If we *glue* the side edges of a square by identifying points on the same hight, then we obtain a cylinder.



21.F. $S^1 \times I/_{[(z,0) \sim (z,1) \text{ for } z \in S^1]}$ is homeomorphic to $S^1 \times S^1$.

Here the partition consists of one-point subsets of $S^1 \times (0,1)$, and pairs of points of the basis circles lying on the same generatrix of the cylinder.

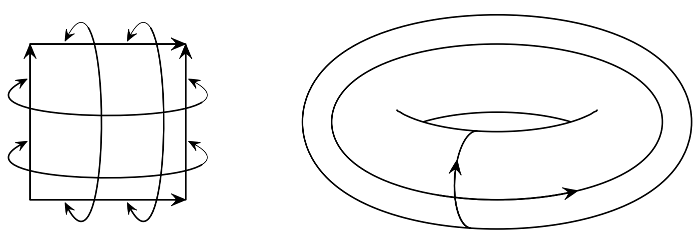
Here is a reformulation of *21.F*: If we *glue* the base circles of a cylinder by identifying points on the same generatrix, then we obtain a torus.

21.G. $I^2/_{[(0,t) \sim (1,t),\ (t,0) \sim (t,1)]}$ is homeomorphic to $S^1 \times S^1$.

In *21.G* , the partition consists of

- one-point subsets of the interior $(0,1) \times (0,1)$ of the square,
- pairs of points on the vertical sides that are the same distance from the bottom side (i.e., pairs $\{(0,t),(1,t)\}$ with $t \in (0,1)$),
- pairs of points on the horizontal sides that lie on the same vertical line (i.e., pairs $\{(t,0),(t,1)\}$ with $t \in (0,1)$),
- the four vertices of the square

Reformulation of *21.G*: Identifying the sides of a square according to the picturewe obtain a torus.



21°4. Transitivity of Factorization

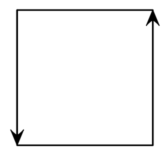
A solution of Problem *21.G* can be based on Problems *21.E* and *21.F* and the following general theorem.

21.H Transitivity of Factorization. *Let S be a partition of a space X, and let S' be a partition of the space $X/_S$. Then the quotient space*

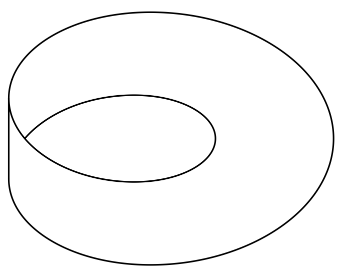
$(X/_S)/_{S'}$ is canonically homeomorphic to $X/_T$, where T is the partition of X into preimages of elements of S' under the projection $X \to X/_S$.

21°5. Möbius Strip

The *Möbius strip* or *Möbius band* is defined as $I^2/_{[(0,t)\sim(1,1-t)]}$. In other words, this is the quotient space of the square I^2 by the partition into centrally symmetric pairs of points on the vertical edges of I^2, and singletons that do not lie on the vertical edges. The Möbius strip is obtained, so to speak, by identifying the vertical sides of a square in such a way that the directions shown on them by arrows are superimposed:



21.I. Prove that the Möbius strip is homeomorphic to the surface that is swept in $\mathbb{R}^3$ by a segment rotating in a half-plane around the midpoint, while the half-plane rotates around its boundary line. The ratio of the angular velocities of these rotations is such that the rotation of the half-plane through 360° takes the same time as the rotation of the segment through 180°. See Figure.



21°6. Contracting Subsets

21.4. Prove that $[0,1]/_{[\frac13,\frac23]}$ is homeomorphic to $[0,1]$, and $[0,1]/_{\{\frac13,1\}}$ is homeomorphic to letter P.

21.5. Prove that the following spaces are homeomorphic:

(a) $\mathbb{R}^2$; (b) $\mathbb{R}^2/_I$; (c) $\mathbb{R}^2/_{D^2}$; (d) $\mathbb{R}^2/_{I^2}$;

(e) $\mathbb{R}^2/_A$, where A is a union of several segments with a common end point;

(f) $\mathbb{R}^2/_B$, where B is a simple finite polygonal line, i.e., a union of a finite sequence of segments $I_1, \dots, I_n$ such that the initial point of I_{i+1} is the final point of I_i.

21.6. Prove that if $f : X \to Y$ is a homeomorphism, then the quotient spaces $X/_A$ and $Y/_{f(A)}$ are homeomorphic.

21.7. Let $A \subset \mathbb{R}^2$ be a ray $\{(x, y) \mid x \geq 0, y = 0\}$. Is $\mathbb{R}^2/_A$ homeomorphic to $\operatorname{Int} D^2 \cup \{(0, 1)\}$?

21°7. Further Examples

21.8. Prove that $S^1/_{[z \sim e^{2\pi i/3}z]}$ is homeomorphic to S^1.

The partition in *21.8* consists of triples of points that are vertices of equilateral inscribed triangles.

21.9. Prove that the following quotient spaces of the disk D^2 are homeomorphic to D^2:

(1) $D^2/_{[(x, y) \sim (-x, -y)]}$,

(2) $D^2/_{[(x, y) \sim (x, -y)]}$,

(3) $D^2/_{[(x, y) \sim (-y, x)]}$.

21.10. Find a generalization of *21.9* with D^n substituted for D^2.

21.11. Describe explicitly the quotient space of line $\mathbb{R}^1$ by equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

21.12. Represent the Möbius strip as a quotient space of cylinder $S^1 \times I$.

21°8. Klein Bottle

Klein bottle is $I^2/_{[(t, 0) \sim (t, 1),\ (0, t) \sim (1, 1 - t)]}$. In other words, this is the quotient space of square I^2 by the partition into

- one-point subsets of its interior,
- pairs of points $(t, 0), (t, 1)$ on horizontal edges that lie on the same vertical line,
- pairs of points $(0, t), (1, 1 - t)$ symmetric with respect to the center of the square that lie on the vertical edges, and
- the quadruple of vertices.

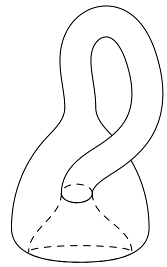
21.13. Present the Klein bottle as a quotient space of

(1) a cylinder;
(2) the Möbius strip.

21.14. Prove that $S^1 \times S^1/_{[(z, w) \sim (-z, \bar{w})]}$ is homeomorphic to the Klein bottle. (Here $\bar{w}$ denotes the complex number conjugate to w.)

21.15. Embed the Klein bottle into $\mathbb{R}^4$ (cf. *21.I* and *19.W*).

21.16. Embed the Klein bottle into $\mathbb{R}^4$ so that the image of this embedding under the orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ would look as follows:



21°9. Projective Plane

Let us identify each boundary point of the disk D^2 with the antipodal point, i.e., factorize the disk by the partition consisting of one-point subsets of the interior of the disk and pairs of points on the boundary circle symmetric with respect to the center of the disk. The result is the *projective plane*. This space cannot be embedded in $\mathbb{R}^3$, too. Thus we are not able to draw it. Instead, we present it in other way.

21.J. A projective plane is a result of gluing together a disk and a Möbius strip via a homeomorphism between their boundary circles.

21°10. You May Have Been Provoked to Perform an Illegal Operation

Solving the previous problem, you did something that did not fit into the theory presented above. Indeed, the operation with two spaces called *gluing* in *21.J* has not appeared yet. It is a combination of two operations: first, we make a single space consisting of disjoint copies of the original spaces, and then we factorize this space by identifying points of one copy with points of another. Let us consider the first operation in detail.

21°11. Set-Theoretic Digression: Sums of Sets

The (*disjoint*) *sum* of a family of sets $\{X_\alpha\}_{\alpha\in A}$ is the set of pairs (x_α, α) such that $x_\alpha \in X_\alpha$. The sum is denoted by $\bigsqcup_{\alpha\in A} X_\alpha$. So, we can write

$$\bigsqcup_{\alpha\in A} X_\alpha = \bigcup_{\alpha\in A} (X_\alpha \times \{\alpha\}).$$

For each $\beta \in A$, we have a natural injection

$$\mathrm{in}_\beta : X_\beta \to \bigsqcup_{\alpha\in A} X_\alpha : x \mapsto (x, \beta).$$

If only two sets X and Y are involved and they are distinct, then we can avoid indices and define the sum by setting

$$X \sqcup Y = \{(x, X) \mid x \in X\} \cup \{(y, Y) \mid y \in Y\}.$$

21°12. Sums of Spaces

21.K. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces. Then the collection of subsets of $\bigsqcup_{\alpha\in A} X_\alpha$ whose preimages under all inclusions in_α, $\alpha\in A$, are open is a topological structure.

The sum $\bigsqcup_{\alpha\in A} X_\alpha$ with this topology is the (*disjoint*) *sum of the topological spaces* X_α ($\alpha\in A$).

21.L. The topology described in *21.K* is the finest topology with respect to which all inclusions in_α are continuous.

21.17. The maps $\mathrm{in}_\beta : X_\beta \to \bigsqcup_{\alpha\in A} X_\alpha$ are topological embedding, and their images are both open and closed in $\bigsqcup_{\alpha\in A} X_\alpha$.

21.18. Which of the standard topological properties are inherited from summands X_α by the sum $\bigsqcup_{\alpha\in A} X_\alpha$? Which are not?

21°13. Attaching Space

Let X and Y be two spaces, A a subset of Y, and $f : A\to X$ a continuous map. The quotient space $X\cup_f Y = (X\sqcup Y)/_{[a\sim f(a) \text{ for } a\in A]}$ is said to be the result of *attaching* or *gluing* the space Y to the space X via f. The map f is the *attaching map*.

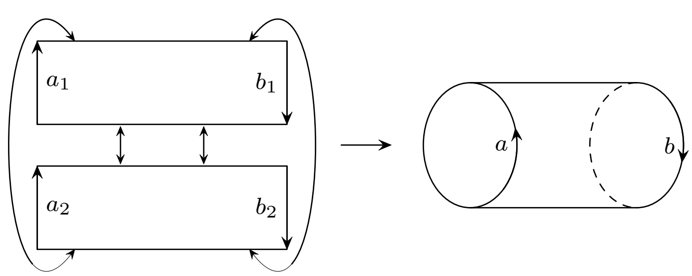
Here the partition of $X\sqcup Y$ consists of one-point subsets of $\mathrm{in}_2(Y\smallsetminus A)$ and $\mathrm{in}_1(X\smallsetminus f(A))$, and sets $\mathrm{in}_1(x)\cup \mathrm{in}_2\big(f^{-1}(x)\big)$ with $x\in f(A)$.

21.19. Prove that the composition of inclusion $X\to X\sqcup Y$ and projection $X\sqcup Y\to X\cup_f Y$ is a topological embedding.

21.20. Prove that if X is a point, then $X\cup_f Y$ is $Y/_A$.

21.M. Prove that attaching the n-disk D^n to its copy via the identity map of the boundary sphere S^{n-1} we obtain a space homeomorphic to S^n.

21.21. Prove that the Klein bottle is a result of gluing together two copies of the Möbius strip via the identity map of the boundary circle.



21.22. Prove that the result of gluing together two copies of a cylinder via the identity map of the boundary circles (of one copy to the boundary circles of the other) is homeomorphic to $S^1\times S^1$.

21.23. Prove that the result of gluing together two copies of the solid torus $S^1 \times D^2$ via the identity map of the boundary torus $S^1 \times S^1$ is homeomorphic to $S^1 \times S^2$.

21.24. Obtain the Klein bottle by gluing two copies of the cylinder $S^1 \times I$ to each other.

21.25. Prove that the result of gluing together two copies of the solid torus $S^1 \times D^2$ via the map

$$S^1 \times S^1 \to S^1 \times S^1 : (x, y) \mapsto (y, x)$$

of the boundary torus to its copy is homeomorphic to S^3.

21.N. Let X and Y be two spaces, A a subset of Y, and $f, g : A \to X$ two continuous maps. Prove that if there exists a homeomorphism $h : X \to X$ such that $h \circ f = g$, then $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

21.O. Prove that $D^n \cup_h D^n$ is homeomorphic to S^n for any homeomorphism $h : S^{n-1} \to S^{n-1}$.

21.26. Classify up to homeomorphism those spaces which can be obtained from a square by identifying a pair of opposite sides by a homeomorphism.

21.27. Classify up to homeomorphism the spaces that can be obtained from two copies of $S^1 \times I$ by identifying the copies of $S^1 \times \{0, 1\}$ by a homeomorphism.

21.28. Prove that the topological type of the space resulting from gluing together two copies of the Möbius strip via a homeomorphism of the boundary circle does not depend on the homeomorphism.

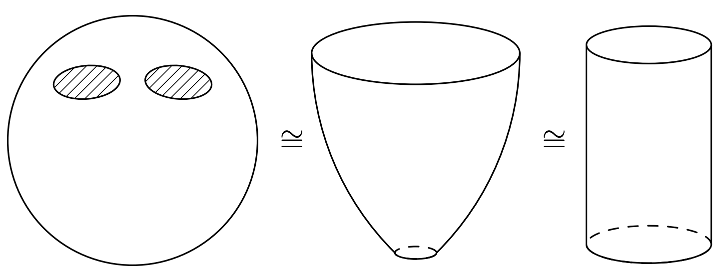
21.29. Classify up to homeomorphism the spaces that can be obtained from $S^1 \times I$ by identifying $S^1 \times 0$ and $S^1 \times 1$ via a homeomorphism.

21°14. Basic Surfaces

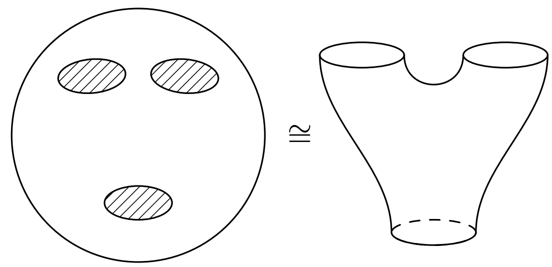
A torus $S^1 \times S^1$ with the interior of an embedded disk deleted is a *handle*. A two-sphere with the interior of n disjoint embedded disks deleted is a *sphere with n holes*.

21.P. A sphere with a hole is homeomorphic to the disk D^2.

21.Q. A sphere with two holes is homeomorphic to the cylinder $S^1 \times I$.



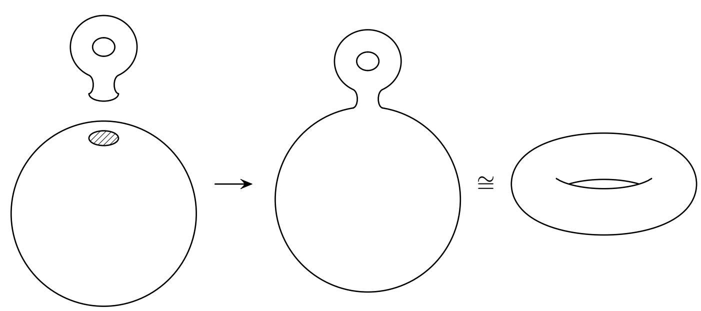
A sphere with three holes has a special name. It is called *pantaloons* or just *pants* .



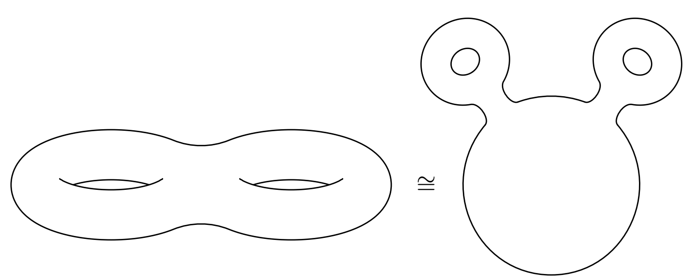
The result of attaching p copies of a handle to a sphere with p holes via embeddings homeomorphically mapping the boundary circles of the handles onto those of the holes is a ***sphere with p handles***, or, in a more ceremonial way (and less understandable, for a while), an ***orientable connected closed surface of genus p***.

> ***21.30.*** Prove that a sphere with p handles is well defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

21.R. A sphere with one handle is homeomorphic to the torus $S^1 \times S^1$.



21.S. A sphere with two handles is homeomorphic to the result of gluing together two copies of a handle via the identity map of the boundary circle.



A sphere with two handles is a ***pretzel***. Sometimes, this word also denotes a sphere with more handles.

The space obtained from a sphere with q holes by attaching q copies of the Möbius strip via embeddings of the boundary circles of the Möbius strips onto the boundary circles of the holes (the boundaries of the holes) is a *sphere with q crosscaps*, or a *nonorientable connected closed surface of genus q*.

> ***21.31.*** Prove that a sphere with q crosscaps is well defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

21.T. A sphere with a crosscap is homeomorphic to the projective plane.

21.U. A sphere with two crosscaps is homeomorphic to the Klein bottle.

A sphere, spheres with handles, and spheres with crosscaps are *basic surfaces*.

21.V. Prove that a sphere with p handles and q crosscaps is homeomorphic to a sphere with $2p + q$ crosscaps (here $q > 0$).

> ***21.32.*** Classify up to homeomorphism those spaces which are obtained by attaching p copies of $S^1 \times I$ to a sphere with $2p$ holes via embeddings of the boundary circles of the cylinders onto the boundary circles of the sphere with holes.

22. Projective Spaces

This section can be considered as a continuation of the previous one. The quotient spaces described here are of too great importance to regard them just as examples of quotient spaces.

22°1. Real Projective Space of Dimension n

This space is defined as the quotient space of the sphere S^n by the partition into pairs of antipodal points, and denoted by $\mathbb{R}P^n$.

22.A. *The space $\mathbb{R}P^n$ is homeomorphic to the quotient space of the n-disk D^n by the partition into one-point subsets of the interior of D^n, and pairs of antipodal point of the boundary sphere S^{n-1}.*

22.B. $\mathbb{R}P^0$ is a point.

22.C. The space $\mathbb{R}P^1$ is homeomorphic to the circle S^1.

22.D. The space $\mathbb{R}P^2$ is homeomorphic to the projective plane defined in the previous section.

22.E. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the quotient space of $\mathbb{R}^{n+1} \smallsetminus 0$ by the partition into one-dimensional vector subspaces of $\mathbb{R}^{n+1}$ punctured at 0.*

A point of the space $\mathbb{R}^{n+1} \smallsetminus 0$ is a sequence of real numbers, which are not all zeros. These numbers are the *homogeneous coordinates* of the corresponding point of $\mathbb{R}P^n$. The point with homogeneous coordinates x_0, x_1, …, x_n is denoted by $(x_0 : x_1 : \dots : x_n)$. Homogeneous coordinates determine a point of $\mathbb{R}P^n$, but are not determined by this point: proportional vectors of coordinates $(x_0, x_1, \dots, x_n)$ and $(\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ determine the same point of $\mathbb{R}P^n$.

22.F. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the metric space, whose points are lines of $\mathbb{R}^{n+1}$ through the origin $0 = (0, \dots, 0)$ and the metric is defined as the angle between lines (which takes values in $[0, \frac{\pi}{2}]$). Prove that this is really a metric.*

22.G. Prove that the map

$$i : \mathbb{R}^n \to \mathbb{R}P^n : (x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$$

is a topological embedding. What is its image? What is the inverse map of its image onto $\mathbb{R}^n$?

22.H. Construct a topological embedding $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$ with image $\mathbb{R}P^n \smallsetminus i(\mathbb{R}^n)$, where i is the embedding from Problem *22.G*.

Therefore the projective space $\mathbb{R}P^n$ can be considered as the result of extending $\mathbb{R}^n$ by adjoining "improper" or "infinite" points, which constitute a projective space $\mathbb{R}P^{n-1}$.

22.1. Introduce a natural topological structure in the set of all lines on the plane and prove that the resulting space is homeomorphic to a) $\mathbb{R}P^2 \smallsetminus \{\text{pt}\}$; b) open Möbius strip (i.e., a Möbius strip with the boundary circle removed).

22.2. Prove that the set of all rotations of the space $\mathbb{R}^3$ around lines passing through the origin equipped with the natural topology is homeomorphic to $\mathbb{R}P^3$.

22°2x. Complex Projective Space of Dimension n

This space is defined as the quotient space of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ by the partition into circles cut by (complex) lines of $\mathbb{C}^{n+1}$ passing through the point 0. It is denoted by $\mathbb{C}P^n$.

22.Ax. $\mathbb{C}P^n$ is homeomorphic to the quotient space of the unit $2n$-disk D^{2n} of the space $\mathbb{C}^n$ by the partition whose elements are one-point subsets of the interior of D^{2n} and circles cut on the boundary sphere S^{2n-1} by (complex) lines of $\mathbb{C}^n$ passing through the origin $0 \in \mathbb{C}^n$.

22.Bx. $\mathbb{C}P^0$ is a point.

The space $\mathbb{C}P^1$ is a *complex projective line*.

22.Cx. The complex projective line $\mathbb{C}P^1$ is homeomorphic to S^2.

22.Dx. The space $\mathbb{C}P^n$ is canonically homeomorphic to the quotient space of the space $\mathbb{C}^{n+1} \smallsetminus 0$ by the partition into complex lines of $\mathbb{C}^{n+1}$ punctured at 0.

Hence, $\mathbb{C}P^n$ can be regarded as the space of complex-proportional nonzero complex sequences $(x_0, x_1, \dots, x_n)$. The notation $(x_0 : x_1 : \dots : x_n)$ and term homogeneous coordinates introduced for the real case are used in the same way for the complex case.

22.Ex. The space $\mathbb{C}P^n$ is canonically homeomorphic to the metric space, whose points are the (complex) lines of $\mathbb{C}^{n+1}$ passing through the origin 0, and the metric is defined as the angle between lines (which takes values in $[0, \frac{\pi}{2}]$).

22°3x. Quaternionic Projective Spaces

Recall that $\mathbb{R}^4$ bears a remarkable multiplication, which was discovered by R. W. Hamilton in 1843. It can be defined by the formula

$$\begin{aligned}(x_1, x_1, x_3, x_4) \times (y_1, y_2, y_3, y_4) = \\ (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4, \quad x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3, \\ x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2, \quad x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1).\end{aligned}$$

It is bilinear, and to describe it in a shorter way it suffices to specify the products of the basis vectors. The latter are traditionally denoted in this case, following Hamilton, as follows:

$$1 = (1, 0, 0, 0), \quad i = (0, 1, 0, 0), \quad j = (0, 0, 1, 0) \quad \text{and} \quad k = (0, 0, 0, 1).$$

In this notation, 1 is really a unity: $(1, 0, 0, 0) \times x = x$ for any $x \in \mathbb{R}^4$. The rest of multiplication table looks as follows:

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i \quad \text{and} \quad ik = -j.$$

Together with coordinate-wise addition, this multiplication determines a structure of algebra in $\mathbb{R}^4$. Its elements are *quaternions*.

22.Fx. Check that the quaternion multiplication is associative.

It is not commutative (e.g., $ij = k \neq -k = ji$). Otherwise, quaternions are very similar to complex numbers. As in $\mathbb{C}$, there is a transformation called *conjugation* acting in the set of quaternions. As the conjugation of complex numbers, it is also denoted by a bar: $x \mapsto \overline{x}$. It is defined by the formula $(x_1, x_2, x_3, x_4) \mapsto (x_1, -x_2, -x_3, -x_4)$ and has two remarkable properties:

22.Gx. We have $\overline{ab} = \overline{b}\,\overline{a}$ for any two quaternions a and b.

22.Hx. We have $a\overline{a} = |a|^2$, i.e., the product of any quaternion a by the conjugate quaternion $\overline{a}$ equals $(|a|^2, 0, 0, 0)$.

The latter property allows us to define, for any $a \in \mathbb{R}^4$, the inverse quaternion

$$a^{-1} = |a|^{-2}\overline{a}$$

such that $aa^{-1} = 1$.

Hence, the quaternion algebra is a *division algebra* or a *skew field*. It is denoted by $\mathbb{H}$ after Hamilton, who discovered it.

In the space $\mathbb{H}^n = \mathbb{R}^{4n}$, there are right quaternionic lines, i.e., subsets $\{(a_1\xi, \dots, a_n\xi) \mid \xi \in \mathbb{H}\}$, and similar left quaternionic lines $\{(\xi a_1, \dots, \xi a_n) \mid \xi \in \mathbb{H}\}$. Each of them is a real 4-dimensional subspace of $\mathbb{H}^n = \mathbb{R}^{4n}$.

***22.I*x.** Find a right quaternionic line that is not a left quaternionic line.

***22.J*x.** Prove that two right quaternionic lines in $\mathbb{H}^n$ either meet only at 0, or coincide.

The quotient space of the unit sphere S^{4n+3} of the space $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4}$ by the partition into its intersections with right quaternionic lines is the (*right*) *quaternionic projective space of dimension* n. Similarly, but with left quaternionic lines, we define the (*left*) *quaternionic projective space of dimension* n.

***22.K*x.** Are the right and left quaternionic projective space of the same dimension homeomorphic?

The left quaternionic projective space of dimension n is denoted by $\mathbb{H}P^n$.

***22.L*x.** $\mathbb{H}P^0$ consists of a single point.

***22.M*x.** $\mathbb{H}P^n$ is homeomorphic to the quotient space of the closed unit disk D^{4n} in $\mathbb{H}^n$ by the partition into points of the interior of D^{4n} and the 3-spheres that are intersections of the boundary sphere S^{4n-1} with (left quaternionic) lines of $\mathbb{H}^n$.

The space $\mathbb{H}P^1$ is the *quaternionic projective line*.

***22.N*x.** Quaternionic projective line $\mathbb{H}P^1$ is homeomorphic to S^4.

***22.O*x.** $\mathbb{H}P^n$ is canonically homeomorphic to the quotient space of $\mathbb{H}^{n+1} \setminus 0$ by the partition to left quaternionic lines of $\mathbb{H}^{n+1}$ passing through the origin and punctured at it.

Hence, $\mathbb{H}P^n$ can be presented as the space of classes of left proportional (in the quaternionic sense) nonzero sequences $(x_0, \dots, x_n)$ of quaternions. The notation $(x_0 : x_1 : \dots : x_n)$ and the term homogeneous coordinates introduced above in the real case are used in the same way in the quaternionic situation.

***22.P*x.** $\mathbb{H}P^n$ is canonically homeomorphic to the set of (left quaternionic) lines of $\mathbb{H}^{n+1}$ equipped with the topology generated by the angular metric (which takes values in $\left[0, \frac{\pi}{2}\right]$).

23x. Finite Topological Spaces

23°1x. Set-Theoretic Digression: Splitting a Transitive Relation Into Equivalence and Partial Order

In the definitions of equivalence and partial order relations, the condition of transitivity seems to be the most important. Below, we supply a formal justification of this feeling by showing that the other conditions are natural companions of transitivity, although they are not its consequences.

23.Ax. *Let $\prec$ be a transitive relation in a set X. Then the relation $\precsim$ defined by*

$$a \precsim b \text{ if } a \prec b \text{ or } a = b$$

is also transitive (and, furthermore, it is certainly reflexive, i.e., $a \precsim a$ for each $a \in X$).

A binary relation $\precsim$ in a set X is a ***preorder*** if it is transitive and reflective, i.e., satisfies the following conditions:

- *Transitivity*. If $a \precsim b$ and $b \precsim c$, then $a \precsim c$.
- *Reflexivity*. We have $a \precsim a$ for any a.

A set X equipped with a preorder is ***preordered***.

If a preorder is antisymmetric, then this is a nonstrict order.

23.1x. Is the relation $a|b$ a preorder in the set $\mathbb{Z}$ of integers?

23.Bx. *If $(X, \precsim)$ is a preordered set, then the relation $\sim$ defined by*

$$a \sim b \text{ if } a \precsim b \text{ and } b \precsim a$$

is an equivalence relation (i.e., it is symmetric, reflexive, and transitive) in X.

23.2x. What equivalence relation is defined in $\mathbb{Z}$ by the preorder $a|b$?

23.Cx. *Let $(X, \precsim)$ be a preordered set and $\sim$ be an equivalence relation defined in X by $\precsim$ according to 23.Bx. Then $a' \sim a$, $a \precsim b$ and $b \sim b'$ imply $a' \precsim b'$ and in this way $\precsim$ determines a relation in the set of equivalence classes $X/_\sim$. This relation is a nonstrict partial order.*

Thus any transitive relation generates an equivalence relation and a partial order in the set of equivalence classes.

23.Dx. How this chain of constructions would degenerate if the original relation was

(1) an equivalence relation, or

(2) nonstrict partial order?

23.Ex. *In any topological space, the relation $\precsim$ defined by*

$$a \precsim b \text{ if } a \in \mathrm{Cl}\{b\}$$

is a preorder.

23.3x. In the set of all subsets of an arbitrary topological space the relation

$$A \precsim B \text{ if } A \subset \mathrm{Cl}\, B$$

is a preorder. This preorder determines the following equivalence relation: sets are equivalent iff they have the same closure.

23.Fx. *The equivalence relation defined by the preorder of Theorem 23.Ex determines the partition of the space into maximal (with respect to inclusion) indiscrete subspaces. The quotient space satisfies the Kolmogorov separation axiom T_0.*

The quotient space of Theorem *23.Fx* is the *maximal T_0-quotient of X*.

23.Gx. A continuous image of an indiscrete space is indiscrete.

23.Hx. Prove that any continuous map $X \to Y$ induces a continuous map of the maximal T_0-quotient of X to the maximal T_0-quotient of Y.

23°2x. The Structure of Finite Topological Spaces

The results of the preceding subsection provide a key to understanding the structure of finite topological spaces. Let X be a finite space. By Theorem *23.Fx*, X is partitioned to indiscrete clusters of points. By *23.Gx*, continuous maps between finite spaces respect these clusters and, by *23.Hx*, induce continuous maps between the maximal T_0-quotient spaces.

This means that we can consider a finite topological space as its maximal T_0-quotient whose points are equipped with multiplicities, that are positive integers: the numbers of points in the corresponding clusters of the original space.

The maximal T_0-quotient of a finite space is a smallest neighborhood space (as a finite space). By Theorem *14.O*, its topology is determined by a partial order. By Theorem *9.Bx*, homeomorphisms between spaces with poset topologies are monotone bijections.

Thus, a finite topological space is characterized up to homeomorphism by a finite poset whose elements are equipped with multiplicities (positive integers). Two such spaces are homeomorphic iff there exists a monotone bijection between the corresponding posets that preserves the multiplicities. To recover the topological space from the poset with multiplicities, we must equip the poset with the poset topology and then replace each of its elements by an indiscrete cluster of points, the number points in which is the multiplicity of the element.

23°3x. Simplicial schemes

Let V be a set, Σ a set of some of subsets of V. A pair (V, Σ) is a *simplicial scheme* with set of *vertices* V and set of *simplices* Σ if

- each subset of any element of Σ belongs to Σ,
- the intersection of any collection of elements of Σ belongs to Σ,
- each one-element subset of V belongs to Σ.

The set Σ is partially ordered by inclusion. When equipped with the poset topology of this partial order, it is called *the space of simplices* of the simplicial scheme (X, Σ).

A simplicial scheme gives rise also to another topological space. Namely, for a simplicial scheme (V, Σ) consider the set $S(V, \Sigma)$ of all functions $c : V \to [0, 1]$ such that

$$\operatorname{Supp}(c) = \{v \in V \mid c(v) \neq 0\} \in \Sigma$$

and $\sum_{v \in V} c(v) = 1$. Equip $S(V, \Sigma)$ with the topology generated by metric

$$\rho(c_1, c_2) = \sup_{v \in V} |c_1(v) - c_2(v)|.$$

The space $S(V, \Sigma)$ is a *simplicial* or *triangulated* space. It is covered by the sets $\{c \in S \mid \operatorname{Supp}(c) = \sigma\}$, where $\sigma \in \Sigma$, which are called its (*open*) *simplices*.

23.4x. Which open simplices of a simplicial space are open sets, which are closed, and which are neither closed nor open?

23.Ix. For each $\sigma \in \Sigma$, find a homeomorphism of the space

$$\{c \in S \mid \operatorname{Supp}(c) = \sigma\} \subset S(V, \Sigma)$$

onto an open simplex whose dimension is one less than the number of vertices belonging to σ. (Recall that the open n-simplex is the set $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_j > 0 \text{ for } j = 1, \dots, n+1 \text{ and } \sum_{i=1}^{n+1} x_i = 1\}$.)

23.Jx. Prove that for any simplicial scheme (V, Σ) the quotient space of the simplicial space $S(V, \Sigma)$ by its partition to open simplices is homeomorphic to the space Σ of simplices of the simplicial scheme (V, Σ).

23°4x. Barycentric Subdivision of a Poset

23.Kx. Find a poset which is not isomorphic to the set of simplices (ordered by inclusion) of whatever simplicial scheme.

Let $(X, \prec)$ be a poset. Consider the set X' of all nonempty finite strictly increasing sequences $a_1 \prec a_2 \prec \dots \prec a_n$ of elements of X. It can also be described as the set of all nonempty finite subsets of X in each of which $\prec$ determines a linear order. It is naturally ordered by inclusion.

The poset $(X', \subset)$ is the *barycentric subdivision of* $(X, \prec)$.

***23.L*x.** For any poset $(X, \prec)$, pair (X, X') is a simplicial scheme.

There is a natural map $X' \to X$ that maps an element of X' (i.e., a nonempty finite linearly ordered subset of X) to its greatest element.

***23.M*x.** Is this map monotone? Strictly monotone? The same questions concerning a similar map that maps a nonempty finite linearly ordered subset of X to its smallest element.

Let (V, Σ) be a simplicial scheme and Σ' be the barycentric subdivision of Σ (ordered by inclusion). The simplicial scheme (Σ, Σ') is the *barycentric subdivision* of the simplicial scheme (V, Σ).

There is a natural mapping $\Sigma \to S(V, \Sigma)$ that maps a simplex $\sigma \in \Sigma$ (i.e., a subset $\{v_0, v_1, \dots, v_n\}$ of V) to the function $b_\sigma : V \to \mathbb{R}$ with $b_\sigma(v_i) = \frac{1}{n+1}$ and $b_\sigma(v) = 0$ for any $v \notin \sigma$.

Define a map $\beta : S(\Sigma, \Sigma') \to S(V, \Sigma)$ that maps a function $\varphi : \Sigma \to \mathbb{R}$ to the function

$$V \to \mathbb{R} : v \mapsto \sum_{\sigma \in \Sigma} \varphi(\sigma) b_\sigma(v).$$

***23.N*x.** Prove that the map $\beta : S(\Sigma, \Sigma') \to S(V, \Sigma)$ is a homeomorphism and constitutes, together with projections $S(V, \Sigma) \to \Sigma$ and $S(\Sigma, \Sigma') \to \Sigma'$ and the natural map $\Sigma' \to \Sigma$ a commutative diagram

$$\begin{array}{ccc} S(\Sigma, \Sigma') & \xrightarrow{\beta} & S(V, \Sigma) \\ \downarrow & & \downarrow \\ \Sigma' & \longrightarrow & \Sigma \end{array}$$

24x. Spaces of Continuous Maps

24°1x. Sets of Continuous Mappings

By $\mathcal{C}(X, Y)$ we denote the set of all continuous maps of a space X to a space Y.

24.1x. Let X be non empty. Prove that $\mathcal{C}(X, Y)$ consists of a single element iff so does Y.

24.2x. Let X be non empty. Prove that there exists an injection $Y \to \mathcal{C}(X, Y)$. In other words, the cardinality $\operatorname{card} \mathcal{C}(X, Y)$ of $\mathcal{C}(X, Y)$ is greater than or equal to $\operatorname{card} Y$.

24.3x. Riddle. Find natural conditions implying that $\mathcal{C}(X, Y) = Y$.

24.4x. Let $Y = \{0, 1\}$ equipped with topology $\{\varnothing, \{0\}, Y\}$. Prove that there exists a bijection between $\mathcal{C}(X, Y)$ and the topological structure of X.

24.5x. Let X be a set of n points with discrete topology. Prove that $\mathcal{C}(X, Y)$ can be identified with $Y \times \dots \times Y$ (n times).

24.6x. Let Y be a set of k points with discrete topology. Find necessary and sufficient condition for the set $\mathcal{C}(X, Y)$ contain k^2 elements.

24°2x. Topologies on Set of Continuous Mappings

Let X and Y be two topological spaces, $A \subset X$, and $B \subset Y$. We define $W(A, B) = \{f \in \mathcal{C}(X, Y) \mid f(A) \subset B\}$,

$$\Delta^{(pw)} = \{W(a, U) \mid a \in X,\ U \text{ is open in } Y\},$$

and

$$\Delta^{(co)} = \{W(C, U) \mid C \subset X \text{ is compact},\ U \text{ is open in } Y\}.$$

24.Ax. $\Delta^{(pw)}$ is a subbase of a topological structure on $\mathcal{C}(X, Y)$.

The topological structure generated by $\Delta^{(pw)}$ is the *topology of pointwise convergence*. The set $\mathcal{C}(X, Y)$ equipped with this structure is denoted by $\mathcal{C}^{(pw)}(X, Y)$.

24.Bx. $\Delta^{(co)}$ is a subbase of a topological structures on $\mathcal{C}(X, Y)$.

The topological structure determined by $\Delta^{(co)}$ is the *compact-open topology*. Hereafter we denote by $\mathcal{C}(X, Y)$ the space of all continuous maps $X \to Y$ with the compact-open topology, unless the contrary is specified explicitly.

24.Cx Compact-Open Versus Pointwise. The compact-open topology is finer than the topology of pointwise convergence.

24.7x. Prove that $\mathcal{C}(I,I)$ is not homeomorphic to $\mathcal{C}^{(pw)}(I,I)$.

Denote by $Const(X,Y)$ the set of all constant maps $f: X \to Y$.

24.8x. Prove that the topology of pointwise convergence and the compact-open topology of $\mathcal{C}(X,Y)$ induce the same topological structure on $Const(X,Y)$, which, with this topology, is homeomorphic Y.

24.9x. Let X be a discrete space of n points. Prove that $\mathcal{C}^{(pw)}(X,Y)$ is homeomorphic $Y \times \ldots \times Y$ (n times). Is this true for $\mathcal{C}(X,Y)$?

24°3x. Topological Properties of Mapping Spaces

24.Dx. Prove that if Y is Hausdorff, then $\mathcal{C}^{(pw)}(X,Y)$ is Hausdorff for any space X. Is this true for $\mathcal{C}(X,Y)$?

24.10x. Prove that $\mathcal{C}(I,X)$ is path connected iff X is path connected.

24.11x. Prove that $\mathcal{C}^{(pw)}(I,I)$ is not compact. Is the space $\mathcal{C}(I,I)$ compact?

24°4x. Metric Case

24.Ex. If Y is metrizable and X is compact, then $\mathcal{C}(X,Y)$ is metrizable.

Let (Y,ρ) be a metric space and X a compact space. For continuous maps $f,g: X \to Y$ put

$$d(f,g) = \max\{\rho(f(x),g(x)) \mid x \in X\}.$$

24.Fx This is a Metric. If X is a compact space and Y a metric space, then d is a metric on the set $\mathcal{C}(X,Y)$.

Let X be a topological space, Y a metric space with metric ρ. A sequence f_n of maps $X \to Y$ *uniformly converges* to $f: X \to Y$ if for any $\varepsilon > 0$ there exists a positive integer N such that $\rho(f_n(x), f(x)) < \varepsilon$ for any $n > N$ and $x \in X$. This is a straightforward generalization of the notion of uniform convergence which is known from Calculus.

24.Gx Metric of Uniform Convergence. Let X be a compact space, (Y,d) a metric space. A sequence f_n of maps $X \to Y$ converges to $f: X \to Y$ in the topology generated by d iff f_n uniformly converges to f.

24.Hx Completeness of $\mathcal{C}(X,Y)$. Let X be a compact space, (Y,ρ) a complete metric space. Then $\big(\mathcal{C}(X,Y), d\big)$ is a complete metric space.

24.Ix Uniform Convergence Versus Compact-Open. Let X be a compact space and Y a metric space. Then the topology generated by d on $\mathcal{C}(X,Y)$ is the compact-open topology.

24.12x. Prove that the space $\mathcal{C}(\mathbb{R},I)$ is metrizable.

24.13x. Let Y be a bounded metric space, X a topological space admitting a presentation $X = \bigcup_{i=1}^{\infty} X_i$, where X_i is compact and $X_i \subset \operatorname{Int} X_{i+1}$ for each $i = 1, 2, \ldots$. Prove that $\mathcal{C}(X,Y)$ is metrizable.

Denote by $\mathcal{C}_b(X,Y)$ the set of all continuous bounded maps from a topological space X to a metric space Y. For maps $f,g \in \mathcal{C}_b(X,Y)$, put

$$d^\infty(f,g) = \sup\{\rho(f(x),g(x)) \mid x \in X\}.$$

24.Jx Metric on Bounded Maps. This is a metric in $\mathcal{C}_b(X,Y)$.

24.Kx d^∞ ***and Uniform Convergence.*** Let X be a topological space and Y a metric space. A sequence f_n of bounded maps $X \to Y$ converges to $f : X \to Y$ in the topology generated by d^∞ iff f_n uniformly converge to f.

24.Lx When Uniform Is Not Compact-Open. Find X and Y such that the topology generated by d^∞ on $\mathcal{C}_b(X,Y)$ is not the compact-open topology.

24°5x. Interactions With Other Constructions

24.Mx. For any continuous maps $\varphi : X' \to X$ and $\psi : Y \to Y'$ the map $\mathcal{C}(X,Y) \to \mathcal{C}(X',Y') : f \mapsto \psi \circ f \circ \varphi$ is continuous.

24.Nx Continuity of Restricting. Let X and Y be two spaces, $A \subset X$. Prove that the map $\mathcal{C}(X,Y) \to \mathcal{C}(A,Y) : f \mapsto f|_A$ is continuous.

24.Ox Extending Target. For any spaces X and Y and any $B \subset Y$, the map $\mathcal{C}(X,B) \to \mathcal{C}(X,Y) : f \mapsto i_B \circ f$ is a topological embedding.

24.Px Maps to Product. For any three spaces X, Y, and Z, the space $\mathcal{C}(X, Y \times Z)$ is canonically homeomorphic to $\mathcal{C}(X,Y) \times \mathcal{C}(X,Z)$.

24.Qx Restricting to Sets Covering Source. Let $\{X_1, \dots, X_n\}$ be a closed cover of X. Prove that for any space Y

$$\phi : \mathcal{C}(X,Y) \to \prod_{i=1}^{n} \mathcal{C}(X_i,Y) \ : \ f \mapsto (f|_{X_1}, \dots, f|_{X_n})$$

is a topological embedding. What if the cover is not fundamental?

24.Rx. Riddle. Can you generalize assertion *24.Qx*?

24.Sx Continuity of Composing. Let X be a space and Y a locally compact Hausdorff space. Prove that the map

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \ : \ (f,g) \mapsto g \circ f$$

is continuous.

24.14x. Is local compactness of Y necessary in *24.Sx*?

24.Tx Factorizing Source. Let S be a closed partition[2] of a Hausdorff compact space X. Prove that for any space Y the map

$$\phi : \mathcal{C}(X/S, Y) \to \mathcal{C}(X, Y)$$

is a topological embedding.

> ***24.15x.*** Are the conditions imposed on S and X in *24.Tx* necessary?

24.Ux The Evaluation Map. Let X and Y be two spaces. Prove that if X is locally compact and Hausdorff, then the map

$$\phi : \mathcal{C}(X, Y) \times X \to Y \ : \ (f, x) \mapsto f(x)$$

is continuous.

> ***24.16x.*** Are the conditions imposed on X in *24.Ux* necessary?

24°6x. Mappings $X \times Y \to Z$ and $X \to \mathcal{C}(Y, Z)$

24.Vx. Let X, Y, and Z be three topological spaces, $f : X \times Y \to Z$ a continuous map. Then the map

$$F : X \to \mathcal{C}(Y, Z) \ : \ F(x) : y \mapsto f(x, y),$$

is continuous.

The converse assertion is also true under certain additional assumptions.

24.Wx. Let X and Z be two spaces, Y a Hausdorff locally compact space, $F : X \to \mathcal{C}(Y, Z)$ a continuous map. Then the map $f : X \times Y \to Z : (x, y) \mapsto F(x)(y)$ is continuous.

24.Xx. If X is a Hausdorff space and the collection $\Sigma_Y = \{U_\alpha\}$ is a subbase of the topological structure of Y, then the collection $\{W(K, U) \mid U \in \Sigma\}$ is a subbase of the compact-open topology in $\mathcal{C}(X, Y)$.

24.Yx. Let X, Y, and Z be three spaces. Let

$$\Phi : \mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$$

be defined by the relation

$$\Phi(f)(x) : y \mapsto f(x, y).$$

Then

(1) if X is a Hausdorff space, then Φ is continuous;
(2) if X is a Hausdorff space, while Y is locally compact and Hausdorff, then Φ is a homeomorphism.

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[2] Recall that a partition is *closed* if the saturation of each closed set is closed.

24.Zx. *Let S be a partition of a space X, and let* $\mathrm{pr} : X \to X/_S$ *be the projection. The space $X \times Y$ bears a natural partition $S' = \{A \times y \mid A \in S,\ y \in Y\}$. If the space Y is Hausdorff and locally compact, then the natural quotient map $f : (X \times Y)/_{S'} \to X/_S \times Y$ of the projection* $\mathrm{pr} \times \mathrm{id}_Y$ *is a homeomorphism.*

24.17x. Try to prove Theorem *24.Zx* directly.

Proofs and Comments

19.A For example, let us prove the second relation:

$$(x, y) \in (A_1 \times B_1) \cap (A_2 \times B_2) \iff x \in A_1,\ y \in B_1,\ x \in A_2,\ y \in B_2$$
$$\iff x \in A_1 \cap A_2,\ y \in B_1 \cap B_2 \iff (x, y) \in (A_1 \cap A_2) \times (B_1 \cap B_2).$$

19.B Indeed,

$$\mathrm{pr}_X^{-1}(A) = \{z \in X \times Y \mid \mathrm{pr}_X(z) \in A\} = \{(x, y) \in X \times Y \mid x \in A\} = A \times Y.$$

19.C $\boxed{\Longrightarrow}$ Indeed, $\Gamma_f \cap (x \times Y) = \{(x, f(x))\}$ is a singleton.
$\boxed{\Longleftarrow}$ If $\Gamma \cap (x \times Y)$ is a singleton $\{(x, y)\}$, then we can put $f(x) = y$.

19.D This follows from *3.A* because the intersection of elementary sets is an elementary set.

19.E Verify that $X \times Y \to Y \times X : (x, y) \mapsto (y, x)$ is a homeomorphism.

19.F In view of a canonical bijection, we can identify two sets and write

$$(X \times Y) \times Z = X \times (Y \times Z) = \{(x, y, z) \mid x \in X, y \in Y, z \in Z\}.$$

However, elementary sets in the spaces $(X \times Y) \times Z$ and $X \times (Y \times Z)$ are different. Check that the collection $\{U \times V \times W \mid U \in \Omega_X,\ V \in \Omega_Y,\ W \in \Omega_Z\}$ is a base of the topological structures in both spaces.

19.G Indeed, for each open set $U \subset X$ the preimage $\mathrm{pr}_X^{-1}(U) = U \times Y$ is an elementary open set in $X \times Y$.

19.H Let Ω' be a topology in $X \times Y$ such that the projections pr_X and pr_Y are continuous. Then, for any $U \in \Omega_X$ and $V \in \Omega_Y$, we have

$$\mathrm{pr}_X^{-1}(U) \cap \mathrm{pr}_Y^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V \in \Omega'.$$

Therefore, each base set of the product topology lies in Ω', whence it follows that Ω' contains the product topology of X and Y.

19.I Clearly, $\mathrm{ab}(\mathrm{pr}_X) : X \times y_0 \to X$ is a continuous bijection. To see that the inverse map is continuous, we must show that each set open in $X \times y_0$ as in a subspace of $X \times Y$ has the form $U \times y_0$. Indeed, if W is open in $X \times Y$, then

$$W \cap (X \times y_0) = \bigcup_\alpha (U_\alpha \times V_\alpha) \cap (X \times y_0) = \bigcup_{\alpha \,:\, y_0 \in V_\alpha} (U_\alpha \times y_0) = \Bigl(\bigcup_{\alpha \,:\, y_0 \in V_\alpha} U_\alpha\Bigr) \times y_0.$$

19.J From the point of view of set theory, we have $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$. The collection of open rectangles is a base of topology in $\mathbb{R}^1 \times \mathbb{R}^1$ (show this), therefore, the topologies in $\mathbb{R}^1 \times \mathbb{R}^1$ and $\mathbb{R}^2$ have one and the same base, and so they coincide. The second assertion is proved by induction and, in turn, implies the third one by *19.7*.

19.K Set $f(z) = (f_1(z), f_2(z))$. If $f(z) = (x, y) \in X \times Y$, then $x = (\operatorname{pr}_X \circ f)(z) = f_1(z)$. We similarly have $y = f_2(z)$.

19.L $\boxed{\Longrightarrow}$ The maps $f_1 = \operatorname{pr}_X \circ f$ and $f_2 = \operatorname{pr}_Y \circ f$ are continuous as compositions of continuous maps (use *19.G*).
$\boxed{\Longleftarrow}$ Recall the definition of the product topology and use *19.20*.

19.M Recall the definition of the product topology and use *19.22*.

19.N Let X and Y be two Hausdorff spaces, $(x_1, y_1), (x_2, y_2) \in X \times Y$ two distinct points. Let, for instance, $x_1 \neq x_2$. Since X is Hausdorff, x_1 and x_2 have disjoint neighborhoods: $U_{x_1} \cap U_{x_2} = \varnothing$. Then, e.g., $U_{x_1} \times Y$ and $U_{x_2} \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2) in $X \times Y$.

19.O If A and B are countable and dense in X and Y, respectively, then $A \times B$ is a dense countable set in $X \times Y$.

19.P See the proof of Theorem *19.Q* below.

19.Q If $\Sigma_X$and Σ_Y are countable bases in X and Y, respectively, then $\Sigma = \{U \times V \mid U \in \Sigma_X,\ V \in \Sigma_Y\}$ is a base in $X \times Y$ by *19.15*.

19.R Show that if ρ_1 and ρ_2 are metrics on X and Y, respectively, then $\rho\big((x_1, y_1), (x_2, y_2)\big) = \max\{\rho_1(x_1, x_2), \rho_2(y_1, y_2)\}$ is a metric in $X \times Y$ generating the product topology. What form have the balls in the metric space $(X \times Y, \rho)$?

19.S For any two points $(x_1, y_1), (x_2, y_2) \in X \times Y$, the set $(X \times y_2) \cup (x_1 \times Y)$ is connected and contains these points.

19.T If u are v are paths joining x_1 with x_2 and y_1 with y_2, respectively, then the path $u \times v$ joins (x_1, y_1) with (x_2, y_2).

19.U It is sufficient to consider a cover consisting of elementary sets. Since Y is compact, each fiber $x \times Y$ has a finite subcovering $\{U_i^x \times V_i^x\}$. Put $W^x = \cap U_i^x$. Since X is compact, the cover $\{W^x\}_{x \in X}$ has a finite subcovering W^{x_j}. Then $\{U_i^{x_j} \times V_i^{x_j}\}$ is the required finite subcovering.

19.V Consider the map $(x, y) \mapsto \Big(\big(\frac{x}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\big), \ln(\sqrt{x^2+y^2}) \Big)$.

20.A First, the preimage $\operatorname{pr}^{-1}\big(\operatorname{pr}(A)\big)$ is saturated, secondly, it is the least because if $B \supset A$ is a saturated set, then $B = \operatorname{pr}^{-1}\big(\operatorname{pr}(B)\big) \supset \operatorname{pr}^{-1}\big(\operatorname{pr}(A)\big)$.

20.C Put $\Omega' = \{U \subset X/_S \mid \operatorname{pr}^{-1}(U) \in \Omega\}$. Let $U_\alpha \in \Omega'$. Since the sets $p^{-1}(U_\alpha)$ are open, the set $p^{-1}(\cup U_\alpha) = \cup p^{-1}(U_\alpha)$ is also open, whence

$\cup U_\alpha \in \Omega'$. Verify the remaining axioms of topological structure on your own.

20.D $\boxed{\Longrightarrow}$ If a set $V \subset X$ is open and saturated, then $V = \operatorname{pr}^{-1}\big(p(V)\big)$, hence, the set $U = \operatorname{pr}(V)$ is open in $X/_S$.

$\boxed{\Longleftarrow}$ Conversely, if $U \subset X/_S$ is open, then $U = \operatorname{pr}\big(\operatorname{pr}^{-1}(U)\big)$, where $V = \operatorname{pr}^{-1}(U)$ is open and saturated.

20.E The set F closed, iff $X/_S \smallsetminus F$ is open, iff $\operatorname{pr}^{-1}(X/_S \smallsetminus F) = X \smallsetminus \operatorname{pr}^{-1}(F)$ is open, iff $p^{-1}(F)$ is closed.

20.F This immediately follows from the definition of the quotient topology.

20.G We must prove that if Ω' is a topology in $X/_S$ such that the factorization map is continuous, then $\Omega' \subset \Omega_{X/S}$. Indeed, if $U \in \Omega'$, then $p^{-1}(U) \in \Omega_X$, whence $U \in \Omega_{X/S}$ by the definition of the quotient topology.

20.H It is connected as a continuous image of a connected space.

20.I It is path-connected as a continuous image of a path-connected space.

20.J It is separable as a continuous image of a separable space.

20.K It is compact as a continuous image of a compact space.

20.L This quotient space consists of two points, one of which is not open in it.

20.M $\boxed{\Longrightarrow}$ Let $a, b \in X/_S$, and let $A, B \subset X$ be the corresponding elements of the partition. If U_a and U_b are disjoint neighborhoods of a and b, then $p^{-1}(U_a)$ and $p^{-1}(U_b)$ are disjoint saturated neighborhoods of A and B. $\boxed{\Longleftarrow}$ This follows from *20.D*.

20.N 1) $\boxed{\Longrightarrow}$ Put $g = f/_S$. $\boxed{\Longleftarrow}$ The set $f^{-1}(y) = p^{-1}(g^{-1}(y))$ is saturated, i.e., it consists of elements of the partition S. Therefore, f is constant at each of the elements of the partition. 2) If A is an element of S, a is the point of the quotient set corresponding to A, and $x \in A$, then $f/_S(a) = f(A) = g(p(x)) = g(a)$.

20.O The map f maps elements of S to those of T iff there exists a map $g : X/_S \to Y/_T$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \operatorname{pr}_X}\big\downarrow & & {\scriptstyle \operatorname{pr}_Y}\big\downarrow \\ X/_S & \xrightarrow{g} & Y/_T \end{array}$$

is commutative. Then we have $f/_{(S,T)} = g$.

20.P This is so because distinct elements of the partition $S(f)$ are preimages of distinct points in Y.

20.Q Since $p^{-1}((f/_S)^{-1}(U)) = (f/_S \circ p)^{-1}(U) = f^{-1}(U)$, the definition of the quotient topology implies that for each $U \in \Omega_Y$ the set $(f/_S)^{-1}(U)$ is open, i.e., the map $f/_S$ is continuous.

20.R See *20.O* and *20.8*.

***20.A*x** Each singleton in $X/_S$ is the image of a singleton in X. Since X satisfies T_1, each singleton in X is closed, and its image, by *20.1*x, is also closed. Consequently, the quotient space also satisfies T_1.

***20.B*x** This follows from *14.25*.

***20.C*x** Let $U_n = p(V_n)$, $n \in \mathbb{N}$, where $\{V_n\}_{n\in\mathbb{N}}$ is a base X. Consider an open set W in the quotient space. Since $\mathrm{pr}^{-1}(W) = \bigcup_{n\in A} V_n$, we have $W = \mathrm{pr}\big(\mathrm{pr}^{-1}(W)\big) = \bigcup_{n\in A} U_n$, i.e., the collection $\{U_n\}$ is a base in the quotient space.

***20.D*x** For an arbitrary point $y \in X/_S$, consider the image of a countable neighborhood base at a certain point $x \in \mathrm{pr}^{-1}(y)$.

***20.E*x** Since the injective factor of a continuous surjection is a continuous bijection, it only remains to prove that the factor is an open map, which follows by *20.7* from the fact that the map $X \times Y \to X/_S \times Y/_T$ is open (see *19.23*).

21.A This follows from *20.P*, *20.Q*, *20.K*, and *16.Y*.

21.B Use *16.Z* instead of *16.Y*.

21.C.1 If $f : t \in [0,1] \mapsto (\cos 2\pi t, \sin 2\pi t) \in S^1$, then $f/_{S(f)}$ is a homeomorphism as a continuous bijection of a compact space onto a Hausdorff space, and the partition $S(f)$ is the initial one.

21.D.1 If $f : x \in \mathbb{R}^n \mapsto (\frac{x}{r}\sin \pi r, -\cos \pi r) \in S^n \subset \mathbb{R}^{n+1}$, then the partition $S(f)$ is the initial one and $f/_{S(f)}$ is a homeomorphism.

21.E Consider the map $g = f \times \mathrm{id} : I^2 = I \times I \to S^1 \times I$ (f is defined as in *21.C.1*). The partition $S(g)$ is the initial one, so that $g/_{S(g)}$ a homeomorphism.

21.F Check that the partition $S(\mathrm{id}_{S^1} \times f)$ is the initial one.

21.G The partition $S(f \times f)$ is the initial one.

21.H Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{p_1} & X/_S \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p_2}\big\downarrow \\ X/_T & \xrightarrow{q} & X/_S/_{S'} \end{array}$$

where the map q is obviously a bijection. The assertion of the problem follows from the fact that a set U is open in $X/_S/_{S'}$ iff $p_1^{-1}\big(p_2^{-1}(U)\big) = p^{-1}\big(q^{-1}(U)\big)$ is open in X iff $q^{-1}(U)$ is open in $X/_T$.

21.I To simplify the formulas, we replace the square I^2 ba a rectangle. Here is a formal argument: consider the map

$$\varphi : [0, 2\pi] \times [-\tfrac{1}{2}, \tfrac{1}{2}] \to \mathbb{R}^3 : (x, y) \mapsto \Big((1 + y \sin \tfrac{x}{2}) \cos x, (1 + y \sin \tfrac{x}{2}) \sin x, y \sin x\Big).$$

Check that φ really maps the square onto the Möbius strip and that $S(\varphi)$ is the given partition. Certainly, the starting point of the argument is not a specific formula. First of all, you should imagine the required map. We map the horizontal midline of the unit square onto the mid-circle of the Möbius strip, and we map each of the vertical segments of the square onto a segment of the strip orthogonal to the the mid-circle. This mapping maps the vertical sides of the square to one and the same segment, but here the opposite vertices of the square are identified with each other (check this).

21.J See the following section.

21.K Actually, it is easier to prove a more general assertion. Assume that we are given topological spaces X_α and maps $f_\alpha : X_\alpha \to Y$. Then $\Omega = \{U \subset Y \mid f_\alpha^{-1}(U) \text{ is open in } X_\alpha\}$ is the finest topological structure in Y with respect to which all maps f_α are continuous.

21.L See the hint to *21.K*.

21.M We map $D_1^n \sqcup D_2^n$ to S^n so that the images of D_1^n and D_2^n are the upper and the lower hemisphere, respectively. The partition into the preimages is the partition with quotient space $D^n \cup_{\mathrm{id}|_{S^{n-1}}} D^n$. Consequently, the corresponding quotient map is a homeomorphism.

21.N Consider the map $F : X \sqcup Y \to X \sqcup Y$ such that $F|_X = \mathrm{id}_X$ and $F|_Y = h$. This mapping maps an element of the partition corresponding to the equivalence relation $z \sim f(x)$ to an element of the partition corresponding to the equivalence relation $x \sim g(x)$. Consequently, there exists a continuous bijection $H : X \cup_f Y \to X \cup_g Y$. Since h^{-1} also is a homeomorphism, H^{-1} is also continuous.

21.O By *21.N*, it is sufficient to prove that any homeomorphism $f : S^{n-1} \to S^{n-1}$ can be extended to a homeomorphism $F : D^n \to D^n$, which is obvious.

21.P For example, the stereographic projection from an inner point of the hole maps the sphere with a hole onto a disk homeomorphically.

21.Q The stereographic projection from an inner point of one of the holes homeomorphically maps the sphere with two holes onto a "disk with a hole". Prove that the latter is homeomorphic to a cylinder. (Another option: if we take the center of the projection in the hole in an appropriate way, then the projection maps the sphere with two holes onto a circular ring, which is obviously homeomorphic to a cylinder.)

21.R By definition, the handle is homeomorphic to a torus with a hole, while the sphere with a hole is homeomorphic to a disk, which precisely fills in the hole.

21.S Cut a sphere with two handles into two symmetric parts each of which is homeomorphic to a handle.

21.T Combine the results of *21.P* *21.J*.

21.U Consider the Klein bottle as a quotient space of a square and cut the square into 5 horizontal (rectangular) strips of equal width. Then the quotient space of the middle strip will be a Möbius band, the quotient space of the union of the two extreme strips will be one more Möbius band, and the quotient space of the remaining two strips will be a ring, i.e., precisely a sphere with two holes. (Here is another, maybe more visual, description. Look at the picture of the Klein bottle: it has a horizontal plane of symmetry. Two horizontal planes close to the plane of symmetry cut the Klein bottle into two Möbius bands and a ring.)

21.V The most visual approach here is as follows: single out one of the handles and one of the films. Replace the handle by a "tube" whose boundary circles are attached to those of two holes on the sphere, which should be sufficiently small and close to each other. After that, start moving one of the holes. (The topological type of the quotient space does not change in the course of such a motion.) First, bring the hole to the boundary of the film, then shift it onto the film, drag it once along the film, shift it from the film, and, finally, return the hole to the initial spot. As a result, we transform the initial handle (a torus with a hole) into a Klein bottle with a hole, which splits into two Möbius bands (see Problem *21.U*), i.e., into two films.

22.A Consider the composition f of the embedding D^n in S^n onto a hemisphere and of the projection $\mathrm{pr} : S^n \to \mathbb{R}P^n$. The partition $S(f)$ is that described in the formulation. Consequently, $f/_{S(f)}$ is a homeomorphism.

22.C Consider $f : S^1 \to S^1 : \ z \mapsto z^2 \in \mathbb{C}$. Then $S^1/_{S(f)} \cong \mathbb{R}P^1$.

22.D See *22.A*.

22.E Consider the composition f of the embedding of S^n in $\mathbb{R}^n \setminus 0$ and of the projection onto the quotient space by the described the partition. It is clear that the partition $S(f)$ is the partition factorizing by which we obtain the projective space. Therefore, $f/_{S(f)}$ is a homeomorphism.

22.F To see that the described function is a metric, use the triangle inequality between the plane angles of a trilateral angle. Now, take each point $x \in S^n$ the line $l(x)$ through the origin with direction vector x. We have thus defined a continuous (check this) map of S^n to the indicated space of lines, whose injective factor is a homeomorphism.

22.G The image of this map is the set $U_0 = \{(x_0 : x_1 : \ldots : x_n) \mid x_0 \neq 0\}$, and the inverse map $j : U_0 \to \mathbb{R}^n$ is defined by the formula

$$(x_0 : x_1 : \ldots : x_n) \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Since both i and j are continuous, i is a topological embedding.

22.H Consider the embedding $S^{n-1} = S^n \cap \{x_{n+1} = 0\} \to S^n \subset \mathbb{R}^{n+1}$ and the induced embedding $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$.

***23.A*x** If $a \precsim b \precsim c$, then we have $a \prec b \prec c$, $a = b = c$, $a \prec b = c$, or $a = b \prec c$. In all four cases, we have $a \precsim c$.

***23.B*x** The relation $\sim$ is obviously reflexive, symmetric, and also transitive.

***23.C*x** Indeed, if $a' \sim a$, $a \precsim b$, and $b \sim b'$, then $a' \precsim a \precsim b \precsim b'$, whence $a' \precsim b'$. Clearly, the relation defined on the equivalence classes is transitive and reflexive. Now, if two equivalence classes $[a]$ and $[b]$ satisfy both $a \precsim b$ and $b \precsim a$, then $[a] = [b]$, i.e., the relation is anti-symmetric, hence, it is a nonstrict order.

***23.D*x** (a) In this case, we obtain the trivial nonstrict order on a singleton; (b) In this case, we obtain the same nonstrict order on the same set.

***23.E*x** The relation is obviously reflexive. Further, if $a \precsim b$, then each neighborhood U of a contains b, and so U also is a neighborhood of b, hence, if $b \precsim c$, then $c \in U$. Therefore, $a \in \mathrm{Cl}\{c\}$, whence $a \precsim c$, and thus the relation is also transitive.

***23.F*x** Consider the element of the partition that consists by definition of points each of which lies in the closure of any other point, so that each open set in X containing one of the points also contains any other. Therefore, the topology induced on each element of the partition is indiscrete. It is also clear that each element of the partition is a maximal subset which is an indiscrete subspace. Now consider two points in the quotient space and two points $x, y \in X$ lying in the corresponding elements of the partition. Since $x \not\sim y$, there is an open set containing exactly one of these points. Since each open set U in X is saturated with respect to the partition, the image of U in $X/_S$ is the required neighborhood.

***23.G*x** Obvious.

***23.H*x** This follows from *23.F*x, *23.G*x, and *20.R*.

***24.A*x** It is sufficient to observe that the sets in $\Delta^{(pw)}$ cover the entire set $\mathcal{C}(X, Y)$. (Actually, $\mathcal{C}(X, Y) \in \Delta^{(pw)}$.)

***24.B*x** Similarly to *24.A*x

***24.C*x** Since each one-point subset is compact, it follows that $\Delta^{(pw)} \subset \Delta^{(co)}$, whence $\Omega^{(pw)} \subset \Omega^{(co)}$.

***24.D*x** If $f \neq g$, then there is $x \in X$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, $f(x)$ and $g(x)$ have disjoint neighborhoods U and V, respectively. The subbase elements $W(x, U)$ and $W(x, V)$ are disjoint neighborhoods of f and g in the space $\mathcal{C}^{(pw)}(X, Y)$. They also are disjoint neighborhoods of f and g in $\mathcal{C}(X, Y)$.

***24.E*x** See assertion *24.I*x.

***24.H*x** Consider functions $f_n \in \mathcal{C}(X, Y)$ such that $\{f_n\}_1^\infty$ is a Cauchy sequence. For every point $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in Y. Therefore, since Y is a complete space, this sequence converges. Put $f(x) = \lim f_n(x)$. We have thus defined a function $f : X \to Y$.
Since $\{f_n\}$ is a Cauchy sequence, for each $\varepsilon > 0$ there exists a positive integer N such that $\rho\big(f_n(x), f_k(x)\big) < \frac{\varepsilon}{4}$ for any $n, k \geq N$ and $x \in X$. Passing to the limit as $k \to \infty$, we see that $\rho\big(f_n(x), f(x)\big) \leq \frac{\varepsilon}{4} < \frac{\varepsilon}{3}$ for any $n \geq N$ and $x \in X$. Thus, to prove that $f_n \to f$ as $n \to \infty$, it remains to show that $f \in \mathcal{C}(X, Y)$. For each $a \in X$, there exists a neighborhood U_a such that $\rho\big(f_N(x), f_N(a)\big) < \frac{\varepsilon}{3}$ for every $x \in U_a$. The triangle inequality implies that for every $x \in U_a$ we have

$$\rho\big(f(x), f(a)\big) \leq \rho\big(f(x), f_N(x)\big) + \rho\big(f_N(x), f_N(a)\big) + \rho\big(f_N(a), f(a)\big) < \varepsilon.$$

Therefore, the function f is a continuous limit of the considered Cauchy sequence.

***24.I*x** Take an arbitrary set $W(K, U)$ in the subbase. Let $f \in W(K, U)$. If $r = \rho(f(K), Y \smallsetminus U)$, then $D_r(f) \subset W(K, U)$. As a consequence, we see that each open set in the compact-open topology is open in the topology generated by the metric of uniform convergence. To prove the converse assertion, it suffices to show that for each map $f : X \to Y$ and each $r > 0$ there are compact sets $K_1, K_2, \dots, K_n \subset X$ and open sets $U_1, U_2, \dots, U_n \subset Y$ such that

$$f \in \bigcap_{i=1}^{n} W(K_i, U_i) \subset D_r(f).$$

Cover $f(X)$ by a finite number of balls with radius $r/4$ centered at certain points $f(x_1), f(x_2), \dots, f(x_n)$. Let K_i be the f-preimage of a closed disk in Y with radius $r/4$, and let U_i be the open ball with radius $r/2$. By construction, we have $f \in W(K_1, U_1) \cap \dots \cap W(K_n, U_n)$. Consider an arbitrary map g in this intersection. For each $x \in K_1$, we see that $f(x)$ and $g(x)$ lie in one and the same open ball with radius $r/2$, whence $\rho(f(x), g(x)) < r$. Since, by construction, the sets $K_1, \dots, K_n$ cover X, we have $\rho(f(x), g(x)) < r$ for all $x \in X$, whence $d(f, g) < r$, and, therefore, $g \in D_r(f)$.

***24.M*x** This follows from the fact that for each compact $K \subset X'$ and $U \subset Y'$ the preimage of the subbase set $W(K, U) \in \Delta^{(co)}(X', Y')$ is the subbase set $W(\varphi(K), \psi^{-1}(U)) \in \Delta^{(co)}(X, Y)$.

***24.N*x** This immediately follows from *24.Mx*.

***24.O*x** It is clear that the indicated map is an injection. To simplify the notation, we identify the space $\mathcal{C}(X, B)$ with its image under this injection. for each compact set $K \subset X$ and $U \in \Omega_B$ we denote by $W^B(K, U)$ the corresponding subbase set in $\mathcal{C}(X, B)$. If $V \in \Omega_Y$ and $U = B \cap V$, then we have $W^B(K, U) = \mathcal{C}(X, B) \cap W(K, V)$, whence it follows that $\mathcal{C}(X, Y)$ induces the compact-open topology on $\mathcal{C}(X, B)$.

***24.P*x** Verify that the natural mapping $f \mapsto (\mathrm{pr}_Y \circ f, \mathrm{pr}_Z \circ f)$ is a homeomorphism.

***24.Q*x** The injectivity of ϕ follows from the fact that $\{X_i\}$ is a cover, while the continuity of ϕ follows from assertion *24.Nx*. Once more, to simplify the notation, we identify the set $\mathcal{C}(X, Y)$ with its image under the injection ϕ. Let $K \subset X$ be a compact set, $U \in \Omega_Y$. Put $K_i = K \cap X_i$ and denote by $W^i(K_i, U)$ the corresponding element in the subbase $\Delta^{(co)}(X_i, Y)$. Since, obviously,

$$W(K, U) = \mathcal{C}(X, Y) \cap \big(W^1(K_1, U) \times \dots \times W^n(K_n, U)\big),$$

the continuous injection ϕ is indeed a topological embedding.

***24.S*x** Consider maps $f : X \to Y$, $g : Y \to Z$, a compact set $K \subset X$ and $V \in \Omega_Z$ such that $g(f(K)) \subset V$, i.e., $\phi(f, g) \in W(K, V)$. Then we have an inclusion $f(K) \subset g^{-1}(V) \in \Omega_Y$. Since Y is Hausdorff and locally compact and the set $f(K)$ is compact, $f(K)$ has a neighborhood U whose closure is compact and also contained in $g^{-1}(V)$ (see, *18.6x*.) In this case, we have $\phi(W(K,U) \times W(\operatorname{Cl} U, V)) \subset W(K,V)$, and, consequently, the map ϕ is continuous.

***24.T*x** The continuity of ϕ follows from *24.Mx*, and its injectivity is obvious. Let $K \subset X/_S$ be a compact set, $U \in \Omega_Y$. The image of the open subbase set $W(K,U) \subset \mathcal{C}(X/_S, Y)$ is the set of all maps $g : X \to Y$ constant on all elements of the partitions and such that $g(\mathrm{pr}^{-1}(K)) \subset U$. It remains to show that the set $W(\mathrm{pr}^{-1}(K), U)$ is open in $\mathcal{C}(X,Y)$. Since the quotient space $X/_S$ is Hausdorff, it follows that the set K is closed. Therefore, the preimage $\mathrm{pr}^{-1}(K)$ is closed, and hence also compact. Consequently, $W(\mathrm{pr}^{-1}(K), U)$ is a subbase set in $\mathcal{C}(X,Y)$.

***24.U*x** Let $f_0 \in \mathcal{C}(X,Y)$ and $x_0 \in X$. To prove that ϕ is continuous at the point (f_0, x_0), consider a neighborhood V of $f_0(x_0)$ in Y. Since the map f_0 is continuous, the point x_0 has a neighborhood U' such that $f_0(U') \subset V$. Since the space X is Hausdorff and locally compact, it follows that x_0 has a neighborhood U such that the closure $\operatorname{Cl} U$ is a compact subset of U'. Since, obviously, $f(x) \in V$ for any map $f \in W = W(\operatorname{Cl} U, V)$ and any point $x \in U$, we see that $\phi(W \times U) \subset V$.

***24.V*x** Assume that $x_0 \in X$, $K \subset Y$ be a compact set, $V \subset \Omega_Z$, and $F(x_0) \in W(K,V)$, i.e., $f(\{x_0\} \times K) \subset V$. Let us show that the map F is continuous. For this purpose, let us find a neighborhood U_0 of x_0 in X such that $F(U_0) \subset W(K,V)$. The latter inclusion is equivalent to the fact that $f(U_0 \times K) \in V$. We cover the set $\{x_0\} \times K$ by a finite number of neighborhoods $U_i \times V_i$ such that $f(U_i \times V_i) \subset V$. It remains to put $U_0 = \bigcap_i U_i$.

***24.W*x** Let $(x_0, y_0) \in X \times Y$, and let G be a neighborhood of the point $z_0 = f(x_0, y_0) = F(x_0)(y_0)$. Since the map $F(x_0) : Y \to Z$ is continuous, y_0 has a neighborhood W such that $F(W) \subset G$. Since Y is Hausdorff and locally compact, y_0 has a neighborhood V with compact closure such that $\operatorname{Cl} V \subset W$ and, consequently, $F(x_0)(\operatorname{Cl} V) \subset G$, i.e., $F(x_0) \in W(\operatorname{Cl} V, G)$. Since the map F is continuous, x_0 has a neighborhood U such that $F(U) \subset W(\operatorname{Cl} V, G)$. Then, if $(x,y) \in U \times V$, we have $F(x) \in W(\operatorname{Cl} V, G)$, whence $f(x,y) = F(x)(y) \in G$. Therefore, $f(U \times V) \subset G$, i.e., f is continuous.

***24.X*x** It suffices to show that for each compact set $K \subset X$, each open set $U \subset Y$, and each $f \in W(K,U)$ there are compact sets $K_1, K_2, \dots, K_m \subset K$ and open sets $U_1, U_2, \dots, U_m \in \Sigma_Y$ such that

$$f \in W(K_1, U_1) \cap W(K_2, U_2) \cap \dots \cap W(K_m, U_m) \subset W(K,U).$$

Let $x \in K$. Since $f(x) \in U$, there are sets $U_1^x, U_2^x, \dots, U_{n_x}^x \in \Sigma_Y$ such that $f(x) \in U_1^x \cap U_2^x \cap \dots \cap U_{n_x} \subset U$. Since f is continuous, x has a neighborhood G_x such that $f(x) \in U_1^x \cap U_2^x \cap \dots \cap U_{n_x}$. Since X is locally compact and Hausdorff, X is regular, consequently, x has a neighborhood

V_x such that $\operatorname{Cl} V_x$ is compact and $\operatorname{Cl} V_x \in G_x$. Since the set K is compact, K is covered by a finite number of neighborhoods V_{x_i}, $i = 1, 2, \dots, n$. We put $K_i = K \cap \operatorname{Cl} V_{x_i}$, $i = 1, 2, \dots, n$, and $U_{ij} = U_j^{x_i}$, $j = 1, 2, \dots, n_{x_i}$. Then the set

$$\bigcap_{i=1}^{n} \bigcap_{j=1}^{n_i} W(K_j, U_{ij})$$

is the required one.

***24.Y*x** First of all, we observe that assertion *24.V*x implies that the map Φ is well defined (i.e., for $f \in \mathcal{C}(X, \mathcal{C}(Y, Z))$ we indeed have $\Phi(f) \in \mathcal{C}(X, \mathcal{C}(Y, Z)))$, while assertion *24.W*x implies that if Y is locally compact and Hausdorff, then Φ is invertible.
1) Let $K \subset X$ and $L \subset Y$ be compact sets, $V \in \Omega_Z$. The sets of the form $W(L, V)$ constitute a subbase in $\mathcal{C}(Y, Z)$. By *24.X*x, the sets of the form $W(K, W(L, V))$ constitute a subbase in $\mathcal{C}(X, \mathcal{C}(Y, Z))$. It remains to observe that $\Phi^{-1}(W(K, W(L, V))) = W(K \times L, V) \in \Delta^{(co)}(X \times Y, Z)$. Therefore, the map Φ is continuous.
2) Let $Q \subset X \times Y$ be a compact set and $G \subset\in \Omega_Z$. Let $\varphi \in \Phi(W(Q, G))$, so that $\varphi(x) : y \mapsto f(x, y)$ for a certain map $f \in W(Q, G)$. For each $q \in Q$, take a neighborhood $U_q \times V_q$ of q such that: the set $\operatorname{Cl} V_q$ is compact and $f(U_q \times \operatorname{Cl} V_q) \subset G$. Since Q is compact, we have $Q \subset \bigcup_{i=1}^n (U_{q_i} \times V_{q_i})$. The sets $W_i = W(\operatorname{Cl} V_{q_i}, G)$ are open in $\mathcal{C}(Y, Z)$, hence, the sets $T_i = W(p_X(Q) \cap \operatorname{Cl} U_{q_i}, W_i)$ are open in $\mathcal{C}(X, \mathcal{C}(Y, Z))$. Therefore, $T = \bigcap_{i=1}^n T_i$ is a neighborhood of φ. Let us show that $T \subset \Phi(W(Q, G))$. Indeed, if $\psi \in T$, then $\psi = \Phi(g)$, and we have $g(x, y) \in G$ for $(x, y) \in Q$, so that $g \in W(Q, G)$, whence $\psi \in \Phi(W(Q, G))$. Therefore, the set $\Phi(W(Q, G))$ is open, and so Φ is a homeomorphism.

***24.Z*x** It is obvious that the quotient map f is a continuous bijection. Consider the factorization map $p : X \times Y \to (X \times Y)/_{S'}$. By *24.V*x, the map $\Phi : X \to \mathcal{C}(Y, (X \times Y)/_{S'})$, where $\Phi(x)(y) = p(x, y)$, is continuous. We observe that Φ is constant on elements of the partition S, consequently, the quotient map $\widetilde{\Phi} : X/_S \to \mathcal{C}(Y, (X \times Y)/_{S'})$ is continuous. By *24.W*x, the map $g : X/_S \times Y \to (X \times Y)/_{S'}$, where $g(z, y) = \widetilde{\Phi}(z)(y)$, is also continuous. It remains to observe that g and f are mutually inverse maps.

Chapter V

Topological Algebra

In this chapter, we study topological spaces strongly related to groups: either the spaces themselves are groups in a nice way (so that all the maps coming from group theory are continuous), or groups act on topological spaces and can be thought of as consisting of homeomorphisms.

This material has interdisciplinary character. Although it plays important roles in many areas of Mathematics, it is not so important in the framework of general topology. Quite often, this material can be postponed till the introductory chapters of the mathematical courses that really require it (functional analysis, Lie groups, etc.). In the framework of general topology, this material provides a great collection of exercises.

In the second part of the book, which is devoted to algebraic topology, groups appear in a more profound way. So, sooner or later, the reader will meet groups. At latest in the next chapter, when studying fundamental groups.

Groups are attributed to Algebra. In the mathematics built on sets, main objects are sets with additional structure. Above, we met a few of the most fundamental of these structures: topology, metric, partial order. Topology and metric evolved from geometric considerations. Algebra studied algebraic operations with numbers and similar objects and introduced into the set-theoretic Mathematics various structures based on operations. One of the simplest (and most versatile) of these structures is the structure of a group. It emerges in an overwhelming majority of mathematical environments. It often appears together with topology and in a nice interaction with it. This interaction is a subject of Topological Algebra.

The second part of this book is called Algebraic Topology. It also treats interaction of Topology and Algebra, spaces and groups. But this is a completely different interaction. The structures of topological space and group do not live there on the same set, but the group encodes topological properties of the space.

25x. Digression. Generalities on Groups

This section is included mainly to recall the most elementary definitions and statements concerning groups. We do not mean to present a self-contained outline of the group theory. The reader is actually assumed to be familiar with groups, homomorphisms, subgroups, quotient groups, etc.

If this is not yet so, we recommend to read one of the numerous algebraic textbooks covering the elementary group theory. The mathematical culture, which must be acquired for mastering the material presented above in this book, would make this an easy and pleasant exercise.

As a temporary solution, the reader can read few definitions and prove few theorems gathered in this section. They provide a sufficient basis for most of what follows.

25°1x. The Notion of Group

Recall that a *group* is a set G equipped with a group operation. A *group operation* in a set G is a map $\omega : G \times G \to G$ satisfying the following three conditions (known as *group axioms*):

- **Associativity.** $\omega(a, \omega(b, c)) = \omega(\omega(a, b), c)$ for any $a, b, c \in G$.
- **Existence of Neutral Element.** There exists $e \in G$ such that $\omega(e, a) = \omega(a, e) = a$ for every $a \in G$.
- **Existence of Inverse Element.** For any $a \in G$, there exists $b \in G$ such that $\omega(a, b) = \omega(b, a) = e$.

25.Ax Uniqueness of Neutral Element. *A group contains a unique neutral element.*

25.Bx Uniqueness of Inverse Element. *Each element of a group has a unique inverse element.*

25.Cx First Examples of Groups. In each of the following situations, check if we have a group. What is its neutral element? How to calculate the element inverse to a given one?

- The set G is the set $\mathbb{Z}$ of integers, and the group operation is addition: $\omega(a, b) = a + b$.
- The set G is the set $\mathbb{Q}_{>0}$ of positive rational numbers, and the group operation is multiplication: $\omega(a, b) = ab$.
- $G = \mathbb{R}$, and $\omega(a, b) = a + b$.
- $G = \mathbb{C}$, and $\omega(a, b) = a + b$.
- $G = \mathbb{R} \smallsetminus 0$, and $\omega(a, b) = ab$.

- G is the set of all bijections of a set A onto itself, and the group operation is composition: $\omega(a, b) = a \circ b$.

25.1x Simplest Group. 1) Can a group be empty? 2) Can it consist of one element?

A group consisting of one element is *trivial*.

25.2x Solving Equations. Let G be a set with an associative operation $\omega : G \times G \to G$. Prove that G is a group iff for any $a, b \in G$ the set G contains a unique element x such that $\omega(a, x) = b$ and a unique element y such that $\omega(y, a) = b$.

25°2x. Additive Versus Multiplicative

The notation above is never used! (The only exception may happen, as here, when the definition of group is discussed.) Instead, one uses either *multiplicative* or *additive* notation.

Under multiplicative notation, the group operation is called *multiplication* and denoted as multiplication: $(a, b) \mapsto ab$. The neutral element is called *unity* and denoted by 1 or 1_G (or e). The element inverse to a is denoted by a^{-1}. This notation is borrowed, say, from the case of nonzero rational numbers with the usual multiplication.

Under additive notation, the group operation is called *addition* and denoted as addition: $(a, b) \mapsto a + b$. The neutral element is called *zero* and denoted by 0. The element inverse to a is denoted by $-a$. This notation is borrowed, say, from the case of integers with the usual addition.

An operation $\omega : G \times G \to G$ is *commutative* if $\omega(a, b) = \omega(b, a)$ for any $a, b \in G$. A group with commutative group operation is *commutative* or *Abelian*. Traditionally, the additive notation is used only in the case of commutative groups, while the multiplicative notation is used both in the commutative and noncommutative cases. Below, we mostly use the multiplicative notation.

25.3x. In each of the following situations, check if we have a group:

(1) a singleton $\{a\}$ with multiplication $aa = a$,
(2) the set $\mathbb{S}_n$ of bijections of the set $\{1, 2, \dots, n\}$ of the first n positive integers onto itself with multiplication determined by composition (the *symmetric group of degree* n),
(3) the sets $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ with coordinate-wise addition,
(4) the set $Homeo(X)$ of all homeomorphisms of a topological space X with multiplication determined by composition,
(5) the set $GL(n, \mathbb{R})$ of invertible real $n \times n$ matrices equipped with matrix multiplication,
(6) the set $M_n(\mathbb{R})$ of all real $n \times n$ matrices with addition determined by addition of matrices,

(7) the set of all subsets of a set X with multiplication determined by the symmetric difference:
$$(A, B) \mapsto A \bigtriangleup B = (A \cup B) \smallsetminus (A \cap B),$$
(8) the set $\mathbb{Z}_n$ of classes of positive integers congruent modulo n with addition determined by addition of positive integers,
(9) the set of complex roots of unity of degree n equipped with usual multiplication of complex numbers,
(10) the set $\mathbb{R}_{>0}$ of positive reals with usual multiplication,
(11) $S^1 \subset \mathbb{C}$ with standard multiplication of complex numbers,
(12) the set of translations of a plane with multiplication determined by composition.

Associativity implies that every finite sequence of elements in a group has a well-defined product, which can be calculated by a sequence of pairwise multiplications determined by any placement of parentheses, say, $abcde = (ab)(c(de))$. The distribution of the parentheses is immaterial. In the case of a sequence of three elements, this is precisely the associativity: $(ab)c = a(bc)$.

25.Dx. Derive from the associativity that the product of any length does not depend on the position of the parentheses.

For an element a of a group G, the powers a^n with $n \in \mathbb{Z}$ are defined by the following formulas: $a^0 = 1$, $a^{n+1} = a^n a$, and $a^{-n} = (a^{-1})^n$.

25.Ex. Prove that raising to a power has the following properties: $a^p a^q = a^{p+q}$ and $(a^p)^q = a^{pq}$.

25°3x. Homomorphisms

Recall that a map $f : G \to H$ of a group to another one is a *homomorphism* if $f(xy) = f(x)f(y)$ for any $x, y \in G$.

25.4x. In the above definition of a homomorphism, the multiplicative notation is used. How does this definition look in the additive notation? What if one of the groups is multiplicative, while the other is additive?

25.5x. Let a be an element of a multiplicative group G. Is the map $\mathbb{Z} \to G : n \mapsto a^n$ a homomorphism?

25.Fx. Let G and H be two groups. Is the constant map $G \to H$ mapping the entire G to the neutral element of H a homomorphism? Is any other constant map $G \to H$ a homomorphism?

25.Gx. *A homomorphism maps the neutral element to the neutral element, and it maps mutually inverse elements to mutually inverse elements.*

25.Hx. The identity map of a group is a homomorphism. The composition of homomorphisms is a homomorphism.

Recall that a homomorphism f is an *epimorphism* if f is surjective, f is a *monomorphism* if f is injective, and f is an *isomorphism* if f is bijective.

25.Ix. *The map inverse to an isomorphism is also an isomorphism.*

Two groups are *isomorphic* if there exists an isomorphism of one of them onto another one.

25.Jx. *Isomorphism is an equivalence relation.*

> ***25.6x.*** Show that the additive group $\mathbb{R}$ is isomorphic to the multiplicative group $\mathbb{R}_{>0}$.

25°4x. Subgroups

A subset A of a group G is a *subgroup* of G if A is invariant under the group operation of G (i.e., for any $a, b \in A$ we have $ab \in A$) and A equipped with the group operation induced by that in G is a group.

For two subsets A and B of a multiplicative group G, we put $AB = \{ab \mid a \in A,\ b \in B\}$ and $A^{-1} = \{a^{-1} \mid a \in A\}$.

25.Kx. A subset A of a multiplicative group G is a subgroup of G iff $AA \subset G$ and $A^{-1} \subset A$.

> ***25.7x.*** The singleton consisting of the neutral element is a subgroup.
>
> ***25.8x.*** Prove that a subset A of a *finite* group is a subgroup if $AA \subset A$. (The condition $A^{-1} \subset A$ is superfluous in this case.)
>
> ***25.9x.*** List all subgroups of the additive group $\mathbb{Z}$.
>
> ***25.10x.*** Is $GL(n, \mathbb{R})$ a subgroup of $M_n(\mathbb{R})$? (See *25.3x* for notation.)

25.Lx. *The image of a group homomorphism $f : G \to H$ is a subgroup of H.*

25.Mx. *Let $f : G \to H$ be a group homomorphism, K a subgroup of H. Then $f^{-1}(K)$ is a subgroup of G. In short:*
The preimage of a subgroup under a group homomorphism is a subgroup.

The preimage of the neutral element under a group homomorphism $f : G \to H$ is called the *kernel* of f and denoted by $\operatorname{Ker} f$.

25.Nx Corollary of 25.Mx. *The kernel of a group homomorphism is a subgroup.*

25.Ox. *A group homomorphism is a monomorphism iff its kernel is trivial.*

25.Px. *The intersection of any collection of subgroups of a group is also a subgroup.*

A subgroup H of a group G is *generated* by a subset $S \subset G$ if H is the smallest subgroup of G containing S.

25.Qx. The subgroup H generated by S is the intersection of all subgroups of G that contain S. On the other hand, H is the set of all elements that are products of elements in S and elements inverse to elements in S.

The elements of a set that generates G are ***generators*** of G. A group generated by one element is ***cyclic***.

25.Rx. A cyclic (multiplicative) group consists of powers of its generator. (I.e., if G is a cyclic group and a generates G, then $G = \{a^n \mid n \in \mathbb{Z}\}$.) Any cyclic group is commutative.

25.11x. A group G is cyclic iff there exists an epimorphism $f : \mathbb{Z} \to G$.

25.Sx. *A subgroup of a cyclic group is cyclic.*

The number of elements in a group G is the ***order*** of G. It is denoted by $|G|$.

25.Tx. Let G be a finite cyclic group, d a positive divisor of $|G|$. Then there exists a unique subgroup H of G with $|H| = d$.

Each element of a group generates a cyclic subgroup, which consists of all powers of this element. The order of the subgroup generated by a (nontrivial) element $a \in G$ is the ***order*** of a. It can be a positive integer or the infinity.

For each subgroup H of a group G, the ***right cosets*** of H are the sets $Ha = \{xa \mid x \in H\}$, $a \in G$. Similarly, the sets aH are the ***left cosets*** of H. The number of distinct right (or left) cosets of H is the ***index*** of H.

25.Ux Lagrange theorem. *If H is a subgroup of a finite group G, then the order of H divides that of G.*

A subgroup H of a group G is ***normal*** if for any $h \in H$ and $a \in G$ we have $aha^{-1} \in H$. Normal subgroups are also called ***normal divisors*** or ***invariant subgroups***.

In the case where the subgroup is normal, left cosets coincide with right cosets, and the set of cosets is a group with multiplication defined by the formula $(aH)(bH) = abH$. The group of cosets of H in G is called the ***quotient group*** or ***factor group*** of G by H and denoted by G/H.

25.Vx. The kernel $\operatorname{Ker} f$ of a homomorphism $f : G \to H$ is a normal subgroup of G.

25.Wx. *The image $f(G)$ of a homomorphism $f : G \to H$ is isomorphic to the quotient group $G/_{\operatorname{Ker} f}$ of G by the kernel of f.*

25.Xx. The quotient group $\mathbb{R}/_{\mathbb{Z}}$ is canonically isomorphic to the group S^1. Describe the image of the group $\mathbb{Q} \subset \mathbb{R}$ under this isomorphism.

25.Yx. Let G be a group, A a normal subgroup of G, and B an arbitrary subgroup of G. Then AB also is a normal subgroup of G, while $A \cap B$ is a normal subgroup of B. Furthermore, we have $AB/_A \cong B/_{A \cap B}$.

26x. Topological Groups

26°1x. Notion of Topological Group

A *topological group* is a set G equipped with both a topological structure and a group structure such that the maps $G \times G \to G : (x, y) \mapsto xy$ and $G \to G : x \mapsto x^{-1}$ are continuous.

***26.1*x.** Let G be a group and a topological space simultaneously. Prove that the maps $\omega : G \times G \to G : (x, y) \mapsto xy$ and $\alpha : G \to G : x \mapsto x^{-1}$ are continuous iff so is the map $\beta : G \times G \to G : (x, y) \mapsto xy^{-1}$.

***26.2*x.** Prove that if G is a topological group, then the inversion $G \to G : x \mapsto x^{-1}$ is a homeomorphism.

***26.3*x.** Let G be a topological group, X a topological space, $f, g : X \to G$ two maps continuous at a point $x_0 \in X$. Prove that the maps $X \to G : x \mapsto f(x)g(x)$ and $X \to G : x \mapsto (f(x))^{-1}$ are continuous at x_0.

***26.A*x.** A group equipped with the discrete topology is a topological group.

***26.4*x.** Is a group equipped with the indiscrete topology a topological group?

26°2x. Examples of Topological Groups

***26.B*x.** The groups listed in *25.C*x equipped with standard topologies are topological groups.

***26.5*x.** The unit circle $S^1 = \{|z| = 1\} \subset \mathbb{C}$ with the standard multiplication is a topological group.

***26.6*x.** In each of the following situations, check if we have a topological group.

(1) The spaces $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ with coordinate-wise addition. ($\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$, while $\mathbb{H}^n$ is isomorphic to $\mathbb{C}^{2n}$.)
(2) The sets $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, and $M_n(\mathbb{H})$ of all $n \times n$ matrices with real, complex, and, respectively, quaternion elements, equipped with the product topology and element-wise addition. (We identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$, and $M_n(\mathbb{H})$ with $\mathbb{H}^{n^2}$.)
(3) The sets $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, and $GL(n, \mathbb{H})$ of invertible $n \times n$ matrices with real, complex, and quaternionic entries, respectively, under the matrix multiplication.
(4) $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $O(n)$, $O(n, \mathbb{C})$, $U(n)$, $SO(n)$, $SO(n, \mathbb{C})$, $SU(n)$, and other subgroups of $GL(n, K)$ with $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

***26.7*x.** Introduce a topological group structure on the additive group $\mathbb{R}$ that would be distinct from the usual, discrete, and indiscrete topological structures.

***26.8*x.** Find two nonisomorphic connected topological groups that are homeomorphic as topological spaces.

***26.9*x.** On the set $G = [0, 1)$ (equipped with the standard topology), we define addition as follows: $\omega(x, y) = x + y \pmod 1$. Is (G, ω) a topological group?

26°3x. Translations and Conjugations

Let G be a group. Recall that the maps $L_a : G \to G : x \mapsto ax$ and $R_a : G \to G : x \mapsto xa$ are *left* and *right translations through* a, respectively. Note that $L_a \circ L_b = L_{ab}$, while $R_a \circ R_b = R_{ba}$. (To "repair" the last relation, some authors define right translations by $x \mapsto xa^{-1}$.)

26.Cx. *A translation of a topological group is a homeomorphism.*

Recall that the *conjugation* of a group G by an element $a \in G$ is the map $G \to G : x \mapsto axa^{-1}$.

26.Dx. *The conjugation of a topological group by any of its elements is a homeomorphism.*

The following simple observation allows a certain "uniform" treatment of the topology in a group: neighborhoods of distinct points can be compared.

26.Ex. If U is an open set in a topological group G, then for any $x \in G$ the sets xU, Ux, and U^{-1} are open.

26.10x. Does the same hold true for closed sets?

26.11x. Prove that if U and V are subsets of a topological group G and U is open, then UV and VU are open.

26.12x. Will the same hold true if we replace everywhere the word *open* by the word *closed*?

26.13x. Are the following subgroups of the additive group $\mathbb{R}$ closed?

(1) $\mathbb{Z}$,

(2) $\sqrt{2}\,\mathbb{Z}$,

(3) $\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$?

26.14x. Let G be a topological group, $U \subset G$ a compact subset, $V \subset G$ a closed subset. Prove that UV and VU are closed.

26.14x.1. Let F and C be two disjoint subsets of a topological group G. If F is closed and C is compact, then 1_G has a neighborhood V such that $CV \cup VC$ does not meet F. If G is locally compact, then V can be chosen so that $\mathrm{Cl}(CV \cup VC)$ be compact.

26°4x. Neighborhoods

26.Fx. *Let Γ be a neighborhood base of a topological group G at 1_G. Then $\Sigma = \{aU \mid a \in G,\ U \in \Gamma\}$ is a base for topology of G.*

A subset A of a group G is *symmetric* if $A^{-1} = A$.

26.Gx. Any neighborhood of 1 in a topological group contains a symmetric neighborhood of 1.

26.Hx. For any neighborhood U of 1 in a topological group, 1 has a neighborhood V such that $VV \subset U$.

***26.15*x.** Let G be a topological group, U a neighborhood of 1_G, and n a positive integer. Then 1_G has a symmetric neighborhood V such that $V^n \subset U$.

***26.16*x.** Let V be a symmetric neighborhood of 1_G in a topological group G. Then $\bigcup_{n=1}^{\infty} V^n$ is an open-closed subgroup.

***26.17*x.** Let G be a group, Σ be a collection of subsets of G. Prove that G carries a unique topology Ω such that Σ is a neighborhood base for Ω at 1_G and (G, Ω) is a topological group, iff Σ satisfies the following five conditions:

(1) each $U \in \Sigma$ contains 1_G,
(2) for every $x \in U \in \Sigma$, there exists $V \in \Sigma$ such that $xV \subset U$,
(3) for each $U \in \Sigma$, there exists $V \in \Sigma$ such that $V^{-1} \subset U$,
(4) for each $U \in \Sigma$, there exists $V \in \Sigma$ such that $VV \subset U$,
(5) for any $x \in G$ and $U \in \Sigma$, there exists $V \in \Sigma$ such that $V \subset x^{-1}Ux$.

26.I*x. *Riddle. In what sense *26.H*x is similar to the triangle inequality?

***26.J*x.** Let C be a compact subset of G. Prove that for every neighborhood U of 1_G the unity 1_G has a neighborhood V such that $V \subset xUx^{-1}$ for every $x \in C$.

26°5x. Separation Axioms

***26.K*x.** A topological group G is Hausdorff, iff G satisfies the first separation axiom, iff the unity 1_G (or, more precisely, the singleton $\{1_G\}$) is closed.

***26.L*x.** A topological group G is Hausdorff iff the unity 1_G is the intersection of its neighborhoods.

***26.M*x.** If the unity of a topological group G is closed, then G is regular (as a topological space).

Use the following fact.

26.M*x*.1. Let G be a topological group, $U \subset G$ a neighborhood of 1_G. Then 1_G has a neighborhood V with closure contained in U: $\operatorname{Cl} V \subset U$.

26.N*x *Corollary. *For topological groups, the first three separation axioms are equivalent.*

***26.18*x.** Prove that a finite group carries as many topological group structures as there are normal subgroups. Namely, each finite topological group G contains a normal subgroup N such that the sets gN with $g \in G$ form a base for the topology of G.

26°6x. Countability Axioms

***26.O*x.** If Γ is a neighborhood base at 1_G in a topological group G and $S \subset G$ is a dense set, then $\Sigma = \{aU \mid a \in S, U \in \Gamma\}$ is a base for the topology of G. (Cf. *26.F*x and *15.J*.)

***26.P*x.** A first countable separable topological group is second countable.

26.19x*. (Cf. *15.Dx*) A first countable Hausdorff topological group G is metrizable. Furthermore, G can be equipped with a right (left) invariant metric.

27x. Constructions

27°1x. Subgroups

27.Ax. Let H be a subgroup of a topological group G. Then the topological and group structures induced from G make H a topological group.

27.1x. Let H be a subgroup of an Abelian group G. Prove that, given a structure of topological group in H and a neighborhood base at 1, G carries a structure of topological group with the same neighborhood base at 1.

27.2x. Prove that a subgroup of a topological group is open iff it contains an interior point.

27.3x. Prove that every open subgroup of a topological group is also closed.

27.4x. Prove that every closed subgroup of finite index is also open.

27.5x. Find an example of a subgroup of a topological group that

(1) is closed, but not open;
(2) is neither closed, nor open.

27.6x. Prove that a subgroup H of a topological group is a discrete subspace iff H contains an isolated point.

27.7x. Prove that a subgroup H of a topological group G is closed, iff there exists an open set $U \subset G$ such that $U \cap H = U \cap \operatorname{Cl} H \neq \varnothing$, i.e., iff $H \subset G$ is locally closed at one of its points.

27.8x. Prove that if H is a non-closed subgroup of a topological group G, then $\operatorname{Cl} H \smallsetminus H$ is dense in $\operatorname{Cl} H$.

27.9x. *The closure of a subgroup of a topological group is a subgroup.*

27.10x. Is it true that the interior of a subgroup of a topological group is a subgroup?

27.Bx. A connected topological group is generated by any neighborhood of 1.

27.Cx. Let H be a subgroup of a group G. Define a relation: $a \sim b$ if $ab^{-1} \in H$. Prove that this is an equivalence relation, and the right cosets of H in G are the equivalence classes.

27.11x. What is the counterpart of *27.Cx* for left cosets?

Let G be a topological group, $H \subset G$ a subgroup. The set of left (respectively, right) cosets of H in G is denoted by G/H (respectively, $H\backslash G$). The sets G/H and $H\backslash G$ carry the quotient topology. Equipped with these topologies, they are called *spaces of cosets*.

27.Dx. For any topological group G and its subgroup H, the natural projections $G \to G/H$ and $G \to H\backslash G$ are open (i.e., the image of every open set is open).

27.Ex. The space of left (or right) cosets of a closed subgroup in a topological group is regular.

27.Fx. The group G is compact (respectively, connected) if so are H and $G/_H$.

27.12x. If H is a connected subgroup of a group G, then the preimage of any connected component of $G/_H$ is a connected component of G.

27.13x. Let us regard the group $SO(n-1)$ as a subgroup of $SO(n)$. If $n \geq 2$, then the space $SO(n)/_{SO(n-1)}$ is homeomorphic to S^{n-1}.

27.14x. The groups $SO(n)$, $U(n)$, $SU(n)$, and $Sp(n)$ are 1) compact and 2) connected for any $n \geq 1$. 3) How many connected components do the groups $O(n)$ and $O(p,q)$ have? (Here, $O(p,q)$ is the group of linear transformations in $\mathbb{R}^{p+q}$ preserving the quadratic form $x_1^2 + \dots + x_p^2 - y_1^2 - \dots - y_q^2$.)

27°2x. Normal Subgroups

27.Gx. Prove that the closure of a normal subgroup of a topological group is a normal subgroup.

27.Hx. The connected component of 1 in a topological group is a closed normal subgroup.

27.15x. The path-connected component of 1 in a topological group is a normal subgroup.

27.Ix. The quotient group of a topological group is a topological group (provided that it is equipped with the quotient topology).

27.Jx. The natural projection of a topological group onto its quotient group is open.

27.Kx. If a topological group G is first (respectively, second) countable, then so is any quotient group of G.

27.Lx. Let H be a normal subgroup of a topological group G. Then the quotient group G/H is regular iff H is closed.

27.Mx. Prove that a normal subgroup H of a topological group G is open iff the quotient group G/H is discrete.

The *center* of a group G is the set $C(G) = \{x \in G \mid xg = gx \text{ for each } g \in G\}$.

27.16x. Each discrete normal subgroup H of a connected group G is contained in the center of G.

27°3x. Homomorphisms

For topological groups, by a *homomorphism* one means a group homomorphism which is *continuous*.

27.Nx. *Let G and H be two topological groups. A group homomorphism $f : G \to H$ is continuous iff f is continuous at 1_G.*

Besides similar modifications, which can be summarized by the following principle: *everything is assumed to respect the topological structures*, the terminology of group theory passes over without changes. In particular, an *isomorphism* in group theory is an invertible homomorphism. Its inverse is a homomorphism (and hence an isomorphism) automatically. In the theory of topological groups, this must be included in the definition: an *isomorphism* of topological groups is an invertible homomorphism whose inverse is also a homomorphism. In other words, an isomorphism of topological groups is a map that is both a group isomorphism and a homeomorphism. Cf. Section 10.

> ***27.17x.*** Prove that the map $[0,1) \to S^1 : \ x \mapsto e^{2\pi i x}$ is a topological group homomorphism.

27.Ox. An epimorphism $f : G \to H$ is an open map iff the injective factor $f/S(f) : G/\operatorname{Ker} f \to H$ of f is an isomorphism.

27.Px. An epimorphism of a compact topological group onto a topological group with closed unity is open.

27.Qx. Prove that the quotient group $\mathbb{R}/\mathbb{Z}$ of the additive group $\mathbb{R}$ by the subgroup $\mathbb{Z}$ is isomorphic to the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with absolute value 1.

27°4x. Local Isomorphisms

Let G and H be two topological groups. A *local isomorphism* from G to H is a homeomorphism f of a neighborhood U of 1_G in G onto a neighborhood V of 1_H in H such that

- $f(xy) = f(x)f(y)$ for any $x, y \in U$ such that $xy \in U$,
- $f^{-1}(zt) = f^{-1}(z)f^{-1}(t)$ for any $z, t \in V$ such that $zt \in V$.

Two topological groups G and H are *locally isomorphic* if there exists a local isomorphism from G to H.

27.Rx. Isomorphic topological groups are locally isomorphic.

27.Sx. The additive group $\mathbb{R}$ and the multiplicative group $S^1 \subset \mathbb{C}$ are locally isomorphic, but not isomorphic.

***27.18*x.** Prove that local isomorphism of topological groups is an equivalence relation.

***27.19*x.** Find neighborhoods of unities in $\mathbb{R}$ and S^1 and a homeomorphism between them that satisfies the first condition in the definition of local isomorphism, but does not satisfy the second one.

***27.20*x.** Prove that if a homeomorphism between neighborhoods of unities in two topological groups satisfies only the first condition in the definition of local isomorphism, then it has a submap that is a local isomorphism between these topological groups.

27°5x. Direct Products

Let G and H be two topological groups. In group theory, the product $G \times H$ is given a group structure.[1] In topology, it is given a topological structure (see Section 19).

***27.T*x.** These two structures are compatible: the group operations in $G\times H$ are continuous with respect to the product topology.

Thus, $G \times H$ is a topological group. It is called the *direct product* of the topological groups G and H. There are canonical homomorphisms related to this: the inclusions $i_G : G \to G \times H : x \mapsto (x, 1)$ and $i_H : H \to G \times H : x \mapsto (1, x)$, which are monomorphisms, and the projections $\mathrm{pr}_G : G \times H \to G : (x, y) \mapsto x$ and $\mathrm{pr}_H : G \times H \to H : (x, y) \mapsto y$, which are epimorphisms.

***27.21*x.** Prove that the topological groups $(G \times H)/i_H(H)$ and G are isomorphic.

***27.22*x.** The product operation is both commutative and associative: $G \times H$ is (canonically) isomorphic to $H \times G$, while $G \times (H \times K)$ is canonically isomorphic to $(G \times H) \times K$.

A topological group G *decomposes into a direct product of two subgroups* A *and* B if the map $A \times B \to G : (x, y) \mapsto xy$ is a topological group isomorphism. If this is the case, the groups G and $A\times B$ are usually identified via this isomorphism.

Recall that a similar definition exists in ordinary group theory. The only difference is that there an isomorphism is just an algebraic isomorphism. Furthermore, in that theory, G decomposes into a direct product of its subgroups A and B iff A and B generate G, A and B are normal subgroups, and $A \cap B = \{1\}$. Therefore, if these conditions are fulfilled in the case of topological groups, then $A \times B \to G : (x, y) \mapsto xy$ is a group isomorphism.

***27.23*x.** Prove that in this situation the map $A \times B \to G : (x, y) \mapsto xy$ is continuous. Find an example where the inverse group isomorphism is not continuous.

[1] Recall that the multiplication in $G \times H$ is defined by the formula $(x, u)(y, v) = (xy, uv)$.

***27.U*x.** Prove that if a compact Hausdorff group G decomposes algebraically into a direct product of two closed subgroups, then G also decomposes into a direct product of these subgroups as a topological group.

***27.24*x.** Prove that the multiplicative group $\mathbb{R}\setminus 0$ of nonzero reals is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^0 = \{1, -1\}$ and $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$.

***27.25*x.** Prove that the multiplicative group $\mathbb{C} \setminus 0$ of nonzero complex numbers is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{R}_{>0}$.

***27.26*x.** Prove that the multiplicative group $\mathbb{H} \setminus 0$ of nonzero quaternions is isomorphic (as a topological group) to the direct product of the multiplicative groups $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ and $\mathbb{R}_{>0}$.

***27.27*x.** Prove that the subgroup $S^0 = \{1, -1\}$ of $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ is not a direct factor.

***27.28*x.** Find a topological group homeomorphic to $\mathbb{R}P^3$ (the three-dimensional real projective space).

Let a group G contain a normal subgroup A and a subgroup B such that $AB = G$ and $A \cap B = \{1_G\}$. If B is also normal, then G is the direct product $A \times B$. Otherwise, G is a *semidirect product* of A and B.

***27.V*x.** Let a topological group G be a semidirect product of its subgroups A and B. If for any neighborhoods of unity, $U \subset A$ and $V \subset B$, their product UV contains a neighborhood of 1_G, then G is homeomorphic to $A \times B$.

27°6x. Groups of Homeomorphisms

For any topological space X, the auto-homeomorphisms of X form a group under composition as the group operation. We denote this group by $\operatorname{Top} X$. To make this group topological, we slightly enlarge the topological structure induced on $\operatorname{Top} X$ by the compact-open topology of $\mathcal{C}(X, X)$.

***27.W*x.** The collection of the sets $W(C, U)$ and $(W(C, U))^{-1}$ taken over all compact $C \subset X$ and open $U \subset X$ is a subbase for the topological structure on $\operatorname{Top} X$.

In what follows, we equip $\operatorname{Top} X$ with this topological structure.

***27.X*x.** If X is Hausdorff and locally compact, then $\operatorname{Top} X$ is a topological group.

27.X*x*.1. If X is Hausdorff and locally compact, then the map $\operatorname{Top} X \times \operatorname{Top} X \to \operatorname{Top} X : (g, h) \mapsto g \circ h$ is continuous.

28x. Actions of Topological Groups

28°1x. Action of a Group on a Set

A *left action* of a group G on a set X is a map $G \times X \to X : (g, x) \mapsto gx$ such that $1x = x$ for any $x \in X$ and $(gh)x = g(hx)$ for any $x \in X$ and $g, h \in G$. A set X equipped with such an action is a *left G-set*. Right G-sets are defined in a similar way.

***28.A*x.** If X is a left G-set, then $G \times X \to X : (x, g) \mapsto g^{-1}x$ is a right action of G on X.

***28.B*x.** If X is a left G-set, then for any $g \in G$ the map $X \to X : x \mapsto gx$ is a bijection.

A left action of G on X is *effective* (or *faithful*) if for each $g \in G \setminus 1$ the map $G \to G : x \mapsto gx$ is not equal to id_G. Let X_1 and X_2 be two left G-sets. A map $f : X_1 \to X_2$ is *G-equivariant* if $f(gx) = gf(x)$ for any $x \in X$ and $g \in G$.

We say that X is a *homogeneous left G-set*, or, what is the same, that G acts on X *transitively* if for any $x, y \in X$ there exists $g \in G$ such that $y = gx$.

The same terminology applies to right actions with obvious modifications.

***28.C*x.** The natural actions of G on $G/_H$ and $_H\backslash G$ transform $G/_H$ and $_H\backslash G$ into homogeneous left and, respectively, right G-sets.

Let X be a homogeneous left G-set. Consider a point $x \in X$ and the set $G^x = \{g \in G \mid gx = x\}$. We easily see that G^x is a subgroup of G. It is called the *isotropy subgroup* of x.

***28.D*x.** Each homogeneous left (respectively, right) G-set X is isomorphic to $G/_H$ (respectively, $_H\backslash G$), where H is the isotropy group of a certain point in X.

28.D*x.*1. All isotropy subgroups G^x, $x \in G$, are pairwise conjugate.

Recall that the *normalizer* $Nr(H)$ of a subgroup H of a group G consists of all elements $g \in G$ such that $gHg^{-1} = H$. This is the largest subgroup of G containing H as a normal subgroup.

***28.E*x.** The group of all automorphisms of a homogeneous G-set X is isomorphic to $N(H)/_H$, where H is the isotropy group of a certain point in X.

28.E*x.*1. If two points $x, y \in X$ have the same isotropy group, then there exists an automorphism of X that sends x to y.

28°2x. Continuous Action

We speak about a *left G-space* X if X is a topological space, G is a topological group acting on X, and the action $G \times X \to X$ is continuous (as a map). All terminology (and definitions) concerning G-sets extends to G-spaces literally.

Note that if G is a discrete group, then any action of G by homeomorphisms is continuous and thus provides a G-space.

28.Fx. Let X be a left G-space. Then the natural map $\phi : G \to \operatorname{Top} X$ induced by this action is a group homomorphism.

28.Gx. If in the assumptions of Problem *28.Fx* the G-space X is Hausdorff and locally compact, then the induced homomorphism $\phi : G \to \operatorname{Top} X$ is continuous.

> ***28.1x.*** In each of the following situations, check if we have a continuous action and a continuous homomorphism $G \to \operatorname{Top} X$:
>
> (1) G is a topological group, $X = G$, and G acts on X by left (or right) translations, or by conjugation;
> (2) G is a topological group, $H \subset G$ is a subgroup, $X = G/_H$, and G acts on X via $g(aH) = (ga)H$;
> (3) $G = GL(n, K)$ (where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$)), and G acts on K^n via matrix multiplication;
> (4) $G = GL(n, K)$ (where $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$), and G acts on KP^{n-1} via matrix multiplication;
> (5) $G = O(n, \mathbb{R})$, and G acts on S^{n-1} via matrix multiplication;
> (6) the (additive) group $\mathbb{R}$ acts on the torus $S^1 \times \dots \times S^1$ according to formula $(t, (w_1, \dots, w_r)) \mapsto (e^{2\pi i a_1 t} w_1, \dots, e^{2\pi i a_r t} w_r)$; this action is an *irrational flow* if $a_1, \dots, a_r$ are linearly independent over $\mathbb{Q}$.

If the action of G on X is not effective, then we can consider its kernel

$$G^{\mathrm{Ker}} = \{g \in G \mid gx = x \quad \text{for all} \quad x \in X\}.$$

This kernel is a closed normal subgroup of G, and the topological group $G/_{G^{\mathrm{Ker}}}$ acts naturally and effectively on X.

28.Hx. The formula $gG^{\mathrm{Ker}}(x) = gx$ determines an effective continuous action of $G/_{G^{\mathrm{Ker}}}$ on X.

A group G acts *properly discontinuously* on X if for any compact set $C \subset X$ the set $\{g \in G \mid (gC) \cap C \neq \varnothing\}$ is finite.

28.Ix. If G acts properly discontinuously and effectively on a Hausdorff locally compact space X, then $\phi(G)$ is a discrete subset of $\operatorname{Top} X$. (Here, as before, $\phi : G \to \operatorname{Top} X$ is the monomorphism induced by the G-action.) In particular, G is a discrete group.

> ***28.2x.*** List, up to similarity, all triangles $T \subset \mathbb{R}^2$ such that the reflections in the sides of T generate a group acting on $\mathbb{R}^2$ properly discontinuously.

28°3x. Orbit Spaces

Let X be a G-space. For $x \in X$, the set $G(x) = \{gx \mid g \in G\}$ is the *orbit* of x. In terms of orbits, the action of G on X is transitive iff it has only one orbit. For $A \subset X$ and $E \subset G$, we put $E(A) = \{ga \mid g \in E,\ a \in A\}$.

28.Jx. *Let G be a compact topological group acting on a Hausdorff space X. Then for any $x \in X$ the canonical map $G/_{G^x} \to G(x)$ is a homeomorphism.*

> ***28.3x.*** Give an example where X is Hausdorff, but $G/_{G_x}$ is not homeomorphic to $G(x)$.

28.Kx. If a compact topological group G acts on a compact Hausdorff space X, then $X/_G$ is a compact Hausdorff space.

> ***28.4x.*** Let G be a compact group, X a Hausdorff G-space, $A \subset X$. If A is closed (respectively, compact), then so is $G(A)$.
>
> ***28.5x.*** Consider the canonical action of $G = \mathbb{R} \smallsetminus 0$ on $X = \mathbb{R}$ (by multiplication). Find all orbits and all isotropy subgroups of this action. Recognize $X/_G$ as a topological space.
>
> ***28.6x.*** Let G be the group generated by reflections in the sides of a rectangle in $\mathbb{R}^2$. Recognize the quotient space $\mathbb{R}^2/_G$ as a topological space. Recognize the group G.
>
> ***28.7x.*** Let G be the group from Problem *28.6x*, and let $H \subset G$ be the subgroup of index 2 constituted by the orientation-preserving elements in G. Recognize the quotient space $\mathbb{R}^2/_H$ as a topological space. Recognize the groups G and H.
>
> ***28.8x.*** Consider the (diagonal) action of the torus $G = (S^1)^{n+1}$ on $X = \mathbb{C}P^n$ via $(z_0, z_1, \dots, z_n) \mapsto (\theta_0 z_0, \theta_1 z_1, \dots, \theta_n z_n)$. Find all orbits and isotropy subgroups. Recognize $X/_G$ as a topological space.
>
> ***28.9x.*** Consider the canonical action (by permutations of coordinates) of the symmetric group $G = \mathbb{S}_n$ on $X = \mathbb{R}^n$ and $X = \mathbb{C}^n$, respectively. Recognize $X/_G$ as a topological space.
>
> ***28.10x.*** Let $G = SO(3)$ act on the space X of symmetric 3×3 real matrices with trace 0 by conjugation $x \mapsto gxg^{-1}$. Recognize $X/_G$ as a topological space. Find all orbits and isotropy groups.

28°4x. Homogeneous Spaces

A G-space is *homogeneous* it the action of G is transitive.

28.Lx. Let G be a topological group, $H \subset G$ a subgroup. Then G is a homogeneous H-space under the translation action of H. The quotient space $G/_H$ is a homogeneous G-space under the induced action of G.

28.Mx. *Let X be a Hausdorff homogeneous G-space. If X and G are locally compact and G is second countable, then X is homeomorphic to $G/_{G^x}$ for any $x \in X$.*

28.Nx. Let X be a homogeneous G-space. Then the canonical map $G/_{G^x} \to X$, $x \in X$, is a homeomorphism iff it is open.

28.11x. Show that $O(n+1)/_{O(n)} = S^n$ and $U(n)/_{U(n-1)} = S^{2n-1}$.

28.12x. Show that $O(n+1)/_{O(n) \times O(1)} = \mathbb{R}P^n$ and $U(n)/_{U(n-1) \times U(1)} = \mathbb{C}P^n$.

28.13x. Show that $Sp(n)/_{Sp(n-1)} = S^{4n-1}$, where

$$Sp(n) = \{A \in GL(\mathbb{H}) \mid AA^* = I\}.$$

28.14x. Represent the torus $S^1 \times S^1$ and the Klein bottle as homogeneous spaces.

28.15x. Give a geometric interpretation of the following homogeneous spaces: 1) $O(n)/_{O(1)^n}$, 2) $O(n)/_{O(k) \times O(n-k)}$, 3) $O(n)/_{SO(k) \times O(n-k)}$, and 4) $O(n)/_{O(k)}$.

28.16x. Represent $S^2 \times S^2$ as a homogeneous space.

28.17x. Recognize $SO(n,1)/_{SO(n)}$ as a topological space.

Proofs and Comments

***26.A*x** Use the fact that any auto-homeomorphism of a discrete space is continuous.

***26.C*x** Any translation is continuous, and the translations by a and a^{-1} are mutually inverse.

***26.D*x** Any conjugation is continuous, and the conjugations by g and g^{-1} are mutually inverse.

***26.E*x** The sets xU, Ux, and U^{-1} are the images of U under the homeomorphisms L_x and R_x of the left and right translations through x and passage to the inverse element (i.e., reversing), respectively.

***26.F*x** Let $V \subset G$ be an open set, $a \in V$. If a neighborhood $U \in \Gamma$ is such that $U \subset a^{-1}V$, then $aU \subset V$. By Theorem *3.A*, Σ is a base for topology of G.

***26.G*x** If U is a neighborhood of 1, then $U \cap U^{-1}$ is a symmetric neighborhood of 1.

***26.H*x** By the continuity of multiplication, 1 has two neighborhoods V_1 and V_2 such that $V_1V_2 \subset U$. Put $V = V_1 \cap V_2$.

***26.J*x** Let W be a symmetric neighborhood such that $1_G \in W$ and $W^3 \subset U$. Since C is compact, C is covered by finitely many sets of the form $W_1 = x_1W, \dots, W_n = x_nW$ with $x_1, \dots, x_n \in C$. Put $V = \bigcap(x_iWx_i^{-1})$. Clearly, V is a neighborhood of 1_G. If $x \in C$, then $x = x_iw_i$ for suitable $i, w_i \in W$. Finally, we have

$$x^{-1}Vx = w_i^{-1}x_i^{-1}Vx_iw_i \subset w_i^{-1}Ww_i \subset W^3 \subset U.$$

***26.K*x** If 1_G is closed, then all singletons in G are closed. Therefore, G satisfies T_1 iff 1_G is closed. Let us prove that in this case the group G is also Hausdorff. Consider $g \neq 1$ and take a neighborhood U of 1_G not containing g. By *26.15x*, 1_G has a symmetric neighborhood V such that $V^2 \subset U$. Verify that gV and V are disjoint, whence it follows that G is Hausdorff.

***26.L*x** $\boxed{\Longrightarrow}$ Use *14.C* $\boxed{\Longleftarrow}$ In this case, each element of G is the intersection of its neighborhoods. Hence, G satisfies the first separation axiom, and it remains to apply *26.Kx*.

26.M*x.*1 It suffices to take a symmetric neighborhood V such that $V^2 \subset U$. Indeed, then for any $g \notin U$ the neighborhoods gV and V are disjoint, whence $\operatorname{Cl} V \subset U$.

***26.O*x** Let W be an open set, $g \in W$. Let V be a symmetric neighborhood of 1_G with $V^2 \subset W$. There 1_G has a neighborhood $U \in \Gamma$ such that $U \subset V$. There exists $a \in S$ such that $a \in gU^{-1}$. Then $g \in aU$ and $a \in gU^{-1} \subset gV^{-1} = gV$. Therefore, $aU \subset aV \subset gV^2 \subset W$.

***26.P*x** This immediately follows from *26.O*x.

***27.B*x** This follows from *26.16*x.

***27.D*x** If U is open, then UH (respectively, HU) is open, see *26.11*x.

***27.E*x** Let G be the group, $H \subset G$ the subgroup. The space G/H of left cosets satisfies the first separation axiom since gH is closed in G for any $g \in G$. Observe that every open set in G/H has the form $\{gH \mid g \in U\}$, where U is an open set in G. Hence, it is sufficient to check that for every open neighborhood U of 1_G in G the unity 1_G has a neighborhood V in G such that $\operatorname{Cl} VH \subset UH$. Pick a symmetric neighborhood V with $V^2 \subset U$, see *26.15*x. Let $x \in G$ belong to $\operatorname{Cl} VH$. Then Vx contains a point vh with $v \in V$ and $h \in H$, so that there exists $v' \in V$ such that $v'x = vh$, whence $x \in V^{-1}VH = V^2H \subset UH$.

***27.F*x** (*Compactness*) First, we check that if H is compact, then the projection $G \to G/H$ is a closed map. Let $F \subset G$ be a closed set, $x \notin FH$. Since FH is closed (see *26.14*x), x has a neighborhood U disjoint with FH. Then UH is disjoint with FH. Hence, the projection is closed. Now, consider a family of closed sets in G with finite intersection property. Their images also form a family of closed sets in G/H with finite intersection property. Since G/H is compact, the images have a nonempty intersection. Therefore, there is $g \in G$ such that the traces of the closed sets in the family on gH have finite intersection property. Finally, since gH is compact, the closed sets in the family have a nonempty intersection.
(*Connectedness*) Let $G = U \cup V$, where U and V are disjoint open subsets of G. Since all cosets gH, $g \in G$, are connected, each of them is contained either in U or in V. Hence, G is decomposed into UH and VH, which yields a decomposition of G/H in two disjoint open subsets. Since G/H is connected, either UH or VH is empty. Therefore, either U or V is empty.

***27.H*x** Let C be the connected component of 1_G in a topological group G. Then C^{-1} is connected and contains 1_G, whence $C^{-1} \subset C$. For any $g \in C$, the set gC is connected and meets C, whence $gC \subset C$. Therefore, C is a subgroup of G. C is closed since connected components are closed. C is normal since gCg^{-1} is connected and contains 1_G, whatever $g \in G$ is.

***27.I*x** Let G be a topological group, H a normal subgroup of G, $a, b \in G$ two elements. Let $\overline{W}$ be a neighborhood of the coset abH in G/H. The preimage of $\overline{W}$ in G is an open set W consisting of cosets of H and containing ab. In particular, W is a neighborhood of ab. Since the multiplication in G is continuous, a and b have neighborhoods U and V, respectively, such that $UV \subset W$. Then $(UH)(VH) = (UV)H \subset WH$. Therefore, multiplication of elements in the quotient group determines a continuous map $G/H \times G/H \to G/H$. Prove on your own that the map $G/H \times G/H : aH \to a^{-1}H$ is also continuous.

***27.J*x** This is special case of *27.Dx*.

***27.K*x** If $\{U_i\}$ is a countable (neighborhood) base in G, then $\{U_iH\}$ is a countable (neighborhood) base in G/H.

***27.L*x** This is a special case of *27.Ex*.

***27.M*x** $\boxed{\Rightarrow}$ In this case, all cosets of H are also open. Therefore, each singleton in G/H is open. $\boxed{\Leftarrow}$ If $1_{G/H}$ is open in G/H, then H is open in G by the definition of the quotient topology.

***27.N*x** $\boxed{\Rightarrow}$ Obvious. $\boxed{\Leftarrow}$ Let $a \in G$, and let $b = f(a) \in H$. For any neighborhood U of b, the set $b^{-1}U$ is a neighborhood of 1_H in H. Therefore, 1_G has a neighborhood V in G such that $f(V) \subset b^{-1}U$. Then aV is a neighborhood of a, and we have $f(aV) = f(a)f(V) = bf(V) \subset bb^{-1}U = U$. Hence, f is continuous at each point $a \in G$, i.e., f is a topological group homomorphism.

***27.O*x** $\boxed{\Rightarrow}$ Each open subset of $G/\operatorname{Ker} f$ has the form $U \cdot \operatorname{Ker} f$, where U is an open subset of G. Since $f/S(f)(U \cdot \operatorname{Ker} f) = f(U)$, the map $f/S(f)$ is open.

$\boxed{\Leftarrow}$ Since the projection $G \to G/\operatorname{Ker} f$ is open (see *27.Dx*), the map f is open if so is $f/S(f)$.

***27.P*x** Combine *27.Ox*, *26.Kx*, and *16.Y*.

***27.Q*x** This follows from *27.Ox* since the exponential map $\mathbb{R} \to S^1 : x \mapsto e^{2\pi xi}$ is open.

***27.S*x** The groups are not isomorphic since only one of them is compact. The exponential map $x \mapsto e^{2\pi xi}$ determines a local isomorphism from $\mathbb{R}$ to S^1.

***27.V*x** The map $A \times B \to G : (a, b) \mapsto ab$ is a continuous bijection. To see that it is a homeomorphism, observe that it is open since for any neighborhoods of unity, $U \subset A$ and $V \subset B$, and any points $a \in A$ and $b \in B$, the product $UaVb = abU'V'$, where $U' = b^{-1}a^{-1}Uab$ and $V' = b^{-1}Vb$, contains abW', where W' is a neighborhood of 1_G contained in $U'V'$.

***27.W*x** This immediately follows from *3.8*.

***27.X*x** The map $\operatorname{Top} X \to \operatorname{Top} X : g \mapsto g^{-1}$ is continuous because it preserves the subbase for the topological structure on $\operatorname{Top} X$. It remains to apply *27.Xx.1*.

27.Xx.1 It suffices to check that the preimage of every element of a subbase is open. For $W(C,U)$, this is a special case of *24.Sx*, where we showed that for any $gh \in W(C,U)$ there is an open U', $h(C) \subset U' \subset g^{-1}(U)$, such that $\operatorname{Cl} U'$ is compact, $h \in W(C,U')$, $g \in W(\operatorname{Cl} U',U)$, and

$$gh \in W(\operatorname{Cl} U',U) \circ W(C,U') \subset W(C,U).$$

The case of $(W(C,U))^{-1}$ reduces to the previous one because for any $gh \in (W(C,U))^{-1}$ we have $h^{-1}g^{-1} \in W(C,U)$, and so, applying the above construction, we obtain an open U' such that $g^{-1}(C) \subset U' \subset h(U)$, $\operatorname{Cl} U'$ is compact, $g^{-1} \in W(C,U')$, $h^{-1} \in W(\operatorname{Cl} U',U)$, and

$$h^{-1}g^{-1} \in W(\operatorname{Cl} U',U) \circ W(C,U') \subset W(C,U).$$

Finally, we have $g \in (W(C,U'))^{-1}$, $h \in (W(\operatorname{Cl} U',U))^{-1}$, and

$$gh \in (W(C,U'))^{-1} \circ (W(\operatorname{Cl} U',U))^{-1} \subset (W(C,U))^{-1}.$$

We observe that the above map is continuous even for the pure compact-open topology on $\operatorname{Top} X$.

28.Gx It suffices to check that the preimage of every element of a subbase is open. For $W(C,U)$, this is a special case of *24.Vx*. Let $\phi(g) \in (W(C,U))^{-1}$. Then $\phi(g^{-1}) \in W(C,U)$, and therefore g^{-1} has an open neighborhood V in G with $\phi(V) \subset W(C,U)$. It follows that V^{-1} is an open neighborhood of g in G and $\phi(V^{-1}) \subset (W(C,U))^{-1}$. (The assumptions about X are needed only to ensure that $\operatorname{Top} X$ is a topological group.)

28.Ix Let us check that 1_G is an isolated point of G. Consider an open set V with compact closure. Let $U \subset V$ be an open subset with compact closure $\operatorname{Cl} U \subset V$. Then, for each of finitely many $g_k \in G$ with $g_k(U) \cap V \neq \varnothing$, let $x_k \in X$ be a point with $g_k(x_k) \neq x_k$, and let U_k be an open neighborhood of x_k disjoint with $g_k(x_k)$. Finally, $G \cap W(\operatorname{Cl} U, V) \cap \bigcap W(x_k, U_k)$ contains only 1_G.

28.Jx The space $G/_{G^x}$ is compact, the orbit $G(x) \subset X$ is Hausdorff, and the map $G/_{G^x} \to G(x)$ is a continuous bijection. It remains to apply *16.Y*.

28.Kx To prove that $X/_G$ is Hausdorff, consider two disjoint orbits, $G(x)$ and $G(y)$. Since $G(y)$ is compact, there are disjoint open sets $U \ni x$ and $V \supset G(y)$. Since $G(x)$ is compact, there is a finite number of elements $g_k \in G$ such that $\bigcup g_k U$ covers $G(x)$. Then $\operatorname{Cl}(\bigcup g_k U) = \bigcup \operatorname{Cl} g_k U = \bigcup g_k \operatorname{Cl} U$ is disjoint with $G(y)$, which shows that $X/_G$ is Hausdorff. (Note that this part of the proof does not involve the compactness of X.) Finally, $X/_G$ is compact as a quotient of the compact space X.

28.Mx It suffices to prove that the canonical map $f : G/_{G^x} \to X$ is open (see *28.Nx*).

Take a neighborhood $V \subset G$ of 1_G with compact closure and a neighborhood $U \subset G$ of 1_G with $\operatorname{Cl} U \cdot \operatorname{Cl} U \subset V$. Since G contains a dense countable set, it follows that there is a sequence $g_n \in G$ such that $\{g_n U\}$ is an open cover of G. It remains to prove that at least one of the sets $f(g_n U) = g_n f(U) = g_n U(x)$ has nonempty interior.
Assume the contrary. Then, using the local compactness of X, its Hausdorff property, and the compactness of $f(g_n \operatorname{Cl} U)$, we construct by induction a sequence $W_n \subset X$ of nested open sets with compact closure such that W_n is disjoint with $g_k U x$ with $k < n$ and $g_n U x \cap W_n$ is closed in W_n. Finally, we obtain nonempty $\bigcap W_n$ disjoint with $G(x)$, a contradiction.

***28.N*x** The canonical map $G/G^x \to X$ is continuous and bijective. Hence, it is a homeomorphism iff it is open (and iff it is closed).