*Part 3*

# Topological Manifolds

This part is devoted to study of the most important topological spaces, the spaces which provide a scene for most of geometric branches in mathematics such as Differential Geometry and Analytical Mechanics.

Chapter X

# Manifolds

## 44. Locally Euclidean Spaces

### 44°1. Definition of Locally Euclidean Space

Let $n$ be a non-negative integer. A topological space $X$ is called a *locally Euclidean space of dimension* $n$ if each point of $X$ has a neighborhood homeomorphic either to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Recall that $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 \geq 0\}$, it is defined for $n \geq 1$.

***44.A.*** The notion of 0-dimensional locally Euclidean space coincides with the notion of discrete topological space.

***44.B.*** Prove that the following spaces are locally Euclidean:

(1) $\mathbb{R}^n$,
(2) any open subset of $\mathbb{R}^n$,
(3) $S^n$,
(4) $\mathbb{R}P^n$,
(5) $\mathbb{C}P^n$,
(6) $\mathbb{R}^n_+$,
(7) any open subset of $\mathbb{R}^n_+$,
(8) $D^n$,
(9) torus $S^1 \times S^1$,
(10) handle,
(11) sphere with handles,

(12) sphere with holes,

(13) Klein bottle,

(14) sphere with crosscaps.

***44.1.*** Prove that an open subspace of a locally Euclidean space of dimension $n$ is a locally Euclidean space of dimension $n$.

***44.2.*** Prove that a bouquet of two circles is not locally Euclidean.

***44.C.*** If $X$ is a locally Euclidean space of dimension $p$ and $Y$ is a locally Euclidean space of dimension $q$ then $X \times Y$ is a locally Euclidean space of dimension $p + q$.

### 44°2. Dimension

***44.D.*** Can a topological space be simultaneously a locally Euclidean space of dimension both 0 and $n > 0$?

***44.E.*** Can a topological space be simultaneously a locally Euclidean space of dimension both 1 and $n > 1$?

***44.3.*** Prove that any nonempty open connected subset of a locally Euclidean space of dimension 1 can be made disconnected by removing two points.

***44.4.*** Prove that any nonempty locally Euclidean space of dimension $n > 1$ contains a nonempty open set, which cannot be made disconnected by removing any two points.

***44.F.*** Can a topological space be simultaneously a locally Euclidean space of dimension both 2 and $n > 2$?

***44.G.*** Let $U$ be an open subset of $\mathbb{R}^2$ and a $p \in U$. Prove that $\pi_1(U \smallsetminus \{p\})$ admits an epimorphism onto $\mathbb{Z}$.

***44.H.*** Deduce from *44.G* that a topological space cannot be simultaneously a locally Euclidean space of dimension both 2 and $n > 2$.

We see that dimension of locally Euclidean topological space is a topological invariant at least for the cases when it is not greater than 2. In fact, this holds true without that restriction. However, one needs some technique to prove this. One possibility is provided by dimension theory, see, e.g., W. Hurewicz and H. Wallman, *Dimension Theory* Princeton, NJ, 1941. Other possibility is to generalize the arguments used in *44.H* to higher dimensions. However, this demands a knowledge of high-dimensional homotopy groups.

***44.5.*** Deduce that a topological space cannot be simultaneously a locally Euclidean space of dimension both $n$ and $p > n$ from the fact that $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$. Cf. *44.H*

### 44°3. Interior and Boundary

A point $a$ of a locally Euclidean space $X$ is said to be an *interior* point of $X$ if $a$ has a neighborhood (in $X$) homeomorphic to $\mathbb{R}^n$. A point $a \in X$, which is not interior, is called a *boundary* point of $X$.

***44.6.*** Which points of $\mathbb{R}^n_+$ have a neighborhood homeomorphic to $\mathbb{R}^n_+$?

***44.I.*** Formulate a definition of boundary point independent of a definition for interior point.

Let $X$ be a locally Euclidean space of dimension $n$. The set of all interior points of $X$ is called the *interior* of $X$ and denoted by $\operatorname{int} X$. The set of all boundary points of $X$ is called the *boundary* of $X$ and denoted by $\partial X$.

These terms (interior and boundary) are used also with different meaning. The notions of boundary and interior points of a set in a topological space and the interior part and boundary of a set in a topological space are introduced in general topology, see Section 6. They have almost nothing to do with the notions discussed here. In both senses the terminology is classical, which is impossible to change. This does not create usually a danger of confusion.

Notations are not as commonly accepted as words. We take an easy opportunity to select unambiguous notations: we denote the interior part of a set $A$ in a topological space $X$ by $\operatorname{Int}_X A$ or $\operatorname{Int} A$, while the interior of a locally Euclidean space $X$ is denoted by $\operatorname{int} X$; the boundary of a set in a topological space is denoted by symbol Fr, while the boundary of locally Euclidean space is denoted by symbol $\partial$.

***44.J.*** For a locally Euclidean space $X$ the interior $\operatorname{int} X$ is an open dense subset of $X$, the boundary $\partial X$ is a closed nowhere dense subset of $X$.

***44.K.*** The interior of a locally Euclidean space of dimension $n$ is a locally Euclidean space of dimension $n$ without boundary (i.e., with empty boundary; in symbols: $\partial(\operatorname{int} X) = \varnothing$).

***44.L.*** The boundary of a locally Euclidean space of dimension $n$ is a locally Euclidean space of dimension $n-1$ without boundary (i.e., with empty boundary; in symbols: $\partial(\partial X) = \varnothing$).

***44.M.*** $\operatorname{int} \mathbb{R}^n_+ \supset \{x \in \mathbb{R}^n : x_1 > 0\}$ and

$$\partial \mathbb{R}^n_+ \subset \{x \in \mathbb{R}^n : x_1 = 0\}.$$

***44.7.*** For any $x, y \in \{x \in \mathbb{R}^n : x_1 = 0\}$, there exists a homeomorphism $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ with $f(x) = y$.

***44.N.*** Either $\partial \mathbb{R}^n_+ = \varnothing$ (and then $\partial X = \varnothing$ for any locally Euclidean space $X$ of dimension $n$), or $\partial \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 = 0\}$.

In fact, the second alternative holds true. However, this is not easy to prove for any dimension.

***44.O.*** Prove that $\partial\mathbb{R}^1_+ = \{0\}$.

***44.P.*** Prove that $\partial\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 \,:\, x_1 = 0\}$. (Cf. *44.G.*)

***44.8.*** Deduce that a $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,:\, x_1 = 0\}$ from $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$. (Cf. *44.P*, *44.5*)

***44.Q.*** Deduce from $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,:\, x_1 = 0\}$ for all $n \geq 1$ that

$$\operatorname{int}(X \times Y) = \operatorname{int} X \times \operatorname{int} Y$$

and

$$\partial(X \times Y) = (\partial(X) \times Y) \cup (X \times \partial Y).$$

The last formula resembles Leibniz formula for derivative of a product.

***44.R. Riddle.*** Can this be a matter of chance?

***44.S.*** Prove that

(1) $\partial(I \times I) = (\partial I \times I) \cup (I \times \partial I)$,
(2) $\partial D^n = S^{n-1}$,
(3) $\partial(S^1 \times I) = S^1 \times \partial I = S^1 \amalg S^1$,
(4) the boundary of Möbius strip is homeomorphic to circle.

***44.T Corollary.*** Möbius strip is not homeomorphic to cylinder $S^1 \times I$.

# 45. Manifolds

## 45°1. Definition of Manifold

A topological space is called a *manifold* of dimension $n$ if it is

- locally Euclidean of dimension $n$,
- second countable,
- Hausdorff.

***45.A.*** Prove that the three conditions of the definition are independent (i.e., there exist spaces not satisfying any one of the three conditions and satisfying the other two.)

***45.A.1.*** Prove that $\mathbb{R} \cup_i \mathbb{R}$, where $i : \{x \in \mathbb{R} : x < 0\} \to \mathbb{R}$ is the inclusion, is a non-Hausdorff locally Euclidean space of dimension one.

***45.B.*** Check whether the spaces listed in Problem *44.B* are manifolds.

A compact manifold without boundary is said to be *closed.* As in the case of interior and boundary, this term coincides with one of the basic terms of general topology. Of course, the image of a closed manifold under embedding into a Hausdorff space is a closed subset of this Hausdorff space (as any compact subset of a Hausdorff space). However absence of boundary does not work here, and even non-compact manifolds may be closed subsets. They are closed in themselves, as any space. Here we meet again an ambiguity of classical terminology. In the context of manifolds the term closed relates rather to the idea of a closed surface.

## 45°2. Components of Manifold

***45.C.*** *A connected component of a manifold is a manifold.*

***45.D.*** *A connected component of a manifold is path-connected.*

***45.E.*** *A connected component of a manifold is open in the manifold.*

***45.F.*** *A manifold is the sum of its connected components.*

***45.G.*** The set of connected components of any manifold is countable. If the manifold is compact, then the number of the components is finite.

***45.1.*** Prove that a manifold is connected, iff its interior is connected.

***45.H.*** The fundamental group of a manifold is countable.

### 45°3. Making New Manifolds out of Old Ones

***45.I.*** Prove that an open subspace of a manifold of dimension $n$ is a manifold of dimension $n$.

***45.J.*** The interior of a manifold of dimension $n$ is a manifold of dimension $n$ without boundary.

***45.K.*** The boundary of a manifold of dimension $n$ is a manifold of dimension $n-1$ without boundary.

> *45.2.* The boundary of a compact manifold of dimension $n$ is a closed manifold of dimension $n-1$.

***45.L.*** If $X$ is a manifold of dimension $p$ and $Y$ is a manifold of dimension $q$ then $X \times Y$ is a manifold of dimension $p+q$.

***45.M.*** Prove that a covering space (in narrow sense) of a manifold is a manifold of the same dimension.

***45.N.*** Prove that if the total space of a covering is a manifold then the base is a manifold of the same dimension.

***45.O.*** Let $X$ and $Y$ be manifolds of dimension $n$, $A$ and $B$ components of $\partial X$ and $\partial Y$ respectively. Then for any homeomorphism $h : B \to A$ the space $X \cup_h Y$ is a manifold of dimension $n$.

> ***45.O.1.*** Prove that the result of gluing of two copy of $\mathbb{R}^n_+$ by the identity map of the boundary hyperplane is homeomorphic to $\mathbb{R}^n$.

***45.P.*** Let $X$ and $Y$ be manifolds of dimension $n$, $A$ and $B$ closed subsets of $\partial X$ and $\partial Y$ respectively. If $A$ and $B$ are manifolds of dimension $n-1$ then for any homeomorphism $h : B \to A$ the space $X \cup_h Y$ is a manifold of dimension $n$.

### 45°4. Double

***45.Q.*** Can a manifold be embedded into a manifold of the same dimension without boundary?

Let $X$ be a manifold. Denote by $DX$ the space $X \cup_{\mathrm{id}_{\partial X}} X$ obtained by gluing of two copies of $X$ by the identity mapping $\mathrm{id}_{\partial X} : \partial X \to \partial X$ of the boundary.

***45.R.*** Prove that $DX$ is a manifold without boundary of the same dimension as $X$.

$DX$ is called the *double* of $X$.

***45.S.*** Prove that a double of a manifold is compact, iff the original manifold is compact.

### 45°5x. Collars and Bites

Let $X$ be a manifold. An embedding $c : \partial X \times I \to X$ such that $c(x, 0) = x$ for each $x \in \partial X$ is called a *collar* of $X$. A collar can be thought of as a neighborhood of the boundary presented as a cylinder over boundary.

***45.Ax.*** *Every manifold has a collar.*

Let $U$ be an open set in the boundary of a manifold $X$. For a continuous function $\varphi : \partial X \to \mathbb{R}_+$ with $\varphi^{-1}(0, \infty) = U$ set

$$B_\varphi = \{(x, t) \in \partial X \times \mathbb{R}_+ \,:\, t \le \varphi(x)\}.$$

A *bite* on $X$ at $U$ is an embedding $b : B_\varphi \to X$ with some $\varphi : \partial X \to \mathbb{R}_+$ such that $b(x, 0) = x$ for each $x \in \partial X$.

This is a generalization of collar. Indeed, a collar is a bite at $U = \partial X$ with $\varphi = 1$.

***45.Ax.1.*** Prove that if $U \subset \partial X$ is contained in an open subset of $X$ homeomorphic to $\mathbb{R}^n_+$, then there exists a bite of $X$ at $U$.

***45.Ax.2.*** Prove that for any bite $b : B \to X$ of a manifold $X$ the closure of $X \smallsetminus b(B)$ is a manifold.

***45.Ax.3.*** Let $b_1 : B_1 \to X$ be a bite of $X$ and $b_2 : B_2 \to \mathrm{Cl}(X \smallsetminus b_1(B_1))$ be a bite of $\mathrm{Cl}(X \smallsetminus b_1(B_1))$. Construct a bite $b : B \to X$ of $X$ with $b(B) = b_1(B_1) \cup b_2(B_2)$.

***45.Ax.4.*** Prove that if there exists a bite of $X$ at $\partial X$ then there exists a collar of $X$.

***45.Bx.*** *For any two collars $c_1, c_2 : \partial X \times I \to X$ there exists a homeomorphism $h : X \to X$ with $h(x) = x$ for $x \in \partial X$ such that $h \circ c_1 = c_2$.*

This means that a collar is unique up to homeomorphism.

***45.Bx.1.*** For any collar $c : \partial X \times I \to X$ there exists a collar $c' : \partial X \times I \to X$ such that $c(x, t) = c'(x, t/2)$.

***45.Bx.2.*** For any collar $c : \partial X \times I \to X$ there exists a homeomorphism

$$h : X \to X \cup_{x \mapsto (x,1)} \partial X \times I$$

with $h(c(x, t)) = (x, t)$.

# 46. Isotopy

## 46°1. Isotopy of Homeomorphisms

Let $X$ and $Y$ be topological spaces, $h, h' : X \to Y$ homeomorphisms. A homotopy $h_t : X \to Y$, $t \in [0,1]$ connecting $h$ and $h'$ (i.e., with $h_0 = h$, $h_1 = h'$) is called an *isotopy* between $h$ and $h'$ if $h_t$ is a homeomorphism for each $t \in [0,1]$. Homeomorphisms $h$, $h'$ are said to be *isotopic* if there exists an isotopy between $h$ and $h'$.

***46.A.*** Being isotopic is an equivalence relation on the set of homeomorphisms $X \to Y$.

***46.B.*** Find a topological space $X$ such that homotopy between homeomorphisms $X \to X$ does not imply isotopy.

This means that isotopy classification of homeomorphisms can be more refined than homotopy classification of them.

***46.1.*** Classify homeomorphisms of circle $S^1$ to itself up to isotopy.

***46.2.*** Classify homeomorphisms of line $\mathbb{R}^1$ to itself up to isotopy.

The set of isotopy classes of homeomorphisms $X \to X$ (i.e. the quotient of the set of self-homeomorphisms of $X$ by isotopy relation) is called the *mapping class group* or *homeotopy group* of $X$.

***46.C.*** For any topological space $X$, the mapping class group of $X$ is a group under the operation induced by composition of homeomorphisms.

***46.3.*** Find the mapping class group of the union of the coordinate lines in the plane.

***46.4.*** Find the mapping class group of the union of bouquet of two circles.

## 46°2. Isotopy of Embeddings and Sets

Homeomorphisms are topological embeddings of special kind. The notion of isotopy of homeomorphism is extended in an obvious way to the case of embeddings. Let $X$ and $Y$ be topological spaces, $h, h' : X \to Y$ topological embeddings. A homotopy $h_t : X \to Y$, $t \in [0,1]$ connecting $h$ and $h'$ (i.e., with $h_0 = h$, $h_1 = h'$) is called an *(embedding) isotopy* between $h$ and $h'$ if $h_t$ is an embedding for each $t \in [0,1]$. Embeddings $h$, $h'$ are said to be *isotopic* if there exists an isotopy between $h$ and $h'$.

***46.D.*** Being isotopic is an equivalence relation on the set of embeddings $X \to Y$.

A family $A_t$, $t \in I$ of subsets of a topological space $X$ is called an *isotopy of the set* $A = A_0$, if the graph $\Gamma = \{(x,t) \in X \times I \,|\, x \in A_t\}$ of the family is fibrewise homeomorphic to the cylinder $A \times I$, i. e. there exists a homeomorphism $A \times I \to \Gamma$ mapping $A \times \{t\}$ to $\Gamma \cap X \times \{t\}$ for any $t \in I$. Such a homeomorphism gives rise to an isotopy of embeddings $\Phi_t : A \to X$, $t \in I$ with $\Phi_0 = \text{in}$, $\Phi_t(A) = A_t$. An isotopy of a subset is also called a *subset isotopy.* Subsets $A$ and $A'$ of the same topological space $X$ are said to be *isotopic in* $X$, if there exists a subset isotopy $A_t$ of $A$ with $A' = A_1$.

***46.E.*** It is easy to see that this is an equivalence relation on the set of subsets of $X$.

As it follows immediately from the definitions, any embedding isotopy determines an isotopy of the image of the initial embedding and any subset isotopy is accompanied with an embedding isotopy. However the relation between the notions of subset isotopy and embedding isotopy is not too close because of the following two reasons:

(1) an isotopy $\Phi_t$ accompanying a subset isotopy $A_t$ starts with the inclusion of $A_0$ (while arbitrary isotopy may start with any embedding);

(2) an isotopy accompanying a subset isotopy is determined by the subset isotopy only up to composition with an isotopy of the identity homeomorphism $A \to A$ (an isotopy of a homeomorphism is a special case of embedding isotopies, since homeomorphisms can be considered as a sort of embeddings).

An isotopy of a subset $A$ in $X$ is said to be *ambient,* if it may be accompanied with an embedding isotopy $\Phi_t : A \to X$ extendible to an isotopy $\tilde{\Phi}_t : X \to X$ of the identity homeomorphism of the space $X$. The isotopy $\tilde{\Phi}_t$ is said to be *ambient* for $\Phi_t$. This gives rise to obvious refinements of the equivalence relations for subsets and embeddings introduced above.

***46.F.*** Find isotopic, but not ambiently isotopic sets in $[0,1]$.

***46.G.*** If sets $A_1, A_2 \subset X$ are ambiently isotopic then the complements $X \smallsetminus A_1$ and $X \smallsetminus A_2$ are homeomorphic and hence homotopy equivalent.

***46.5.*** Find isotopic, but not ambiently isotopic sets in $\mathbb{R}$.

***46.6.*** Prove that any isotopic compact subsets of $\mathbb{R}$ are ambiently isotopic.

***46.7.*** Find isotopic, but not ambiently isotopic compact sets in $R^3$.

***46.8.*** Prove that any two embeddings $S^1 \to \mathbb{R}^3$ are isotopic. Find embeddings $S^1 \to \mathbb{R}^3$ that are not ambiently isotopic.

### 46°3. Isotopies and Attaching

***46.A*x.** Any isotopy $h_t : \partial X \to \partial X$ extends to an isotopy $H_t : X \to X$.

***46.B*x.** Let $X$ and $Y$ be manifolds of dimension $n$, $A$ and $B$ components of $\partial X$ and $\partial Y$ respectively. Then for any isotopic homeomorphisms $f, g : B \to A$ the manifolds $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

***46.C*x.** Let $X$ and $Y$ be manifolds of dimension $n$, let $B$ be a compact subset of $\partial Y$. If $B$ is a manifold of dimension $n-1$ then for any embeddings $f, g : B \to \partial X$ ambiently isotopic in $\partial X$ the manifolds $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

### 46°4. Connected Sums

***46.H.*** Let $X$ and $Y$ be manifolds of dimension $n$, and $\varphi : \mathbb{R}^n \to X$, $\psi : \mathbb{R}^n \to Y$ be embeddings. Then

$$X \smallsetminus \varphi(\operatorname{Int} D^n) \cup_{\psi(S^n) \to X \smallsetminus \varphi(\operatorname{Int} D^n) : \psi(a) \mapsto \varphi(a)} Y \smallsetminus \psi(\operatorname{Int} D^n)$$

is a manifold of dimension $n$.

This manifold is called a *connected sum* of $X$ and $Y$.

***46.I.*** Show that the topological type of the connected sum of $X$ and $Y$ depends not only on the topological types of $X$ and $Y$.

***46.J.*** Let $X$ and $Y$ be manifolds of dimension $n$, and $\varphi : \mathbb{R}^n \to X$, $\psi : \mathbb{R}^n \to Y$ be embeddings. Let $h : X \to X$ be a homeomorphism. Then the connected sums of $X$ and $Y$ defined via $\psi$ and $\varphi$, on one hand, and via $\psi$ and $h \circ \varphi$, on the other hand, are homeomorphic.

***46.9.*** Find pairs of manifolds connected sums of which are homeomorphic to

(1) $S^1$,
(2) Klein bottle,
(3) sphere with three crosscaps.

***46.10.*** Find a disconnected connected sum of connected manifolds. Describe, under what circumstances this can happen.

# Proofs and Comments

***44.A*** Each point in a 0-dimensional locally Euclidean space has a neighborhood homeomorphic to $\mathbb{R}^0$ and hence consisting of a single point. Therefore each point is open.

*Chapter XI*

# Classifications in Low Dimensions

In different geometric subjects there are different ideas which dimensions are low and which high. In topology of manifolds low dimension means at most 4. However, in this chapter only dimensions up to 2 will be considered, and even most of two-dimensional topology will not be touched. Manifolds of dimension 4 are the most mysterious objects of the field. Dimensions higher than 4 are easier: there is enough room for most of the constructions that topology needs.

## 47. One-Dimensional Manifolds

### 47°1. Zero-Dimensional Manifolds

This section is devoted to topological classification of manifolds of dimension one. We could skip the case of 0-dimensional manifolds due to triviality of the problem.

***47.A.*** *Two* 0*-dimensional manifolds are homeomorphic iff they have the same number of points.*

The case of 1-dimensional manifolds is also simple, but requires more detailed considerations. Surprisingly, many textbooks manage to ignore 1-dimensional manifolds absolutely.

### 47°2. Reduction to Connected Manifolds

***47.B.*** *Two manifolds are homeomorphic iff there exists a one-to-one correspondence between their components such that the corresponding components are homeomorphic.*

Thus, for topological classification of $n$-dimensional manifolds it suffices to classify only *connected* $n$-dimensional manifolds.

### 47°3. Examples

***47.C.*** What connected 1-manifolds do you know?

(1) Do you know any *closed* connected 1-manifold?
(2) Do you know a connected *compact* 1-manifold, which is not closed?
(3) What *non-compact* connected 1-manifolds do you know?
(4) Is there a *non-compact* connected 1-manifolds with boundary?

### 47°4. How to Distinguish Them From Each Other?

***47.D.*** Fill the following table with pluses and minuses.

| Manifold $X$ | Is $X$ compact? | Is $\partial X$ empty? |
|---|---|---|
| $S^1$ | | |
| $\mathbb{R}^1$ | | |
| $I$ | | |
| $\mathbb{R}^1_+$ | | |

### 47°5. Statements of Main Theorems

***47.E.*** *Any connected manifold of dimension 1 is homeomorphic to one of the following for manifolds:*

- *circle* $S^1$,
- *line* $\mathbb{R}^1$,
- *interval* $I$,
- *half-line* $\mathbb{R}^1_+$.

This theorem may be splitted into the following four theorems:

***47.F.*** *Any closed connected manifold of dimension 1 is homeomorphic to circle* $S^1$.

***47.G.*** *Any non-compact connected manifold of dimension 1 without boundary is homeomorphic to line* $\mathbb{R}^1$.

***47.H.*** *Any compact connected manifold of dimension 1 with nonempty boundary is homeomorphic to interval $I$.*

***47.I.*** *Any non-compact connected manifold of dimension one with nonempty boundary is homeomorphic to half-line $\mathbb{R}^1_+$.*

### 47°6. Lemma on 1-Manifold Covered with Two Lines

***47.J Lemma.*** *Any connected manifold of dimension 1 covered with two open sets homeomorphic to $\mathbb{R}^1$ is homeomorphic either to $\mathbb{R}^1$, or $S^1$.*

Let $X$ be a connected manifold of dimension 1 and $U, V \subset X$ be its open subsets homeomorphic to $\mathbb{R}$. Denote by $W$ the intersection $U \cap V$. Let $\varphi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$ be homeomorphisms.

***47.J.1.*** Prove that each connected component of $\varphi(W)$ is either an open interval, or an open ray, or the whole $\mathbb{R}$.

***47.J.2.*** Prove that a homeomorphism between two open connected subsets of $\mathbb{R}$ is a (strictly) monotone continuous function.

***47.J.3.*** Prove that if a sequence $x_n$ of points of $W$ converges to a point $a \in U \smallsetminus W$ then it does not converge in $V$.

***47.J.4.*** Prove that if there exists a bounded connected component $C$ of $\varphi(W)$ then $C = \varphi(W)$, $V = W$, $X = U$ and hence $X$ is homeomorphic to $\mathbb{R}$.

***47.J.5.*** In the case of connected $W$ and $U \neq V$, construct a homeomorphism $X \to \mathbb{R}$ which takes:

- $W$ to $(0, 1)$,
- $U$ to $(0, +\infty)$, and
- $V$ to $(-\infty, 1)$.

***47.J.6.*** In the case of $W$ consisting of two connected components, construct a homeomorphism $X \to S^1$, which takes:

- $W$ to $\{z \in S^1 : -1/\sqrt{2} < \operatorname{Im}(z) < 1/\sqrt{2}\}$,
- $U$ to $\{z \in S^1 : -1/\sqrt{2} < \operatorname{Im}(z)\}$, and
- $V$ to $\{z \in S^1 : \operatorname{Im}(z) < 1/\sqrt{2}\}$.

### 47°7. Without Boundary

***47.F.1.*** Deduce Theorem *47.F* from Lemma *47.I*.

***47.G.1.*** Deduce from Lemma *47.I* that for any connected non-compact one-dimensional manifold $X$ without a boundary there exists an embedding $X \to \mathbb{R}$ with open image.

***47.G.2.*** Deduce Theorem *47.G* from *47.G.1*.

### 47°8. With Boundary

***47.H.1.*** Prove that any compact connected manifold of dimension 1 can be embedded into $S^1$.

***47.H.2.*** List all connected subsets of $S^1$.

***47.H.3.*** Deduce Theorem *47.H* from *47.H.2*, and *47.H.1*.

***47.I.1.*** Prove that any non-compact connected manifold of dimension 1 can be embedded into $\mathbb{R}^1$.

***47.I.2.*** Deduce Theorem *47.I* from *47.I.1*.

### 47°9. Corollaries of Classification

***47.K.*** Prove that connected sum of closed 1-manifolds is defined up homeomorphism by topological types of summands.

***47.L.*** Which 0-manifolds bound a compact 1-manifold?

### 47°10. Orientations of 1-manifolds

*Orientation* of a *connected non-closed* 1-manifold is a linear order on the set of its points such that the corresponding interval topology (see. *7.P.* ) coincides with the topology of this manifold.

*Orientation* of a *connected closed* 1-manifold is a cyclic order on the set of its points such that the topology of this cyclic order (see **??**) coincides with the topology of the 1-manifold.

*Orientation* of an *arbitrary* 1-manifold is a collection of orientations of its connected components (each component is equipped with an orientation).

***47.M.*** *Any* 1*-manifold admits an orientation.*

***47.N.*** An orientation of 1-manifold induces an orientation (i.e., a linear ordering of points) on each subspace homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$. Vice versa, an orientation of a 1-manifold is determined by a collection of orientations of its open subspaces homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$, if the subspaces cover the manifold and the orientations agree with each other: the orientations of any two subspaces define the same orientation on each connected component of their intersection.

***47.O.*** Let $X$ be a cyclicly ordered set, $a \in X$ and $B \subset X \smallsetminus \{a\}$. Define in $X \smallsetminus \{a\}$ a linear order induced, as in **??**, by the cyclic order on $X \smallsetminus \{a\}$, and equip $B$ with the linear order induced by this linear order on $X \smallsetminus \{a\}$. Prove that if $B$ admits a bijective monotone map onto $\mathbb{R}$, or $[0;1]$, or $[0;1)$, or $(0;1]$, then this linear order on $B$ does not depend on $a$.

The construction of *47.O* allows one to define an orientation on any 1-manifold which is a subspace of an *oriented closed* 1-manifold. A 1-manifold, which is a subspace of an *oriented non-closed* 1-manifold $X$, inherits from $X$ an orientation as a linear order. Thus, any 1-manifold, which is a subspace of an *oriented* 1-manifold $X$, inherits from $X$ an orientation. This orientation is said to be *induced* by the orientation of $X$.

A topological embedding $X \to Y$ of an oriented 1-manifold to another one is said to *preserve* the orientation if it maps the orientation of $X$ to the orientation induced on the image by the orientation of $Y$.

***47.P.*** *Any two orientation preserving embeddings of an oriented connected 1-manifold $X$ to an oriented connected 1-manifold $Y$ are isotopic.*

***47.Q.*** *If two embeddings of an oriented 1-manifold $X$ to an oriented 1-manifold $Y$ are isotopic and one of the embeddings preserves the orientation, then the other one also preserves the orientation*

***47.R. [Corollary] Orientation of a closed segment is determined by the ordering of its end points.***

An orientation of a segment is shown by an arrow directed from the initial end point to the final one.

***47.S.*** *A connected 1-manifold admits two orientations. A 1-manifold consisting of $n$ connected components admits $2^n$ orientations.*

### 47°11. Mapping Class Groups

***47.T.*** Find the mapping class groups of

(1) $S^1$,
(2) $\mathbb{R}^1$,
(3) $\mathbb{R}^1_+$,
(4) $[0,1]$,
(5) $S^1 \amalg S^1$,
(6) $\mathbb{R}^1_+ \amalg \mathbb{R}^1_+$.

***47.1.*** Find the mapping class group of an arbitrary 1-manifold with finite number of components.

# 48. Two-Dimensional Manifolds: General Picture

## 48°1. Examples

***48.A.*** What connected 2-manifolds do you know?

(1) List *closed* connected 2-manifold that you know.
(2) Do you know a connected *compact* 2-manifold, which is not closed?
(3) What *non-compact* connected 2-manifolds do you know?
(4) Is there a *non-compact* connected 2-manifolds with non-empty boundary?

***48.1.*** Construct non-homeomorphic non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group.

For notions relevant to this problem see what follows.

## 48°2x. Ends and Odds

Let $X$ be a non-compact Hausdorff topological space, which is a union of an increasing sequence of its compact subspaces

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X.$$

Each connected component $U$ of $X \smallsetminus C_n$ is contained in some connected component of $X \smallsetminus C_{n-1}$. A decreasing sequence $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$ of connected components of

$$(X \smallsetminus C_1) \supset (X \smallsetminus C_2) \supset \dots \supset (X \smallsetminus C_n) \supset \dots$$

respectively is called an *end of $X$ with respect to $C_1 \subset \dots \subset C_n \subset \dots$.*

***48.Ax.*** Let $X$ and $C_n$ be as above, $D$ be a compact set in $X$ and $V$ a connected component of $X \smallsetminus D$. Prove that there exists $n$ such that $D \subset C_n$.

***48.Bx.*** Let $X$ and $C_n$ be as above, $D_n$ be an increasing sequence of compact sets of $X$ with $X = \cup_{n=1}^{\infty} D_n$. Prove that for any end $U_1 \supset \dots \supset U_n \supset \dots$ of $X$ with respect to $C_n$ there exists a unique end $V_1 \supset \dots \supset V_n \supset \dots$ of $X$ with respect to $D_n$ such that for any $p$ there exists $q$ such that $V_q \subset U_p$.

***48.Cx.*** Let $X$, $C_n$ and $D_n$ be as above. Then the map of the set of ends of $X$ with respect to $C_n$ to the set of ends of $X$ with respect to $D_n$ defined by the statement of *48.Bx* is a bijection.

Theorem *48.Cx* allows one to speak about *ends* of $X$ without specifying a system of compact sets

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X$$

with $X = \cup_{n=1}^{\infty} C_n$. Indeed, *48.Bx* and *48.Cx* establish a canonical one-to-one correspondence between ends of $X$ with respect to any two systems of this kind.

***48.Dx.*** Prove that $\mathbb{R}^1$ has two ends, $\mathbb{R}^n$ with $n > 1$ has only one end.

***48.Ex.*** Find the number of ends for the universal covering space of the bouquet of two circles.

***48.Fx.*** Does there exist a 2-manifold with a finite number of ends which cannot be embedded into a compact 2-manifold?

***48.Gx.*** Prove that for any compact set $K \subset S^2$ with connected complement $S^2 \smallsetminus K$ there is a natural map of the set of ends of $S^2 \smallsetminus K$ to the set of connected components of $K$.

Let $W$ be an open set of $X$. The set of ends $U_1 \supset \dots \supset U_n \supset \dots$ of $X$ such that $U_n \subset W$ for sufficiently large $n$ is said to be *open.*

***48.Hx.*** Prove that this defines a topological structure in the set of ends of $X$.

The set of ends of $X$ equipped with this topological structure is called the *space of ends* of $X$. Denote this space by $\mathcal{E}(X)$.

***48.1.1.*** Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with non-homeomorphic spaces of ends.

***48.1.2.*** Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with different number of ends.

***48.1.3.*** Construct non-compact connected manifolds of dimension two without boundary with isomorphic infinitely generated fundamental group and the same number of ends, but with different topology in the space of ends.

***48.1.4.*** Let $K$ be a completely disconnected closed set in $S^2$. Prove that the map $\mathcal{E}(S^2 \smallsetminus K) \to K$ defined in *48.Gx* is continuous.

***48.1.5.*** Construct a completely disconnected closed set $K \subset S^2$ such that this map is a homeomorphism.

***48.Ix.*** Prove that there exists an uncountable family of pairwise nonhomeomorphic connected 2-manifolds without boundary.

The examples of non-compact manifolds dimension 2 presented above show that there are too many non-compact connected 2-manifolds. This makes impossible any really useful topological classification of non-compact 2-manifolds. Theorems reducing the homeomorphism problem for 2-manifolds of this type to the homeomorphism problem for their spaces of ends do not seem to be useful: spaces of ends look not much simpler than the surfaces themselves.

However, there is a special class of non-compact 2-manifolds, which admits a simple and useful classification theorem. This is the class of *simply connected* non-compact 2-manifolds without boundary. We postpone its consideration to section $53^\circ 4$x. Now we turn to the case, which is the simplest and most useful for applications.

## $48^\circ 3$. Closed Surfaces

***48.B.*** *Any connected closed manifold of dimension two is homeomorphic either to sphere $S^2$, or sphere with handles, or sphere with crosscaps.*

Recall that according to Theorem *43.O* the basic surfaces represent pairwise distinct topological (and even homotopy) types. Therefore, *43.O* and *48.B* together give topological and homotopy classifications of closed 2-dimensional manifolds.

We do not recommend to have a try at proving Theorem *48.B* immediately and, especially, in the form given above. All known proofs of *48.B* can be decomposed into two main stages: firstly, a manifold under consideration is equipped with some additional structure (like triangulation or smooth structure); then using this structure a required homeomorphism is constructed. Although the first stage appears in the proof necessarily and is rather difficult, it is not useful outside the proof. Indeed, any closed 2-manifold, which we meet in a concrete mathematical context, is either equipped, or can be easily equipped with the additional structure. The methods of imposing the additional structure are much easier, than a general proof of existence for such a structure in an arbitrary 2-manifold.

Therefore, we suggest for the first case to restrict ourselves to the second stage of the proof of Theorem *48.B*, prefacing it with general notions related to the most classical additional structure, which can be used for this purpose.

## $48^\circ 4$. Compact Surfaces with Boundary

As in the case of one-dimensional manifolds, classification of compact two-dimensional manifolds with boundary can be easily reduced to the classification of closed manifolds. In the case of one-dimensional manifolds it was very useful to double a manifold. In two-dimensional case there is a construction providing a closed manifold related to a compact manifold with boundary even closer than the double.

***48.C.*** *Contracting to a point each connected component of the boundary of a two-dimensional compact manifold with boundary gives rise to a closed two-dimensional manifold.*

> ***48.2.*** A space homeomorphic to the quotient space of *48.C* can be constructed by attaching copies of $D^2$ one to each connected component of the boundary.

***48.D.*** *Any connected compact manifold of dimension 2 with nonempty boundary is homeomorphic either to sphere with holes, or sphere with handles and holes, or sphere with crosscaps and holes.*

# 49. Triangulations

### 49°1. Triangulations of Surfaces

By an *Euclidean triangle* we mean the convex hall of three non-collinear points of Euclidean space. Of course, it is homeomorphic to disk $D^2$, but it is not solely the topological structure that is relevant now. The boundary of a triangle contains three distinguished points, its *vertices,* which divide the boundary into three pieces, its *edges.* A *topological triangle* in a topological space $X$ is an embedding of an Euclidean triangle into $X$. A *vertex* (respectively, *edge*) of a topological triangle $T \to X$ is the image of a vertex ( respectively, edge) of $T$ in $X$.

A set of topological triangles in a 2-manifold $X$ is a *triangulation* of $X$ provided the images of these triangles form a fundamental cover of $X$ and any two of the images either are disjoint or intersect in a common side or in a common vertex.

***49.A.*** Prove that in the case of compact $X$ the former condition (about fundamental cover) means that the number of triangles is finite.

***49.B.*** Prove that the condition about fundamental cover means that the cover is locally finite.

### 49°2. Triangulation as cellular decomposition

***49.C.*** A triangulation of a 2-manifold turns it into a cellular space, 0-cells of which are the vertices of all triangles of the triangulation, 1-cells are the sides of the triangles, and 2-cells are the interiors of the triangles.

This result allows us to apply all the terms introduced above for cellular spaces. In particular, we can speak about skeletons, cellular subspaces and cells. However,in the latter two cases we rather use terms *triangulated subspace* and *simplex.* Triangulations and terminology related to them appeared long before cellular spaces. Therefore in this context the adjective *cellular* is replaced usually by adjectives *triangulated* or *simplicial.*

### 49°3. Two Properties of Triangulations of Surfaces

***49.D Unramified.*** Let $E$ be a side of a triangle involved into a triangulation of a 2-manifold $X$. Prove that there exist at most two triangles of this triangulation for which $E$ is a side. Cf. *44.G*, *44.H* and *44.P*.

***49.E Local strong connectedness.*** Let $V$ be a vertex of a triangle involved into a triangulation of a 2-manifold $X$ and $T$, $T'$ be two triangles of the triangulation adjacent to $V$. Prove that there exists a sequence

$T = T_1, T_2, \dots, T_n = T'$ of triangles of the triangulation such that $V$ is a vertex of each of them and triangles $T_i$, $T_{i+1}$ have common side for each $i = 1, \dots, n-1$.

Triangulations present a surface combinatorially.

### 49°4x. Scheme of Triangulation

Let $X$ be a 2-manifold and $\mathcal{T}$ a triangulation of $X$. Denote the set of vertices of $\mathcal{T}$ by $V$. Denote by $\Sigma_2$ the set of triples of vertices, which are vertices of a triangle of $\mathcal{T}$. Denote by $\Sigma_1$ the set of pairs of vertices, which are vertices of a side of $\mathcal{T}$. Put $\Sigma_0 = S$. This is the set of vertices of $\mathcal{T}$. Put $\Sigma = \Sigma_2 \cup \Sigma_1 \cup \Sigma_0$. The pair $(V, \Sigma)$ is called the *(combinatorial) scheme* of $\mathcal{T}$.

***49.Ax.*** Prove that the combinatorial scheme $(V, \Sigma)$ of a triangulation of a 2-manifold has the following properties:

(1) $\Sigma$ is a set consisting of subsets of $V$,
(2) each element of $\Sigma$ consists of at most 3 elements of $V$,
(3) three-element elements of $\Sigma$ cover $V$,
(4) any subset of an element of $\Sigma$ belongs to $\Sigma$,
(5) intersection of any collection of elements of $\Sigma$ belongs to $\Sigma$,
(6) for any two-element element of $\Sigma$ there exist exactly two three-element elements of $\Sigma$ containing it.

Recall that objects of this kind appeared above, in Section 23°3x. Let $V$ be a set and $\Sigma$ is a set of finite subsets of $V$. The pair $(V, \Sigma)$ is called a *triangulation scheme* if

- any subset of an element of $\Sigma$ belongs to $\Sigma$,
- intersection of any collection of elements of $\Sigma$ belongs to $\Sigma$,
- any one element subset of $V$ belongs to $\Sigma$.

For any simplicial scheme $(V, \Sigma)$ in 23°3x a topological space $S(V, \Sigma)$ was constructed. This is, in fact, a cellular space, see *40.Ax*.

***49.Bx.*** Prove that if $(V, \Sigma)$ is the combinatorial scheme of a triangulation of a 2-manifold $X$ then $S(V, \Sigma)$ is homeomorphic to $X$.

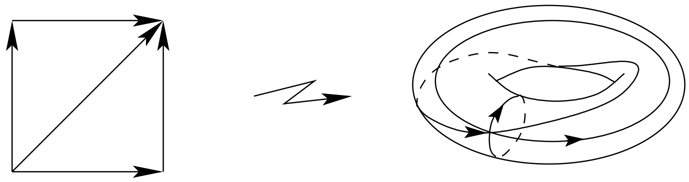
***49.Cx.*** Let $(V, \Sigma)$ be a triangulation scheme such that

(1) $V$ is countable,
(2) each element of $\Sigma$ consists of at most 3 elements of $V$,
(3) three-element elements of $\Sigma$ cover $V$,
(4) for any two-element element of $\Sigma$ there exist exactly two three-element elements of $\Sigma$ containing it

Prove that $(V, \Sigma)$ is a combinatorial scheme of a triangulation of a 2-manifold.

### 49°5. Examples

***49.1.*** Consider the cover of torus obtained in the obvious way from the cover of the square by its halves separated by a diagonal of the square.



Is it a triangulation of torus? Why not?

***49.2.*** Prove that the simplest triangulation of $S^2$ consists of 4 triangles.

***49.3*.*** Prove that a triangulation of torus $S^1 \times S^1$ contains at least 14 triangles, and a triangulation of the projective plane contains at least 10 triangles.

## 49°6. Subdivision of a Triangulation

A triangulation $\mathcal{S}$ of a 2-manifold $X$ is said to be a *subdivision* of a triangulation $\mathcal{T}$, if each triangle of $\mathcal{S}$ is contained in some triangle[1] of $\mathcal{T}$. Then $\mathcal{S}$ is also called a *refinement* of $\mathcal{T}$.

There are several standard ways to subdivide a triangulation. Here is one of the simplest of them. Choose a point inside a triangle $\tau$, call it a new vertex, connect it by disjoint arcs with vertices of $\tau$ and call these arcs new edges. These arcs divide $\tau$ to three new triangles. In the original triangulation replace $\tau$ by these three new triangles. This operation is called a *star subdivision centered at* $\tau$. See Figure 1.

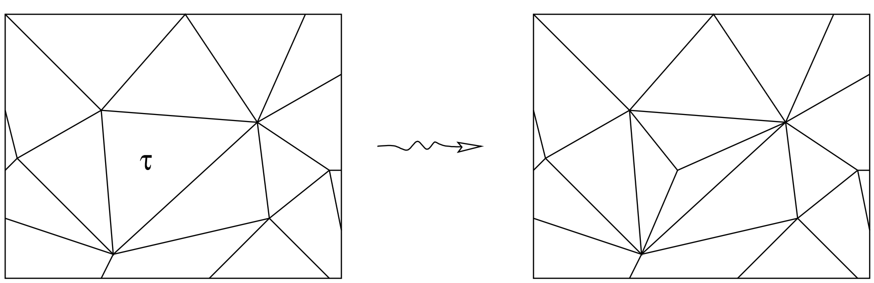



**Figure 1.** Star subdivision centered at triangle $\tau$

.

[1]Although triangles which form a triangulation of $X$ have been defined as topological embeddings, we hope that a reader guess that when one of such triangles is said to be contained in another one this means that the image of the embedding which is the former triangle is contained in the image of the other embedding which is the latter.

***49.F.*** Give a formal description of a star subdivision centered at a triangle $\tau$. I.e., present it as a change of a triangulation thought of as a collection of topological triangles. What three embeddings of Euclidean triangles are to replace $\tau$? Show that the replacement gives rise to a triangulation. Describe the corresponding operation on the combinatorial scheme.

Here is another subdivision defined locally. One adds a new vertex taken on an edge $\varepsilon$ of a given triangulation. One connects the new vertex by two new edges to the vertices of the two tringles adjacent to $\varepsilon$. The new edges divide these triangles, each to two new triangles. The rest of triangles of the original triangulation are not affected. This operation is called a *star subdivision centered at* $\varepsilon$. See Figure 2.

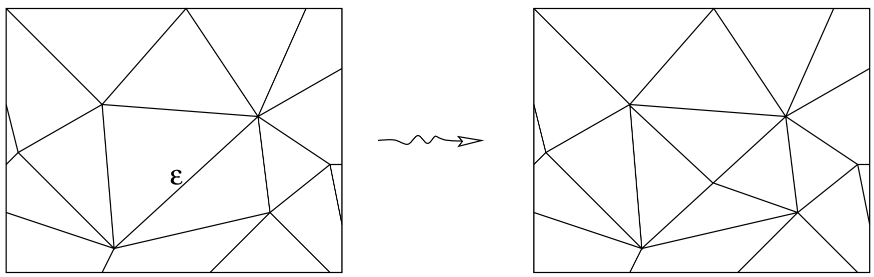



**Figure 2.** Star subdivision centered at edge $\varepsilon$.

***49.G.*** Give a formal description of a star subdivision centered at edge $\varepsilon$. What four embeddings of Euclidean triangles are to replace the topological triangles with edge $\varepsilon$? Show that the replacement gives rise to a triangulation. Describe the corresponding operation on the combinatorial scheme.

> ***49.4.*** Find a triangulation and its subdivision, which cannot be presented as a composition of star subdivisions at edges or triangles.
>
> ***49.5*.*** Prove that any subdivision of a triangulation of a compact surface can be presented as a composition of a finite sequences of star subdivisions centered at edges or triangles and operations inverse to such subdivisions.

By a *baricentric subdivision* of a triangle we call a composition of a star subdivision centered at this tringle followed by star subdivisions at each of its edges. See Figure 3.

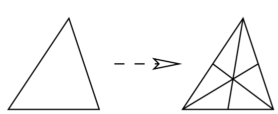


**Figure 3.** Baricentric subdivision of a triangle.

*Baricentric subdivision* of a triangulation of 2-manifold is a subdivision which is a simultaneous baricentric subdivision of all triangles of this triangulation. See Figure 4.

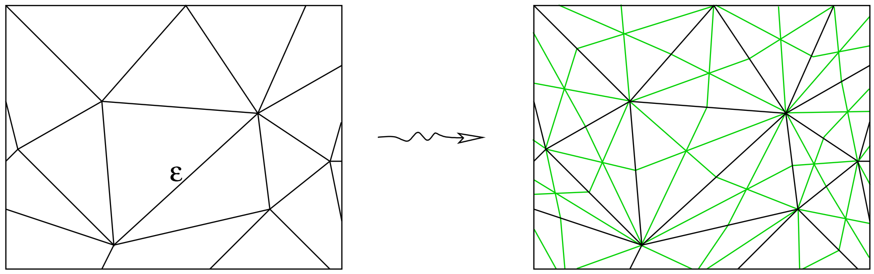



**Figure 4.** Baricentric subdivision of a triangulation.

***49.H.*** Establish a natural one-to-one correspondence between vertices of a baricentric subdivision a simplices (i.e., vertices, edges and triangles) of the original tringulation.

***49.I.*** Establish a natural one-to-one correspondence between triangles of a baricentric subdivision and triples each of which is formed of a triangle of the original triangulation, an edge of this triangle and a vertex of this edge.

The expression *baricentric subdivision* has appeared in a diiferent context, see Section 20. Let us relate the two notions sharing this name .

***49.Dx Baricentric subdivision of a triangulation and its scheme.*** Prove that the combinatorial scheme of the baricentric subdivision of a triangulation of a 2-manifold coincides with the baricentric subdivision of the scheme of the original triangulation (see 23°4x).

### 49°7. Homotopy Type of Compact Surface with Non-Empty Boundary

***49.J.*** Any compact connected triangulated 2-manifold with non-empty boundary collapses to a one-dimensional simplicial subspace.

***49.K.*** Any compact connected triangulated 2-manifold with non-empty boundary is homotopy equivalent to a bouquet of circles.

***49.L.*** The Euler characteristic of a triangulated compact connected 2-manifold with non-empty boundary does not depend on triangulation. It is equal to $1-r$, where $r$ is the rank of the one-dimensional homology group of the 2-manifold.

***49.M.*** The Euler characteristic of a triangulated compact connected 2-manifold with non-empty boundary is not greater than 1.

***49.N.*** The Euler characteristic of a triangulated closed connected 2-manifold with non-empty boundary is not greater than 2.

### 49°8. Triangulations in dimension one

By an *Euclidean segment* we mean the convex hall of two different points of a Euclidean space. It is homeomorphic to $I$. A *topological segment* or *topological edge* in a topological space $X$ is a topological embedding of an Euclidean segment into $X$. A set of topological segments in a 1-manifold $X$ is a *triangulation* of $X$ if the images of these topological segments constitute a fundamental cover of $X$ and any two of the images either are disjoint or intersect in one common end point.

Traingulations of 1-manifolds are similar to triangulations of 2-manifolds considered above.

***49.O.*** Find counter-parts for theorems above. Which of them have no counter-parts? What is a counter-part for the property *49.D*? What are counter-parts for star and baricentric subdivisions?

***49.P.*** Find homotopy classification of triangulated compact 1-manifolds using arguments similar to the ones from Section 49°7. Compare with the topological classification of 1-manifolds obtained in Section 47.

***49.Q.*** What values take the Euler characteristic on compact 1-manifolds?

***49.R.*** What is relation of the Euler characteristic of a compact triangulated 1-manifold $X$ and the number of $\partial X$?

***49.S.*** *Triangulation of a 2-manifold $X$ gives rise to a triangulation of its boundary $\partial X$. Namely, the edges of the triangualtion of $\partial X$ are the sides of triangles of the original triangulation which lie in $\partial X$.*

### 49°9. Triangualtions in higher dimensions

***49.T.*** Generalize everything presented above in this section to the case of manifolds of higher dimensions.

# 50. Handle Decomposition

## 50°1. Handles and Their Anatomy

Together with triangulations, it is useful to consider representations of a manifold as a union of balls of the same dimension, but adjacent to each other as if they were thickening of cells of a cellular space

A space $D^p \times D^{n-p}$ is called a *(standard) handle of dimension $n$ and index $p$*. Its subset $D^p \times \{0\} \subset D^p \times D^{n-p}$ is called the *core* of handle $D^p \times D^{n-p}$, and a subset $\{0\} \times D^{n-p} \subset D^p \times D^{n-p}$ is called its cocore. The boundary $\partial(D^p \times D^{n-p}) =$ of the handle $D^p \times D^{n-p}$ can be presented as union of its *base* $D^p \times S^{n-p-1}$ and *cobase* $S^{p-1} \times D^{n-p}$.

***50.A.*** Draw all standard handles of dimensions $\leq 3$.

A topological embedding $h$ of the standard handle $D^p \times D^{n-p}$ of dimension $n$ and index $p$ into a manifold of the same dimansion $n$ is called a *handle of dimension $n$ and index $p$.* The image under $h$ of $\operatorname{Int} D^p \times \operatorname{Int} D^{n-p}$ is called the *interior* of $h$, the image of the core $h(D^p \times \{0\})$ of the standard handle is called the *core* of $h$, the image $h(\{0\} \times D^{n-p})$ of cocore, the *cocore,* etc.

## 50°2. Handle Decomposition of Manifold

Let $X$ be a manifold of dimension $n$. A collecton of $n$-dimensional handles in $X$ is called a *handle decomposition of $X$*, if

(1) the images of these handles constitute a locally finite cover of $X$,
(2) the interiors of these handles are pairwise disjoint,
(3) the base of each of the handles is contained in the union of cobases of the handles of smaller indices.

Let $X$ be a manifold of dimension $n$ with boundary. A collection of $n$-dimensional handles in $X$ is called a *handle decomposion of $X$ modulo boundary,* if

(1) the images of these handles constitute a locally finite cover of $X$,
(2) the interiors of these handles are pairwise disjoint,
(3) the base of each of the handles is contained in the union of $\partial X$ and cobases of the handles of smaller indices.

A composition of a handle $h : D^p \times D^{n-p} \to X$ with the homeomorphism of transposition of the factors $D^p \times D^{n-p} \to D^{n-p} \times D^p$ turns the handle $h$ of index $p$ into a handle of the same dimension $n$, but of the complementary index $n - p$. The core of the handle turns into the cocore, while the base, to cobase.

***50.B.*** Composing each handle with the homeomorphism transposing the factors turns a handle decomposition of manifold into a handle decomposition modulo boundary of the same manifold. Vice versa, a handle decomposition modulo boundary turns into a handle decomposition of the same manifold.

Handle decompositions obtained from each other in this way are said to be *dual* to each other.

***50.C. Riddle.*** For $n$-dimensional manifold with boundary split into two $(n-1)$-dimensional manifolds with disjoint closures, define handle decomposition modulo one of these manifolds so that the dual handle decomposition would be modulo the complementary part of the boundary.

***50.1.*** Find handle decompositions with a minimal number of handles for the following manifolds:

| | | | | | |
|---|---|---|---|---|---|
| (a) | circle $S^1$; | (b) | sphere $S^n$; | (c) | ball $D^n$ |
| (d) | torus $S^1 \times S^1$; | (e) | handle; | (f) | cylinder $S^1 \times I$; |
| (g) | Möbius band; | (h) | projective plane $\mathbb{R}P^2$; | (i) | projective space $\mathbb{R}P^n$; |
| (j) | sphere with $p$ handles; | (k) | sphere with $p$ cross-caps; | (l) | sphere with $n$ holes. |

### 50°3. Handle Decomposition and Triangulation

Let $X$ be a 2-manifold, $\tau$ its triangulation, $\tau'$ its baricentric subdivision, and $\tau''$ the baricentric subdivision of $\tau'$. For each simplex $S$ of $\tau$ denote by $H_S$ the union of all simplices of $\tau''$ which contain the unique vertex of $\tau'$ that lies in $\int S$. Thus, if $S$ is a vertex then $H_S$ is the union of all triangles of $\tau''$ containing this vertex, if $S$ is an edge then $H_S$ is the union all of the triangles of $\tau''$ which intersect with $S$ but do not contain any of its vertices, and, finally, if $S$ is a triangle of $\tau$ then $H_S$ is the union of all triangles of $\tau''$ which lie in $S$ but do not intersect its boundary.

***50.D Handle Decomposition out of a Triangulation.*** Sets $H_S$ constitute a handle decomposition of $X$. The index of $H_S$ equals the dimension of $S$.

***50.E.*** Can every handle decomposition of a 2-manifold be constructed from a triangulation as indicated in *50.D*?

***50.F.*** How to triangulate a 2-manifold which is equipped with a handle decomposition?

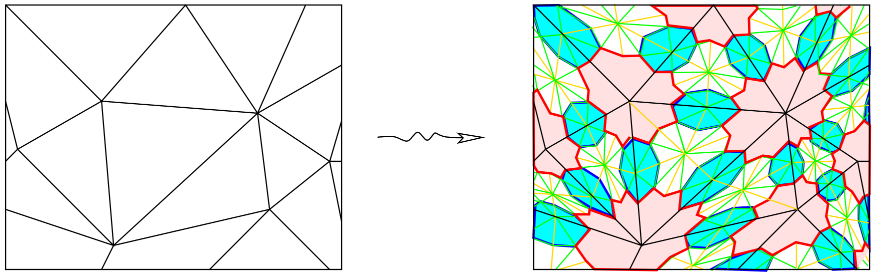


**Figure 5.** Construction of a handle decomposition from a triangulation.

## 50°4. Regular Neighborhoods

Let $X$ be a 2-manifold, $\tau$ its triangulation, and $A$ be a simplicial subspace of $X$. The union of all those simplices of the double baricentric subdivision $\tau''$ of $\tau$ which intersect $A$ is called the *regular* or *second baricentric neighborhood* of $A$ (with respect to $\tau$).

Of course, usually regular neighborhood is not open in $X$, since it is the union of simplices, which are closed. So, it is a neighborhood of $A$ only in wide sense (its interior contains $A$).

***50.G.*** A regular neighborhood of $A$ in $X$ is a 2-manifold. It coincides with the union of handles corresponding to the simplices contained in $A$. These handles constitute a handle decomposition of the regular neighborhood.

***50.H Collaps Induces Homemorphism.*** Let $X$ be a triangulated 2-manifold and $A \subset X$ be its triangulated subspace. If $X \searrow A$ then $X$ is homeomorphic to a regular neighborhood of $A$.

***50.I.*** Any triangulated compact connected 2-manifold with non-empty boundary is homeomorphic to a regular neighborhood of some of its 1-dimensional triangulated subspaces.

***50.J.*** In a triangulated 2-manifold, any triangulated subspace which is a tree has regular neighborhood homeomorphic to disk.

***50.K.*** In a triangulated 2-manifold, any triangulated subspace homeomorphic to circle has regular neighborhood homeomorphic either to the Möbius band or cylinder $S^1 \times I$.

In the former case the circle is said to be *one-sided*, in the latter, *two-sided*.

## 50°5. Cutting 2-Manifold Along a Curve

***50.L Cut Along a Curve.*** Let $F$ be a triangulated surface and $C \subset F$ be a compact one-dimensional manifold contained in the 1-skeleton of $F$ and satisfying condition $\partial C = \partial F \cap C$. Prove that there exists a 2-manifold $T$ and surjective map $p : T \to F$ such that:

(1) $p| : T \smallsetminus p^{-1}(C) \to F \smallsetminus C$ is a homeomorphism,

(2) $p| : p^{-1}(C) \to C$ is a two-fold covering.

***50.M Uniqueness of Cut.*** The 2-manifold $T$ and map $p$ which exist according to Theorem *50.L*, are unique up to homeomorphism: if $\tilde{T}$ and $\tilde{p}$ are other 2-manifold and map satisfying the same hypothesis then there exists a homeomorphism $h : \tilde{T} \to T$ such that $p \circ h = \tilde{p}$.

The 2-manifold $T$ described in *50.L* is called the result of *cutting of $F$ along $C$*. It is denoted by $F \text{✂} C$. This is not at all the complement $F \smallsetminus C$, although a copy of $F \smallsetminus C$ is contained in $F \text{✂} C$ as a dense subset homotopy equivalent to the whole $F \text{✂} C$.

***50.N Triangulation of Cut Result.*** $F \text{✂} C$ possesses a unique triangulation such that the natural map $F \text{✂} C \to F$ maps homeomorphically edges and triangles of this triangulation onto edges and, respectivly, triangles of the original triangulation of $F$.

***50.O.*** Let $X$ be a triangulated 2-manifold, $C$ be its triangulated subspace homeomorphic to circle, and let $F$ be a regular neighborhood of $C$ in $X$. Prove

(1) $F \text{✂} C$ consists of two connected components, if $C$ is two-sided on $X$, it is connected if $C$ is one-sided;

(2) the inverse image of $C$ under the natural map $X \text{✂} C \to X$ consists of two connected components if $C$ is two-sided on $X$, it is connected if $C$ is one-sided on $X$.

This proposition discloses the meaning of words *one-sided and two-sided* circle on a 2-manifold. Indeed, both connected components of the result of cutting of a regular neighborhood, and connected components of the inverse image of the circle can claim its right to be called a *side* of the circle or a *side of the cut*.

***50.2.*** Describe the topological type of $F \text{✂} C$ for the following $F$ and $C$:

(1) $F$ is sphere $S^2$, and $C$ is its equator;
(2) $F$ is a Möbius strip, and $C$ is its middle circle (deformation retract);
(3) $F = S^1 \times S^1$, $C = S^1 \times 1$;
(4) $F$ is torus $S^1 \times S^1$ standardly embedded into $\mathbb{R}^3$, and $C$ is the trefoil knot lying on $F$, that is $\{(z, w) \in S^1 \times S^1 \mid z^2 = w^3\}$;
(5) $F$ is a Möbius strip, $C$ is a segment: findtwo topologically different position of $C$ on $F$ and describe $F \text{✂} C$ for each of them;
(6) $F = \mathbb{R}P^2$, $C = \mathbb{R}P^1$.

(7) $F = \mathbb{R}P^2$, $C$ is homeomorphic to circle: find two topologically different position $C$ on $F$ and describe $F \mathbin{\text{✂}} C$ for each of them.

***50.P Euler Characteristic and Cut.*** *Let $F$ be a triangulated compact 2-manifold and $C \subset \int F$ be a closed one-dimensional contained in the 1-skeleton of the triangulation of $F$. Then $\chi(F \mathbin{\text{✂}} C) = \chi F$.*

***50.Q.*** Find the Euler characteristic of $F \mathbin{\text{✂}} C$, if $\partial C \neq \varnothing$.

***50.R Generalized Cut (Incise).*** Let $F$ be a triangulated 2-manifold and $C \subset F$ be a compact 1-dimensional manifold contained in 1-skeleton of $F$ and satisfying condition $\partial F \cap C \subset \partial C$. Let $D = C \smallsetminus (\partial C \smallsetminus \partial F)$. Prove that there exist a 2-manifold $T$ and sujective continuous map $p : T \to F$ such that:

(1) $p| : T \smallsetminus p^{-1}(D) \to F \smallsetminus D$ is a homeomorphism,
(2) $p| : p^{-1}(D) \to D$ is a two-fold covering.

***50.S Uniqueness of Cut.*** The 2-manifold $T$ and map $p$, which exist according to Theorem *50.R*, ae unique up to homeomorphism: if $\tilde{T}$ and $\tilde{p}$ are other 2-manifold and map satisfying the same hypothesis then there exists a homeomorphism $h : \tilde{T} \to T$ such that $p \circ h = \tilde{p}$.

The 2-Manifold $T$ described in *50.R* is also called the result of *cutting of $F$ along $C$* and denoted by $F \mathbin{\text{✂}} C$.

***50.3.*** Show that if $C$ is a segment contained in the interior of a 2-manifold $F$ then $F \mathbin{\text{✂}} C$ is homeomorphic to $F \smallsetminus \operatorname{Int} B$, where $B$ is the subset of $\int F$ homeomorphic to disk.

***50.4.*** Show that if $C$ is a segment such that one of its end points is in $\int F$ and the other one is on $\partial F$ then $F \mathbin{\text{✂}} C$ is homeomorphic to $F$.

### 50°6. Orientations

Recall that an *orientation of a segment* is a linear order of the set of its points. It is determined by its restriction to the set of its end points, see *47.R*. To describe an orientation of a segment it suffices to say which of its end points is initial and which is final.

Similarly, orientation of a triangle can be described in a number of ways, each of which can be chosen as the definition. By an *orientation of a triangle* one means a collection of orientations of its edges such that each vertex of the triangle is the final point for one of the edges adjacent to it and initial point for the other edge. Thus, an orientation of a triangle defines an orientation on each of its sides.

A segment admits two orientations. A triangle also admits two orientations: one is obtained from another one by change of the orientation on each side of the triangle. Therefore an orientation of any side of a triangle defines an orientation of the triangle.

Vertices of an oriented triangle are cyclicly ordered: a vertex $A$ follows immediately the vertex $B$ which is the initial vertex of the edge which finishes at $A$. Similarly the edges of an oriented triangle are cyclicly ordered: a side $a$ follows immediately the side $b$ which final end point is the initial point of $a$.

Vice versa, each of these cyclic orders defines an orientation of the triangle.

An *orientation of a triangulation* of a 2-manifold is a collection of orientations of all triangles constituting the triangulation such that for each edge the orientations defined on it by the orientations of the two adjacent triangles are opposite to each other. A triangulation is said to be *orientable*, if it admits an orientation.

***50.T Number of Orientations.*** *A triangulation of a connected 2-manifold is either non-orientable or admits exactly two orientations. These two orientations are opposite to each other. Each of them can be recovered from the orientation of any triangle involved in the triangulation.*

***50.U Lifting of Triangulation.*** Let $B$ be a triangulated surface and $p : X \to B$ be a covering. Can you equip $X$ with a triangulation?

***50.V Lifting of Orientation.*** Let $B$ be an oriented triangulated surface and $p : X \to B$ be a covering. Equip $X$ with a triangulation such that $p$ maps each simplex of this triangulation homeomorphically onto a simplex of the original triangulation of $B$. Is this triangulation orientable?

***50.W.*** Let $X$ be a triangulated surface, $C \subset X$ be a 1-dimensional manifold contained in 1-skeleton of $X$. If the triangulation of $X$ is orientable, then $C$ is two-sided.

# 51. Topological Classification of Compact Triangulated 2-Manifolds

## 51°1. Spines and Their Regular Neighborhoods

Let $X$ be a triangulated compact connected 2-manifold with non-empty boundary. A simplicial subspace $S$ of the 1-skeleton of $X$ is a *spine* of $X$ if $X$ collapses to $S$.

***51.A.*** *Let $X$ be a triangulated compact connected 2-manifold with non-empty boundary. Then a regular neighborhood of its spine is homeomorphic to $X$.*

***51.B Corollary.*** *A triangulated compact connected 2-manifold with non-empty boundary admits a handle decomposition without handles of index 2.*

A *spine* of a *closed* connected 2-manifold is a spine of this manifold with an interior of a triangle from the triangulation removed.

***51.C.*** *A triangulated closed connected 2-manifold admits a handle decomposition with exactly one handle of index 2.*

***51.D.*** *A spine of a triangulated closed connected 2-manifold is connected.*

***51.E Corollary.*** *The Euler characteristic of a closed connected triangulated 2-manifold is not greater than 2. If it is equal to 2, then the 2-manifold is homeomorphic to $S^2$.*

***51.F Corollary: Extremal Case.*** *Let $X$ be a closed connected triangulated 2-manifold $X$. If $\chi(X) = 2$, then $X$ is homeomorphic to $S^2$.*

## 51°2. Simply connected compact 2-manifolds

***51.G.*** *A simply connected compact triangulated 2-manifold with non-empty boundary collapses to a point.*

***51.H Corollary.*** *A simply connected compact triangulated 2-manifold with non-empty boundary is homeomorphic to disk $D^2$.*

***51.I Corollary.*** *Let $X$ be a compact connected triangulated 2-manifold $X$ with $\partial X \neq \varnothing$. If $\chi(X) = 1$, then $X$ is homeomorphic to $D^2$.*

## 51°3. Splitting off crosscaps and handles

***51.J.*** *A non-orientable triangulated 2-manifold $X$ is a connected sum of $\mathbb{R}P^2$ and a triangulated 2-manifold $Y$. If $X$ is connected, then $Y$ is also connected.*

***51.K.*** Under conditions of Theorem *51.J*, if $X$ is compact then $Y$ is compact and $\chi(Y) = \chi(X) + 1$.

***51.L.*** *If on an orientable connected triangulated* 2*-manifold* $X$ *there is a simple closed curve* $C$ *contained in the* 1*-skeleton of* $X$ *such that* $X \smallsetminus C$ *is connected, then* $C$ *is contained in a simplicial subspace* $H$ *of* $X$ *homeomorphic to torus with a hole and* $X$ *is a connected sum of a torus and a triangulated connected orientable* 2*-manifold* $Y$.

*If* $X$ *is compact, then* $Y$ *is compact and* $\chi(Y) = \chi(X) + 2$.

***51.M.*** *A compact connected triangulated* 2*-manifold with non-empty connected boundary is a connected sum of a disk and some number of copies of the projective plane and/or torus.*

***51.N Corollary.*** *A simply connected closed triangulated* 2*-manifold is homeomorphic to* $S^2$.

***51.O.*** *A compact connected triangulated* 2*-manifold with non-empty boundary is a connected sum of a sphere with holes and some number of copies of the projective plane and/or torus.*

***51.P.*** *A closed connected triangulated* 2*-manifold is a connected sum of some number of copies of the projective plane and/or torus.*

## 51°4. Splitting of a Handle on a Non-Orientable 2-Manifold

***51.Q.*** *A connected sum of torus and projective plane is homeomorphic to a connected sum of three copies of the projective plane.*

***51.Q.1.*** *On torus there are 3 simple closed curves which meet at a single point transversal to each other.*

***51.Q.2.*** *A connected sum of a surface* $S$ *with* $\mathbb{R}P^2$ *can be obtained by deleting an open disk from* $S$ *and identifying antipodal points on the boundary of the hole.*

***51.Q.3.*** *On a connected sum of torus and projective plane there exist three disjoint one-sided simple closed curves.*

## 51°5. Final Formulations

***51.R.*** *Any connected closed triangulated* 2*-manifold is homeomorphic either to sphere, or sphere with handles, or sphere with crosscaps.*

***51.S.*** *Any connected compact triangulated* 2*-manifold with non-empty boundary is homeomorphic either to sphere with holes, or sphere with holes and handles, or sphere with holes and crosscaps.*

***51.1.*** Find the place for the Klein Bottle in the above classification.

***51.2.*** Prove that any closed triangulated surface with non-orientable triangulation is homeomorphic either to projective plane number of handles or Klein bottle with handles. (Here the number of handles is allowed to be null.)

# 52. Cellular Approach to Topological Classification of Compact surfaces

In this section we consider another, more classical and detailed solution of the same problem. We classify compact triangulated 2-manifolds in a way which provides also an algorithm building a homeomorphism between a given surface and one of the standard surfaces.

### 52°1. Families of Polygons

Triangulations provide a combinatorial description of 2-dimensional manifolds, but this description is usually too bulky. Here we will study other, more practical way to present 2-dimensional manifolds combinatorially. The main idea is to use larger building blocks.

Let $\mathcal{F}$ be a collection of convex polygons $P_1, P_2, \dots$. Let the sides of these polygons be oriented and paired off. Then we say that this is a *family of polygons.* There is a natural quotient space of the sum of polygons involved in a family: one identifies each side with its pair-mate by a homeomorphism, which respects the orientations of the sides. This quotient space is called just the *quotient of the family.*

***52.A.*** Prove that the quotient of the family of polygons is a 2-manifold without boundary.

***52.B.*** Prove that the topological type of the quotient of a family does not change when the homeomorphism between the sides of a distinguished pair is replaced by other homeomorphism which respects the orientations.

***52.C.*** Prove that any triangulation of a 2-manifold gives rise to a family of polygons whose quotient is homeomorphic to the 2-manifold.

A family of polygons can be described combinatorially: Assign a letter to each distinguished pair of sides. Go around the polygons writing down the letters assigned to the sides and equipping a letter with exponent $-1$ if the side is oriented against the direction in which we go around the polygon. At each polygon we write a word. The word depends on the side from which we started and on the direction of going around the polygon. Therefore it is defined up to cyclic permutation and inversion. The collection of words assigned to all the polygons of the family is called a *phrase associated with the family of polygons.* It describes the family to the extend sufficient to recovering the topological type of the quotient.

***52.1.*** Prove that the quotient of the family of polygons associated with phrase $aba^{-1}b^{-1}$ is homeomorphic to $S^1 \times S^1$.

***52.2.*** Identify the topological type of the quotient of the family of polygons associated with phrases

(1) $aa^{-1}$;
(2) $ab$, $ab$;
(3) $aa$;
(4) $abab^{-1}$;
(5) $abab$;
(6) $abcabc$;
(7) $aabb$;
(8) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$;
(9) $a_1a_1a_2a_2\dots a_ga_g$.

***52.D.*** A collection of words is a phrase associated with a family of polygons, iff each letter appears twice in the words.

A family of polygons is called *irreducible* if the quotient is connected.

***52.E.*** A family of polygons is irreducible, iff a phrase associated with it does not admit a division into two collections of words such that there is no letter involved in both collections.

### 52°2. Operations on Family of Polygons

Although any family of polygons defines a 2-manifold, there are many families defining the same 2-manifold. There are simple operations which change a family, but do not change the topological type of the quotient of the family. Here are the most obvious and elementary of these operations.

(1) Simultaneous reversing orientations of sides belonging to one of the pairs.
(2) Select a pair of sides and subdivide each side in the pair into two sides. The orientations of the original sides define the orderings of the halves. Unite the first halves into one new pair of sides, and the second halves into the other new pair. The orientations of the original sides define in an obvious way orientations of their halves. This operation is called 1-*subdivision.* In the quotient it effects in subdivision of a 1-cell (which is the image of the selected pair of sides) into two 1-cells. This 1-cells is replaced by two 1-cells and one 0-cell.
(3) The inverse operation to 1-subdivision. It is called 1-*consolidation.*
(4) Cut one of the polygons along its diagonal into two polygons. The sides of the cut constitute a new pair. They are equipped with an orientation such that gluing the polygons by a homeomorphism respecting these orientations recovers the original polygon. This operation is called 2-*subdivision.* In the quotient it effects in subdivision of a 2-cell into two new 2-cells along an arc whose end-points

are 0-cells (may be coinciding). The original 2-cell is replaced by two 2-cells and one 1-cell.

(5) The inverse operation to 2-subdivision. It is called 2-*consolidation.*

## 52°3. Topological and Homotopy Classification of Closed Surfaces

***52.F Reduction Theorem.*** *Any finite irreducible family of polygons can be reduced by the five elementary operations to one of the following standard families:*

(1) $aa^{-1}$

(2) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$

(3) $a_1a_1a_2a_2\dots a_ga_g$ *for some natural* $g$.

***52.G Corollary, see 51.R.*** *Any triangulated closed connected manifold of dimension 2 is homeomorphic to either sphere, or sphere with handles, or sphere with crosscaps.*

Theorems *52.G* and *43.O* provide classifications of triangulated closed connected 2-manifolds up to homeomorphisms and homotopy equivalence.

***52.F.1 Reduction to Single Polygon.*** Any finite irreducible family of polygons can be reduced by elementary operations to a family consisting of a single polygon.

***52.F.2 Cancellation.*** A family of polygons corresponding to a phrase containing a fragment $aa^{-1}$ or $a^{-1}a$, where $a$ is any letter, can be transformed by elementary operations to a family corresponding to the phrase obtained from the original one by erasing this fragment, unless the latter is the whole original phrase.

***52.F.3 Reduction to Single Vertex.*** An irreducible family of polygons can be turned by elementary transformations to a polygon such that all its vertices are projected to a single point of the quotient.

***52.F.4 Separation of Crosscap.*** A family corresponding to a phrase consisting of a word $XaYa$, where $X$ and $Y$ are words and $a$ is a letter, can be transformed to the family corresponding to the phrase $bbY^{-1}X$.

***52.F.5.*** If a family, whose quotient has a single vertex in the natural cell decomposition, corresponds to a phrase consisting of a word $XaYa^{-1}$, where $X$ and $Y$ are nonempty words and $a$ is a letter, then $X = UbU'$ and $Y = Vb^{-1}V'$.

***52.F.6 Separation of Handle.*** A family corresponding to a phrase consisting of a word $UbU'aVb^{-1}V'a^{-1}$, where $U$, $U'$, $V$, and $V'$ are words and $a$, $b$ are letters, can be transformed to the family presented by phrase $dcd^{-1}c^{-1}UV'VU'$.

***52.F.7 Handle plus Crosscap Equals 3 Crosscaps.*** A family corresponding to phrase $aba^{-1}b^{-1}ccX$ can be transformed by elementary transformations to the family corresponding to phrase $abdbadX$.

# 53. Recognizing Closed Surfaces

***53.A.*** What is the topological type of the 2-manifold, which can be obtained as follows: Take two disjoint copies of disk. Attach three parallel strips connecting the disks and twisted by $\pi$. The resulting surface $S$ has a connected boundary. Attach a copy of disk along its boundary by a homeomorphism onto the boundary of the $S$. This is the space to recognize.

***53.B.*** *Euler characteristic of the cellular space obtained as quotient of a family of polygons is invariant under homotopy equivalences.*

***53.1.*** How can *53.B* help to solve *53.A*?

***53.2.*** Let $X$ be a closed connected surface. What values of $\chi(X)$ allow to recover the topological type of $X$? What ambiguity is left for other values of $\chi(X)$?

## 53°1. Orientations

By an *orientation of a polygon* one means orientation of all its sides such that each vertex is the final end point for one of the adjacent sides and initial for the other one. Thus an orientation of a polygon includes orientation of all its sides. Each segment can be oriented in two ways, and each polygon can be oriented in two ways.

An orientation of a family of polygons is a collection of orientations of all the polygons comprising the family such that for each pair of sides one of the pair-mates has the orientation inherited from the orientation of the polygon containing it while the other pair-mate has the orientation opposite to the inherited orientation. A family of polygons is said to be *orientable* if it admits an orientation.

***53.3.*** Which of the families of polygons from Problem *52.2* are orientable?

***53.4.*** Prove that a family of polygons associated with a word is orientable iff each letter appear in the word once with exponent $-1$ and once with exponent 1.

***53.C.*** *Orientability of a family of polygons is preserved by the elementary operations.*

A surface is said to be *orientable* if it can be presented as the quotient of an orientable family of polygons.

***53.D.*** A surface $S$ is orientable, iff any family of polygons whose quotient is homeomorphic to $S$ is orientable.

***53.E.*** Spheres with handles are orientable. Spheres with crosscaps are not.

### 53°2. More About Recognizing Closed Surfaces

***53.5.*** How can the notion of orientability and *53.C* help to solve *53.A*?

***53.F.*** *Two closed connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic and either are both orientable or both non-orientable.*

### 53°3. Recognizing Compact Surfaces with Boundary

***53.G. Riddle.*** Generalize orientabilty to the case of nonclosed manifolds of dimension two. (Give as many generalization as you can and prove that they are equivalent. The main criterium of success is that the generalized orientability should help to recognize the topological type.)

***53.H.*** *Two compact connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic, are both orientable or both nonorientable and their boundaries have the same number of connected components.*

### 53°4x. Simply Connected Surfaces

***53.Ax Theorem***.* *Any simply connected non-compact manifold of dimension two without boundary is homeomorphic to* $\mathbb{R}^2$.

***53° 4x.1.*** Any simply connected triangulated non-compact manifold without boundary can be presented as the union of an increasing sequence of compact simplicial subspaces $C_0 \supset C_1 \supset C_2 \supset \dots \supset C_n \supset \dots$ such that each of them is a 2-manifold with boundary and $\operatorname{Int} C_n \subset C_{n+1}$ for each $n$.

***53° 4x.2.*** Under conditions of *53° 4x.1* the sequence $C_n$ can be modified in such a way that each $C_n$ becomes simply connected.

***53.Bx Corollary.*** *The universal covering of any surface with empty boundary and infinite fundamental group is homeomorphic to* $R^2$.

# Proofs and Comments

***47.A*** Indeed, any 0-dimensional manifold is just a countable discrete topological space, and the only topological invariant needed for topological classification of 0-manifolds is the number of points.

***47.B*** Each manifold is the sum of its connected components.

***47.C***

(1) $S^1$,

(2) $I$,

(3) $\mathbb{R}$, $\mathbb{R}_+$,

(4) $\mathbb{R}_+$.

***47.D***

| Manifold $X$ | Is $X$ compact? | Is $\partial X$ empty? |
|---|---|---|
| $S^1$ | $+$ | $+$ |
| $\mathbb{R}^1$ | $-$ | $+$ |
| $I$ | $+$ | $-$ |
| $\mathbb{R}^1_+$ | $-$ | $-$ |

***48.Fx*** Yes, for example, a plane with infinite number of handles.

***49.Q*** All non-negative inetegers.

***49.R*** $\chi(X) = \frac{1}{2}\chi(\partial X) = \frac{1}{2}\sharp(\partial X)$. To prove this, consider double $DX$ of $X$, and observe that $\chi(DX) = 2\chi(X) - \chi(\partial X)$, while $\chi(DX) = 0$, since $DX$ is a closed 1-manifold.

***50.V*** Yes, it is orientable. An orientation can be obtain by taking on each triangle of $X$ the orientation which is mapped by $p$ to the orientation of its image.

***51.Q.1*** Represent the torus as the quotient space of the unit square. Take the images of a diagonal of the square and the two segments connecting the midpoints of the opposite sides of the square.

*Chapter XII*

# Surfaces Beyond Classification

In most of the textbooks which present topological classification of compact surfaces the classification is the top result. However the topology of 2-manifolds does not stop, but rather begins with it. Below we discuss few topics which are not included usually.

# 54. Curves and Graphs on Surfaces

## 54°1. Genus of Surface

In mathematical literature one of the most frequently mentioned invariants of compact 2-manifolds cannot be seriously discussed without the classification theorem, although it was introduced by Riemann before the was formulated.

The *genus* of a surface $X$ is the maximal number of disjoint simple closed curves $C_1, \dots, C_g$ on $X$ which do not divide $X$ (i.e., such that $X \setminus \cup_{i=1}^{g} C_i$ is connected). The genus of $X$ is denoted by $g(X)$.

In what follows we assume all the surfaces triangulated and curves simplicial. Let us calculate genus for closed surfaces.

***54.A Genus of Sphere with Handles.*** *The genus of sphere with $h$ handles is $h$.*

***54.A.1.*** Find $h$ disjoint simple closed curves which do not divide a sphere with $h$ handles.

***54.A.2.*** Cutting an orientable 2-manifold along a system of $k$ disjoint simple closed curves creates $2k$ new connected components of the boundary and does not change the Euler characteristic.

***54.A.3.*** Attaching a disk to a 2-manifold along a boundary component homeomorphic to $S^1$ by a homeomorphism of the boundary circle of the disk to the boundary component increases the Euler characteristic of the surface by 1.

***54.A.4.*** The Euler characteristic of a closed connected surface cannot be greater than 2.

***54.B Genus of Sphere with Crosscaps.*** *The genus of sphere with $h$ crosscaps is $h$.*

***54.B.1.*** Find $h$ disjoint simple closed curves which do not divide a sphere with $h$ crosscaps.

***54.B.2.*** Cutting a 2-manifold along a collection of $k$ disjoint one-sided simple closed curves creates $k$ new connected components of the boundary and does not change the Euler characteristic.

***54.1.*** A collection of $k$ disjoint simple closed curves on a connected 2-manifold of genus $g$ divides the 2-manifold to at least $\max(1, k-g+1)$ connected pieces.

***54.2.*** To what number of connected pieces does a collection of disjoint simple closed curves can divide a connected 2-manifold of genus $g$, if the collection consists of $p$ two-sided and $q$ one-sided curves?

### 54°2x. Polygonal Jordan, Schönflies and Annulus Theorems

The following two famous theorems which in a simplicial case are straightforward corollaries of the topological classification of compact 2-manifolds.

***54.Ax Jordan Theorem.*** *The complement of any simple close curve on the plane consists of two connected components.*

***54.Bx Schönflies Theorem.*** *Under conditions of the Jordan Theorem the closure of one of the components of the compliment is homeomorphic to $D^2$, the other one is homeomorphic to $D^2 \smallsetminus 0$.*

Without assumption of simpliciality of the simple closed curve these theorems can be deduced from their simplicial versions and appropriate versions of approximation theorems, or can be proven independently. The simplest proof of the general Schönflies Theorem is based on the Riemann mapping theorem.

***Information:*** Jordan Theorem is a very special corollary of general homological duality theorems (Alexander duality). Its straightforward generalizations hold true in higher dimensions.

Schönflies Theorem is much more delicate. Its literal generalizations without additional assumptions just in general topological setup do not hold true in dimensions $\geq 3$. For any $n \geq 3$ there is a topological embedding $i : S^{n-1} \to \mathbb{R}^n$ such that none of the connected components of $\mathbb{R}^n \smallsetminus i(S^{n-1})$ is simply connected. The first examples of this kind were constructed by J. W. Alexander, they are known as *Alexander horned spheres*.

Here is another classical theorem of the same flavor. As for the Jordan and Schönflies theorems, the tools provided by the material given above allows one to prove only its simplicial version, although a they hold true as formulated below, without any assumption of triangulability.

***54.Cx Annulus Theorem.*** *For any two disjoint simple closed curves $A$ and $B$ on $S^2$, the complement $S^2 \smallsetminus (A \cup B)$ consists of three connected components. The closure of one of them is homeomorphic to the annulus $S^1 \times I$, the closures of the others are homeomorphic to disk $D^2$.*

### 54°3x. Planarity of Graphs

A one-dimensional cellular space is *planar* if it can be embedded to $\mathbb{R}^2$.

***54.Dx.*** A one-dimensional cellular space is planar iff it can be embedded to $S^2$.

***54.1x.*** Find a non-planar 1-dimensional cellular space.

Denote by $G_n$ a one-dimensional cellular space formed by $n$ vertices and $\binom{n}{2}$ edges, with an edge connecting each pair of vertices. This space is called a *complete graph* with $n$ vertices. This is the 1-skeleton of an $(n-1)$-dimensional simplex.

***54.E*x.** Space $G_n$ is planar iff $n \le 4$.

***54.E*x.*1.*** $G_4$ is planar. Any its topological embedding to $S^2$ is equivalent to the embedding of 1-skeleton to 2-skeleton of a tetrahedron.

Denote by $G_{m,n}$ a one-dimensional cellular space formed by $m+n$ vertices divided to two sets consisting of $m$ and $n$ vertices respectively, in which any vertex from one set connected with a single edge to each vertex of another one, while no vertices of the same set are connected with an edge.

***54.F*x.** $G_{3,3}$ is not a planar graph.

***54.F*x.*1.*** $G_{3,2}$ is a planar graph. Any two its topological embeddings to $S^2$ are equivalent.

***54.2*x.** Which $G_{m,n}$ are planar, which are not?

***54.G*x *Kuratowski Theorem*.*** A one-dimensional cellular space $X$ is not a planar graph iff either $G_5$ or $G_{3,3}$ can be embedded to $X$.

***54.3*x.** Does there exist a connected 2-manifold $U$ such that any connected finite 1-dimensional cellular space can be topologically embedded to $U$?

***54.4*x.** Does there exist a connected compact 2-manifold $U$ such that any connected finite 1-dimensional cellular space can be topologically embedded to $U$?

***54.5*x.** Find a 1-dimensional cellular space which is not embeddable to torus $S^1 \times S^1$.

# 55x. Coverings and Branched Coverings

## 55°1x. Finite Coverings of Closed Surfaces

*For which closed connected* 2*-manifolds* $X$ *and* $Y$ *does there exist a covering* $X \to Y$*?*

***55.Ax Revise and Recollect.*** We have done some steps towards solution of this problem. Examine the material above and find relevant results.

***55.Bx Coverings of Torus.*** *Any covering space of torus* $S^1 \times S^1$ *with finite number of sheets is homeomorphic to* $S^1 \times S^1$*. There exists a covering* $S^1 \times S^1 \to S^1 \times S^1$ *with any finite number of sheets.*

***55.Cx Euler Characteristic of Covering Space.*** *If* $X$ *and* $Y$ *are finite simplicial spaces and* $X \to Y$ *is a simplicial map which is an* $n$*-fold covering, then* $\chi(X) = n\chi(Y)$*.*

***55.Dx Coverings of Orientable Closed Surface.*** *Let* $X$ *and* $Y$ *be closed connected orientable triangulated* 2*-manifolds. Covering* $X \to Y$ *exits iff* $\chi(X)$ *divides* $\chi(Y)$*.*

***55.1x.*** Let $X$ and $Y$ be closed connected orientable triangulated 2-manifolds with $\chi(Y) = d\chi(X)$. Prove that there exist a regular $d$-fold covering $X \to Y$ in a narrow sense with the automorphism group $\mathbb{Z}_d$.

***55.2x.*** Let $X$ and $Y$ be closed connected orientable triangulated 2-manifolds and $p, q : X \to Y$ be regular coverings (in narrow sense) with the automorphism group $\mathbb{Z}_d$. Then there exist homeomorphisms $f : X \to X$ and $g : Y \to Y$ such that $q \circ f = g \circ p$, that is the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{g} & Y \end{array}$$

is commutative.

***55.3x.*** Find regular coverings $p, q : X \to Y$ in narrow sense with the same number of sheets, where $X$ and $Y$ are orientable closed connected 2-manifolds, for which there exist no homeomorphisms $f : X \to X$ and $g : Y \to Y$ with $q \circ f = g \circ p$. What is the minimal possible number of sheets?

***55.Ex Corollary.*** *Sphere with two handles does not cover any orientable closed surface.*

***55.4x.*** Does sphere with two handles cover a non-orientable closed surface?

Let $X$ be a triangulated 2-manifold. For any triangle $\tau$ of its triangulation consider two copies of $\tau$ equipped with orientations opposite each other.

These copies are marked by the orientations, so we may think about them as about pairs $(\tau, o_1)$ and $(\tau, o_2)$, where $o_i$ is an orientation of $\tau$, but we rather need to think about them as about duplicates of the triangles marked with the orientations.

Let us factorize the sum of all these duplicates according to the following rule: we identify sides of duplicates of two triangles if the sides are dupplicates of the same edge in $X$ and the orientations associated with the duplicates of the triangles induce opposite orientations on the side. Denote the quotient space by $X^{or}$ (warning: this notation is not commonly accepted, but a commonly accepted notation does not exist). The identity maps of the copies of triangles to the original triangles induce map $X^{or} \to X$. It is called *orientation covering* of $X$. See Figure 1.

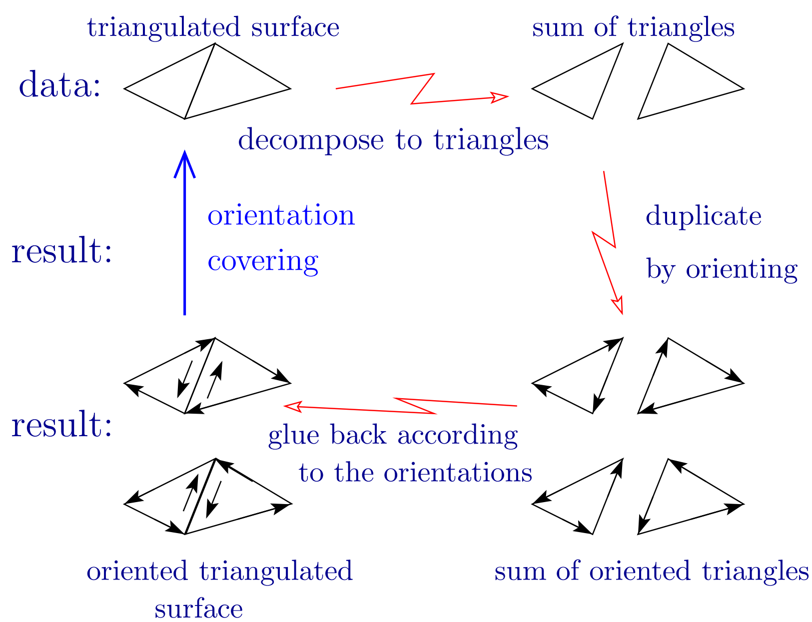



**Figure 1.** Construction of the orientation covering.

***55.Fx Theorem on Orientation Covering.*** *For any triangulated 2-manifold $X$ the construction above gives an* oriented *triangulated 2-manifold $X^{or}$ and a 2-fold covering $X^{or} \to X$. The non-trivial automorphism of this covering reverses the orientation.*

***55.Gx Orientability Versus Orientation Covering.*** *A triangulated 2-manifold is orientable iff its orientation covering is trivial.*

***55.Hx.*** *Any covering $p : X \to B$ of a non-orientable connected triangulated 2-manifold $B$ with orientable covering space $X$ can be factorize through the* *orientation covering of $B$: there exists a covering $q : X \to B^{or}$ such that the composition $X \xrightarrow{q} B^{or} \to B$ is $p : X \to B$.*

***55.Ix*** Corollary. *Let $X$ be an orientable closed connected 2-manifold and $Y$ be a non-orientable closed connected 2-manifold. There exists a covering $X \to Y$ iff $\chi(X)$ divides $2\chi(Y)$.*

***55.Jx.*** *Let $X$ and $Y$ be non-orientable closed connected 2-manifolds. There exists a covering $X \to Y$ iff $\chi(X)$ divides $\chi(Y)$.*

***55.Jx.1.*** There is a covering of Klein bottle by itself with any number of sheets.

## 55°2x. Branched Coverings

The notion of branched covering is more general and more classical than the notion of covering. Branched coverings are not that useful for calculation of fundamental groups and higher homotopy groups. This is why it would be pointless to study them in part 2 of this book, where the main goal was to calculate fundamental groups.

Let $U$ and $V$ be 2-manifolds and $m$ a natural number. A map $p : V \to U$ is called a *model $m$-fold branched covering*, if there exist homeomorphisms $g : U \to \mathbb{C}$ and $h : V \to \mathbb{C}$ such that $h \circ p \circ h^{-1}(z) = z^m$.

A map $p : Y \to X$ is called a *branched covering*, if for any $a \in X$ there exists a neighborhood $U$ of $a$ in $X$ such that $p^{-1}(U)$ is the union of disjoint open sets $V_\alpha$ such that for each $\alpha$ the submap $V_\alpha \to U$ of $p : Y \to X$ is a standard branched covering. The manifold $X$ is called the *base* of the branched covering $p : Y \to X$ and $Y$ the covering space. A point of the base is called *branch point*, if among model branched coverings of its neighborhood there is an $m$-fold covering with $m > 1$.

***55.Kx.*** A branched covering without branch points is a covering.

Branched coverings appear first in Complex Analysis. The following theorem provides a good reason for this.

***55.Lx.*** *For any analytic function $f : \mathbb{C} \to \mathbb{C}$ with $f(a) = b$ there exist neighborhoods $U$ and $V$ of $a$ and $b$, respectively, and homeomorphisms $\alpha : U \to D^2$ and $\beta : V \to D^2$ such that $\beta \circ f \circ \alpha^{-1}(z) = z^m$ for some natural $m$.*

***55.Mx Corollary 1.*** *Any non-constant complex polynomial $p$ in one variable defines a branched covering $\mathbb{C} \to \mathbb{C}$.*

***55.Nx Corollary 2.*** *Let $X$ and $Y$ be closed complex 1-manifolds (closed Riemann surfaces). Any holomorphic map $Y \to X$ is a branched covering.*

A branched covering without branch points is a covering.

# Proofs and Comments

***54.A.1*** Is the collection of one meridian curve taken from each handle good for this?

***54.B.1*** Is the collection of one middle one-sided curve taken from each crosscap good for this?

***55.A*x** According to the solution of Problem *50.V*, if $Y$ is orientable than $X$ is orientable, too. By Theorem *39.A*, $\pi_1(X)$ is isomorphic to a subgroup of $\pi_1(Y)$. Since $X$ is closed, it is compact and hence the fiber of a covering, being a discrete subspace of a compact Hausdorff space $X$ should be finite. Therefore, by Theorem *39.G*, the subgroup of $\pi_1(Y)$ isomorphic to $\pi_1(X)$ has a finite index. Vice versa, according to Theorems *45.M* and *39.D*x, for any subgroup $G \subset \pi_1(Y)$ of finite index there exists a covering $X \to Y$ with $\pi_1(X)$ isomorphic to $G$.

***55.B*x** Among closed surfaces only torus has commutative fundamental group. Therefore only torus can cover torus. The map

$$S^1 \times S^1 \to S^1 \times S^1 : (z, w) \mapsto (z^n, w)$$

is an $n$-fold covering.

***55.D*x** $\boxed{\Longrightarrow}$ It follows from Theorem . $\boxed{\Longleftarrow}$ Set $d = \chi(X) : \chi(Y)$. Represent $Y$ as a connected sum of torus with some other closed surface. I.e., find a simple closed curve on $Y$ which divides $Y$ into a handle $H$ and a disk with handles $D$. Take a $d$-fold covering of $H$ (say, the one induced by a $d$-fold covering of the torus which is obtained from $H$ by attaching a disk to the boundary). The covering space has $d$ boundary components. Fill each of them with a copy of $D$ and extend the covering by the homeomorphisms of these copies to $D$. Calculate the Euler characteristic of the covering space. It equals $\chi(X)$. Since the covering space and $X$ are orientable closed connected orientable 2-surfaces with the same Euler characteristic, they are homeomorphic.

***55.J*x** $\boxed{\Longrightarrow}$ See Theorem 54°1. $\boxed{\Longleftarrow}$ A non-orientable closed connected 2-manifold either is homeomorphic to $\mathbb{R}P^2$ or is a connected sum of the Klein bottle with some closed non-orientable manifold. If $Y$ is homeomorphic to $\mathbb{R}P^2$, then $\chi(Y) = 1$ and $\chi(X) = 1$. Hence $Y$ is homeomorphic to $\mathbb{R}P^2$ and for the covering one can take the identity map. For the other cases, it suffices to construct a covering of Klein bottle by itself with any natural number of sheets.

*Chapter XIII*

# One-Dimensional Homology

## 56x. One-Dimensional Homology and Cohomology

### 56°1x. Why and What for

Sometimes the fundamental group contains too much information to deal with, and it is handy to ignore a part of this information. A regular way to do this is to use instead of the fundamental group some of its natural quotient groups. One of them, the abelianized fundamental group, was introduced and used in Section 43 to prove, in particular, that spheres with different numbers of handles are not homotopy equivalent, see Problems *43.M*, *43.M.1*-*43.N.1* and *43.O*.

In this Section we will study the one-dimensional homology and its closest relatives. Usually they are studied in the framework of homology theory together with their high-dimensional generalizations. This general theory requires much more algebra and takes more time and efforts. On the other hand, one-dimensional case is useful on its own, involves a lot of specific details and provides a geometric intuition, which is useful, in particular, for studying the high-dimensional homology.

### 56°2x. One-Dimensional Integer Homology

Recall that for a path-connected space $X$ the abelianized fundamental group of $X$ is called its one-dimensional homology group and denoted by $H_1(X)$. If $X$ is an arbitrary topological space then $H_1(X)$ is the direct sum of the one-dimensional homology groups of all the connected components of $X$.

***56.1x.*** Find $H_1(X)$ for the following spaces

(1) Möbius strip,
(2) handle,
(3) sphere with $p$ handles and $r$ holes,
(4) sphere with $p$ crosscaps $r$ holes,
(5) the complement in $\mathbb{R}^3$ of the circles $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x, y, z) \in \mathbb{R}^3 \mid x = 0, z^2 + (y - 1)^2 = 1\}$,
(6) the complement in $\mathbb{R}^3$ of the circles $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x, y, z) \in \mathbb{R}^3 \mid z = 1, x^2 + y^2 = 1\}$,

The name of $H_1(X)$ appears often with the adjective *integer* or expression *with coefficients in* $\mathbb{Z}$, so it comes as *one-dimensional integer homology group* of $X$, or *one-dimensional homology group of $X$ with coefficients in* $\mathbb{Z}$. This is done to distinguish $H_1(X)$ from its genegalizations, one-dimensional homology groups with coefficients in any abelian group $G$. The case of $G = \mathbb{Z}_2$ is considered below, but we will not study these generalizations in full generality.

The group operation in $H_1(X)$ (as well as in other homology groups) is written additively and called *addition.* Thus the product of loops represents the *sum* of the homology classes represented by the loops multiplied.

Few more new words. An element of a homology group is called a *homology class.* The homology classes really admit several interpretations as equivalence classes of objects of various nature. For example, according to the definition we start with, a homology class is a coset consisting of elements of the fundamental group. In turn, each element of the fundamental group consists of loops. Thus, we can think of a homology class as of a set of loops.

### 56°3x. Null-Homologous Loops and Disks with Handles

A loop which belongs to the zero homology class is said to be *null-homologous.* Loops, which belong to the same homology class, are said to be *homologous* to each other.

***56.Ax Null-Homologous Loop.*** *Let $X$ be a topological space. A circular loop $s : S^1 \to X$ is null-homologous, iff there exist a continuous map $f$ of a disk $D$ with handles (i.e., a sphere with a hole and handles) to $X$ and a homeomorphism $h$ of $S^1$ onto the boundary circle of $D$ such that $f \circ h = s$.*

***56.Ax.1.*** In the fundamental group of a disk with handles, a loop, whose homotopy class generates the fundamental group of the boundary circle, is homotopic to a product of commutators of meridian and longitude loops of the handles.

A homotopy between a loop and a product of commutators of loops can be thought of as an extension of the loop to a continuous map of a sphere with handles and a hole.

## 56°4x. Description of $H_1(X)$ in Terms of Free Circular Loops

Factorization by the commutator subgroup kills the difference between translation maps defined by different paths. Therefore the abelianized fundamental groups of a path-connected space can be naturally identified. Hence each free loop defines a homology class. This suggests that $H_1(X)$ can be defined starting with free loops, rather than loops at a base point.

***56.Bx.*** On the sphere with two handles and three holes shown in Figure 1 the sum of the homology classes of the three loops, which go counterclockwise arround the three holes, is zero.

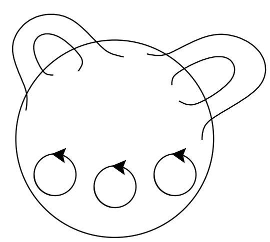


**Figure 1.** Sphere with two handles and three holes. The boundary circles of the holes are equipped with arrows showing the counter-clockwise orientation.

***56.Cx Zero-Homologous Collections of Loops.*** Let $X$ be a pathwise connected space and $s_1, \dots, s_n : S^1 \to X$ be a collection of $n$ free loops. Prove that the sum of homology classes of $s_1, \dots, s_n$ is equal to zero, iff there exist a continuous map $f : F \to X$, where $F$ is a sphere with handles and $n$ holes, and embeddings $i_1, \dots, i_n : S^1 \to F$ parametrizing the boundary circles of the holes in the counter-clockwise direction (as in Figure 1) such that $s_k = f \circ i_k$ for $k = 1, \dots, n$.

***56.Dx Homologous Collections of Loops.*** In a topological space $X$ any class $\xi \in H_1(X)$ can be represented by a finite collection of free circular loops. Collections $\{u_1, \dots, u_p\}$ and $\{v_1, \dots, v_q\}$ of free circular loops in $X$ define the same homology class, iff there exist a continuous map $f : F \to X$, where $F$ is a disjoint sum of several spheres with handles and holes with the total number of holes equal $p + q$, and embeddings $i_1, \dots, i_{p+q} : S^1 \to F$

parametrizing the boundary circles of all the holes of $F$ in the counter-clockwise direction such that $u_k = f \circ i_k$ for $k = 1, \dots, p$ and $v_k^{-1} = f \circ i_{k+p}$ for $k = 1, \dots, q$.

### 56°5x. Homology and Continuous Maps

Let $X$ be a path connected topological space with a base point $x_0 \in X$. The factorization map $\pi_1(X, x_0) \to H_1(X)$ is usually called the *Hurewicz homomorphism*[1] and denoted by $H$. If $X$ is not path connected and $X_0$ is its path connected component containing $x_0$, then the inclusion $X_0 \hookrightarrow X$ defines an isomorphism in : $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$. On the other hand, $H_1(X_0)$ is contained in $H_1(X)$ as a direct summand. This allows one to define the Hurewicz homomorphism $\pi_1(X, x_0) \to H_1(X)$ as a composition of the Hurewicz homomorphism $H : \pi_1(X_0, x_0) \to H_1(X_0)$ (which is already defined above), isomorphism $\text{in}^{-1} : \pi_1(X, x_0) \to \pi_1(X_0, x_0)$ (inverse to the inclusion isomorphism), and inclusion $H_1(X_0) \hookrightarrow H_1(X)$.

***56.Ex.*** Let $f : (X, x_0) \to (Y, y_0)$ be a continuous map. If $X$ is path connected, then the diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow[f_*]{} & \pi(Y, y_0) \\ {\scriptstyle H}\big\downarrow & & {\scriptstyle H}\big\downarrow \\ H_1(X) & & H_1(Y) \end{array}$$

is completed in a unique way to a commutative diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow[f_*]{} & \pi(Y, y_0) \\ {\scriptstyle H}\big\downarrow & & {\scriptstyle H}\big\downarrow \\ H_1(X) & \longrightarrow & H_1(Y) \end{array}$$

The homomorphism $H_1(X) \to H_1(Y)$ completing the diagram in *56.Ex* is denoted by the same symbol $f_*$ as the homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. It is also called a *homomorphism induced by* $f$.

***56.Fx.*** Extend the definition of $f_* : H_1(X) \to H_1(Y)$ given in *56.Ex* to the case when $X$ is not path connected.

***56.Gx.*** For any continuous map $f : X \to Y$ and any loop $\varphi : S^1 \to X$, the image under $f_* : H_1(X) \to H_1(Y)$ of the homology class represented by $\varphi$ is the homology class represented by $f \circ \varphi$.

---

[1] Witold Hurewicz has introduced a high dimensional generalization of this homomorphism, $\pi_n(X, x_0) \to H_n(X)$, which we cannot discuss here for you are not assumed to be familiar with $H_n(X)$. The homomorphism $\pi_1(X, x_0) \to H_1(X)$ should be rather attributed to Henry Poincaré, although the group $H_1(X)$ was introduced long after he died.

***56.2*x.** Look through 36, 37, 38, 39 and 43 and find all the theorems about homomorphisms of fundamental groups which gives rise to similar theorems about homomorphisms of one-dimensional homology groups. In which applications the fundamental groups can be replaced by one-dimensional homology groups?

***56.3*x *Homology Group of a Cellular Space.*** Deduce from the calculation of the fundamental group of a cellular space (see 43) an algorithm for calculation of $H_1(X)$ for a cellular space $X$.

## 56°6x. One-Dimensional Cohomology

Let $X$ be a path-connected topological space and $G$ a commutative group.

***56.H*x.** The homomorphisms $\pi_1(X, x_0) \to G$ comprise a commutative group in which the group operation is the pointwise addition.

The group $\mathrm{Hom}(\pi_1(X, x_0), G)$ of all the homomorphisms $\pi_1(X, x_0) \to G$ is called *one-dimensional cohomology group of $X$ with coefficients in $G$* and denoted by $H^1(X; G)$.

For an arbitrary topological space $X$, the one-dimensional cohomology group of $X$ with coefficients in $G$ is defined as the direct product of one-dimensional cohomology group with coefficients in $G$ of all the path-connected components of $X$.

***56.I*x *Cohomology via Homology.*** $H^1(X; G) = \mathrm{Hom}(H_1(X), G)$.

***56.J*x *Cohomology and Regular Coverings.*** This map is a bijection of the set of all the regular $G$-coverings of $X$ onto $H^1(X; G)$.

***56.4*x *Addition of $G$-Coverings.*** What operation on the set of regular $G$-coverings corresponds to addition of cohomology classes?

## 56°7x. Integer Cohomology and Maps to $S^1$

Let $X$ be a topological space and $f : X \to S^1$ a continuous map. It induces a homomorphism $f_* : H_1(X) \to H_1(S^1) = \mathbb{Z}$. Therefore it defines an element of $H^1(X; \mathbb{Z})$.

***56.K*x.** This construction defines a bijection of the set of all the homotopy classes of maps $X \to S^1$ onto $H^1(X; \mathbb{Z})$.

***56.L*x *Addition of Maps to Circle.*** What operation on the set of homotopy classes of maps to $S^1$ corresponds to the addition in $H^1(X; \mathbb{Z})$?

***56.M*x.** What regular $\mathbb{Z}$-covering of $X$ corresponds to a homotopy class of mappings $X \to S^1$ under the compositions of the bijections described in *56.K*x and *56.J*x

## 56°8x. One-Dimensional Homology Modulo 2

Here we define yet another natural quotient group of the fundamental group. It is even simpler than $H_1(X)$.

For a path-connected $X$, consider the quotient group of $\pi_1(X)$ by the normal subgroup generated by squares of all the elements of $\pi(X)$. It is denoted by $H_1(X;\mathbb{Z}_2)$ and called *one-dimensional homology group of $X$ with coefficients in $\mathbb{Z}_2$* or *the first $\mathbb{Z}_2$-homology group* of $X$. For an arbitrary $X$, the group $H_1(X;\mathbb{Z}_2)$ is defined as the sum of one-dimensional homology group with coefficients in $\mathbb{Z}_2$ of all the path-connected components of $X$.

Elements of $H_1(X;\mathbb{Z}_2)$ are called *one-dimensional homology classes modulo 2* or *one-dimensional homology classes with coefficients in $\mathbb{Z}_2$*. They can be thought of as classes of elements of the fundamental groups or classes of loops. A loop defining the zero homology class modulo 2 is said to be *null-homologous modulo 2.*

***56.N*x.** In a disk with crosscaps the boundary loop is null-homologous modulo 2.

***56.O*x *Loops Zero-Homologous Modulo 2.*** Prove that a circular loop $s : S^1 \to X$ is null-homologous modulo 2, iff there exist a continuous map $f$ of a disk with crosscaps $D$ to $X$ and a homeomorphism $h$ of $S^1$ onto the boundary circle of $D$ such that $f \circ h = s$.

***56.P*x.** If a loop is null-homologous then it is null-homologous modulo 2.

***56.Q*x *Homology and Mod 2 Homology.*** $H_1(X;\mathbb{Z}_2)$ is commutative for any $X$, and can be obtained as the quotient group of $H_1(X)$ by the subgroup of all even homology classes, i.e. elements of $H_1(X)$ of the form $2\xi$ with $\xi \in H_1(X)$. Each element of $H_1(X;\mathbb{Z}_2)$ is of order 2 and $H_1(X;\mathbb{Z}_2)$ is a vector space over the field of two elements $\mathbb{Z}_2$.

***56.5*x.** Find $H_1(X;\mathbb{Z}_2)$ for the following spaces

(1) Möbius strip,
(2) handle,
(3) sphere with $p$ handles,
(4) sphere with $p$ crosscaps,
(5) sphere with $p$ handles and $r$ holes,
(6) sphere with $p$ crosscaps and $r$ holes,
(7) the complement in $\mathbb{R}^3$ of the circles $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x,y,z) \in \mathbb{R}^3 \mid x = 0, z^2 + (y-1)^2 = 1\}$,
(8) the complement in $\mathbb{R}^3$ of the circles $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x,y,z) \in \mathbb{R}^3 \mid z = 1, x^2 + y^2 = 1\}$,

***56.6*x *$\mathbb{Z}_2$-Homology of Cellular Space.*** Deduce from the calculation of the fundamental group of a cellular space (see Section 43) an algorithm for calculation of the one-dimensional homology group with $\mathbb{Z}_2$ coefficients of a cellular space.

***56.Rx Collections of Loops Homologous Mod 2.*** Let $X$ be a topological space. Any class $\xi \in H_1(X;\mathbb{Z}_2)$ can be represented by a finite collection of free circular loops in $X$. Collections $\{u_1,\dots,u_p\}$ and $\{v_1,\dots,v_q\}$ of free circular loops in $X$ define the same homology class modulo 2, iff there exist a continuous map $f : F \to X$, where $F$ is a disjoint sum of several spheres with crosscaps and holes with the total number of holes equal $p+q$, and embeddings $i_1,\dots,i_{p+q} : S^1 \to F$ parametrizing the boundary circles of all the holes of $F$ such that $u_k = f \circ i_k$ for $k = 1,\dots,p$ and $v_k = f \circ i_{k+p}$ for $k = 1,\dots,q$.

> ***56.7x.*** Compare *56.Rx* with *56.Dx*. Why in *56.Rx* the counter-clockwise direction has not appeared? In what other aspects *56.Rx* is simpler than *56.Dx* and why?

***56.Sx Duality Between Mod 2 Homology and Cohomology.***

$$H^1(X;\mathbb{Z}_2) = \operatorname{Hom}(H_1(X;\mathbb{Z}_2),\mathbb{Z}_2) = \operatorname{Hom}_{\mathbb{Z}_2}(H_1(X;\mathbb{Z}_2),\mathbb{Z}_2)$$

for any space $X$. If $H_1(X;\mathbb{Z}_2)$ is finite then $H_1(X;\mathbb{Z}_2)$ and $H^1(X;\mathbb{Z}_2)$ are finite-dimensional vector spaces over $\mathbb{Z}_2$ dual to each other.

> ***56.8x.*** A loop is null-homologous modulo 2 in $X$, iff it is covered by a loop in any two-fold covering space of $X$.

***56.Tx. Riddle.*** *Homology Modulo $n$?* Generalize all the theory above about $\mathbb{Z}_2$-homology to define and study $\mathbb{Z}_n$-homology for any natural $n$.

# 57. One-Dimensional mod2-Homology of Surfaces

### 57°1. Polygonal Paths on Surface

Let $F$ be a triagulated surface. A path $s : I \to F$ is said to be *polygonal* if $s(I)$ is contained in the one-dimensional skeleton of the triangulation of $F$, the preimage of any vertex of the triangulation is finite, and the restriction of $s$ to a segment between any two consequitive points which are mapped to vertices is an affine homeomorphism onto an edge of the triangulation. In terms of kinematics, a polygonal path represents a moving point, which goes only along edges, does not stay anywhere, and, whenever it appears on an edge, it goes along the edge with a constant speed to the opposite end-point. A circular loop $l : S^1 \to F$ is said to be *polygonal* if the corresponding path $I \xrightarrow{t \mapsto \exp(2\pi it)} S^1 \xrightarrow{l} F$ is polygonal.

***57.A.*** Let $F$ be a triagulated surface. Any path $s : I \to F$ connecting vertices of the triangulation is homotopic to a polygonal path. Any circular loop $l : S^1 \to F$ is freely homotopic to a polygonal one.

A polygonal path is a combinatorial object:

***57.B.*** To describe a polygonal path up to homotopy, it is enough to specify the order in which it passes through vertices.

On the other hand, pushing a path to the one-dimensional skeleton can create new double points. Some edges may appear several time in the same edge.

***57.1.*** Let $F$ be a triangulated surface and $\alpha$ be an element of $\pi_1(F)$ different from 1. Prove that there exists a natural $N$ such that for any $n \geq N$ each polygonal loop representing $\alpha^n$ passes through some edge of the triangulation more than once.

### 57°2. Bringing Loops to General Position

To avoid a congestion of paths on edges, one can add new edges, i.e., subdivide the triangulation, see Section 49°6.

***57.C.*** Let $F$ be a triangulated and $u$, $v$ polygonal circular loops on $F$. Then there exist a subdivision of the triangulation of $F$ and polygonal loops $u'$, $v'$ homotopic to $u$ and $v$, respectively, such that $u'(I) \cap v'(I)$ is finite.

***57.D.*** Let $F$ be a triangulated and $u$ a polygonal circular loop on $F$. Then there exist a subdivision of the triangulation of $F$ and a polygonal loop $v$

homotopic to $u$ such that $v$ maps the preimage $v^{-1}(\varepsilon)$ of any edge $\varepsilon \subset v(I)$ homeomorphically onto $\varepsilon$. (In other words, $v$ passes along each edge at most once).

Let $u$, $v$ be polygonal circular loops on a triangulated surface $F$ and $a$ be an isolated point of $u(I) \cap v(I)$. Suppose $u^{-1}(a)$ and $v^{-1}(a)$ are one point sets. One says that $u$ intersects $v$ *translversally* at $a$ if there exist a neighborhood $U$ of $a$ in $F$ and a homeomorphism $U \to \mathbb{R}^2$ which maps $u(I) \cap U$ onto the $x$-axes and $v(I) \cap U$ to $y$-axes.

Polygonal circular loops $u$, $v$ on a triangulated surface are said to be in *general position* to with respect each other, if $u(I) \cap v(i)$ is finite, for each point $a \in u(i) \cap v(I)$ each of the sets $u^{-1}(a)$ and $v^{-1}(a)$ contains a single point and $u$, $v$ are transversal at $a$.

***57.E.*** Any two circular loops on a triangulated surface are homotopic to circular loops, which are polygonal with respect to some subdivision of the triangulation and in general position with respect to each other.

For a map $f : X \to Y$ denote by $S_k(f)$ the set

$$\{a \in X \mid f^{-1}f(a) \text{ consists of } k \text{ elements}\}$$

and put

$$S(f) = \{a \in X \mid f^{-1}f(a) \text{ consists of more than 1 element}\}.$$

A polygonal circular loop $l$ on a triangulated surface $F$ is said to be *generic* if

(1) $S(l)$ is finite,
(2) $S(l) = S_2(l)$,
(3) at each $a \in l(S_2(l))$ the two branches of $s(I)$ intersecting at $a$ are transversal, that is $a$ has a neighborhood $U$ in $F$ such that there exists a homeomorphism $U \to \mathbb{R}^2$ mapping the images under $s$ of the connected components of $s^{-1}(U)$ to the coordinate axis.

***57.F.*** Any circular loop on a triangulated surface is homotopic to a circular loop, which is polygonal with respect to some subdivision of the triangulation and generic.

Generic circular loops are especially suitable for graphic representation, because the image of a circular loop defines it to a great extend:

***57.G.*** Let $l$ be a generic polygonal loop on a triangulated surface. Then any generic polygonal loop $k$ with $k(S^1) = l(S^1)$ is homotopic in $l(S^1)$ to either $l$ or $l^{-1}$.

Thus, to describe a generic circular loop up to a reparametrization homotopic to identity, it is sufficient to draw the image of the loop on the surface and specify the direction in which the loop runs along the image.

The image of a generic polygonal loop is called a *generic (polygonal) closed connected curve.* A union of a finite collection of generic closed connected polygonal curves is called a *generic (polygonal) closed curve.* A generic closed connected curve without double points (i.e., an embedded oriented circle contained in the one-dimensional skeleton of a triangulated surface) is called a *simple polygonal closed curve.*

The adjective *closed* in the definitions above appears because there is a version of the definitions with (non-closed) paths instead of loops.

***57.H. Riddle.*** What modifications in Problems *57.C* – *57.G* and corresponding definitions should be done to replace loops by paths everywhere?

By a *generic polygonal curve* we will mean a union of a finite collection of pairwise disjoint images of generic polygonal loops and paths.

### 57°3. Curves on Surfaces and Two-Fold Coverings

Let $F$ be a two-dimensional triangulated surface and $C \subset F$ a manifold of dimension one contained in the 1-skeleton of the triangulation of $F$. Let $\partial C = \partial F \cap C$. Since the preimage $\tilde{C}$ of $C$ under the natural projection $F \between C \to F$ is a two-fold covering space of $C$, there is an involution $\tau : \tilde{C} \to \tilde{C}$ which is the only nontrivial automorphism of this covering. Take two copies of $F \between C$ and identify each $x \in \tilde{C}$ in one of them with $\tau(x)$ in the other copy. The resulting space is denoted by $F^{\approx C}$.

***57.I.*** The natural projection $F \between C \to F$ defines a continuous map $F^{\approx C} \to F$. This is a two-fold covering. Its restriction over $F \smallsetminus C$ is trivial.

### 57°4. One-Dimensional $\mathbb{Z}_2$-Cohomology of Surface

By *56.Jx*, a two-fold covering of $F$ can be thought of as an element of $H^1(F; \mathbb{Z}_2)$. Thus any one-dimensional manifold $C$ contained in the 1-skeleton of $F$ and such that $\partial C = \partial F \cap C$ defines a cohomology class of $F$ with coefficients in $\mathbb{Z}_2$. This class is said to be *realized* by $C$.

***57.J.*** The cohomology class with coefficients in $\mathbb{Z}_2$ realized by $C$ in a compact surface $F$ is zero, iff $C$ divides $F$, that is, $F = G \cup H$, where $G$ and $H$ are compact two-dimensional manifolds with $G \cap H = C$.

Recall that the cohomology group of a path-connected space $X$ with coefficients in $\mathbb{Z}_2$ is defined above in Section 56x as $\operatorname{Hom}(\pi_1(X), \mathbb{Z}_2)$.

***57.K.*** Let $F$ be a triangulated connected surface, let $C \subset F$ be a manifold of dimension one with $\partial C = \partial F \cap C$ contained in the 1-skeleton of $F$. Let $l$ be a polygonal loop on $F$ which is in general position with respect to $C$. Then the value which the cohomology class with coefficients in $\mathbb{Z}_2$ defined by $C$ takes on the element of $\pi_1(F)$ realized by $l$ equals the number of points of $l \cap C$ reduced modulo 2.

### 57°5. One-Dimensional $\mathbb{Z}_2$-Homology of Surface

***57.L $\mathbb{Z}_2$-Classes via Simple Closed Curves.*** Let $F$ be a triangulated connected two-dimensional manifold. Every homology class $\xi \in H_1(F; \mathbb{Z}_2)$ can be represented by a polygonal simple closed curve.

***57.M.*** A $\mathbb{Z}_2$-homology class of a triangulated two-dimensional manifold $F$ represented by a polygonal simple closed curve $A \subset F$ is zero, iff there exists a compact two-dimensional manifold $G \subset F$ such that $A = \partial G$.

Of course, the "if" part of *57.M* follows straightforwardly from *56.Ox*. The "only if" part requires trickier arguments.

***57.M.1.*** If $A$ is a polygonal simple closed curve on $F$, which does not bound in $F$ a compact 2-manifold, then there exists a connected compact 1-manifold $C \subset F$ with $\partial C = \partial F \cap C$, which intersects $A$ in a single point transversally.

***57.M.2.*** Let $F$ be a two-dimensional triangulated surface and $C \subset F$ a manifold of dimension one contained in the 1-skeleton of the triangulation of $F$. Let $\partial C = \partial F \cap C$. Any polygonal loop $f : S^1 \to F$, which intersects $C$ in an odd number of points and transversally at each of them, is covered in $F^{\approx C}$ by a path with distinct end-points.

***57.M.3.*** See *56.8x*.

### 57°6. Poincaré Duality

*To be written!*

### 57°7. One-Sided and Two-Sided Simple Closed Curves on Surfaces

*To be written!*

### 57°8. Orientation Covering and First Stiefel-Whitney Class

*To be written!*

### 57°9. Relative Homology

*To be written!*