Hints, Comments, Advises, Solutions, and Answers

1.1 The set $\{\varnothing\}$ consists of one element, which is the empty set $\varnothing$. Of course, this element itself is the empty set and contains no elements, but the set $\{\varnothing\}$ consists of a single element $\varnothing$.

1.2 1) and 2) are correct, while 3) is not.

1.3 Yes, the set $\{\{\varnothing\}\}$ is a singleton, its single element is the the set $\{\varnothing\}$.

1.4 2, 3, 1, 2, 2, 2, 1, 2 for $x \neq \frac{1}{2}$ and 1 if $x = \frac{1}{2}$.

1.5 (a) $\{1, 2, 3, 4\}$; (b) $\{\}$; (c) $\{-1, -2, -3, -4, -5, -6, \dots\}$

1.8 The set of solutions for a system of equations is equal to the intersection of the sets of solutions of individual equations belonging to the system.

2.1 The solution involves the equality $\cup(a_\alpha; +\infty) = (\inf a_\alpha; +\infty)$. Prove it. By the way, the collection of closed rays $[a; +\infty)$ is not a topological structure since it may happen that $\cup[a_\alpha; +\infty) = (a_0; +\infty)$ (give an example).

2.2 Yes, it is. A proof coincides almost literally with the solution of the preceding problem.

2.3 The main point here is to realize that the axioms of topological structure are conditions on the *collection* of subsets, and if these conditions are fulfilled, then the collection is a topological structure. The second collection is not a topological structure because it contains the sets $\{a\}$, $\{b, d\}$, but does not contain $\{a, b, d\} = \{a\} \cup \{b, d\}$. Find two elements of the third collection such that their intersection does not belong to it. By this you would prove that this is not a topology. Finally, we easily see that all unions and intersections of elements of the first collection still belong to the first collection.

2.10 The following sets are closed

(1) in a discrete space: all sets;

(2) in an indiscrete space: only the sets that are also open, i.e., the empty set and the whole space;

(3) in the arrow: $\varnothing$, the whole space and segments of the form $[0, a]$;

(4) in $\curlyvee$: the sets $X, \varnothing, \{b, c, d\}, \{a, c, d\}, \{b, d\}, \{d\}, and \{c, d\}$;

(5) in $\mathbb{R}_{T_1}$: all finite sets and the whole $\mathbb{R}$.

2.11 Here it is important to overcome the feeling that the question is completely obvious. Why is not $(0, 1]$ open? If $(0, 1] = \cup(a_\alpha, b_\alpha)$, then $1 \in (a_{\alpha_0}, b_{\alpha_0})$ for some α_0, whence $b_{\alpha_0} > 1$, and it follows that $\cup(a_\alpha, b_\alpha) \neq (0, 1]$. The set

$$\mathbb{R} \smallsetminus (0, 1] = (-\infty, 0] \cup (1, +\infty)$$

is not open for similar reasons. On the other hand, we have

$$(0, 1] = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = \bigcap_{n=1}^{\infty} \left(0, \frac{n+1}{n}\right).$$

2.13 Verify that $\Omega = \{U \mid X \smallsetminus U \in \mathcal{F}\}$ is a topological structure.

2.14 A control sum: the number of such collections is 14.

2.15 By this point, you must already know everything needed for solving this problem, so solve it on your own. Please, don't be lazy.

3.1 Certainly not! A topological structure is recovered from its base as the set of unions of all collections of sets belonging to the base.

3.2

(1) A discrete space admits the base consisting of all one-point subsets of the space and this base is minimal. (Why?)

(2) For a base in $\curlyvee$, we can take, say, $\{\{a\}, \{b\}, \{a, c\}, \{a, b, c, d\}\}$.

(3) The minimal base in indiscrete space is formed by a single set: the whole space.

(4) In the arrow, $\{[0, +\infty), (r, +\infty)\}_{r \in \mathbb{Q}_+}$ is a base.

3.3 We will show that, removing any element from any base of the standard topology of the line, we obtain a base of the same topology! Let U be an arbitrary element of a base. It can be presented as a union of open intervals that are shorter than the distance between some two points of U. We would need at least two such intervals. Each of the intervals, in turn, is a union of sets of the base under consideration. U is not involved into these unions since U is not contained in so short intervals. Hence, U is a union of elements of the base distinct from U, and it can be replaced by this union in a presentation of an open set as a union of elements of the base.

3.4 The whole topological structure is its own base. So, the question is when this is the only base. No open set in such a space is a union of two open sets distinct from it. Hence, open sets are linearly ordered by inclusion. Furthermore, the space should contain no increasing infinite sequence of open sets since otherwise an open set could be obtained as a union of sets in such a sequence.

3.5, 3.6 In solution of each of these problems the following easy lemma may be of use: $A = \bigcup B_\alpha$, where $B_\alpha \in \mathcal{B}$ iff $\forall\, x \in A\ \exists\, B_x \in \mathcal{B} : x \in B_x \subset A$.

3.7 The statement: "$\mathcal{B}$ is a base of a topological structure" is equivalent to the following: the set of unions of all collections of sets belonging to $\mathcal{B}$ is a topological structure. Σ^1 is a base of some topology by *3.B* and *3.6*. So, you must to prove analogs of *3.6* for Σ^2 and Σ^∞. To prove the coincidence of the structures determined, say, by the bases Σ^1 and Σ^2, you need to prove that a union of disks can be presented as a union of squares, and vice versa. Is it sufficient to prove that a disk is a union of squares? What is the simplest way to do this? (Cf. our advice concerning *3.5* and *3.6*.)

3.9 Observe that the intersection of several arithmetic progressions is an arithmetic progression.

3.10 Since the sets $\{i, i+d, i+2d, \dots\}$, $i = 1, \dots, d$, are open, pairwise disjoint and cover the whole $\mathbb{N}$, it follows that each of them is closed. In particular, for each prime number p the set $\{p, 2p, 3p, \dots\}$ is closed. All together, the sets of the form $\{p, 2p, 3p, \dots\}$ cover $\mathbb{N} \smallsetminus \{1\}$. Hence, if the set of prime numbers were finite, then the set $\{1\}$ would be open. However, it is not a union of arithmetic progressions.

3.11 The inclusion $\Omega_1 \subset \Omega_2$ means that a set open in the first topology (i.e., belonging to Ω_1) also belongs to Ω_2. Therefore, you must only prove that $\mathbb{R} \smallsetminus \{x_i\}_{i=1}^n$ is open in the canonical topology of the line.

4.2 Cf. *4.B*.

4.4 Look for the answer to *4.7*.

4.7 Squares with sides parallel to the coordinate axes and bisectors of the coordinate angles, respectively.

4.8 We have $D_1(a) = X$, $D_{1/2}(a) = \{a\}$, and $S_{1/2}(a) = \varnothing$.

4.9 For example, let $X = D_1(0) \subset \mathbb{R}^1$. Then $D_{3/2}(5/6) \subset D_1(0)$.

4.10 Three points suffice.

4.11 Let $R > r$ and $D_R(b) \subset D_r(a)$. Take $c \in D_R(b)$ and use the triangle inequality $\rho(b,c) \le \rho(b,a) + \rho(a,c)$.

4.12 Put $u = b - x$ and $t = x - a$. The Cauchy inequality becomes an equality iff the vectors u and t have the same direction, i.e., x lies on the segment connecting a and b.

4.13 For the metric $\rho^{(p)}$ with $p > 1$, this set is the segment connecting a and b, while for the metric $\rho^{(1)}$ it is a rectangular parallelepiped whose opposite vertices are a and b.

4.14 See the proof of *4.F*.

4.19 The discrete one.

4.20 Just recall that you need to prove that $X \smallsetminus D_r(a) = \{x \mid \rho(x,a) > r\}$ is open.

4.23 Use the obvious equality $X \smallsetminus S_r(a) = B_r(a) \cup (X \smallsetminus D_r(a))$ and the result of *4.20*.

4.25 Only the line and discrete spaces.

4.26 By *3.7*, for $n = 2$ metrics $\rho^{(2)}$, $\rho^{(1)}$, and $\rho^{(\infty)}$ are equivalent; similar arguments work for $n > 2$, too. Cf. *4.30*.

4.27 First, we prove that $\Omega_2 \subset \Omega_1$ provided that $\rho_2(x,y) \le C\rho_1(x,y)$. Indeed, the inequality $\rho_2 \le C\rho_1$ implies $B_r^{(\rho_1)}(a) \subset B_{Cr}^{(\rho_2)}$. Now let us use Theorem *4.I*. The inequality $c\rho_1(x,y) \le \rho_2(x,y)$ can written as $\rho_1(x,y) \le \frac{1}{c}\rho_2(x,y)$. Hence, $\Omega_1 \subset \Omega_2$.

4.28 The metrics $\rho_1(x,y) = |x-y|$ and $\rho_2(x,y) = \arctan|x-y|$ on the line are equivalent, but obviously there is no constant C such that $\rho_1 \le C\rho_2$.

4.29 Two metrics ρ_1 and ρ_2 are equivalent if there exist $c, C, d > 0$ such that $\rho_1(x,y) \le d$ implies $c\rho_1(x,y) \le \rho_2(x,y) \le C\rho_1(x,y)$.

4.30 Use the result of Problem *4.27*. Show that for any pair of metrics $\rho^{(p)}$, $1 \le p \le \infty$ there exist appropriate constants c and C.

4.31 We have $\Omega_1 \subset \Omega_C$ because $\rho_1(f,g) \le \rho_C(f,g)$. On the other hand, there is no ρ_1-ball centered at the origin is contained in $B_1^{(\rho_C)}(0)$ since for each $\varepsilon > 0$ there exists a function f such that $\int_0^1 |f(x)|\,dx < \varepsilon$ and $\max\limits_{[0,1]} |f(x)| \ge 1$, so $\Omega_C \not\subset \Omega_1$.

4.32 Clearly, in all five cases the only thing which is to be proved and is not completely obvious is the triangle inequality. It is also obvious for $\rho_1 + \rho_2$. Furthermore,

$$\rho_1(x,y) \le \rho_1(x,z) + \rho_1(z,y) \le \\ \max\{\rho_1(x,z), \rho_2(x,z)\} + \max\{\rho_1(y,z), \rho_2(y,z)\}.$$

A similar inequality holds true for $\rho_2(x,y)$, therefore $\max\{\rho_1,\rho_2\}$ is a metric. Construct examples which would prove that neither $\min\{\rho_1,\rho_2\}$, nor $\frac{\rho_1}{\rho_2}$, nor $\rho_1\rho_2$ is a metric. (To do this, it would be suffice to find three points with appropriate pairwise distances.)

4.33 Assertion (c) is quite obvious. Assertions (a) and (b) follow from (c) for $f(t) = \frac{t}{1+t}$ and $f(t) = \min\{1,t\}$, respectively. Thus, it suffices to check that these functions satisfy the assumptions of the assertion (c).

4.34 Since $\frac{\rho}{1+\rho} \le \rho$, and the inequality $\frac{1}{2}\rho(x,y) \le \frac{\rho(x,y)}{1+\rho(x,y)}$ holds true for $\rho(x,y) \le 1$, the statement follows from the result of *4.29*.

5.1 In the same way as the relative topology: if Σ is a base in X, then $\Sigma_A = \{A \cap V \mid V \in \Sigma\}$ is a base of the relative topology in A.

5.2

(1) Discrete, because $(n-1, n+1) \cap \mathbb{N} = \{n\}$;
(2) $\Omega_{\mathbb{N}} = \{(k, k+1, k+2...)\}_{k\in\mathbb{N}}$;
(3) discrete;
(4) $\Omega = \{\varnothing, \{2\}, \{1,2\}\}$.

5.3 Yes, it is open since $[0,1) = (-1,1) \cap [0,2]$, and $(-1,1)$ is open on the line.

5.5 $\boxed{\Longrightarrow}$ Set $V = U$. $\boxed{\Longleftarrow}$ Use Problem *5.E*.

5.6 Consider the interval $(-1,1) \subset \mathbb{R} \subset \mathbb{R}^2$ and the open disk with radius 1 and center at $(0,0)$ on the plane $\mathbb{R}^2$. Another solution is suggested by the following general statement: any open set is locally closed. Indeed, if U is open in X, then U is a neighborhood of each of its points, while $U \cap U$ is closed in U.

5.7 The metric topology in A is determined by the base $\Sigma_1 = \{B_r^A(a) \mid a \in A\}$, where $B_r^A(a) = \{x \in A \mid \rho(x,a) < r\}$ is the open ball in A with center a and radius r. The second topology is determined by the base $\Sigma_2 = \{A \cap B_r(x) \mid x \in X\}$, where $B_r(x)$ is an open ball in X. Obviously, $B_r^A(a) = A \cap B_r(a)$ for $a \in A$. Therefore $\Sigma_1 \subset \Sigma_2$, whence $\Omega_1 \subset \Omega_2$. However, it may happen that $\Sigma_1 \ne \Sigma_2$. It remains to prove that elements of

Σ_2 are open in the topology determined by Σ_1. For this purpose, check that for each point x of an element $U \in \Sigma_2$, there is $V \in \Sigma_1$ such that $x \in V \subset U$.

6.1 We have $\operatorname{Int}\{a, b, d\} = \{a, b\}$ since this is really the greatest set that is open in  and contained in $\{a, b, d\}$.

6.2 The interior of the interval $(0, 1)$ on the line with the Zariski topology is empty because no nonempty open set of this space is contained in $(0, 1)$.

6.3 Indeed,

$$\operatorname{Cl}_A B = \bigcap_{\substack{F \supset B, \\ A \smallsetminus F \in \Omega_A}} F = \bigcap_{\substack{H \supset B, \\ X \smallsetminus H \in \Omega}} (H \cap A) = A \cap \bigcap_{\substack{H \supset B, \\ X \smallsetminus H \in \Omega}} H = A \cap \operatorname{Cl}_X B.$$

The second equality may be obviously violated. Indeed, let $X = \mathbb{R}^2$, $A = B = \mathbb{R}^1$. Then $\operatorname{Int}_A B = \mathbb{R}^1 \neq \varnothing = (\operatorname{Int}_X B) \cap A$.

6.4 $\operatorname{Cl}\{a\} = \{a, c, d\}$.

6.5 $\operatorname{Fr}\{a\} = \{c, d\}$.

6.6 1) This follows from *6.K*. 2) See *6.7*.

6.8 In (X, Ω_1) there are less open sets, and hence less closed sets than in (X, Ω_2). Therefore the intersection of all sets closed in (X, Ω_1) and containing A cannot be smaller than the intersection of all sets closed in (X, Ω_2) and containing A.

6.9 $\operatorname{Int}_1 A \subset \operatorname{Int}_2 A$.

6.10 Since $\operatorname{Int} A$ is an open set contained in B, it is contained in $\operatorname{Int} B$, which is the greatest one of such sets.

6.11 Since the set $\operatorname{Int} A$ is open, it coincides with its interior.

6.12 (8) Obvious inclusion $\operatorname{Int} A \cap \operatorname{Int} B \subset A \cap B$ implies $\operatorname{Int} A \cap \operatorname{Int} B \subset \operatorname{Int}(A \cap B)$. Further, we have $\operatorname{Int} A \supset \operatorname{Int}(A \cap B)$ since $A \supset A \cap B$. Similarly, $\operatorname{Int} A \supset \operatorname{Int}(A \cap B)$. Therefore, $\operatorname{Int} A \cap \operatorname{Int} B \supset \operatorname{Int}(A \cap B)$. (9) The second statement is not correct, see Problem *6.13*.

6.13 $\operatorname{Int}([-1, 0] \cup [0, 1]) = (-1, 1) \neq (-1, 0) \cup (0, 1) = \operatorname{Int}[-1, 0] \cup \operatorname{Int}[0, 1]$.

6.14 $\operatorname{Int} A \cup \operatorname{Int} B$ is an open set contained in $A \cup B$, hence $\operatorname{Int} A \cup \operatorname{Int} B$ is contained in the interior of $A \cup B$. Thus, $\operatorname{Int} A \cup \operatorname{Int} B \subset \operatorname{Int}(A \cup B)$.

6.15 If $A \subset B$, then we have $\operatorname{Cl} A \subset \operatorname{Cl} B$, $\operatorname{Cl}\operatorname{Cl} A = \operatorname{Cl} A$, $\operatorname{Cl} A \cup \operatorname{Cl} B = \operatorname{Cl}(A \cup B)$, and $\operatorname{Cl} A \cap \operatorname{Cl} B \supset \operatorname{Cl}(A \cap B)$.

6.16 $\operatorname{Cl}\{1\} = [0, 1]$, $\operatorname{Int}[0, 1] = \varnothing$, $\operatorname{Fr}(2, +\infty) = [0, 2]$.

6.17 $\operatorname{Int}((0, 1] \cup \{2\}) = (0, 1)$, $\operatorname{Cl}\{\frac{1}{n} \mid n \in \mathbb{N}\} = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $\operatorname{Fr} \mathbb{Q} = \mathbb{R}$.

6.18 $\operatorname{Cl}\mathbb{N} = \mathbb{R}$, $\operatorname{Int}(0,1) = \varnothing$, and $\operatorname{Fr}[0,1] = \mathbb{R}$. Indeed, in $\mathbb{R}_{T_1}$ closed sets are either a finite set or the whole line. Therefore the closure of any infinite set is ...

6.19 Yes, it does. Indeed, since $D_r(x)$ is closed, we have $\operatorname{Cl} B_r(x) \subset D_r(x)$, whence

$$\operatorname{Fr} B_r(x) = \operatorname{Cl} B_r(x) \smallsetminus B_r(x) \subset D_r(x) \smallsetminus B_r(x) = S_r(x).$$

6.20 Yes, it does. Indeed, since since $B_r(x)$ is open, we have $\operatorname{Int} D_r(x) \supset B_r(x)$, whence

$$\operatorname{Fr} D_r(x) = D_r(x) \smallsetminus \operatorname{Int} D_r(x) \subset D_r(x) \smallsetminus B_r(x) = S_r(x).$$

6.21 Let $X = [0,1] \cup \{2\}$ with metric $\rho(x,y) = |x-y|$. Then $S_2(0) = \{2\}$ and $\operatorname{Cl} B_2(0) = [0,1]$.

6.22.1 For instance, $A = [0,1)$.

6.22.2 Take $A = [0,1) \cup (1,2] \cup \big(\mathbb{Q} \cap [3,4]\big) \cup \{5\}$.

6.22.3 Since $\operatorname{Int} A \subset \operatorname{Cl}\operatorname{Int} A$ and $\operatorname{Int} A$ is open, it follows that $\operatorname{Int} A \subset \operatorname{Int}\operatorname{Cl}\operatorname{Int} A$. Therefore, $\operatorname{Cl}\operatorname{Int} A \subset \operatorname{Cl}\operatorname{Int}\operatorname{Cl}\operatorname{Int} A$.
Since $\operatorname{Int}\operatorname{Cl}\operatorname{Int} A \subset \operatorname{Cl}\operatorname{Int} A$ and $\operatorname{Cl}\operatorname{Int} A$ is closed, it follows that $\operatorname{Cl}\operatorname{Int} A \supset \operatorname{Cl}\operatorname{Int}\operatorname{Cl}\operatorname{Int} A$.

6.23 Let us consecutively construct sets J_n, $n \geq 1$, such that J_n is a union of intervals of length 3^{-n}. Put $J_0 = \bigcup_{n\in\mathbb{Z}}(2n, 2n+1)$. If the sets $J_0, \dots, J_{n-1}$ are constructed, then let J_n be the union of the middle thirds of the segments constituting $\mathbb{R} \smallsetminus \bigcup_{k=0}^{n-1} J_k$. If $A = \bigcup_{k=0}^{\infty} J_{3k}$, $B = \bigcup_{k=0}^{\infty} J_{3k+1}$, and $C = \bigcup_{k=0}^{\infty} J_{3k+2}$, then $\operatorname{Fr} A = \operatorname{Fr} B = \operatorname{Fr} C = \operatorname{Cl}\big(\bigcup_{k=0}^{\infty} \operatorname{Cl} J_k\big)$. (In a similar way, we easily construct an infinite family of open sets with common boundary.)

6.24 If the endpoints of two segments are close to each other, then each point on one of them is close to a point on the other one. If two points belong to the interior of a convex set, then the convex set contains a cylindric neighborhood of the segment connecting the points.

6.27 By (1), $X \in \Omega$. From (2) it follows that $\operatorname{Cl}_* X = X$, whence $\varnothing \in \Omega$. For $U_1, U_2 \in \Omega$, (3) implies that $U_1 \cap U_2 \in \Omega$. Prior to checking that the 1st axiom of topological structure is fulfilled, show that it implies monotonicity of Cl_*: if $A \subset B$, then $\operatorname{Cl}_* A \subset \operatorname{Cl}_* B$, and deduce that $\operatorname{Cl}_*(\cap_\alpha A_\alpha) \subset \cap_\alpha \operatorname{Cl}_* A_\alpha$ for any family of sets A_α.
To prove that the operations Cl_* and the closure coincide, we recommend, as usual, to replace equality of sets by two inclusions and use the fact that a set F is closed iff $F = \operatorname{Cl}_* F$. (You must use property (4) somewhere!)

6.29 1) Nonempty sets; 2) unbounded sets; 3) infinite sets.

6.30 $\boxed{\Rightarrow}$ In a discrete space, each set is closed, hence the only everywhere-dense set is the whole space. $\boxed{\Leftarrow}$ Argue by contradiction. If the space X is not discrete, then there exists a point x such that the singleton $\{x\}$ is not open, and hence $X \smallsetminus x$ is everywhere dense, as well as the entire X.

6.31 There are many ways to formulate this property. For example, the intersection of all nonempty open sets is nonempty. See *2.6*.

6.32 1) Yes, it is. This follows from monotonicity of closure. 2) No, it is not. The easiest counter-example can be constructed in an indiscrete space. We recommend to construct a counter-example in $\mathbb{R}$ and take $\mathbb{Q}$ as one of the sets.

6.33 Let A and B be two open everywhere-dense sets, U an open set. Hint: $U \cap (A \cap B) = (U \cap A) \cap B$.

6.34 Only one of two sets needs to be open.

6.35 1) Let $\{U_k\}$ be a countable family of open everywhere-dense sets, V a nonempty open set on the line. Construct a sequence of nested segments $[a_1, b_1] \supset \dots \supset [a_n, b_n] \supset \dots$ such that $[a_n, b_n] \subset V \cap \bigcap\limits_{k=1}^{n} U_k$ and $b_n - a_n \to 0$. The point $\sup\{a_n\} = \inf b_n$ belongs to each of the segments. Therefore, $V \cap \bigcap\limits_{k=1}^{\infty} U_k \neq \varnothing$, and hence $\bigcap\limits_{k=1}^{\infty} U_k$ is everywhere dense. 2) The second question is answered in the negative.

6.36 Let $U_n \supset \mathbb{Q}$, $n \in \mathbb{N}$, be open sets. Since they contain $\mathbb{Q}$, all of them are everywhere dense. First, we enumerate all rational numbers: let $\mathbb{Q} = \{x_n \mid n \in \mathbb{N}\}$. After that, we find a segment $[a_1, b_1] \subset U_1$ such that $x_1 \notin U_1$. Since U_2 is everywhere dense, it contains a segment $[a_2, b_2] \subset [a_1, b_1] \cap U_2$ such that $x_2 \notin [a_2, b_2]$. Proceeding further in this way, we obtain a nested sequence $\{[a_n, b_n]\}$ of segments such that 1) $[a_n, b_n] \subset U_n$ and 2) $x_n \notin [a_n, b_n]$. By a standard theorem of Calculus, there exists a point $c \in \bigcap_1^\infty [a_n, b_n]$. Obviously, $c \in (\bigcap U_n) \smallsetminus \mathbb{Q}$.

6.37 Of course, it cannot, because the exterior of an everywhere dense set is empty (We assume that $X \neq \varnothing$).

6.38 It is empty.

6.39 Yes, it is.

6.40 It suffices to observe we have $X \smallsetminus \operatorname{Int}\operatorname{Cl} A = \operatorname{Cl}(X \smallsetminus \operatorname{Cl} A) = \operatorname{Cl}(\operatorname{Int}(X \smallsetminus A)) = X$.

6.41 1) Let F be a closed set in a space X. Then $\operatorname{Fr} F$ has the exterior $X \smallsetminus \operatorname{Int}\operatorname{Fr} F = (X \smallsetminus F) \cup \operatorname{Int} F$. Therefore, $\operatorname{Cl}(X \smallsetminus \operatorname{Int}\operatorname{Fr} F) = \operatorname{Cl}((X \smallsetminus F) \cup$

$\operatorname{Int} F) = X$ because $\operatorname{Cl}(X \smallsetminus F) = (X \smallsetminus F) \cup \operatorname{Fr} F$.
2) Yes, this is also true. The boundary of an open set U is nowhere dense since $\operatorname{Fr} U$ is also the boundary of the closed set $X \smallsetminus U$.
3) For arbitrary sets the statement is not true, in general: for instance, $\operatorname{Fr} \mathbb{Q} = \mathbb{R}$.

6.42 Clearly,

$$X \smallsetminus \operatorname{Cl}(\cup A_i) = X \smallsetminus \cup \operatorname{Cl} A_i = \cap (X \smallsetminus \operatorname{Cl} A_i).$$

Now the result follows from *6.33*.

6.43 This set is $\operatorname{Int} \operatorname{Cl} A$.

6.44 Let $Y_n \subset \mathbb{R}$, $n \in \mathbb{N}$, be nowhere-dense sets. Since Y_1 is nowhere dense, there is a segment $[a_1, b_1] \subset \mathbb{R} \smallsetminus Y_1$. Since Y_2 also is nowhere dense, $[a_1, b_1]$ contains a segment $[a_2, b_2] \subset \mathbb{R} \smallsetminus Y_2$, and so on. Proceeding further in this way, we obtain a sequence of nested segments $\{[a_n, b_n]\}$ such that $[a_n, b_n] \subset \mathbb{R} \smallsetminus Y_n$. By a standard theorem of Calculus, there exists a point $c \in \cap_1^\infty [a_n, b_n]$. Obviously, $c \in \mathbb{R} \smallsetminus \bigcap_{n=1}^\infty Y_n \neq \varnothing$.

6.45 For example, each point of a finite subset A of the line is an adherent point of A, but not a limit point.

6.47 The set of limit points of $\mathbb{N}$ in $\mathbb{R}_{T_1}$ is the whole $\mathbb{R}_{T_1}$.

6.48 (1) $\Longrightarrow$ (2): Consider $V = \bigcup_{x \in A} U_x$, where U_x are the neighborhoods that exist by the definition of local closeness, and show that $A = V \cap \operatorname{Cl} A$.
(2) $\Longrightarrow$ (3): Use the definition of the relative topology induced on a subset.
(3) $\Longrightarrow$ (1): For neighborhoods U_x, one can take a set independent on x.

7.1 No, because it is not antisymmetric. Indeed, $-1|1$ and $1|-1$, but $-1 \neq 1$.

7.2 The hypotheses of Theorem *7.J* turn into the following restrictions on C: C is closed with respect to addition, contains the zero, and no non-identity translation maps C bijectively onto itself.

7.6 1) Obviously, the greatest element is maximal and the smallest one is minimal, but the converse statements are not true. 2) These notions coincide for any subset of a poset, iff any two elements of the poset are comparable (i.e., one of them is greater than the other). $\boxed{\Rightarrow}$ Indeed, consider, e.g., a two-element subset. If the two elements were incomparable, then each of them would be maximal, and hence, by assumption, the greatest. However, the greatest element is unique. A contradiction. $\boxed{\Leftarrow}$ Comparability of any two elements obviously implies that in any subset any maximal element is the greatest one, and any minimal element is the smallest one.

7.9 The relation of inclusion in the set of all subsets of X is a linear order iff X is either empty or one-point.

7.11 Consider, say, the following condition: for arbitrary a, b, and c such that $a \prec c$ and $b \prec c$, there exists an element d such that $a \preceq d$, $b \preceq d$, and $d \prec c$. Show that this condition implies that the right rays form a base of a topology; show that it holds true in any linearly ordered set. Also show that this condition holds true if the right rays form a base of a topology.

7.13 A point open in the poset topology is maximal in the entire poset. Similarly, a point closed in the poset topology is minimal in the entire poset.

7.14 Rays of the forms (a, ∞) and $[a, \infty)$, the empty set, and the whole line.

7.16 The lower cone of the point.

7.17 A singleton consisting of an element that is greater than any other element of the entire poset.

8.1 Yes, they do. Let us prove, for example, the latter equality. Let $x \in f^{-1}(Y \smallsetminus A)$. Then $f(x) \in Y \smallsetminus A$, whence $f(x) \notin A$. Therefore, $x \notin f^{-1}(A)$ and $x \in X \smallsetminus f^{-1}(A)$. We have thus proved that $f^{-1}(Y \smallsetminus A) \subset X \smallsetminus f^{-1}(A)$. Each step in this argument is reversible. The reversing gives rise to the opposite inclusion.

8.2 Let us prove (13). If $y \in f(A \cup B)$, then we can find $x \in A \cup B$ such that $f(x) = y$. If $x \in A$, then $y \in f(A)$, while if $x \in B$, then $y \in f(B)$. In both cases we have $y \in f(A) \cup f(B)$. The inverse inclusion has even simpler proof. Inclusion $A \subset A \cup B$ implies $f(A) \subset f(A \cup B)$. Similarly, $f(B) \subset f(A \cup B)$. Thus $f(A) \cup f(B) \subset f(A \cup B)$. The other two equalities may happen to be wrong, see *8.3* and *8.4*.

8.3 Consider the constant map $f : \{0, 1\} \to \{0\}$. Let $A = \{0\}$ and $B = \{1\}$. Then $f(A) \cap f(B) = \{0\}$, while $f(A \cap B) = f(\varnothing) = \varnothing$. Similarly, $f(X \smallsetminus A) = f(B) = \{0\} \neq \varnothing$, although $Y \smallsetminus f(A) = \varnothing$.

8.4 We have $f(A \cap B) \subset f(A) \cap f(B)$. (Prove this!) However, there is no natural inclusion between $f(X \smallsetminus A)$ and $Y \smallsetminus f(A)$. Indeed, we can arbitrarily change a map on $X \smallsetminus A$ without changing it on A, and hence without changing $Y \smallsetminus f(A)$.

8.5 The bijectivity of f suffices for any equality of this kind. The Injectivity is necessary and sufficient for (14), but the surjectivity is necessary for (15). Thus, the bijectivity of f is necessary to make correct all equalities of *8.2*.

8.6 We probe only the inclusion $\subset$. Let $y \in B \cap f(A)$. Then $y = f(x)$, where $x \in A$. On the other hand, $x \in f^{-1}(B)$, whence $x \in f^{-1}(B) \cap A$, and therefore $y \in f(f^{-1}(B) \cap A)$. Prove the opposite inclusion on your own.

8.7 No, not necessarily. Example: $f : \{0\} \to \{0, 1\}$, $g : \{0, 1\} \to \{0\}$. Surely, f must be injective (see *8.K*), and g surjective (see *8.M*).

9.1 The map id is continuous iff $U = \mathrm{id}^{-1}(U) \in \Omega_1$ for each $U \in \Omega_2$, i.e., $\Omega_2 \subset \Omega_1$.

9.2 (a), (d): Yes, it is. (b), (c): Not necessarily.

9.3 1) Any map $X \to Y$ is continuous. 2) A map $Y \to X$ is continuous iff the preimage of each point is open. Only constant maps $Y \to X$ (i.e., the maps that map the whole Y to a single point of X) can be surely said to be continuous.

9.4 1) A map $X \to Y$ is continuous iff its image is indiscrete. Therefore only constant maps $X \to Y$ are continuous independently on the topology in Y. 2) All maps $Y \to X$ are continuous.

9.5 $\Omega' = \{f^{-1}(U) \mid U \in \Omega\}$ is a topology in A. Furthermore, this is the coarsest topology in A with respect to which f is continuous.

9.6 $\boxed{\Longrightarrow}$ $A \subset \mathrm{Cl}\, A$ for any A. Hence $f^{-1}(A) \subset f^{-1}(\mathrm{Cl}\, A)$. If f is continuous, then $f^{-1}(\mathrm{Cl}\, A)$ is closed, and $f^{-1}(A) \subset f^{-1}(\mathrm{Cl}\, A)$ implies $\mathrm{Cl}\, f^{-1}(A) \subset f^{-1}(\mathrm{Cl}\, A)$. $\boxed{\Longleftarrow}$ For A closed, we have $\mathrm{Cl}\, f^{-1}(A) \subset f^{-1}(A)$. Therefore, $f^{-1}(A)$ coincides with its closure, and hence is closed. Thus the preimage of any closed set is closed. By *9.A*, the map f is continuous.

9.7 f is continuous, iff

- $\mathrm{Int}\, f^{-1}(A) \supset f^{-1}(\mathrm{Int}\, A)$ for any $A \subset Y$, iff
- $\mathrm{Cl}\, f(A) \supset f(\mathrm{Cl}\, A)$ for any $A \subset X$, iff
- $\mathrm{Int}\, f(A) \subset f(\mathrm{Int}\, A)$ for any $A \subset X$.

9.8 $\boxed{\Longrightarrow}$ By definition. $\boxed{\Longleftarrow}$ Use the fact that the preimage of an open set is a union of preimages of base sets.

9.9 An experience with continuous functions gained in Calculus and a natural expectation that the continuity studied in Calculus is not too different from the continuity studied here give a strong evidence in favor of a negative answer. The following argument based on the above definition also provides it: the set $U = (1, 2]$ is open in $[0, 2]$, but its preimage $f^{-1}((1, 2]) = [1, 2)$ is not.

9.10 Yes, f is continuous. Consider what a set $f^{-1}(a, +\infty)$ (i.e., the preimage of a set open in the arrow) can be. By the way, what about continuity of map g coinciding with f everywhere besides at $x = 1$, and with $g(1) = 2$?

9.11 Constant maps. If, for instance, $0, 1 \in f(\mathbb{R}_Z)$, then consider the sets $f^{-1}\left(-\infty, \frac{1}{2}\right)$ and $f^{-1}\left(\frac{1}{2}, +\infty\right)$. Can both of them be open?

9.12 Constant maps and maps such that the preimage of each point is finite.

9.13 The functions that are monotonically increasing and continuous from the left. (Recall that a monotonically increasing function f is continuous from the left if $\sup\{f(x) \mid x < a\} = f(a)$ for each a.)

9.14 The map f is continuous, while g^{-1} is not. Indeed, the topology on $\mathbb{Z}_+$ is discrete, while the singleton $\{0\}$ is not open in the topology on $f(\mathbb{Z}_+)$.

9.15 Let A be an everywhere dense subset of a space X, and let $f : X \to Y$ be a continuous surjection. By Theorem *6.M*, it suffices to prove that $f(A)$ meets any nonempty open subset U of Y. Since f is surjective and continuous, the preimage $f^{-1}(U)$ of such a set is also nonempty and open. Therefore, its intersection with everywhere dense subset A of X is nonempty. Hence, $U \cap f(A)$ is nonempty.

9.16 Of course, it is not true. For example, the projection $\mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ maps the line $\{(x, y) \in \mathbb{R}^2 \mid y = 0\}$, which is nowhere dense in $\mathbb{R}^2$, onto the whole target space.

9.17 Yes, such a set exists. Take for A the Cantor set and consider the map that sends the number $\sum\limits_{i=1}^{+\infty} \frac{a_i}{3^i}$, where $a_i = 0; 2$, to the number $\sum\limits_{i=1}^{+\infty} \frac{a_i}{2^{i+1}}$. It must be checked that this map is continuous. Please, do this on your own.

9.18 Let us prove the first statement. Let U_a be a neighborhood of $a \in X$ such that $f(U_a) \subset \left(-\frac{\varepsilon}{2} + f(a), f(a) + \frac{\varepsilon}{2}\right)$, and let V_a be a similar neighborhood for g. Taking $W_a = U_a \cap V_a$, we obtain $(f + g)(W_a) \subset (-\varepsilon + f(a), f(a) + \varepsilon)$.

9.20 Put

$$f_i(x) = \begin{cases} 0 & x \le 0, \\ ix & 0 \le x \le \frac{1}{i}, \\ 1 & x \ge \frac{1}{i}. \end{cases}$$

Then the formula $x \mapsto \sup\{\, f_i(x) \mid i \in \mathbb{N}\,\}$ determines a function that takes value 0 at $x \le 0$ and 1 at $x > 0$.

9.21 The topology in $\mathbb{R}^n$ is generated by the metric

$$\rho^{(\infty)}(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

(see *4.26*). Observe that $\rho^{(\infty)}(f(x), f(a)) < \varepsilon$ iff $|f_i(x) - f_i(a)| < \varepsilon$ for all $i = 1, 2, \dots, n$.

9.22 Use *9.21* and *9.18*.

9.23 Use *9.21*, *9.18*, and *9.19*.

9.24 If Ω' is a topology such that the map $x \mapsto \rho(x, A)$ is continuous for each A, then Ω' contains all open balls. Therefore, Ω' contains all sets open in the metric topology.

9.25 If $\rho(x, a) < \varepsilon$, then $\rho\big(f(x), f(a)\big) \le \alpha\varepsilon < \varepsilon$.

9.27 Where we deal with closed sets.

9.28 Use the following property of polynomials: a polynomial P with real coefficients that takes value 0 on a nonempty open set identically vanishes. For polynomials in one variable, this property easily follows from the Bezout theorem, while for polynomials in many variables it is proved by induction on the number of variables. The continuity of the function $x \mapsto P(x)$ on R^n implies that the set of zeros $\{x \mid P(x) = 0\}$ of P is closed. Cf. *9.O*.

9.29 In cases (a), (c), and (d), this is not true. Consider functions constant on each element of these covers, but not constant on the whole space.
In case (b), this is true. Try to prove this using arguments that you know from calculus. (Cf. *9.T*.)

9.31 If the intersection of a set U with each element of Γ is open in this element, then the same is true for any element of Γ'. Since, by assumption, Γ' is a fundamental cover, it follows that U is open in the whole space. Thus, the cover Γ is fundamental.

9.32 If $B \cap U$ is open in U for each $U \in \Gamma$, and $A \in \Delta$, then $(B \cap U) \cap A = (B \cap A) \cap (U \cap A)$ is open in $U \cap A$. Hence, $B \cap A$ is open in A. Since the cover Δ is fundamental, B is open in X.

9.33 This follows from the preceding statement. What cover should be taken as Δ?

***9.1*x** Consider map $f : [0, 2] \to \mathbb{R}$ with $f(x) = x$ for $x \in [0, 1]$ and $f(x) = x + 1$ for $x \in (1, 2]$.

***9.2*x** No. Here are two of many counterexamples. First, the map $f : \{\pm\frac{1}{n}, 0\}_{n=1}^{\infty} \to \{-1, 0, 1\}$, which maps positive numbers to 1, negative, to -1, and 0 to 0. Secondly, consider $\mathbb{R}^2$ with relation

$$(a, b) \prec (a', b') \text{ if } a < a' \text{ or } a = a' \text{ and } b < b'$$

This is a linear order (check!). The projection $\mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ is monotone (but not strictly monotone) with respect to $\prec$ and $<$, but the preimage of any proper open subset $U \subset \mathbb{R}$ is not open in the interval topology determined by $\prec$.

***9.3*x** Yes, it is. Furthermore, it suffices to require only that f be nonstrictly monotone.

10.1 Statements *10.C–10.E* imply that homeomorphism is an equivalence relation: *10.C* implies reflexivity of homeomorphism, *10.D* implies transitivity, and *10.E* implies symmetricity.

10.2 Show that $\tau \circ \tau = \mathrm{id}$, whence $\tau^{-1} = \tau$. To see that the inversion is continuous, write τ down in coordinates and use *9.18*, *9.19*, and *9.21*.

10.3 Show that $\mathrm{Im}(f(x+iy)) = (ad-bc)y/|cz+d|^2$, whence $f(\mathcal{H}) \subset \mathcal{H}$. Find the inverse map (it is determined by a similar formula). Use *9.18*, *9.19*, and *9.21* to prove the continuity.

10.4 $\boxed{\Longrightarrow}$ Use Intermediate Value Theorem. $\boxed{\Longleftarrow}$ Use *10.M*.

10.5 Cf. *10.H*. 1), 2) This is obvious. 3) Any bijection $\mathbb{R}_Z \to \mathbb{R}_Z$ establishes a one-to-one correspondence between finite (i.e., closed!) subsets.

10.6 Only the identity map of  is a homeomorphism.

10.7 Use *9.13* and *10.M*.

10.8 Let $X = Y = \bigcup_{k=0}^{\infty}[2k, 2k+1)$ and consider the bijection

$$X \to Y : \ x \mapsto \begin{cases} \frac{x}{2} & \text{if } x \in [0,1), \\ \frac{x-1}{2} & \text{if } x \in [2,3), \\ x-2 & \text{if } x \geq 4. \end{cases}$$

10.10 To solve all assertions, except (f) and (i), apply maps used in the solution of Problem *10.O*. To solve (f) and (i), use polar coordinates.

10.11 In assertion (b): each nonempty open convex set in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

10.12 Every such a set is homeomorphic to one of the following sets: a point, a segment, a ray, a disk, a strip, a half-plane, a plane. (Prove this!)

10.13 In Problems *10.T* and *10.11*, it is sufficient to replace the 2-disk D^2 by the n-disk D^n and the open 2-disk B^2 by the open n-ball B^n. The situation with Problem *10.12* is more complicated. Let $K \subset \mathbb{R}^n$ be a closed convex set. First, we can assume that $\mathrm{Int}\, K \neq \varnothing$ because otherwise K is isometric to a subset of $\mathbb{R}^k$ with $k < n$. Secondly, we assume that K is unbounded. (Otherwise, K is homeomorphic to a closed disk, see above.) If K does not contain a line, then K is homeomorphic to a half-space. If K contains a line, then K is isometric to a "cylinder" with convex closed "base" in $\mathbb{R}^{n-1}$ and "elements" parallel to the nth coordinate axis, which allows us to use induction on dimension. Try to formulate a complete answer.

10.14 Map each link of the polygon homeomorphically to a suitable arc of the circle.

10.15 Map each link of the polyline homeomorphically to a suitable part of the segment. (Cf. the preceding problem. The homeomorphism can easily be chosen piecewise linear.)

10.16 Accurately plug in the definitions!

10.17 Combining the techniques of Problems *10.S* and *10.O* (assertion (e)), show that the "infinite cross" is homeomorphic to the set $K = \{|x| + |y| \le 2\} \smallsetminus \{(0, \pm 2), (\pm 2, 0)\}$ (another square without vertices).

10.18 The proof is elementary, but rather complicated!

10.19 Using homeomorphisms of Problem *10.O*, you can construct, e.g., the following homeomorphisms: (a) $\cong$ (d) $\cong$ (f), (d) $\cong$ (e) $\cong$ (h) $\cong$ (b), (h) $\cong$ (g) $\cong$ (c).

10.20 Using homeomorphisms of Problem *10.O*, you can construct, e.g., the following homeomorphisms: (c) $\cong$ (b) $\cong$ (a) $\cong$ (d) $\cong$ (e) $\cong$ (g). The prove that, e.g., (d) $\cong$ (f).

10.21 For the case of one segment, this is assertion *10.20* (f). In the general case, use *10.19* (i.e., the fact that (l) $\cong$ (h); observe that the homeomorphism can be fixed on the boundary of the square). Surround the segments by disjoint rhombi and apply the above homeomorphism in each of them.

10.22 Use induction on the number of links of the polyline X. Each time, applying the argument used in the solution of the Problem *10.21* to the outer link of X, we homeomorphically map $\mathbb{R}^2 \smallsetminus X$ onto the complement of a polyline with smaller number of links.

10.23 Prove that for any $p, q \in \operatorname{Int} D^2$ there is a homeomorphism $f : D^2 \to D^2$ such that $f(p) = f(q)$ and $\operatorname{ab}(f) : S^1 \to S^1$ is the identity. After that, use induction.
Here is a more explicit construction. Let $K = \{(x_i, y_i)\}_{i=1}^n$. We can assume that x_i's are pairwise distinct. (Why?) Take any continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x_i) = y_i$, $i = 1, \dots, n$. Then $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, y - f(x))$ is a homeomorphism with $F(K) \subset \mathbb{R}^1$. There is a homeomorphism $g : \mathbb{R} \to \mathbb{R}$ such that $g(x_i) = i$, $i = 1, \dots, n$. Consider the homeomorphism $G : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (g(x), y)$. Then $(G \circ F)(K) = \{1, \dots, n\}$, whence $\mathbb{R}^2 \smallsetminus K \cong \mathbb{R}^2 \smallsetminus \{1, \dots, n\}$.

10.24 Use the homeomorphism (b) $\cong$ (c) in Problem *10.20*.

10.25 Use Problems *10.24* and *10.23*.

10.26 Use the homeomorphism $(x, t) \mapsto (x, (1 - t)f(x) + tg(x))$.

10.27 The first question is as follows: what is the mug from the mathematical point of view? How is it presented? Actually, there is a precise approach to describing similar objects and introduce the corresponding class of spaces ("manifolds"), but for now we use the "common sense". We start with a cylinder, which is homeomorphic to a closed 3-disk, which in turn is homeomorphic to a half-disk, is not it? Further, if we delete from the half-disk a concentric half-disk of smaller radius, then the rest (i.e., the "skin of a half of a water-mellon") is still homeomorphic to the half-disk. (We can prove this quite rigorously, and even give the required formulas.) The remaining "skin" is a mug without a handle, which is thus homeomorphic to a cylinder. Furthermore, we can assume that the "disks" along which the handle adjoins the mug correspond to the bases of the cylinder, cf. *10.25*, while the handle is a (deformed) cylinder itself. "Pasting together" two cylinders, we certainly obtain a doughnut as a result!

10.28 The following objects are homeomorphic to a coin: a saucer, a glass, a spoon, a fork, a knife, a plate, a nail, a screw, a bolt, a nut, a drill. The remaining objects are homeomorphic to a wedding ring: a cup, a flower pot, a key.

10.29 Formulate and prove the plane version of the problem. After that use rotation. An intermediate shape here is a 3-disk in which a thin cylinder is drilled out. We can also single out the following useful lemma. Let C_0 be a cylinder, $C \subset C_0$ a smaller cylinder with upper base lying inside that of C_0. Then there exists a homeomorphism $f : \mathrm{Cl}(C_0 \smallsetminus C) \to C_0$ identical on $\operatorname{Fr} C_0 \smallsetminus C$.

10.30 Our argument will be close to that used in the solution of Problem *10.27*. Repeating the first step of the solution to Problem *10.29*, we "get rid" of the large spherical hole at the end of the "tube". After that, we observe that the knotted tube has a neighborhood homeomorphic to a cylinder. Applying the lemma formulated in the above solution, we obtain a homeomorphism between the ball with a knotted hole and the whole ball.

10.31 In Figure, we have a sequence of images, where any two neighboring ones are connected by an (easy to imagine) homeomorphism. (The latter is actually a result of a "deformation".) It remains to take the composition.

10.32 Use the sequence of images depicted in Figure. (Cf. the solution to the previous problem.)

10.33 Both spaces are homeomorphic to $S^3 \smallsetminus (S^1 \cup \text{point})$. To see this, use the homeomorphism $\mathbb{R}^3 \cong S^3 \smallsetminus \text{point}$ of Problem *10.R*. (The second time, take the point to be deleted on the circle S^1.) In the general case of $\mathbb{R}^n$, this argument also works. But what happens if we replace S^1 by S^k?

10.34 The stereographic projection $S^n \smallsetminus (0, \dots, 0, 1) \to \mathbb{R}^n$ maps our set to a (spherically symmetric) neighborhood of S^{k-1}, which is easily seen to be homeomorphic to $\mathbb{R}^n \smallsetminus \mathbb{R}^{n-k}$.

10.35 Here are properties that distinguish each of the spaces from the remaining ones: $\mathbb{Z}$ is discrete, $\mathbb{Q}$ is countable, each proper closed subset of $\mathbb{R}_{T_1}$ is finite, and, finally, any two nonempty open sets in the arrow have nonempty intersection.

10.36 Set $X = \{k\}_{-\infty}^{-1} \cup \bigcup_{k=0}^{\infty}[2k; 2k+1)$ and $Y = X \cup \{1\}$ and consider the bijections

$$X \to Y : x \mapsto \begin{cases} x+1 & \text{if } x \le -2, \\ 1 & \text{if } x = -1, \\ x & \text{if } x \ge 0; \end{cases} \quad Y \to X : x \mapsto \begin{cases} x & \text{if } x < 0, \\ \frac{x}{2} & \text{if } x \in [0,1], \\ \frac{x-1}{2} & \text{if } x \in [2,3), \\ x-2 & \text{if } x \ge 4. \end{cases}$$

Similar tricks are called "Hilbert's hotel". Guess why.

10.37 This is indeed very simple. Take $[0,1]$ and $\mathbb{R}$. (Actually, any two nonhomeomorphic subsets of $\mathbb{R}$ with nonempty interiors would do.)

10.38 The topology in $\mathbb{Q}$ is not discrete.

10.39 1), 2) If the discrete space is not one-point, this is impossible.

10.40 See *10.35*.

11.1 1)–3) Yes: in each of these spaces, two nonempty open sets always have nonempty intersection.

11.2 The empty space and a singleton.

11.3 A disconnected two-point space is obviously discrete.

11.4 1) No, $\mathbb{Q}$ is not connected since, for instance, $\mathbb{Q} = \big(\mathbb{Q} \cap (-\infty, \sqrt{2})\big) \cup \big(\mathbb{Q} \cap (\sqrt{2}, +\infty)\big)$. 2) $\mathbb{R} \smallsetminus \mathbb{Q}$ is also disconnected for a similar (and even simpler) reason.

11.5 1) Yes, if (X, Ω_1) is connected, then so is (X, Ω_2): if $X = U \cup V$, where $U, V \in \Omega_1$, then $U, V \in \Omega_2$. 2) No, the connectedness of (X, Ω_1) does not imply that of (X, Ω_2): consider the case where Ω_1 is indiscrete, Ω_2 is discrete, and X contains more than one point.

11.6 A subset A of a space X is disconnected iff there exist open subsets $U, V \subset X$ such that $A \subset U \cup V$, $U \cap V \cap A = \varnothing$, $U \cap A \ne \varnothing$, and $V \cap A \ne \varnothing$.

11.7 1), 3): No, it is not, because the relative topology on $\{0,1\}$ is discrete (see *11.2*). 2): Yes, it is, because the relative topology on $\{0,1\}$ is not discrete (see *11.3*).

11.8 1) Every subset of the arrow is connected. 2) A subset of $\mathbb{R}_{T_1}$ is connected iff it is empty, one-point, or infinite.

11.9 Show that $[0,1]$ is both open and closed in $[0,1] \cup (2,3]$.

11.10 Given $x, y \in A \subset \mathbb{R}$, $z \in (x, y)$, and $z \notin A$, produce two nonempty sets open in A that partition A. Cf. *11.4*.

11.11 $\boxed{\Longrightarrow}$ Let B and C be two nonempty subsets of A open in A that partition A. $\boxed{\Longleftarrow}$ Use the fact that if $B \cap \operatorname{Cl}_X C = \varnothing$, then $B = A \cap (X \smallsetminus \operatorname{Cl}_X C)$.

11.12 Let $X = A \cup x_*$, $x_* \notin A$, and let Ω_* consist of the empty set and all sets containing x_*. Is this a topological structure in X? What topology does it induce on A?

11.13 Let A be disconnected, and let B and C satisfy the hypothesis of *11.11*. Then we can put

$$U = \{x \in \mathbb{R}^n \mid \rho(x, B) < \rho(x, C)\} \quad \text{and} \quad V = \{x \in \mathbb{R}^n \mid \rho(x, B) > \rho(x, C)\}.$$

Notice that the conclusion of *11.13* would still hold true if in the hypothesis we replaced $\mathbb{R}^n$ by an arbitrary space where every open subspace is normal, see Section 14.

11.15 Obvious. (Cf. *11.6*.)

11.15 The set A is dense in B equipped with the relative topology induced from the ambient space. Therefore, we can apply *11.B*.

11.16 Assume the contrary: let $A \cup B$ be disconnected. Then there exist open subsets U and V of the ambient space such that $A \cup B \subset U \cup V$, $U \cap (A \cup B) \neq \varnothing$, $V \cap (A \cup B) \neq \varnothing$, and $U \cap V \cap (A \cup B) = \varnothing$ (cf. the solution of Problem *11.6*). Since $A \cup B \subset U \cup V$, the set A meets at least one of the sets U and V. Without loss of generality, we can assume that $A \cap U \neq \varnothing$. Then $A \cap V = \varnothing$ by the connectedness of A, whence $A \subset U$. Since U is a neighborhood of any point of $A \cap \operatorname{Cl} B$, it meets B. The set V also meets B since $V \cap (A \cup B) \neq \varnothing$, while $A \cap V = \varnothing$. This contradicts the connectedness of B since $B \cap U$ and $B \cap V$ form a partition of B into two nonempty sets open in B.

11.17 If $A \cup B$ is disconnected, then there exist sets U and V open in X such that $U \cup V \supset A \cup B$, $U \cap (A \cup B) \neq \varnothing$, $V \cap (A \cup B) \neq \varnothing$, and $U \cap V \cap (A \cup B) = \varnothing$. Since A is connected, A is contained in U or V. Without loss of generality we may assume that $A \subset U$. Set $B_1 = B \cap V$. Since B is open in $X \smallsetminus A$ and $V \subset X \smallsetminus A$, the set B_1 is open in V. Therefore, B_1 is open in X. Furthermore, we have $B_1 \subset X \smallsetminus U \subset X \smallsetminus A$, therefore B_1 is closed in $X \smallsetminus U$ and hence also in X. Thus B_1 is both open and closed in X, contrary to the connectedness of X.

11.18 No, it does not. Example: put $A = \mathbb{Q}$ and $B = \mathbb{R} \smallsetminus \mathbb{Q}$.

11.19 1) If A and B are open and A is disconnected, then $A = U \cup V$, where U and V are disjoint nonempty sets open in A. Since $A \cap B$ is connected, then either $A \cap B \subset U$, or $A \cap B \subset V$. Without loss of generality,

we can assume that $A \cap B \subset U$. Then $\{V, U \cup B\}$ is a partition of $A \cup B$ into nonempty open sets. (U and V are open in $A \cup B$ because an open subset of an open set is open.) This contradicts the connectedness of $A \cup B$.
2) In the case of closed A and B, the same arguments work if openness is everywhere replaced by closedness.

11.20 Not necessarily. Consider the closed sets $K_n = \{(x, y) \mid x \geq 0,\ y \in \{0, 1\}\} \cup \{(x, y) \mid x \in \mathbb{N},\ x \geq n,\ y \in [-1, 1]\}$, $n \in \mathbb{N}$. (An infinite ladder, railroad, fence, hedge, handrail, balustrade, or banisters, whichever you prefer.) Their intersection is the union of the rays $\{y = 1, x \geq 0\}$ and $\{y = -1, x \geq 0\}$.

11.21 Let C be a connected component of X, $x \in C$ an arbitrary point. If U_x is a connected neighborhood of x, then U_x lies entirely in C, and so x is an interior point of C, which is thus open.

11.22 Theorem *11.I* allows us to transform the statement under consideration into the following obvious statement: if a set M is connected and A is both open and closed, then either $M \subset A$, or $M \subset X \smallsetminus A$.

11.23 See the next problem.

11.24 Prove that any two points in the Cantor set cannot belong to the same connected component.

11.25 If $\operatorname{Fr} A = \varnothing$, then $A = \operatorname{Cl} A = \operatorname{Int} A$ is a nontrivial open-closed set.

11.26 If $F \cap \operatorname{Fr} A = \varnothing$, then $F = (F \cap \operatorname{Cl} A) \cup (F \cap \operatorname{Cl}(X \smallsetminus A))$ and $F \cap \operatorname{Cl} A \cap \operatorname{Cl}(X \smallsetminus A) = \varnothing$.

11.27 If $\operatorname{Cl} A$ is disconnected, then $\operatorname{Cl} A = F_1 \cup F_2$, where F_1 and F_2 are nonempty disjoint sets closed in X. Each of them meets A since $F_1 \cup F_2$ is the smallest closed set containing A. Therefore A splits into the union of nonempty sets $A_1 = A \cap F_1$ and $A_2 = A \cap F_2$, whose boundaries $\operatorname{Fr} A_1$ and $\operatorname{Fr} A_2$ are nonempty by *11.25*. This contradicts the connectedness of $\operatorname{Fr} A = \operatorname{Fr} A_1 \cup \operatorname{Fr} A_2$.

11.29 Combine *11.N* and *11.10*.

11.30 Let M be the connected component of unity. For each $x \in M$, the set $x \cdot M$ is connected and contains $x = x \cdot 1$. Therefore $x \cdot M$ meets M, whence $x \cdot M \subset M$. Thus M is a subgroup of X. Furthermore, for each $x \in X$ the set $x^{-1} \cdot M \cdot x$ is connected and contains the unity. Consequently $x^{-1} \cdot M \cdot x \subset M$. Hence the subgroup M is normal.

11.31 Let $U \subset \mathbb{R}$ be an open set. For each $x \in U$, let $(m_x, M_x) \subset U$ be the largest open interval containing x. (Take the union of all open intervals in U that contain x.) Any two such intervals either coincide or are disjoint, i.e., they constitute a partition of U.

11.32 1) Certainly, it is connected because if l is the spiral, then $\operatorname{Cl} l = l \cup S^1$. 2) Obviously, the answer will not change if we add to the spiral only a part of the limit circle.

11.33 (a) This set is disconnected since, for example, so is its projection to the x axis.
(b) This set is connected because any two of its points are joined by a broken line (with at most two segments).
(c) This set is connected. Consider the set $X \subset \mathbb{R}^2$ defined as the union of lines $y = kx$ with $k \in \mathbb{Q}$. Clearly, the coordinates (x, y) of any point of X are either both rational or both irrational. Obviously X is connected, while the set under consideration is contained in the closure of X (coinciding with the whole plane).

13.17 Let $A \subset \mathbb{R}^n$ be the connected set. Use the fact that balls in $\mathbb{R}^n$ are connected by *11.U* (or by *11.V*) and apply *11.E* to the family $\{A\} \cup \{B_\varepsilon(x)\}_{x \in A}$.

11.35 For $x \in A$, let $V_x \subset U$ be a spherical neighborhood of x. Consider the neighborhood $\bigcup_{x \in A} V_x$ of A. To show that it is connected, use the fact that balls in $\mathbb{R}^n$ are connected by *11.U* (or by *11.V*) and apply *11.E* to the family $\{A\} \cup \{V_x\}_{x \in A}$.

11.36 Let

$$X = \{(0,0), (0,1)\} \cup \left\{(x, y) \mid x \in [0,1],\ y = \tfrac{1}{n},\ n \in \mathbb{N}\right\} \subset \mathbb{R}^2.$$

Prove that any open and closed set contains both points $A(0,0)$ and $B(1,0)$.

12.1 This is an immediate corollary of Theorem *12.A*. Indeed any real polynomial of odd degree takes both positive and negative values (for values of the argument with sufficiently large absolute values).

12.2 Combine *11.Z*, *12.B*, and *12.E*.

12.3 There are nine topological types, namely: (1) A, R; (2) B; (3) C, G, I, J, L, M, N, S, U, V, W, Z; (4) D, O; (5) E, F, T, Y; (6) H, K; (8) P; (9) Q; (7) X. Notice that the answer depends on the graphics of the letters. For example, we can draw letter R homeomorphic not to A, but to Q. To prove that letters of different types are not homeomorphic, use arguments similar to that in the solution of *12.E*.

12.4 A square with any of its points removed is still connected (prove this!), while the segment is not. (We emphasize that the sentence “Because a square cannot be partitioned into two nonempty open sets.” cannot serve as a proof of the mentioned fact. The simplest approach would be to use *11.I*.)

12.5 Use *10.R*.

***12.2*x** This is so because for any $x_0 \in X$ the set $\{x \mid f(x) = f(x_0)\}$ is both open and closed (prove this!). Here is another version of the argument. For each point y in the source space the preimage $f^{-1}(y)$ is open.

***12.4*x** Fix $h \in H$ and consider the map $x \mapsto xhx^{-1}$. Since H is a normal subgroup, the image of G is contained in H. Since H is discrete, this map is locally constant. Therefore, by *12.2*x, it is constant. Since the unity is mapped to h, it follows that $xhx^{-1} = h$ for any $x \in G$. Therefore $gh = hg$ for any $g \in G$ and $h \in H$.

***12.5*x** Consider the union of all sets with property $\mathcal{E}$ containing a point a. (Is not it natural to call this set a *component of a in the sense of $\mathcal{E}$?*) Prove that such sets constitute an open partition of X. Therefore, if X is connected, any such a set is the whole X.

***12.7*x** Introduce a coordinate system with y-axis l, and consider the function f sending $t \in \mathbb{R}$ to the area of the part A that lies to the left of the line $x = t$. Prove that f is continuous. Observe that the set of values of f is the segment $[0; S]$, where S is the area of A, and apply the Intermediate Value Theorem.

***12.8*x** If A is connected, then the function introduced in the solution of Problem *12.7*x is strictly monotone on $f^{-1}\big((0, S)\big)$.

***12.9*x** Fix a Cartesian coordinate system on the plane and, for any $\varphi \in [0, \pi]$, consider also the coordinate system obtained by rotating the fixed one through an angle of φ around the origin. Let f_A and f_B be functions defined by the following property: the line defined by $x = f_A(\varphi)$ (respectively, $x = f_B(\varphi)$) in the corresponding coordinate system divides A (respectively, B) into two parts of equal areas. Put $g(\varphi) = f_A(\varphi) - f_B(\varphi)$. Clearly, $g(\pi) = -g(0)$. Hence, if we proved the continuity of f_A and f_B, then Intermediate Value Theorem would imply existence of φ_0 such that $g(\varphi_0) = 0$. The corresponding line $x = f_A(\varphi_0)$ divides each of the figures into two parts of equal areas. Prove continuity of f_A and f_B!

***12.10*x** The idea of solution is close to the idea of solution of the preceding problem. Find an appropriate function whose zero would give rise to the desired lines, while the existence of a zero follows from Intermediate Value Theorem.

13.1 Combine *11.R* and *11.N*.

13.2 Combine *13.1* and *11.26*.

13.3 $\boxed{\Longrightarrow}$ This is obvious since in_A is continuous.
$\boxed{\Longleftarrow}$ Indeed, u is continuous as a submap of the continuous map $\mathrm{in}_A \circ u$.

13.4 A one-point discrete space, an indiscrete space, the arrow, and $\mathbb{R}_{T_1}$ are path-connected. Also notice that the points a and c in  can be connected by a path!

13.5 Use *13.3*.

13.6 Combine (the formula of) *13.C* and *13.5*.

13.8 Indeed, let $u : I \to X$ be a path. Then any two points $u(x), u(y) \in u(I)$ are connected by the path defined as the composition of u and $I \to I : t \mapsto (1-t)x + ty$.

13.9 A path in the space of polygons looks as a deformation of a polygon. Let us join an arbitrary polygon P with a regular triangle T. We take a vertex V of P and move it to (say, the midpoint of) the diagonal of P joining the neighboring vertices of V, thus reducing the number of vertices of P. Proceeding by induction, we come to a triangle, which is easy to deform into T.
It is also easy to see that any convex n-gon can be deformed to a regular n-gon in the space of convex n-gons.

13.11 We consider the case where A and B are open and prove that A is path-connected. Let $x, y \in A$, and let u be a path joining x and y in $A \cup B$. If $u(I) \not\subset A$, then we set $\bar{t} = \sup\{t \mid u([0,t]) \subset A\}$. Since A is open, $u(\bar{t}) \in B$. Since B is open, there is $t_0 < \bar{t}$ with $u(t_0) \in B$, whence $u(t_0) \in A \cap B$. In a similar way, we find $t_1 \in I$ such that $u(t_1) \in A \cap B$ and $u([t_1, 1]) \subset A$. It remains to join $u(t_0)$ and $u(t_1)$ by a path in $A \cap B$.

13.12 1), 2) The assertion about the boundary is trivial, and an example is easy to find in $\mathbb{R}^1$. It is also easy to find a path-connected set in $\mathbb{R}^2$ with disconnected interior. (Why are there no such examples in $\mathbb{R}^1$?)

13.13 Let $x, y \in \operatorname{Cl} A$. Assume that $x, y \in \operatorname{Int} A$. (Otherwise, the argument becomes even simpler.) Then we join x and y with points $x', y' \in \operatorname{Fr} A$ by segments and join x' and y' by a path in $\operatorname{Fr} A$.

13.16 $\boxed{\Longrightarrow}$ This is *13.M*. $\boxed{\Longleftarrow}$ Combine the result of *11.Y* with *13.6* (or *13.B*).

13.17 Combine Problem *11.34* and Theorem *13.U*.

13.18 Combine Problem *11.35* and Theorem *13.U*.

***13.1*x** Use multiplication of paths.

***13.2*x** Obvious.

***13.3*x** Obvious.

***13.4*x** Define polygon-connected components and show that they are open for open sets in $\mathbb{R}^n$.

***13.5*x** For example, set $A = S^1$.

***13.6*x** Let $x, y \in \mathbb{R}^2 \smallsetminus X$. Draw two nonparallel lines through x and y that do not intersect X.

13.7x Let $x, y \in \mathbb{R}^n \smallsetminus X$. Draw a plane through x and y that intersects each of the affine subspaces at most at one point and apply Problem *13.6x*. (In order to find such a plane, use the orthogonal projection of $\mathbb{R}^n$ to the orthogonal complement of the line through x and y.)

13.8x Let $w_1, w_2 \in \mathbb{C}^n \smallsetminus X$. Observe that the complex line through w_1 and w_2 intersects each of the algebraic subsets at a finite number of points and apply Problem *13.6x*.

13.9x The set $Symm(n; \mathbb{R}) = \{A \mid {}^tA = A\}$ is a linear subspace in the space of all matrices, hence, it is path-connected. To handle the other subspaces, use the function $A \mapsto \det A$. Since (obviously) it is continuous and takes in each case both positive and negative values, but never vanishes, it immediately follows that $GL(n; \mathbb{R})$, $O(n; \mathbb{R})$, $Symm(n; \mathbb{R}) \cap GL(n; \mathbb{R})$, and $\{A \mid A^2 = \mathbb{E}\}$ are disconnected. In fact, each of them has two path-connected components. Let us show, for example, that $GL_+(n; \mathbb{R}) = \{A \mid \det A > 0\}$ is path-connected. The following assertion is of use here, as well as below. For each basis $\{e_i\}$ in $\mathbb{R}^n$ there exist paths $e_i : I \to \mathbb{R}^n$ such that: 1) for each $t \in [0, 1]$ the collection $\{e_i(t)\}$ is a basis; 2) $e_i(0) = e_i$, $i = 1, \dots, n$; 3) $\{e_i(1)\}$ is an orthonormal basis. (Prove this.)

13.10x $GL(n, \mathbb{C})$ is even polygon-connected by *13.8x* since $\det A = 0$ is an algebraic equation in $\mathbb{C}^{n^2}$. The other spaces are path-connected.

14.1 Only the discrete space is Hausdorff (and, formally, indiscrete singletons).

14.2 Read the following formula written with quantifiers: $\exists\, U_b\ \forall\, N \in \mathbb{N}\ \exists\, n > N :\ a_n \in X \smallsetminus U_b$.

14.4 Let $f, g : X \to Y$ be two continuous maps and let Y be a Hausdorff space. To prove that the coincidence set $C(f, g)$ is closed, we show that its complement is open. If $x \in X \smallsetminus C(f, g)$, then $f(x) \neq g(x)$. Since Y is Hausdorff, $f(x)$ and $g(x)$ have disjoint neighborhoods U and V. For each $y \in f^{-1}(U) \cap g^{-1}(V)$, we obviously have $f(y) \neq g(y)$, whence $f^{-1}(U) \cap g^{-1}(V) \subset X \smallsetminus C(f, g)$. Since f and g are continuous, this intersection is a neighborhood of y.

14.5 Consider the following two maps from I to the arrow: $x \mapsto 1$ and $x \mapsto \operatorname{sgn} x$. (Here, $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ is the function that takes negative numbers to -1, 0 to 0, and positive numbers to 1.)

14.6 This follows from *14.4* because, obviously, the fixed point set of f is $C(f, \mathrm{id}_X)$.

14.7 Let X be the arrow. Consider the map $f : X \to X : x \mapsto x + \sin x$. What is the fixed point set of f? Is it closed in X?

14.8 By *14.4*, the coincidence set $C(f,g)$ of f and g is closed in X. Since $C(f,g)$ contains the everywhere-dense set A, it coincides with the entire X.

14.10 Only the first two properties are hereditary.

14.11 We have $\{x\} = \bigcap_{U \ni x} U$ iff for each $y \neq x$ the point x has a neighborhood U that does not contain y, which is precisely T_1.

14.12 This is obvious.

14.13 See *14.J*.

14.14 Consider a neighborhood of $f(a)$ that does not contain $f(b)$ and take its preimage.

14.15 Otherwise, the indiscrete space would contain nontrivial closed subsets (preimages of singletons).

14.16 This is a complete analog of the topology on $\mathbb{R}_{T_1}$: only finite sets and the entire space are closed.

14.17 Consider the coarsest topology on $\mathbb{R}$ that contains the usual topology and is such that the set $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is closed. Show that in this space the point 0 and the set A cannot be separated by neighborhoods.

14.18 An obvious example is the indiscrete space. A more instructive example is the "real line with two zeros", which is also of interest in some other cases: let $X = \mathbb{R} \cup 0'$, and let the base of the topology in X consist of all usual open intervals $(a,b) \subset \mathbb{R}$ and of "modified intervals" $(a,b)' := (a,0) \cup 0' \cup (0,b)$, where $a < 0 < b$. (Verify that this is indeed a base.) Axiom T_3 is fulfilled, but 0 and $0'$ have no disjoint neighborhoods in X.

14.19 $\boxed{\Longrightarrow}$ Let a space X satisfy T_3. If $b \in X$ and W is a neighborhood of b, then, applying T_3 to b and $X \smallsetminus W$, we obtain disjoint open sets U and V such that $b \in U$ and $X \smallsetminus W \subset V$. Obviously, $\mathrm{Cl}(U) \subset X \smallsetminus V \subset W$.
$\boxed{\Longleftarrow}$ Let X be the space, let $F \subset X$ be a closed set, and let $b \in X \smallsetminus F$. Then $X \smallsetminus F$ is a neighborhood of x, and we can find a neighborhood U of x with $\mathrm{Cl}(U) \subset X \smallsetminus F$. Then $X \smallsetminus \mathrm{Cl}(U)$ is the required neighborhood of F disjoint with U.

14.20 Let X be a space, $A \subset X$ a subspace, B a closed subset of A. If $x \notin B$, then $x \notin F$, where F is closed in X and $F \cap A = B$. The rest is obvious.

14.21 For example, consider an indiscrete space or the arrow.

14.22 Cf. the proof of assertion *14.19*. $\boxed{\Longrightarrow}$ Let a space X satisfy T_4. If $F \subset X$ is a closed set and W is a neighborhood of F, then, applying T_4 to F and $X \smallsetminus W$, we obtain disjoint open sets U and V such that $F \subset U$ and $X \smallsetminus W \subset V$. Obviously, $\mathrm{Cl}(U) \subset X \smallsetminus V \subset W$.
$\boxed{\Longleftarrow}$ Let X be the space, and let $F, G \subset X$ be two disjoint closed sets.

Then $X \smallsetminus G$ is a neighborhood of F, and we can find a neighborhood U of F with $\mathrm{Cl}(U) \subset X \smallsetminus G$. Then $X \smallsetminus \mathrm{Cl}(U)$ is the required neighborhood of F disjoint with U.

14.23 Use the fact that a closed subset of a closed subspace is closed in the entire space and recall the definition of the relative topology.

14.24 For example, consider $A = \mathbb{N}$ and $B = \left\{n + \frac{1}{n}\right\}_1^\infty$ in $\mathbb{R}$.

14.25 Let $F_1, F_2 \subset Y$ be disjoint closed sets. Since f is continuous, their preimages $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are also closed in X. Since X satisfies T_4, the preimages have disjoint neighborhoods W_1 and W_2. By assumption, the closed sets $A_i = X \smallsetminus W_i$, $i = 1, 2$, have closed images B_i. Since $B_1 \cup B_2 = f(A_1) \cup f(A_2) = f(A_1 \cup A_2) = f(X) = Y$, the open sets $U_1 = Y \smallsetminus B_1$ and $U_2 = Y \smallsetminus B_2$ are disjoint. Check that $F_i \subset U_i$, $i = 1, 2$.

***14.1*x** Let $x, y \in \mathcal{N}$ be two distinct points. If at least one of them lies in $\mathcal{H}$, then, obviously, they have disjoint neighborhoods. Now if $x, y \in \mathbb{R}^1$, then they are separated by certain disjoint disks D_x and D_y.

***14.2*x** Verify that if an open disk $D \subset \mathcal{H}$ touches $\mathbb{R}^1$ at a point x, then $\mathrm{Cl}(D \cup x) = \mathrm{Cl}\, D$. After that, use *14.19*.

***14.3*x** The discrete structure.

***14.4*x** Since $\mathbb{R}^1$ is closed in $\mathcal{N}$ and the relative topology on $\mathbb{R}^1$ is discrete, each subset of $\mathbb{R}^1$ is closed in $\mathcal{N}$. Let us prove that the closed sets $\{(x, 0) \mid x \in \mathbb{Q}\}$ and $\{(x, 0) \mid x \in \mathbb{R} \smallsetminus \mathbb{Q}\}$ have no disjoint neighborhoods in $\mathcal{N}$. Let U be a Niemytski neighborhood of $\mathbb{R}^1 \smallsetminus \mathbb{Q}$. For each $x \in \mathbb{R}^1 \smallsetminus \mathbb{Q}$, fix an $r(x)$ such that an open disk $D_{r(x)} \subset U$ of radius $r(x)$ touches $\mathbb{R}^1$ at x. Put $Z_n = \{x \in \mathbb{R}^1 \mid r(x) > 1/n\}$. Since, obviously, $\mathbb{Q} \cup \bigcup_{n=1}^\infty Z_n = \mathbb{R}^1$, the result of *6.44* implies that there is (sufficiently large) n such that Z_n is *not* nowhere dense. Therefore, $\mathrm{Cl}\, Z_n$ contains a segment $[a, b] \subset \mathbb{R}^1$, whence it follows that $U \cup [a, b]$ contains a whole neighborhood of $[a, b]$, which meets any neighborhood in $\mathcal{N}$ of any rational in $[a, b]$. Hence, U meets each neighborhood of $\mathbb{Q}$, and so, indeed, $\mathcal{N}$ is not normal.

***14.6*x** Add a point x_* to $\mathcal{N}$: $\mathcal{N}^* = \mathcal{N} \cup x_*$. The topology Ω^* on $\mathcal{N}^*$ is obtained from the topology Ω on $\mathcal{N}$ by adding sets of the form $x_* \cup U$, where $U \in \Omega$ contains all points in $\mathbb{R}^1$ except a finite number. Verify that $(\mathcal{N}^*, \Omega^*)$is a normal space.

***14.8*x** Set $f(x) = \dfrac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}$.

14.9*x*.1 Set $A = f^{-1}\big([-1, -\frac{1}{3}]\big)$ and $B = f^{-1}\big([\frac{1}{3}, 1]\big)$. Use *14.8x* to prove that there exists a function $g : X \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ such that $g(A) = -\frac{1}{3}$ and $g(B) = \frac{1}{3}$.

14.9x By *14.9x.1*, there is a function $g_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $|f(x) - g_1(x)| \le \frac{2}{3}$ for every $x \in F$. Put $f_1(x) = f(x) - g_1(x)$. Slightly modifying the proof of *14.9x.1* we obtain a function $g_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$ such that $|f_1(x) - g_2(x)| \le \frac{4}{9}$ for every $x \in F$, i.e. $|f(x) - g_1(x) - g_2(x)| \le \frac{4}{9}$. Repeating this process, we construct a sequence of functions $g_n : X \to \left[-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}\right]$ such that

$$|f(x) - g_1(x) - \ldots - g_n(x)| \le \tfrac{2^n}{3^n}.$$

Use *24.Hx* to prove that the sum $g_1(x)+\ldots+g_n(x)$ converges to a continuous function $g : X \to [-1, 1]$. Obviously, $g|_F = f$.

15.1 This is obvious.

15.2 Sending each curve C in Σ to a pair of points in $\mathbb{Q}^2 \subset \mathbb{R}^2$ lying inside two "halves" of C, we obtain an injection $\Sigma \to \mathbb{Q}^4$. It remains to observe that $\mathbb{Q}^4$ is countable and use *15.1*. (In order to show that $\mathbb{Q}^4$ is countable, use *15.F* and *15.E*.)

15.3 The arrow is second countable: $\{(x, +\infty) \mid x \in \mathbb{Q}\}$ is a countable base. (Use *15.F*.) Use *15.G* to show that $\mathbb{R}_{T_1}$ is not second countable.

15.4 Yes, they are: $\mathbb{N}$ is dense both in the arrow and in $\mathbb{R}_{T_1}$.

15.5 Consider the space from Problem *2.6*.

15.6 Take an uncountable set (e.g., $\mathbb{R}$) with all distances between distinct points equal to 1. (See *4.A*.)

15.7 Let X be a separable space, let $\{U_\alpha\}_{\alpha \in J}$ be the collection of pairwise disjoint open sets of X, and let $A \subset X$ be a countable everywhere-dense subset. Taking for each $\alpha \in J$ a point $p(\alpha) \in A \cap U_\alpha \ne \varnothing$, we obtain an injection $J \to A$.

15.8 Use *11.H*, *13.U*, *13.S*, *15.M*, and *15.7*.

15.9 Consider $\mathrm{id} : \mathbb{R} \to \mathbb{R}_{T_1}$ and use *15.M* and the result of *15.3*.

15.10 Let X be the space, B_0 a countable base of X, and B an arbitrary base of X. By the Lindelöf Theorem *15.O*, each set in B_0 is the union of a countable collection of sets in B. It remains to use *15.E*.

15.12 Obviously, it suffices to prove only the last assertion. If U is an open set and $a \in U$, then there is $r > 0$ such that $B_r(a) \subset U$. Since $r_n \to 0$, there is $k \in \mathbb{N}$ such that $r_k < r$, whence $B_{r_k}(a) \subset U$.

15.13 If X is a discrete (respectively, indiscrete) space, then the minimal base at a point $x \in X$ is $\{\{x\}\}$ (respectively, $\{X\}$).

15.14 All spaces except $\mathbb{R}_{T_1}$, cf. *15.3*.

15.15 Equip $\mathbb{R}$ with the topology determined by the base $\{[a,b) \mid a,b \in \mathbb{R},\ a < b\}$.

15.16 If $\{V_i\}_1^\infty$ is a countable neighborhood base, then put $U_i = \bigcap_1^n V_i$.

15.17 In this space, $x_n \to a$ iff $x_n = a$ for all sufficiently large n. It follows that $\operatorname{SCl} A = A$ for each $A \subset \mathbb{R}$. Check that $\operatorname{SCl}[0,1] = [0,1] \neq \operatorname{Cl}[0,1] = \mathbb{R}$.

15.18 Consider the identical map of the space from Problem *15.17* to $\mathbb{R}$.

16.1 1) If (X,Ω_2) is compact, then, obviously, so is (X,Ω_1). 2) The converse is wrong in general.

16.2 The arrow is compact. (Which set must belong to each cover of the arrow?) The space $\mathbb{R}_{T_1}$ is also compact: if Γ is an open cover of $\mathbb{R}_{T_1}$, then any nonempty element of Γ covers the entire $\mathbb{R}_{T_1}$ except a finite number of points, each of which, in turn, is covered by an element of Γ.

16.3 This set is not compact in $\mathbb{R}$ since, e.g., the cover $\{(0, 2 - \frac{1}{n})\}_{n\in\mathbb{N}}$ contains no finite subcovering.

16.4 The set $[1,2)$ is compact in the arrow because any open set containing 1 (i.e., a ray $(a,+\infty)$ with $a < 1$, or even $[0,+\infty)$ itself) contains the entire $[1,2)$. Notice that the set $(1,2]$ is not compact (to prove this, use *16.D*).

16.5 A is compact in the arrow iff $\inf A \in A$.

16.6 See the solution of *16.2*.

16.7 1) If Γ covers $A \cup B$, then Γ covers both A and B. Therefore, Γ contains both a finite subcovering of A and a finite subcovering of B, whose union is a finite cover of $A \cup B$. 2) The set $A \cap B$ is not necessarily compact (use *16.5* to construct the corresponding example). Unfortunately, sometimes students present a "proof" of the fact that $A \cap B$ is compact. Here is a typical argument. "Since A is compact, A has a finite cover, and since B is compact, B also has a finite cover. Taking pairwise intersections of the elements of these covers, we obtain a finite cover of the intersection $A \cap B$." Why does not this argument imply in any way that the intersection of two compact sets is compact?

16.8 Take an open cover Γ of A, and let $U_0 \in \Gamma$ be an open set containing 0. Then U_0 covers the entire A except a finite number of points, each of which, in turn, is covered by an element of Γ. (Cf. the solution of *16.2*.)

16.9 Consider an indiscrete two-point space and its one-point subset.

16.10 Combine *16.K*, *2.F*, and *16.J*.

16.11 Take any $\lambda_0 \in \Lambda$. Then $\{X \smallsetminus K_\lambda\}_{\lambda \in \Lambda}$ is an open cover of the compact set $K_{\lambda_0} \smallsetminus U$. If $\{X \smallsetminus K_{\lambda_i}\}_1^n$ is a finite subcovering, then $U \supset \bigcap_1^n K_{\lambda_i}$.

16.12 By *16.K*, all sets K_n are closed subsets of K_1. Since the collection $\{K_n\}$ obviously has the finite intersection property and K_1 is compact, we have $\bigcap_1^\infty K_n \neq \varnothing$ is nonempty (see Theorem *16.G*). Assume the contrary: let $\bigcap K_n = F_1 \cup F_2$, where F_1 and F_2 are two disjoint nonempty closed sets. By Theorem 13.17 and *16.O*, they have disjoint neighborhoods U_1 and U_2. Applying *16.11* to $U_1 \cup U_2$, we see that for some n we have $U_1 \cup U_2 \supset K_n \supset F_1 \cup F_2$, which contradicts the connectedness of K_n.

16.13 Only if the space is finite.

16.14 From *16.T* it follows that S^1, S^n, and the ellipsoid are compact. The remaining sets are not compact: $[0,1)$ and $[0,1) \cap \mathbb{Q}$ are not closed in $\mathbb{R}$, while the ray and the hyperboloid are unbounded.

16.15 $GL(n)$ is not even closed in $L(n,n) = \mathbb{R}^{n^2}$, while $SL(n)$ and space (d) are not bounded. Therefore, only $O(n)$ is compact because it is both closed and bounded (check this).

16.16 By *12.C* and Theorems *16.P* and *16.U*, $f(I)$ is a compact interval, i.e., a segment.

16.17 $\boxed{\Longrightarrow}$ This is *16.V*. $\boxed{\Longleftarrow}$ Since the function $A \to \mathbb{R} : x \mapsto \rho(0,x)$ is bounded, A is bounded. Let us prove that A is closed. Assume the contrary: let $x_0 \in \operatorname{Cl} A \smallsetminus A$. Then the function $A \to \mathbb{R} : x \mapsto 1/\rho(x,x_0)$ is unbounded, a contradiction. Since A is closed and bounded, it is compact by *16.T*.

16.18 Consider the function $f : G \to \mathbb{R} : x \mapsto \rho(x,F)$. By *4.35*, f is continuous. Since $\rho(G,F) = \inf_{x \in G} f(x)$, it remains to apply *16.V*. Recall that f takes only positive values! (See *4.L*.)

16.19 Use *16.18* and, e.g., put $\varepsilon = \rho(A, X \smallsetminus U)$.

16.20 Prove that if $A \subset \mathbb{R}^n$ is a closed set, then for each $x \in \mathbb{R}^n$ there is $y \in A$ such that $\rho(x,y) = \rho(x,A)$, whence $V = \bigcup_{x \in A} D_\varepsilon(x)$. The set $\bigcup_{x \in A} B_\varepsilon(x)$ is path-connected as a connected open subset of $\mathbb{R}^n$, which implies that V is also path-connected.

16.22 Consider the function $\varphi : X \to \mathbb{R} : x \mapsto \rho(x, f(x))$. If $f(x) \neq x$, then, by assumption, we have $\varphi(f(x)) = \rho(f(x), f(f(x))) < \rho(x, f(x)) = \varphi(x)$. Prove that φ is continuous. Since X is compact, φ attains its minimal value at a certain point x_0 by *16.V*. However, if $f(x_0) \neq x_0$, then $\varphi(f(x_0)) < \varphi(x_0)$, and so $\varphi(x_0)$ is not the minimal value of φ, a contradiction.

16.23 Let $U_1, \dots, U_n$ be a finite subcovering of the initial cover. We put $f_i(x) = \rho(x, X \smallsetminus U_i)$. Since the functions $f_i(x)$ are continuous, so is the function $\varphi : x \mapsto \max\{f_i(x)\}_1^n$. Since X is compact, φ attains its minimal value r. Since U_i cover X, we have $r > 0$.

16.24 Obvious.

16.25 If X is not compact, then use, e.g., *10.B*. If Y is not Hausdorff, then consider, e.g., the identical map id of I with the usual topology to I with the Zariski topology, or simply the identical map of a discrete space to the same set with indiscrete topology.

16.26 No, there is no such subspace. Let $A \subset \mathbb{R}^n$ be a noncompact set. If A is not closed, then the inclusion $\text{in} : A \to \mathbb{R}^n$ is not a closed map. If $A = \mathbb{R}^n$, then there exists a homeomorphism $\mathbb{R}^n \to \{x \in \mathbb{R}^n \mid x_1 > 0\}$. If A is closed, but not bounded, then take $x_0 \notin A$ and consider an inversion with center x_0.

16.27 Use *5.F*: closed sets of a closed subspace are closed in the ambient space.

***16.1*x** Let $p : \mathbb{R}^n \to \mathbb{R}$ be a norm. The inequality

$$p(x) = p\left(\sum x_i e_i\right) \le \sum p(x_i e_i) = \sum |x_i| p(e_i) = \sum \lambda_i |x_i|$$

implies that p is continuous at zero (here, $\{e_i\}$ is the standard basis in $\mathbb{R}^n$). Show that p is also continuous at any other point of $\mathbb{R}^n$.

***16.2*x** Since the sphere is compact, there are real numbers $c, C > 0$ such that $c|x| \le p(x) \le C|x|$, where $|\cdot|$ is the usual Euclidean norm. Now use *4.27*.

***16.3*x** Certainly not!

***16.4*x** Consider a cover of X by neighborhoods on which f is bounded.

17.1 This obviously follows from *17.E*.

17.2 By the Zorn lemma, there exists a maximal set in which the distances between the points are at least ε; this set will be the required ε-net.

***17.1*x** No, they are not compact. Consider the sequence $\{e_n\}$, where e_n is the unit basis vector. What are the pairwise distances between these points?

***17.2*x** This set is compact because the set

$$A = \{x \in l^\infty \mid |x_n| \le 2^{-n} \text{ for } n \le k,\ x_n = 0 \text{ for } n > k\}$$

is a 2^{-k}-net in the set.

***17.4*x** No, there does not exist such normed space. Prove that if E is a finite-dimensional subspace of a normed space (X, p), $x \notin E$, and $y \in E$ is a point in E closest to x , then the point $x_0 = \frac{x-y}{|x-y|}$ is such that $p(x_0 - z) \ge 1$.

(This fact is called the "Lemma on a Perpendicular".) Using this assertion, we can construct by induction a sequence $x_n \in X$ such that $p(x_n) = 1$, $p(x_n - x_k) \geq 1$ for $n \neq k$. It is clear that it has no convergent subsequence.

***17.5*x** See *4.Ix*.

***17.6*x** If $x = a_0 + a_1 p + \dots$ and $y = a_0 + a_1 p + \dots + a_k p^k$, then $\rho(x, y) \leq p^{-k-1}$.

***17.7*x** Yes, $\mathbb{Z}_p$ is complete. To prove this, use the following assertion: if $x = a_0 + a_1 p + \dots$, $y = b_0 + b_1 p + \dots$, and $\rho(x, y) < p^{-k}$, then $a_i = b_i$ for all $i = 1, \dots, k$.

***17.8*x** Yes, $\mathbb{Z}_p$ is compact. Since the finite set $A = \{y = a_0 + a_1 p + \dots + a_k p^k\}$ is a p^{-k-1}-net in $\mathbb{Z}_p$, the completeness of Z_p proved in *17.7x* implies that it is compact.

***17.9*x** Use the Hausdorff metric.

***17.10*x** We can view $\mathbb{R}^{2n}$ as the space of n-tuples of points in the plane. Each n-tuple has a convex hull, which is a convex polygon with at most n vertices. Let $\mathcal{K} \subset \mathbb{R}^{2n}$ be the set of all n-tuples with convex hulls contained in $\mathcal{P}_n$. We easily see that $\mathcal{K}$ is bounded and closed, i.e., $\mathcal{K}$ is compact. The map $\mathcal{K} \to \mathcal{P}_n$ taking an n-tuple to its convex hull is obviously continuous and surjective, whence it follows that $\mathcal{P}_n$ is compact.

***17.11*x** Use the fact that $\mathcal{P}_n$ is compact and the area determines a continuous function $S : \mathcal{P}_n \to \mathbb{R}$.

***17.12*x** It is sufficient to show that if a polygon $P \subset D$ is not regular, then we can find a polygon $P' \subset D$ that has perimeter at most p and area greater than that of P, or perimeter less than p and area at least that of P. 1) First, it is convenient to assume that P (as well as P') contains the center of D. 2) If P has two neighboring sides of different length, then we can make them equal of smaller length without changing the area. 3) If P is equilateral, but has different angles, we once more enlarge the area, this time even decreasing the perimeter.

***17.13*x** As in *17.9x*, the Hausdorff metric would do.

***17.14*x** Consider a sequence consisting of regular polygons of perimeter p with increasing number of vertices. Show that this sequence has no limit in $\mathcal{P}_\infty$. Therefore, no such a sequence contains a convergent sequence, and so $\mathcal{P}_\infty$ is not even sequentially compact.

***17.15*x** Once more, use the Hausdorff metric, as in *17.9x* and *17.13x*.

***17.16*x** By *17.N*, it suffices to show that 1) $\mathcal{P}$ contains a compact ε-net for each (arbitrarily small) $\varepsilon > 0$, and 2) $\mathcal{P}$ is complete. 1) $\mathcal{P}_n$

with sufficiently large n would do. (What finite ε-net would you suggest?) 2) Let $K_1, K_2, \dots$ be a Cauchy sequence in $\mathcal{P}$. Show that $K_* := \operatorname{Cl}\left(\bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty} K_i\right)\right)$ is a convex set in $\mathcal{P}$, and $K_i \to K_*$ as $i \to \infty$.

***17.17*x** This follows from *17.16x* and the continuity of the area function $S : \mathcal{P} \to \mathbb{R}$. (Cf. *17.11x*.)

***17.18*x** Similarly to *17.12x*, it suffices to show that we can increase the area of a compact set X distinct from a disk without increasing the perimeter of X. 1) First, we take two points $A, B \in \operatorname{Fr} X$ that divide $\operatorname{Fr} X$ in two parts of equal length. 2) The line AB splits X into two parts, X_1 and X_2. Suppose that the area of X_1 is at least that of X_2. Then, if we replace X_2 by a mirror reflection of X_1, we do not decrease $S(X)$. If X_1 is not a half-disk, then there is a point $C \in X_1 \cap \operatorname{Fr} X$ such that $\angle ACB \neq \pi/2$, and we easily increase $S(X)$.

***18.1*x** Obvious.

***18.2*x** All of them, except $\mathbb{Q}$.

***18.3*x** Let $A = \bigcup_{n=1}^{\infty}\left(\frac{1}{n+1}, \frac{1}{n}\right)$ and $B = \{0\}$. The sets A and B are discrete and so locally compact, but the point $0 \in A \cup B$ has no neighborhood with compact closure (in $A \cup B$).

***18.4*x** See *18.Lx*.

***18.7*x** This is obvious since an open set U meets an $A \in \Gamma$ iff U meets $\operatorname{Cl} A$.

***18.8*x** This immediately follows from *18.Qx*.

***18.9*x** Use *18.8x*.

***18.11*x** Let X be a locally compact space. Then X has a base consisting of open sets with compact closures. By the Lindelöf theorem, the base (being an open cover of X) contains a countable subcovering of X. It remains to use assertion *18.Xx*.

***18.12*x** Repeat the proof of a similar fact about compactness.

***18.13*x** This is obvious. (Recall the definitions.)

***18.14*x** Consider the cover $\Gamma' = \{X \smallsetminus F, U_\alpha\}$ of X. Let $\{V_\alpha\}$ be a locally finite refinement of Γ'. Then $\Delta = \{V_\alpha \mid V_\alpha \cap F \neq \varnothing\}$ is cover of F. Put $W = \bigcup_{V_\alpha \in \Delta} V_\alpha$. Since Δ is locally finite, $K = \bigcup_{V_\alpha \in \Delta} \operatorname{Cl} V_\alpha$ is a closed set. Then W and $X \smallsetminus K$ are the required disjoint neighborhoods of F and M.

***18.15*x** This immediately follows from *18.14x* (or *18.16x*).

***18.16*x** This immediately follows from *18.14x*.

***18.17*x** Since X is Hausdorff and locally compact, each point $x \in U_\alpha \in \Gamma$ has a neighborhood $V_{\alpha,x}$ with compact closure lying in U_α. Since X is paracompact, the open cover $\{V_{\alpha,x}\}$ of X has a locally finite refinement Δ, as required.

***18.18*x** The argument involves the Zorn lemma. Consider the set $\mathcal{M}$ of all open covers Δ of X such that for each $V \in \Delta$ either $V \in \Gamma$, or $\operatorname{Cl} V$ is contained in an element of Γ. We assign to $\Delta \in \mathcal{M}$ the subset $A_\Delta = \{V_\alpha \mid \operatorname{Cl} V_\alpha \subset U_\alpha\} \subset \Gamma$. Introduce a natural order on the set $\{A_\Delta \mid \Delta \in \mathcal{M}\}$, show that this set has a largest element A_{Δ_0}, which coincides with the entire Γ, and, therefore, Δ_0 is the required cover.

***18.20*x** Next to obvious.

19.1 $\operatorname{pr}_Y^{-1}(B) = X \times B$.

19.2 We have:

$$\operatorname{pr}_Y(\Gamma_f \cap (A \times Y)) = \operatorname{pr}_Y(\{(x, f(x)) \mid x \in A\}) = \{f(x) \mid x \in A\} = f(A).$$

Prove the second identity on your own.

19.3 Indeed, $(A \times B) \cap \Delta = \{(x, y) \mid x \in A,\ y \in B,\ x = y\} = \{(x, x) \mid x \in A \cap B\}$.

19.4 $\operatorname{pr}_X|_{\Gamma_f} : (x, f(x)) \leftrightarrow x$.

19.5 Indeed, $f(x_1) = f(x_2)$ iff $\operatorname{pr}_Y(x_1, f(x_1)) = \operatorname{pr}_Y(x_2, f(x_2))$.

19.6 This obviously follows from the relation $T(x, f(x)) = (f(x), x) = (y, f^{-1}(y))$.

19.7 Use the formula

$$(A \times B) \cap \bigcup_\alpha (U_\alpha \times V_\alpha) = \bigcup_\alpha ((A \times B) \cap (U_\alpha \times V_\alpha)) = \bigcup_\alpha ((A \cap U_\alpha) \times (B \cap V_\alpha)).$$

19.8 Use the third formula of *19.A*:

$$(X \times Y) \smallsetminus (A \times B) = ((X \smallsetminus A) \times Y) \cup (X \times (Y \smallsetminus B)) \in \Omega_{X \times Y}.$$

19.9 As usual, we check the two inclusions. $\boxed{\subset}$ Use *19.8*.

$\boxed{\supset}$ If x and y are adherent points of A and B, respectively, then, obviously, (x, y) is an adherent point of $A \times B$.

19.10 Yes, this is true. Once more, we check two inclusions. $\boxed{\subset}$ This is obvious. $\boxed{\supset}$ If $z = (x, y) \in \operatorname{Int}(A \times B)$, then z has an elementary neighborhood: $z \in W = U \times V \subset A \times B$, which means that x has a neighborhood $U_x \subset A$ and y has a neighborhood $V_y \subset B$, i.e., $x \in \operatorname{Int} A$ and $y \in \operatorname{Int} B$, whence $z = (x, y) \in \operatorname{Int} A \times \operatorname{Int} B$).

19.11 Certainly not! For instance, the boundary of the square $I \times I \subset \mathbb{R}^2$ is the contour of the square, while the product $\operatorname{Fr} I \times \operatorname{Fr} I$ consists of four points.

19.12 No, it is not in general; consider the set $(-1,1)\times(-1,1)\subset\mathbb{R}^2$.

19.13 Since A and B are closed, we have $\operatorname{Fr}A = A\smallsetminus\operatorname{Int}A$ and $\operatorname{Fr}B = B\smallsetminus\operatorname{Int}B$. The set $A\times B$ is also closed by *19.8*, whence by the third formula in *19.A* we have

$$\operatorname{Fr}(A\times B) = (A\times B)\smallsetminus\operatorname{Int}(A\times B) = (A\times B)\smallsetminus(\operatorname{Int}A\times\operatorname{Int}B)$$
$$= \big((A\smallsetminus\operatorname{Int}A)\times B\big)\cup\big(A\times(B\smallsetminus\operatorname{Int}B)\big) = (\operatorname{Fr}A\times B)\cup(A\times\operatorname{Fr}B). \tag{23}$$

19.14 Using *19.9*, *19.10*, and the third formula of *19.A*, we obtain

$$\operatorname{Fr}(A\times B) = \operatorname{Cl}(A\times B)\smallsetminus\operatorname{Int}(A\times B) = (\operatorname{Cl}A\times\operatorname{Cl}B)\smallsetminus(\operatorname{Int}A\times\operatorname{Int}B)$$
$$= \big((\operatorname{Cl}A\smallsetminus\operatorname{Int}A)\times\operatorname{Cl}B\big)\cup\big(\operatorname{Cl}A\times(\operatorname{Cl}B\smallsetminus\operatorname{Int}B)\big) = (\operatorname{Fr}A\times\operatorname{Cl}B)\cup(\operatorname{Cl}A\times\operatorname{Fr}B)$$
$$= \big(\operatorname{Fr}A\times(B\cup\operatorname{Fr}B)\big)\cup\big((A\cup\operatorname{Fr}A)\times\operatorname{Fr}B\big) = (\operatorname{Fr}A\times B)\cup(\operatorname{Fr}A\times\operatorname{Fr}B)\cup(A\times\operatorname{Fr}B).$$

19.15 It is sufficient to show that each elementary set in the product topology of $X\times Y$ is a union of sets of such form. Indeed,

$$\bigcup_\alpha U_\alpha\times\bigcup_\beta V_\beta = \bigcup_{\alpha,\beta}(U_\alpha\times V_\beta).$$

19.16 $\boxed{\Longrightarrow}$ The restriction $\operatorname{pr}_X|_{\Gamma_f}$ is obviously a continuous bijection. The inverse map $X\to\Gamma_f : x\mapsto(x,f(x))$ is continuous iff so is the map $g: X\to X\times Y : x\mapsto(x,f(x))$, which is true because $g^{-1}(U\times V) = U\cap f^{-1}(V)$.

$\boxed{\Longleftarrow}$ Use the relation $f = \operatorname{pr}_Y\circ\big(\operatorname{pr}_X|_{\Gamma_f}\big)^{-1}$.

19.17 Indeed, $\operatorname{pr}_X(W) = \operatorname{pr}_X\big(\bigcup_\alpha(U_\alpha\times V_\alpha)\big) = \bigcup_\alpha\operatorname{pr}_X(U_\alpha\times V_\alpha) = \bigcup_\alpha U_\alpha$. (We assumed that $V_\alpha\neq\varnothing$.)

19.18 No, it is not; consider the projection of the hyperbola $A = \{(x,y)\mid xy=1\}\subset\mathbb{R}^2$ to the x axis.

19.19 Let $F\subset X\times Y$ be a closed set and let $x\notin\operatorname{pr}_X(F)$. Then $(x\times Y)\cap F=\varnothing$, and for each $y\in Y$ the point (x,y) has an elementary neighborhood $U_x(y)\times V_y\subset(X\times Y)\smallsetminus F$. Since the fiber $x\times Y$ is compact, there is a finite subcovering $\{V_{y_i}\}_{i=1}^n$. The neighborhood $U=\bigcap_1^n U_x(y_i)$ is obviously disjoint with $\operatorname{pr}_X(F)$. Therefore, the complement of $\operatorname{pr}_X(F)$ is open, and so $\operatorname{pr}_X(F)$ is closed.

19.20 Plug in the definitions.

19.21 This is rather straightforward.

19.22 This is also quite straightforward.

19.23 Recall the definition of the product topology and use *19.21*.

19.24 Let us check that ρ is continuous at each point $(x_1, x_2) \in X \times X$. Indeed, let $d = \rho(x_1, x_2)$, $\varepsilon > 0$. Then, using the triangle inequality, we easily see that $\rho\big(B_{\varepsilon/2}(x_1) \times B_{\varepsilon/2}(x_2)\big) \subset (d - \varepsilon, d + \varepsilon)$.

19.25 This is quite straightforward.

19.26 $\boxed{\Longrightarrow}$ Let $(x, y) \notin \Delta$. Then the points x and y are distinct, and so they have disjoint neighborhoods: $U_x \cap V_y = \varnothing$. Then $(U_x \times V_y) \cap \Delta = \varnothing$ by *19.3*, i.e., $U_x \times V_y \subset X \times X \smallsetminus \Delta$. Therefore, $(X \times X) \smallsetminus \Delta$ is open.
$\boxed{\Longleftarrow}$ Let x and y be two distinct points of X. Then $(x, y) \in (X \times X) \smallsetminus \Delta$, and, since Δ is closed, (x, y) has an elementary neighborhood $U_x \times V_y \subset X \times X \smallsetminus \Delta$. It follows that $U_x \times V_y$ is disjoint with Δ, whence $U_x \cap V_y = \varnothing$ by *19.3*, as required.

19.27 Combine *19.26* and *19.25*.

19.28 The projection $\mathrm{pr}_X : X \to Y$ is a closed map by *19.19*. Therefore, the restriction $\mathrm{pr}_X|_\Gamma : \Gamma \to X$ is also closed by *16.27*, it is a homeomorphism by *16.24*, and so f is continuous by *19.16*.
Another option: use *19.19* and the identity $f^{-1}(F) = \mathrm{pr}_X\big(\Gamma_f \cap (X \times F)\big)$.

19.29 Consider the map $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ 1/x, & \text{otherwise.} \end{cases}$

19.32 Only the path-connectedness implies the continuity. The functions described in the Problem *19.31* provide counterexamples to other assertions.

19.36 No, they are not.

19.37 It is convenient to use the following property, which is equivalent to the regularity of a space (see *14.19*). For each neighborhood W of (x, y), there is a neighborhood U of (x, y) such that $\mathrm{Cl}\, U \subset W$. It is sufficient to consider the case where W is an elementary neighborhood. Use the regularity of X and Y and Problem *19.9*.

19.38.1 Let A and B be disjoint closed sets. For each $a \in A$, there exists an open set $U_a = [a, x_a) \subset X \smallsetminus B$. Put $U = \bigcup_{a \in A} U_a$. The neighborhood $V \supset B$ is defined similarly. If $U \cap V \neq \varnothing$, then for some $a \in A$ and $b \in B$ we have $[a, x_a) \cap [b, y_b) \neq \varnothing$. Let, say, $a < b$. Then $b \in [x, x_b)$, a contradiction.

19.38.2 The set ∇ is closed in $\mathbb{R}^2$, a fortiori ∇ is closed in $\mathcal{R} \times \mathcal{R}$. Since $\{(x, -x)\} = \nabla \cap \big([x, x+1) \times [-x, -x+1)\big)$, it follows that each point of ∇ is open in ∇.

19.38.3 See *14.4x*.

19.39 Modify the argument used in the proof of assertion *19.S*.

19.40 This follows from *19.U* and *19.9*.

19.43 $\mathbb{R}^n \smallsetminus \mathbb{R}^k \cong (\mathbb{R}^{n-k} \smallsetminus 0) \times \mathbb{R}^k \cong (S^{n-k-1} \times \mathbb{R}) \times \mathbb{R}^k \cong S^{n-k-1} \times \mathbb{R}^{k+1}$.

19.45 The space $O(n)$ is the union of $SO(n)$ and a disjoint open subset homeomorphic to $SO(n)$. Therefore, $O(n)$ is homeomorphic to $SO(n) \times \{-1, 1\} \cong SO(n) \times O(1)$.

19.46 It is sufficient to show that $GL_+(n) = \{A \mid \det A > 0\}$ is homeomorphic to $SL(n) \times (0, +\infty)$. The required homeomorphism takes a matrix $A \in GL_+(n)$ to the pair $(\frac{1}{\sqrt[n]{\det A}} A, \det A)$.

19.48 The existence of such a homeomorphism is directly connected with the existence of quaternions (see the last subsection in 22), and therefore in the proof we also use properties of quaternions. Let $\{x_0, x_1, x_2, x_3\}$ be a quadruple of pairwise orthogonal unit quaternions determining a point in $SO(4)$. The required homeomorphism takes this quadruple to the pair consisting of the unit quaternion $x_0 \in S^3$ and the triple $\{x_0^{-1}x_1, x_0^{-1}x_2, x_0^{-1}x_3\}$ of pairwise orthogonal vectors in $\mathbb{R}^3$, which determines an element in $SO(3)$. (Notice that, e.g., $SO(5)$ is not homeomorphic to $S^4 \times SO(4)$!)

20.2 The map pr takes each point to the element of the partition (regarded as an element of the quotient set) containing the point, and so the preimage $\mathrm{pr}^{-1}(\text{point}) = \mathrm{pr}^{-1}(\mathrm{pr}(x))$ is also the element of the partition containing the point $x \in X$.

20.3 Let $X/_S = \{a, b, c\}$, where $p^{-1}(a) = [0, \frac{1}{3}]$, $p^{-1}(b) = (\frac{1}{3}, \frac{2}{3}]$, and $p^{-1}(c) = (\frac{2}{3}, 1]$. Then $\Omega_{X/_S} = \{\varnothing, \{c\}, \{b, c\}, \{a, b, c\}\}$.

20.4 All elements of the partition are open in X.

20.6 Let $X = \mathbb{N} \times I$. Let the partition S consist of the fiber $N = \mathbb{N} \times 0$ and singletons. Let $\mathrm{pr}(N) = x_* \in X/_S$, let us prove that the point x_* has no countable neighborhood base. Assume the contrary: let $\{U_k\}$ be a countable neighborhood base at x_*. Each of the sets $\mathrm{pr}^{-1}(U_k)$ is open in X and contains each of the points $x_n = (n, 0) \in X$. For each of these points, X contains an open set V_n such that $x_n \in V_n \subset \mathrm{pr}^{-1}(U_n)$. It remains to observe that $W = \mathrm{pr}\big(\cup V_n\big)$ is a neighborhood of x_* that is not contained in any of the neighborhoods U_n of x_*, a contradiction.

20.7 For each open set $U \subset X/_S$, the image $f/_S(U) = f\big(\mathrm{pr}^{-1}(U)\big)$ is open as the image of the open set $\mathrm{pr}^{-1}(U)$ under the open map f.

20.1x $\boxed{\Rightarrow}$ If F is a closed set in X, then $F = \mathrm{pr}^{-1}\big(\mathrm{pr}(F)\big)$, hence, $\mathrm{pr}(F)$ is closed. $\boxed{\Leftarrow}$ This follows from the fact that for each closed set F in X the set $\mathrm{pr}^{-1}\big(\mathrm{pr}(F)\big)$, first, is closed, because pr is continuous, and, secondly, is a saturation of F.

20.2x Let A be the closed element of the partition that is not one-point. The saturation of any closed set F is either F itself, or the union $F \cup A$, i.e., a closed set.

20.3x This is similar to *20.1x*.

20.4x If A is saturated, then for each subset $U \subset A$ the saturation of U is also a subset of A. Consequently, the saturation of $\operatorname{Int} A$ lies in A, and, since the saturation is open, it coincides with $\operatorname{Int} A$. Since $X \smallsetminus A$ is also saturated, $\operatorname{Int}(X \smallsetminus A) = X \smallsetminus \operatorname{Cl} A$ is saturated, too, and so $\operatorname{Cl} A$ is also saturated.

21.1 Here is a partition of the segment with quotient space homeomorphic to the letter A. It consists of the two-point sets $\{\frac{1}{6}, 1\}$, $\{\frac{2}{3} - x, \frac{2}{3} + x\}$ for $x \in (0, \frac{1}{6}]$; the other elements are singletons. The idea of the proof is the same as that used in *21.2*: we construct a continuous surjection of the segment onto the letter A. Consider the map defined by the following formulas:

$$f(t) = \begin{cases} (3t, 6t) & \text{if } x \in [0, \frac{1}{3}], \\ (3t, 4 - 6t) & \text{if } x \in [\frac{1}{3}, \frac{1}{2}], \\ (\frac{9}{2} - 6t, 1) & \text{if } x \in [\frac{1}{2}, \frac{2}{3}], \\ (6t - \frac{7}{2}, 1) & \text{if } x \in [\frac{2}{3}, \frac{5}{6}], \\ (3t - 1, 6 - 6t) & \text{if } x \in [\frac{5}{6}, 1]. \end{cases}$$

Show that $f(I)$ is precisely the letter A, and the partition into the preimages under f is the partition described in the beginning of the solution.

21.2 Let $u : I \to I \times I$ be a Peano curve, i.e., a continuous surjection. Then the injective factor of the map u is a homeomorphism of a certain quotient space of the segment onto the square.

21.3 Let S be the partition of A into $A \cap B$ and singletons in $X \smallsetminus B = A \smallsetminus B$, T the partition of X into B and singletons in $X \smallsetminus B$, $\operatorname{pr}_A : A \to A/_S$ and $\operatorname{pr}_X : X \to X/_T$ the projections. Since the quotient map $q : A/_{A \cap B} \to X/_B$ is obviously a continuous bijection, to prove that q is a homeomorphism, it suffices to check that q is an open map. Let $U \subset A/_{A \cap B}$ be an open set, $V = \operatorname{pr}_A^{-1} U$. Then V is open in A and saturated in X. If $V \cap B = \varnothing$, then V is also open in X because $\{A, B\}$ is a fundamental cover of X, and so $q(U) = \operatorname{pr}_X(V)$ is open in $X/_T$. If $V \cap B \neq \varnothing$, then, obviously, $V \supset A \cap B$, and so the saturated set $W = V \cup B$ is open in X. In this case, $q(U) = \operatorname{pr}_X(W)$ is also open in $X/_B$.

21.4 Consider the map $f: I \to I$, where

$$f(x) = \begin{cases} \frac{3}{2}x & \text{if } x \in \left[0, \frac{1}{3}\right], \\ \frac{1}{2} & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ \frac{3x-1}{2} & \text{if } x \in \left[\frac{2}{3}, 1\right], \end{cases}$$

and prove that $S(f)$ is the given partition. Therefore, $f/_{S(f)} : I/_{S(f)} \cong I$.

21.5 Consider the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ that vanishes for $t \in [0, 1]$ and is equal to $t-1$ for $t \geq 1$ and the map $f : \mathbb{R}^2 \to \mathbb{R}^2$, where $f(x, y) = \left(\frac{\varphi(r)}{r}x, \frac{\varphi(r)}{r}y\right)$; here, as before, $r = \sqrt{x^2 + y^2}$. By construction, $\mathbb{R}^2/_{D^2} = \mathbb{R}^2/_{S(f)}$. The map $f/_{S(f)}$ is a continuous bijection. In order to see that $f/_{S(f)}$ is a homeomorphism, use *18.Ox* (*18.Px*). In order to see that $\mathbb{R}^2$ is also homeomorphic to other spaces, use the constructions described in the solutions of Problems *10.20–10.22*.

21.6 Let S be the partition of X into A and singletons in $X \setminus A$. Let T be the partition of Y into $f(A)$ and singletons in $Y \setminus f(A)$. Show that $f/_{(S,T)}$ is a homeomorphism.

21.7 No, it is not. The quotient space $\mathbb{R}^2/_A$ has no countable base at the image of A, while $\operatorname{Int} D^2 \cup \{(0, 1)\}$ is first countable as a subspace of $\mathbb{R}^2$. We can construct a continuous map $\mathbb{R}^2 \to \operatorname{Int} D^2 \cup \{(0, 1)\}$ that maps A to $(0, 1)$ and determines a homeomorphism $\mathbb{R}^2 \smallsetminus A \to \operatorname{Int} D^2$. This map determines a continuous map $\mathbb{R}^2/_A \to D^2 \cup \{(0, 1)\}$, but the inverse map is not continuous.

21.8 The partition $S(\varphi)$, where $\varphi : S^1 \to S^1 \subset \mathbb{C} : \ z \mapsto z^3$, is precisely the partition into given triples, whence $S^1/_S \cong S^1$.

21.9 For the first equivalence relation, consider the map $\varphi(z) = z^2$.

21.10 Notice: the quotient space of D^n by the equivalence relation $x \sim y \iff x_i = -y_i$ is not homeomorphic to D^n!

21.11 Consider $f : \mathbb{R} \to S^1$: $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. It is clear that $x \sim y \iff f(x) = f(y)$, and so the partition $S(f)$ is the given one. Unfortunately, here we cannot simply apply Theorem *16.Y* because $\mathbb{R}$ is not compact. Prove, that, nevertheless, this quotient space is compact.

21.12 The quotient space of the cylinder by the equivalence relation $(x, p) \sim (y, q)$ if $x + y = 1$ and $p = -q$ (here $x, y \in [0, 1]$ and $p, q \in S^1$), is homeomorphic to the Möbius strip.

21.13 Use the transitivity of factorization (Theorem *21.H*). Let S be the partition of the square into pairs of points on vertical sides lying on one horizontal line; all the remaining elements of the partition are singletons. We see that the quotient space I^2/S is homeomorphic to the cylinder. Now let S' be the partition of the cylinder into pairs of points on the bases symmetric with respect to the center of the cylinder; the other elements are singletons. Then the partition T of the square into the preimages under the map $p : I^2 \to I^2/S$ of the preimages of elements of S' coincides with the partition the quotient space by which is the Klein bottle.

21.17 The first assertion follows from the fact that the open sets in the topology induced from $\bigsqcup_{\alpha\in A} X_\alpha$ on the image $\mathrm{in}_\beta(X_\beta)$ have the form $\{(x,\beta) \mid x \in U\}$, where U is an open set in X_β, and so ab $\mathrm{in}_\beta : X_\beta \to \mathrm{in}_\beta(X_\beta)$ a homeomorphism. Furthermore, each of these images is open in the sum of the spaces (because each of its in_α-preimages is either empty, or equal to X_β), and hence is also closed.

21.18 The separation axioms and the first axiom of countability are inherited. The separability and the second axiom of countability require that the index set be countable. The space $\bigsqcup_{\alpha\in A} X_\alpha$ is disconnected if the number of summands is greater than one. The space is compact if the number of summands is finite and each of the summands is compact.

21.19 The composition $\varphi = \mathrm{pr} \circ \mathrm{in}_2$ is injective because each element of the partition in $X_1 \sqcup X_2$ contains at most one point in $\mathrm{in}_2(X_2)$. The continuity of φ is obvious. Consider an open set $U \subset X_2$. The set $\mathrm{in}_1(X_1) \cup \mathrm{in}_2(U)$ is open in $X_1 \sqcup X_2$ and saturated, and so its image W is open in $X_2 \cup_f X_1$. Since the intersection $W \cap \varphi(X_2) = \varphi(U)$ is open in $\varphi(X_2)$, it follows that φ is a topological embedding.

21.20 Thus, $X = \{*\}$. Put $Y' = Y \sqcup \{*\}$ and $A' = A \sqcup \{*\}$. It is clear that the factor $g : Y/A \to Y'/A'$ of the injection $\mathrm{in} : Y \to Y'$ is a continuous bijection. Prove that the map g is open.

21.21 Cut the square in order to obtain (after factorization) two Möbius strips, which must be glued together along their boundary circles.

21.22 Use the map

$$(\mathrm{id}_{S^1} \times i_+) \sqcup (\mathrm{id}_{S^1} \times i_-) : (S^1 \times I) \sqcup (S^1 \times I) \to S^1 \times S^1,$$

where $i_\pm$ are embeddings of I in S^1 onto the upper and, respectively, lower semicircle.

21.23 See *21.M* and *21.22*.

21.24 If the square, whose quotient space is the Klein bottle, is cut by a vertical segment in two rectangles, then gluing together the horizontal sides we obtain two cylinders.

21.25 Let $S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$. The subset of the sphere determined by the equation $|z_1| = |z_2|$ consists of all pairs (z_1, z_2) such that $|z_1| = |z_2| = \frac{1}{\sqrt{2}}$, therefore, the set is a torus. Now consider the subset T_1 determined by the inequality $|z_1| \le |z_2|$ and the map taking $(z_1, z_2) \in T_1$ to $(u, v) = \left(\frac{z_1}{|z_2|}, \frac{z_2}{|z_2|}\right) \in \mathbb{C}^2$. Show that this map is a homeomorphism of T_1 onto $D^2 \times S^1$ and complete the argument on your own.

21.26 The cylinder or the Möbius strip. Consider a homeomorphism g between the vertical sides of the square, let $g : (0, x) \mapsto (1, f(x))$. The map f is a homeomorphism $I \to I$, therefore, f is a (strictly) monotone function. Assume that the function f is increasing, in particular, $f(0) = 0$ and $f(1) = 1$. Let us show that there is a homeomorphism $h : I^2 \to I^2$ such that $h(0, x) = x$ and $h(1, x) = (1, f(x))$ for all $x \in I$. For this purpose, we subdivide the square by the diagonals in four parts, and define h on the right-hand triangle by the formula

$$h\left(\tfrac{1+t}{2}, \tfrac{1-t}{2} + tx\right) = \left(\tfrac{1+t}{2}, \tfrac{1-t}{2} + tf(x)\right),$$

$t, x \in I$. On the remaining three triangles, h is identical. It is clear that that the homeomorphism takes the element $\{(0, x), (1, x)\}$ of the partition to the element $\{(0, x), (1, f(x))\}$, therefore, there exists a continuous bijection of the cylinder (consequently, a homeomorphism) onto the result of gluing together the square by the homeomorphism g of its vertical sides. If the function f is decreasing, then, arguing in a similar way, we see that the result of this gluing is the Möbius strip.

21.27 The torus and the Klein bottle; similarly to *21.26*.

21.28 Show that any homeomorphism of the boundary circle extends to the entire Möbius strip.

21.29 See *21.27*.

21.30 Show that each auto-homeomorphism of the boundary circle of a handle extends to an auto-homeomorphism of the entire handle. (Compare Problem *21.28*. When solving both problems, it is convenient to use the following fact: each auto-homeomorphism of the outer boundary circle of a ring extends to an auto-homeomorphism of the entire ring that is fixed on the inner boundary circle or determines a mirror symmetry of it.)

21.31 See the solutions to Problems *21.28* *21.30*.

21.32 We can assume that the holes are split into the pairs of holes connected by “tubes”. (Compare the solution to Problem *21.V*.) Together with a disk surrounding such a pair, each tube either forms a handle or a Klein bottle with a hole. If each of the tubes forms a handle, then we obtain a sphere with handles. Otherwise, we transform all handles into Klein bottles with holes (see the solution to Problem *21.V*) and obtain a sphere with films.

22.1 There exists a natural one-to-one correspondence between lines in the plane that are determined by equations of the form $ax + by + c = 0$ and points $(a : b : c)$ in $\mathbb{R}P^2$. Observe that the complement of the image of the set of all lines is the singleton $\{(0 : 0 : 1)\}$.

***23.1*x** Yes, it is. A number a always divides a (formally speaking, even 0 divides 0). Further, if a divides b and b divides c, then a divides c.

***23.2*x** $a \sim b$ iff $a = \pm b$.

***23.3*x** This is obvious because $A \subset \operatorname{Cl} B$ iff $\operatorname{Cl} A \subset \operatorname{Cl} B$.

***24.1*x** This is obvious. (Cf. Problem *24.2*x.)

***24.2*x** Taking each point $y \in Y$ to the constant map $X \to Y : x \mapsto y$, we obtain an injection $Y \to \mathcal{C}(X, Y)$.

***24.4*x** The correspondence $f \mapsto f^{-1}(0)$ determines a bijection $\mathcal{C}(X, Y) \to \Omega_X$.

***24.5*x** Since X is a discrete space, each map $f : X \to Y$ is continuous. If $X = \{x_1, x_2, \dots, x_n\}$, then f is uniquely determined by the collection $\{f(x_1), \dots, f(x_n)\} \in Y^n$.

***24.6*x** The set X has two connected components.

***24.7*x** It is clear (prove this) that the topological structures $\mathcal{C}(I, I)$ and $\mathcal{C}^{(pw)}(I, I)$ are distinct, and, consequently, the identical map of the set $\mathcal{C}(I, I)$ is not a homeomorphism. In order to prove that the spaces considered are not homeomorphic, we must find a topological property such that one of the spaces satisfies it, while the other does not. Show that $\mathcal{C}(I, I)$ satisfies the first axiom of countability, while $\mathcal{C}^{(pw)}(I, I)$ does not.

***24.8*x** We identify Y with $\operatorname{Const}(X, Y)$ via the map $y \mapsto f_y : x \mapsto y$. Consider the intersections of sets in the subbase with the image of Y under the above map. We have $W(x, U) \cap \operatorname{Const}(X, Y) = U$, hence, the intersection of Y with any subbase set in the topology of pointwise convergence is open in Y. Conversely, for each open set U in Y and for each $x \in X$ we have $U = W(x, U) \cap \operatorname{Const}(X, Y)$. The same argument is also valid in the case of the compact-open topology.

***24.9*x** The mapping $f \mapsto (f(x_1), f(x_2), \dots, f(x_n))$ maps the subbase set $W(x_1, U_1) \cap W(x_2, U_2) \cap \dots \cap W(x_n, U_n)$ to the base set $U_1 \times U_2 \times \dots \times U_n$ of the product topology. Finally, it is clear that if X is finite, then the topologies $\Omega^{co}(X, Y)$ and $\Omega^{pw}(X, Y)$ coincide.

***24.10*x** $\boxed{\Rightarrow}$ Use *24.W*x. $\boxed{\Leftarrow}$ Since X is a path-connected space, any two paths in X are freely homotopic. Consider a homotopy $h : I \times I \to X$. By *24.V*x, the map $\widetilde{h} : I \to \mathcal{C}(I, X)$ defined by the formula $\widetilde{h}(t)(s) = h(t, s)$, is continuous. Therefore, any two paths in X are joined by a path in the space of paths, which precisely means that the space $\mathcal{C}(I, X)$ is path-connected.

***24.11*x** The space $\mathcal{C}^{(pw)}(I, I)$ is noncompact since the sequence of functions $f_n(x) = x^n$ has no accumulation points in this space. The same sequence has no limit points in $\mathcal{C}(I, I)$, and, hence, this space also is not compact.

***24.12*x** Let

$$d_n(f, g) = \max\{|f(x) - g(x)| : x \in [-n, n]\},\ n \in \mathbb{N}.$$

Put

$$d(f, g) = \sum_{n=1}^{\infty} \frac{d_n(f, g)}{2^n(1 + d_n(f, g))}.$$

We easily see that d is a metric. Show that d generates the compact-open topology.

***24.13*x** The proof is similar to that of assertion *24.12*x. We only need to observe that since, obviously, $X = \bigcup_{i=1}^{\infty} \operatorname{Int} X_i$, for each compact set $K \subset X$ there is n such that $K \subset X_n$.

***25.1*x** 1) No, it cannot. 2) Yes, it can.

***26.1*x** Use the fact that 1) $\beta(x, y) = \omega(x, \alpha(y))$, and 2) $\alpha(x) = \beta(1, x)$ and $\omega(x, y) = \beta(x, \alpha(y))$. $\boxed{\Longrightarrow}$ Use the continuity of compositions. $\boxed{\Longleftarrow}$ Write $b^{-1} = 1 \cdot b^{-1}$ and $ab = a \cdot (1 \cdot b^{-1})^{-1}$.

***26.2*x** In the notation used in the proof of assertion *26.1*x, α is a continuous map inverse to itself. Therefore, α is a homeomorphism.

***26.3*x** Use the fact that the former map is the composition $\omega \circ (f \times g)$, while the latter is the composition $\alpha \circ f$ (in the notation used in the proof of *26.1*x).

***26.4*x** Yes, it is. In order to prove this, use the fact that any auto-homeomorphism of an indiscrete space is continuous.

***26.5*x** If the topology in a group is induced by the standard topology of the Euclidean space, then in order to verify that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous it suffices to check that they are determined by continuous functions. If $x = a + ib$ and $y = c + id$, then $xy = (ac - bd) + i(ad + bc)$. Therefore, the multiplication is determined by the function $(a, b, c, d) \mapsto (ac - bd, ad + bc)$, which is obviously continuous. The passage to the inverse element is also determined by the continuous function (on $\mathbb{R}^2 \smallsetminus 0$)

$$\mathbb{R}^2 \smallsetminus 0 \to \mathbb{R}^2 \smallsetminus 0 : (a, b) \mapsto \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).$$

26.6x Use the idea of the solution to Problem *26.5x* and the fact that addition, multiplication, and their compositions are continuous.

26.7x Consider, e.g., the *cofinite* topology of Problem *2.5*, or, what would be more interesting, the topology of an irrational flow $\mathbb{R} \to T^2$. (See *28.1x* (f).)

26.8x Consider any two (algebraically) nonisomorphic discrete finite groups of equal order. Here is a more meaningful example: the topological group $GL_+(2,\mathbb{R}) \subset GL(2,\mathbb{R})$ of invertible 2×2 matrices with positive determinant is homeomorphic to $O_+(2)\times\mathbb{R}^3$. (Here, $O_+(2) = O(2)\cap GL_+(2,\mathbb{R})$.) The two groups are not isomorphic because the first one is not Abelian, while the second one is.

26.10x Yes, it does. (For the same reason as in *26.Ex*.)

26.11x Use the fact that $UV = \bigcup_{x\in V} Ux$ and $VU = \bigcup_{x\in V} xU$.

26.12x No, it will not. A counterexample is given by a point by point sum $U + V$ of a singleton $U \subset \mathbb{R}$ with an open interval $V \in \mathbb{R}$. A counterexample where both U and V are closed is given in *26.13x*

26.13x (a), (b) Yes. (c) No. This group is everywhere dense, but obviously does not coincide with $\mathbb{R}$. (For example, because it is countable, while $\mathbb{R}$ is not.)

26.14x Let $x \notin UV$. Then U and xV^{-1} are disjoint. Apply *26.14x.1* and take a neighborhood W of 1_G such that WU does not meet xV^{-1}. Then $W^{-1}x$ does not meet UV.

26.14x.1 For each $x \in C$, the unity 1_G has a neighborhood V_x such that xV_x does not meet F. By *26.Hx*, 1_G has a neighborhood W_x such that $W_x^2 \subset V_x$. Since C is compact, C is covered by finitely many sets of the form $W_1 = x_1W_{x_1}, \dots, W_n = x_nW_{x_n}$. Put $V_1 = \bigcap_1^n W_{x_i}$. Then $CV_1 \subset \bigcup W_iV \subset \bigcup x_iW_{x_i}^2 \subset \bigcup x_iV_{x_i}$, so that CV does not meet F. In a similar way, we construct a neighborhood V_2 of 1_G such that V_2C does not meet F. The neighborhood $V = V_1 \cap V_2$ possesses the required property. If G is a locally compact group, then we choose the neighborhood V_x with compact closure and then proceed as before.

26.15x By *26.Hx*, 1_G has a neighborhood V' with $V'V' \subset U$. By *26.Gx*, V' contains a symmetric neighborhood V_2 of 1_G. Then $V_2V_2 \subset V'V' \subset U$. After that, proceed by induction, replacing U by V_2 and choosing as V_n a symmetric neighborhood V of 1_G such that $V^{n-1} \subset V_2$. Then $V^n \subset V_2^2 \subset U$. Observe that $V \subset VV$.

26.16x The set $H = \bigcup_{n=1}^\infty V^n$ is open. Clearly, $1 \in H$, $H^{-1} \subset H$, and $HH \subset H$. Hence, H is a subgroup. It remains to observe that an open subgroup is always closed (see *27.3x*).

***26.18*x** Let N be the intersection of all neighborhoods of 1_G. Since G is finite, there are only finitely many neighborhoods involved, and hence N is open. From *26.Gx* and *26.Hx* it follows that $N = N^{-1}$ and $N^2 = N$. Hence, N is a subgroup. It is normal since otherwise $N \cap gNg^{-1}$ would be a smaller neighborhood of 1_G than N.

***27.2*x** $\boxed{\Rightarrow}$ Obvious. (Consider the unity.) $\boxed{\Leftarrow}$ Let H be the subgroup, U an open set, $g \in U \subset H$. Then $h \in hg^{-1}U \subset H$ for each $h \in H$, therefore, each point of H is inner.

***27.3*x** For any subgroup H and any $g \notin H$, the sets H and gH are disjoint. Hence, the complement of H is the union of gH over all $g \notin H$. Therefore, the complement of H is open if H is open.

***27.4*x** Use the same argument as in the solution to Problem *27.3x* and observe that in the case of finite index there are only finitely many distinct cosets gH such that $g \notin H$.

***27.5*x** Consider $\mathbb{Z} \subset \mathbb{R}$ and, respectively, $\mathbb{Q} \subset \mathbb{R}$.

***27.6*x** Show that if H contains an isolated point, then all points of H are isolated.

***27.7*x** Let $U \subset G$ be an open set such that $U \cap H = U \cap \operatorname{Cl} H \neq \varnothing$. If $g \notin H$ and $gH \cap U \neq \varnothing$, then g belongs to the open set $\bigcup_{h \in H} h(U \smallsetminus H)$ disjoint with H. If gH is disjoint with U, take $h' \in H \cap U$ and a symmetric open neighborhood V of 1 such that $Vh' \subset U$. Then Vg is an open neighborhood of g disjoint with H. (Otherwise, $vg = h$ implies $gh^{-1}h' = v^{-1}h' \in Vh'$.)

***27.8*x** By *27.7x*, the closure of $\operatorname{Cl} H \smallsetminus H$ contains H.

***27.9*x** Use the fact that $(\operatorname{Cl} H)^{-1} = \operatorname{Cl} H^{-1}$ and $\operatorname{Cl} H \cdot \operatorname{Cl} H \subset \operatorname{Cl}(H \cdot H) = \operatorname{Cl} H$.

***27.10*x** This is true if the interior is nonempty, see *27.2x*.

***27.12*x** Repeat the argument used in the solution to *27.Fx*.

***27.13*x** We identify elements of $SO(n)$ with positively oriented orthonormal bases in $\mathbb{R}^n$. The map $p : SO(n) \to S^{n-1}$ sends each basis to its last vector. The preimage of a point $x \in S^{n-1}$ is the right coset of $SO(n-1)$ (prove this). Clearly, p is continuous. The quotient map of p is a continuous bijection $\widehat{p} : SO(n)/_{SO(n-1)} \to S^{n-1}$. Since $SO(n)$ is compact and S^{n-1} is Hausdorff, $\widehat{p}$ is a homeomorphism.

***27.14*x** 1) The groups $SO(n)$, $U(n)$, $SU(n)$, and $Sp(n)$ are bounded closed subsets of the corresponding matrix spaces. Therefore, they are compact.

2) To check that $SO(n)$ is connected, combine *27.13x* and *27.Fx*, and then use induction (we observe that the group $SO(2) \cong S^1$ is connected).

(Another, more hand-operated, method consists in using normal forms. For example, for any $x \in SO(n)$ there is $g \in SO(n)$ such that the matrix gxg^{-1} consists of diagonal blocks of $SO(1)$ and $SO(2)$ matrices. The latter block matrices belong to the connected component C of the unity in $SO(n)$. Since C is a normal subgroup (see *27.Hx*), it follows that $x \in C$.) In order to prove that $U(n)$, $SU(n)$, and $Sp(n)$ are connected, state and prove the corresponding counterparts of *27.13x* and then use *27.Fx*.

3) The group $O(n)$ has two connected components: $SO(n)$ and its complement (the only nontrivial coset of $SO(n)$). The group $O(p, q)$ has four connected components if $p > 0$ and $q > 0$. To check this, use induction on p and q, at each step using *27.12x* and *18.Ox*.

***27.15*x** See the solution to *27.Hx*.

***27.16*x** Let $h \in H$. Since H is normal, we have a map $\eta : G \to H : g \mapsto ghg^{-1}$. Since G is connected, the image of η is a connected subset of H. Since H is discrete, it is a point, and so η is constant. Since $\eta(1) = h$, we have $ghg^{-1} = \eta(g) = h$ for all $g \in G$. Therefore, $gh = gh$ for all $g \in G$, i.e., $h \in C(G)$.

***27.19*x** Consider the exponential map $\mathbb{R} \to S^1 : x \mapsto e^{2\pi xi}$ and an open interval in $\mathbb{R}$ containing 0 and $\frac{1}{2}$.

***27.20*x** Let U and V be neighborhoods of unity in topological groups G and H, respectively. Let $f : U \to V$ be a homeomorphism such that $f(xy) = f(x)f(y)$ for any $x, y \in U$. By *26.Hx*, 1_G has a neighborhood $\widehat{U}$ in G such that $\widehat{U}^2 \subset U$. Since $\widehat{U} \subset U$, we have $f(xy) = f(x)f(y)$ for any $x, y \in \widehat{U}$ with $xy \in \widehat{U}$. Put $\widehat{V} = f(\widehat{U})$ and consider $z, t \in \widehat{V}$ with $zt \in \widehat{V}$. Then $z = f(x)$ and $t = f(y)$, where $x, y \in \widehat{U}$, whence $xy \in U$, and so $f(xy) = f(x)f(y) = zt$. Therefore, we have $x = f^{-1}(z)$ and $y = f^{-1}(t)$, whence $f^{-1}(z)f^{-1}(t) = xy = f^{-1}(zt)$.

***27.21*x** This follows from *27.Ox* because the projection $\mathrm{pr}_G : G \times H \to G$ is an open map.

***27.23*x** The map is continuous as a restriction of the continuous map $G \times G \to G : (x, y) \mapsto xy$. As an example, consider the case where $G = \mathbb{R}$, $A = \mathbb{Q}$, and B is generated by the irrational elements of a Hamel basis of $\mathbb{R}$ (i.e., a basis of $\mathbb{R}$ as of a vector space over $\mathbb{Q}$). The inverse group isomorphism $\mathbb{R} \to A \times B$ here is not continuous since, e.g., $\mathbb{R}$ is connected, while $A \times B$ is not.

***27.U*x** Let a compact Hausdorff group G be the direct product of two closed subgroups A and B. Then A and B are compact and Hausdorff, and so $A \times B \to G : (a, b) \mapsto ab$ is a continuous bijection from a compact space to a Hausdorff one. By *16.Y*, it is a homeomorphism.

***27.24*x** An isomorphism is $S^0 \times \mathbb{R}_{>0} \to \mathbb{R} \smallsetminus 0 : (s, r) \mapsto rs$.

***27.25*x** An isomorphism is $S^1 \times \mathbb{R}_{>0} \to \mathbb{C} \smallsetminus 0 : (s, r) \mapsto rs$.

***27.26*x** An isomorphism is $S^3 \times \mathbb{R}_{>0} \to \mathbb{H} \smallsetminus 0 : (s, r) \mapsto rs$.

***27.27*x** This is obvious because the 3-sphere S^3 is connected, while S^0 is not. However, the subgroup $S^0 = \{1, -1\}$ of $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ is not a direct factor even group-theoretically. Indeed, otherwise any value ± 1 of the projection $S^3 \to S^0$ on the standard generators i, j, and k would lead to a contradiction.

***27.28*x** Take the quotient group in *27.27x*.

***28.1*x** In (1) and (2), the map $G \to \operatorname{Top} X$ is continuous (see the solution to *28.Gx*). However, if we require $\operatorname{Top} X$ to be a topological group, then we need additional assumptions, e.g., the Hausdorff axiom and local compactness.

***28.2*x** Each of the angles has the form $\pi/n, n \in \mathbb{N}$. Therefore, there are only two solutions: $(\pi/2, \pi/3, \pi/6)$ and $(\pi/3, \pi/3, \pi/3)$.

***28.3*x** Such examples are given by the irrational flow (see *28.1x* (f)), or by the action of $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ regarded as a discrete group acting by translations on $\mathbb{R}$. In the latter case, we have $G = G/_{G^x}$, while $G(x)$ is not discrete. (Cf. *26.13x*.)

***28.4*x** Let A be closed. In order to prove that $G(A)$ is closed, consider an orbit $G(x)$ disjoint with $G(A)$. For each $g \in G$, let $U(g) \subset X$ and $V(g) \subset G$ be neighborhoods of x and g, respectively, such that $V(g)U(g)$ is disjoint with $G(A)$. Since G is compact, there is a finite number of elements $g_k \in G$ such that $V(g_k)$ cover G. Then the saturation of $\bigcap U(g_k)$ is an open set disjoint with $G(A)$ and containing $G(x)$.
If A is compact, then so is $G(A)$ as the image of the compact space $G \times A$ under the continuous action $G \times A \to X$.

***28.5*x** There are two orbits: $\{0\}$ and $\mathbb{R} \smallsetminus 0$. The corresponding isotropy subgroups are G and $\{1_G\}$. The quotient space is a two-point set, say $\{0, 1\}$, with nontrivial topology (neither discrete, nor indiscrete).

***28.6*x** The quotient space is canonically homeomorphic to the rectangle itself. A homeomorphism is induced by the inclusion of the rectangle to $\mathbb{R}^2$ (a continuous section of the quotient map). The group G is described in Problem *28.7x*.

***28.7*x** Using the transitivity of factorization, replace $\mathbb{R}^2/_G$ by the quotient of two adjacent rectangles that is obtained by identifying the points on their distinct edges via the reflection in their common edge. The latter

quotient is homeomorphic to S^2 (a "pillow").
The group G is the direct square $C \times C$ of the free product C of two copies of $\mathbb{Z}/2$ (see 43°7x), and $H \subset G$ is a subgroup of elements of even degree.

***28.8*x** Two points belong to the same orbit iff their vectors of absolute values $|z_0|, \dots, |z_n|$ are proportional. In other words, the orbits correspond in a one-to-one manner to "positive quadrant" directions in $\mathbb{R}^{n+1}$. The isotropy subgroups are coordinate subtori, i.e., the subtori of G where some of the coordinates vanish: the same coordinates as the zero coordinates of the points in the orbit. By transitivity of factorization, $X/_G$ is homeomorphic to the projectivization of the "positive quadrant" $\mathbb{R}^{n+1}_{>0}/\mathbb{R}_{>0}$. The latter is a closed n-simplex.

***28.9*x** Two points belong to the same orbit iff all symmetric functions of their coordinates coincide. Thus, at least set-theoretically, the Vieta map evaluating the unitary (i.e., with leading coefficient 1) polynomial equation of degree n with given n roots identifies $X/_G$ with the space of unitary polynomials of degree n, i.e., $\mathbb{C}^n$. Since both spaces are locally compact and the group $G = \mathbb{S}_n$ is compact (even finite), the quotient map $X/_G \to \mathbb{C}^n$ is a homeomorphism.

***28.10*x** Two such matrices belong to the same orbit iff the matrices have the same eigenvalues, counting the multiplicities. Thus, at least set-theoretically, the map evaluating the eigenvalues in decreasing order, $\lambda_1 \ge \lambda_2 \ge \lambda_3$, identifies $X/_G$ with the subspace of $\mathbb{R}^3$ determined by the above inequalities and the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Since this map has a continuous section (that given by diagonal matrices), it follows that $X/_G$ is homeomorphic to the above subspace of $\mathbb{R}^3$, which is a plane region bounded by two rays making an angle of $\frac{2\pi}{3}$. The isotropy group of an interior point in the region is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. For interior points of the rays, the isotropy group is the normalizer of $SO(2)$, and the orbits are real projective planes. For $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the isotropy group is the entire $SO(3)$, while the orbit is one-point.

***28.11*x** The sphere $S^n \subset \mathbb{R}^{n+1}$ (respectively, $S^{2n-1} \subset \mathbb{C}^n$) is a Hausdorff homogeneous G-space, on which $G = O(n+1)$ (respectively, $G = U(n)$) acts naturally. For any point $x \in S^n$ (respectively, $x \in S^{2n-1}$), the isotropy group is a standardly embedded $O(n) \subset O(n+1)$ (respectively, $U(n-1) \subset U(n)$). So, it remains to apply *28.M*x.

***28.12*x** The above action of $O(n+1)$ (respectively, $U(n)$) descends to $\mathbb{R}P^n$ (respectively, $\mathbb{C}P^{n-1}$). For any point $x \in S^n$ (respectively, $x \in S^{2n-1}$), the isotropy group is $O(n) \times O(1)$ (respectively, $U(n-1) \times U(1)$).

***28.13*x** Similarly to *28.11*x, this follows from the representation of $S^{4n-1} \subset \mathbb{H}^n$ as a homogeneous $Sp(n)$-space.

28.14x The torus is $\mathbb{R}^2/_H$, where $H = \mathbb{Z}^2 \subset \mathbb{R}^2$. To obtain the Klein bottle in the form $\mathbb{R}^2/_G$, add to H the reflection $(x, y) \mapsto (1 - x, y)$.

28.15x 1) The space of n-tuples $(L_1, \dots, L_n)$ of pairwise orthogonal vector lines L_k in $\mathbb{R}^n$.
2) The Grassmannian of (non-oriented) vector k-planes in $\mathbb{R}^n$.
3) The Grassmannian of oriented vector k-planes in $\mathbb{R}^n$.
4) The Stiefel variety of $(n-k)$-orthogonal unit frames in $\mathbb{R}^n$.

28.16x 1) Use the fact that the product of two homogeneous spaces is a homogeneous space. (Over what group?) 2) A more interesting option: show that $S^2 \times S^2$ is homeomorphic to the Grassmannian of oriented vector 2-planes in $\mathbb{R}^4$.

28.17x By definition, the group $SO(n, 1)$ acts transitively on the quadric Q in $\mathbb{R}^{n+1}$ given by the equation $-x_0^2 + x_1^2 + \dots + x_n^2 = 0$. The isotropy group of any point of Q is the standardly embedded $SO(n) \subset SO(n, 1)$. By *28.Mx*, the quotient space $SO(n, 1)/_{SO(n)}$ is homeomorphic to Q, which in turn is homeomorphic to a disjoint sum of two open n-balls.

29.1 For each continuous map $f : X \to I$, the map $H : \ H(x, t) = (1 - t)f(x)$ is a homotopy between f and the constant map $h_0 : \ x \mapsto 0$.

29.2 Let $f_0, f_1 : Z \to X$ be two constant maps with $f_0(Z) = \{x_0\}$ and $f_1(Z) = \{x_1\}$. $\boxed{\Longrightarrow}$ If H is a homotopy between f_0 and f_1, then for any $z_* \in Z$ the path $u : t \mapsto H(z_*, t)$ joins x_0 and x_1, which thus lie in one path-connected component of X.
$\boxed{\Longleftarrow}$ If x_0 and x_1 are joined by a path $u : I \to X$, then $Z \times I \to X : (z, t) \mapsto u(t)$ is a homotopy between f_0 and f_1.

29.3 Let us show that an arbitrary map $f : I \to Y$ is null-homotopic. Indeed, if $H(s, t) = f(s \cdot (1 - t))$, then $H(s, 0) = f(s)$ and $H(s, 1) = f(0)$. Consider two continuous maps $f, g : I \to Y$. We show that if $f(I)$ and $g(I)$ lie in one and the same path-connected component of Y, then they are homotopic. Each of the maps f and g is null-homotopic, therefore, they are homotopic due to the transitivity of the homotopy relation and the result of Problem *29.2*. To make the picture complete, we present an explicit homotopy joining f and g:

$$H(s, t) = \begin{cases} f(s \cdot (1 - 3t)) & \text{for } t \in \left[0, \frac{1}{3}\right], \\ u(3s - 1) & \text{for } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ g(s \cdot (3t - 2)) & \text{for } t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

29.4 Prove that each continuous map to a star-shaped set is homotopic to the constant map with image equal to the center of the star.

29.5 Let $f : C \to X$ be a continuous map. Let a be the center of the set C. Then the required homotopy $H : C \times I \to X$ is defined by the formula $H(c,t) = f(ta + (1-t)c)$.

29.6 The space X is path-connected.

29.7 Use assertion *29.F* and the fact that $S^n \setminus \text{point} \cong \mathbb{R}^n$.

29.8 If a path $u : I \to \mathbb{R}^n \setminus 0$ joins $x = f(0)$ and $y = g(0)$, then u determines a homotopy between f and g because $0 \times I \cong I$.

29.9 Consider the maps f and g defined by the formulas $f(0) = -1$ and $g(0) = 1$. They are not homotopic because the points 1 and -1 lie in distinct path-connected components of $\mathbb{R} \setminus 0$.

29.10 If $n > 1$, then there is a unique homotopy class. For $n = 1$, there are $(k+1)^m$ such classes.

29.11 Since for each point $x \in X$ and each real $t \in I$ we have the inequality

$$|(1-t)f(x) + tg(x)| = \big|f(x) + t\big(g(x) - f(x)\big)\big| \geq |f(x)| - |g(x) - f(x)| > 0,$$

it follows that the image of the rectilinear homotopy joining f and g lies in $\mathbb{R}^n \setminus 0$, therefore, these maps are homotopic.

29.12 For the simplicity, we assume that the leading coefficients of p and q are equal to 1. Use *29.11* to show that the maps determined by the polynomial $p(x)$ of degree n and the monomial z^n are homotopic.

29.13 The required homotopy is given by the formula

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}.$$

How do you think, where have we used the assumption $|f(x) - g(x)| < 2$?

29.14 This immediately follows from *29.13*.

30.1 To shorten the notation, put $\alpha = (uv)w$ and $\beta = u(vw)$; by assumption, $\alpha(s) = \beta(s)$ for all $s \in [0,1]$. In the proof of assertion *30.E.2*, we construct a function φ such that $\alpha \circ \varphi = \beta$. Consequently, $\alpha(s) = \alpha(\varphi(s))$, whence $\alpha(s) = \alpha(\varphi^n(s))$ for all $s \in [0,1]$ and $n \in \mathbb{N}$ (here φ^n is the n-fold composition of φ). Since $\varphi(s) < s$ for $s \in (0,1)$, it follows that the sequence $\varphi^n(s)$ is monotone decreasing, and we easily see that it tends to zero for each $s \in (0,1)$. By assumption, $\alpha : I \to X$, therefore, $\alpha(s) = \alpha(\varphi^n(s)) \to \alpha(0) = x_0$ for all $s \in [0,1)$, whence $\alpha(s) = x_0$ also for all $s \in [0,1)$. Consequently, we also have $\alpha(1) = x_0$.

30.2 The solution of Problem *30.D* implies that we must construct three paths u, v, and w in a certain space such that $\alpha(\varphi(s)) = \alpha(s)$ for all $s \in [0,1]$ (here, as in *30.1*, $\alpha = (uv)w$). Consider, for example, the paths $I \to [0,3]$ defined by the formulas $u(s) = s$, $v(s) = s+1$, and $w(s) = s+2$;

the path $\alpha : [0,1] \to [0,3]$ is a bijection. We introduce in $[0,3]$ the following equivalence relation: $x \sim y$ if there are $n, k \in \mathbb{N}$ such that $x = \alpha(\varphi^k(s))$ and $y = \alpha(\varphi^n(s))$. Let X be the quotient space of $[0,3]$ by this relation. Then the paths $u' = \mathrm{pr} \circ u$, $v' = \mathrm{pr} \circ v$, and $w' = \mathrm{pr} \circ w$ satisfy $(u'v')w' = u'(v'w')$.

30.4 If $u(s) = e_a u(s)$, then

$$u(s) = \begin{cases} a & \text{if } s \in \left[0, \frac{1}{2}\right], \\ u(2s-1) & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Thus, $u(s) = a$ for all $s \in \left[0, \frac{1}{2}\right]$. Further, if $s \in \left[\frac{1}{2}, \frac{3}{4}\right]$, then $2s - 1 \in \left[0, \frac{1}{2}\right]$, whence it follows that $u(s) = u(2s-1) = a$ also for all $s \in \left[\frac{1}{2}, \frac{3}{4}\right]$. Reasoning further in a similar way, we see as a result that $u(s) = a$ for all $s \in [0, 1)$. If we put no restrictions on the space X, then it is quite possible that $u(1) = x \neq a$ (show this). Also show that the assumptions of the problem imply that $u(1) = a$ (cf. *30.1*).

30.5 This is quite obvious.

31.1 The homotopies h such that $h(0,t) = h(1,t)$ for all $t \in I$.

31.2 See Problem *31.3*.

31.3 If $z = e^{2\pi i s}$, then

$$uv\left(e^{2\pi i s}\right) = \begin{cases} u\left(e^{4\pi i s}\right) & \text{if } s \in \left[0, \frac{1}{2}\right], \\ v\left(e^{4\pi i s}\right) & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} U(z^2) & \text{if } Imz \geq 0, \\ V(z^2) & \text{if } Imz \leq 0. \end{cases}$$

31.4 Consider the set of homotopy classes of circular loops at a certain point x_0, where the operation is defined as in Problem *31.3*.

31.5 The group is trivial because any map to such a space is continuous, and so any two loops (at the same point) are homotopic.

31.6 This group is trivial because the quotient space in question is homeomorphic to D^2.

31.7 Up to homeomorphism, a two-point set admits only three topological structures: the indiscrete one, the discrete one, and the topology where only one point of the two is open. The first case is considered in *31.5*, while the discrete space is not path-connected. Therefore, we should only consider the case where $\Omega_X = \{\varnothing, X, \{a\}\}$, $a \in X$. Let u be a loop at a. The formula

$$h(s,t) = \begin{cases} u(s) & \text{if } t = 0, \\ a & \text{if } t \in (0,1] \end{cases}$$

determines a homotopy between u and a constant loop. Indeed, the continuity of h follows from the fact that the set $h^{-1}(a) = \big(u^{-1}(a) \times I\big) \cup \big(I \times (0,1]\big)$ is open in the square $I \times I$.

31.9 Use Theorem *31.H*, the fact that $\mathbb{R}^n \smallsetminus 0 \cong \mathbb{R} \times S^{n-1}$, and Theorem *31.G*.

31.10 A discrete space is simply connected iff it is a singleton. An indiscrete space, $\mathbb{R}^n$, a convex set, and a star-shaped set are simply connected. The sphere S^n is simply connected iff $n \geq 2$. The space $\mathbb{R}^n \smallsetminus 0$ is simply connected iff $n \geq 3$.

31.11 We observe that since the space X is path-connected, we have $U \cap V \neq \varnothing$. Consider a loop $u : I \to X$, for the sake of definiteness, let $u(0) = u(1) = x_0 \in U$. By *31.G.3*, there is a sequence of points $a_1, \dots a_N \in I$, where $0 = a_1 < a_2 < \dots < a_{N-1} < a_N = 1$, such that for each i the image $u([a_i, a_{i+1}])$ is contained in U or in V. Furthermore, (uniting the segments) we can assume that if $u([a_{k-1}, a_k]) \not\subset U$ (or V), then $u([a_k, a_{k+1}]) \subset U$ (respectively, U), whence $u(a_k) \in U \cap V$ for all $k = 1, 2, \dots, N-1$. Consider the segment $[a_k, a_{k+1}]$ such that $u([a_k, a_{k+1}]) \subset V$. The points $u(a_k)$ and $u(a_{k+1})$ are joined by a path $v_k : [a_k, a_{k+1}] \to U \cap V$. Since V is simply connected, there exists a homotopy $h_k : [a_k, a_{k+1}] \times I \to V$ joining $u|_{[a_k, a_{k+1}]}$ and v_k, consequently, u is homotopic to a loop $v : I \to U$. Since the set U is also simply connected, it follows that v is null-homotopic, thus, X is simply connected.

31.12 Actually, at the moment we cannot give a complete solution of the problem because up to now we have not seen any example of a non-simply connected space. In what follows, we prove, e.g., that the circle is not simply connected. Put

$$U = \{(x,y) \in S^1 \mid y > 0\} \cup \{(1,0)\}, \quad V = \{(x,y) \in S^1 \mid y \leq 0\}.$$

Each of the sets is homeomorphi to an interval, therefore, they are simply connected, and their intersection is a singleton, which is path-connected. However, the space $U \cup V = S^1$ is not is simply connected.

31.13 Consider an arbitrary loop $s : I \to U$. Since $U \cup V$ is simply connected, it follows that this loop is null-homotopic in $U \cup V$, therefore, there exists a homotopy $H : I \times I \to U \cup V$ between s and a constant path. We subdivide the unit square $I \times I$ by segments parallel to its sides into smaller squares K_n so that the image of each of these squares be entirely contained in U or V. Consider the union K of those squares of the partition whose images are contained in V. Let L be a contour consisting of the boundaries of the squares in K, enclosing a certain part of K. It is clear that $L \subset U \cap V \subset U$, therefore, the homotopy H extends from L to the set bounded by L so that the image of the set be contained in U. Reasoning further in a similar way, we obtain a homotopy $H' : I \times I \to U$.

32.1 It is easy to describe a family of loops a_t constituting a free homotopy between the loop a and a loop representing the element $T_s(\alpha)$. Namely, the loop a_t starts at $s(t)$, it reaches the point $x_0 = s(0)$ at the moment $\frac{t}{3}$, after that it runs along the path a and returns to the point x_0 at the moment $1 - \frac{t}{3}$, and, finally, returns to the point $s(t)$. In this case, the loop a_0 is the initial loop a. The loop a_1 is defined by the formulas

$$a_1(\tau) = \begin{cases} s(1-3\tau) & \text{if } \tau \in \left[0, \frac{1}{3}\right], \\ a(3\tau - 1) & \text{if } \tau \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ s(3\tau - 2) & \text{if } \tau \in \left[\frac{2}{3}, 1\right], \end{cases}$$

and, consequently, the homotopy class of a_1 is that of $\sigma^{-1}\alpha\sigma$. To complete the argument, we present a formula for the above homotopy:

$$H(\tau, t) = \begin{cases} s(t - 3\tau) & \text{if } \tau \in \left[0, \frac{t}{3}\right], \\ a\left(\frac{3\tau - t}{3-2t}\right) & \text{if } \tau \in \left[\frac{t}{3}, \frac{3-t}{3}\right], \\ s(3\tau + t - 3) & \text{if } \tau \in \left[\frac{3-t}{3}, 1\right]. \end{cases}$$

32.2 Consider the homotopy defined by the formula

$$H'(\tau, t) = \begin{cases} s(1 - 3\tau) & \text{if } \tau \in \left[0, \frac{1-t}{3}\right], \\ H\left(\frac{3\tau + t - 1}{2t+1}, t\right) & \text{if } \tau \in \left[\frac{1-t}{3}, \frac{t+2}{3}\right], \\ s(3\tau - 2) & \text{if } \tau \in \left[\frac{t+2}{3}, 1\right], \end{cases}$$

and verify that $H'(\tau, 1) = b(\tau)$, and the correspondence $\tau \mapsto H'(\tau, 0)$ determines a path in the homotopy class $[s^{-1}as]$.

***32.1*x** This immediately follows from assertion *32.B*x.

33.1 If $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism, then p homeomorphically maps $V_\alpha \cap p^{-1}(U')$ onto U'.

33.2 See the proof of assertion *33.F*; the coverings p and q are said to be isomorphic.

33.3 This follows from *33.H* and *33.E* because $\mathbb{C} \smallsetminus 0 \cong S^1 \times \mathbb{R}$ and $p' : \mathbb{R} \to \mathbb{R} : x \mapsto nx$ is a trivial covering. Also sketch a trivially covered neighborhood of a point $z \in \mathbb{C} \smallsetminus 0$.

33.4 Consider the following two partitions of the rectangle $K = [0, 2] \times [0, 1]$. The partition R consists of the two-point sets $\{(0, y), (2, y) \mid y \in [0, 1]\}$, all the remaining elements of R are singletons. The partition R' consists of the two-point sets $\{(x, y), (x + 1, 1 - y) \mid x \in (0, 1), y \in [0, 1]\}$

and the three-point sets $\{(0,y),(1,1-y),(2,y) \mid x \in (0,1),\ y \in [0,1]\}$. Since each element of the first partition is contained in a certain element of the second partition, it follows that a quotient map $p : K/_R \to K/_{R'}$ is defined, which is the required covering of the Möbius strip by a cylinder. There is also a simpler option. We introduce an equivalence relation on $S^1 \times I:\ (z,t) \sim (-z, 1-t)$. Verify that the quotient space by this relation is homeomorphi to the Möbius strip, and the factorization projection is a covering.

33.5 The solution is similar to that of Problem *33.4*. Consider two partitions of the rectangle $K = [0,3] \times [0,1]$. The two-point elements of the first of them are the pairs $\{(0,y),(3,1-y) \mid y \in [0,1]\}$, and the four-point elements of the second one are quadruples $\{(0,y),(1,1-y),(2,y),(3,1-y) \mid x \in (0,1), y \in [0,1]\}$.

33.6 Modify the solution of Problem *33.4*, including into the partition R the quadruple of the vertices of the rectangle K and the pairs $\{(x,0),(x,1) \mid x \in (0,2)\}$. Another approach to constructing the same covering involves introducing the following equivalence relation in $S^1 \times S^1$: $(z,w) \sim (-z,\overline{w})$ (see the solution of Problem *33.4*).

33.7 There are standard coverings $\mathbb{R} \times S^1 \to S^1 \times S^1$ and $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ such that their compositions with the covering whose construction was outlined in the solution of Problem *33.6* are coverings of the Klein bottle by a cylinder and by the plane. Modifying the solution of Problem *33.5*, we obtain a nontrivial covering of the Klein bottle by the Klein bottle. We also present a more geometric description of the required covering. Let $q : M \to M$ be a covering of the Möbius strip by the Möbius strip, let M_1 and M_2 be two copies of the Möbius strip, and let $q_1 : M_1 \to M_1$ and $q_2 : M_2 \to M_2$ be two copies of q. If we paste M_1 and M_2 together along their common boundary, then we obtain the Klein bottle. It is clear that as a result we construct a covering of the Klein bottle by the Klein bottle.

33.8 The preimages of points have the form $\big\{\big(x+k, \frac{1}{2} + (-1)^{k-1}(\frac{1}{2} - y) + l\big) \mid k,l \in \mathbb{Z}\big\}$.

33.9 We already have coverings $S^2 \to \mathbb{R}P^2$ and $S^1 \times S^1 \to K$, where K is the Klein bottle, thus, we have coverings of the sphere with k crosscaps by a sphere with $k-1$ handles for $k = 1,2$. We prove that such a covering exists for each k. Let S_1 and S_2 be two copies of the sphere with k holes. Denote by S the "basic" sphere with k holes and consider the map $p' : S_1 \sqcup S_2 \to S$. Now we fill the holes in S by crosscaps (i.e., by Möbius strips), and we fill the pairs of holes in S_1 and, respectively, S_2 by the cylinders $S^1 \times I$. As a result, we obtain K, which is a sphere with k crosscaps, and $S_1 \sqcup S_2$ with k attached cylinders is homeomorphi to the sphere M with $k-1$ handles.

Since the Möbius strip is covered by a cylinder, p' extends to a two-fold covering $p : M \to K$.

33.10 Actually, we prove that each local homeomorphism is an open map, and, as it follows from *33.11*, each covering is a local homeomorphism. So, let the set V be open in X, $V' = p(V)$. Consider a point $b = p(x) \in V'$, where $x \in V$. By the definition of a local homeomorphism, x has a neighborhood U such that $p(U)$ is an open set and $p| : U \to p(U)$ is a homeomorphism. Therefore, the set $p(U \cap V)$ is open in V', thus, it is open in B, and hence it is a neighborhood of b lying in $p(V)$. Thus, $p(U)$ is an open set.

33.11 If $x \in X$, U is a trivially covered neighborhood of the point $b = p(x)$, and $p^{-1}(U) = \bigcup V_\alpha$, then there is a set V_α containing x. By the definition of a covering, $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

33.12 See, e.g., *33.K*.

33.13 Let $f : X \to Y$ be a local homeomorphism, let G be an open subset of X, and let $x \in G$. Assume that U is a neighborhood of x (in X) such that $f(U)$ is open in Y and the restriction $f|_U : U \to f(U)$ is a homeomorphism. If $V = W \cap U$, then $f(W)$ is open in $f(U)$, therefore, $f(W)$ is also open in Y. It is clear that $f|_W : W \to f(W)$ is a homeomorphism.

33.14 Only for the entire line. We show that if A is a proper subset of $\mathbb{R}$, then $p|_A : A \to S^1$ is not a covering. Indeed, A has a boundary point x_0, let $b_0 = p(x_0)$. We easily see that b_0 has no trivially covered (for $p|_A$) neighborhood.

33.15 See, for example, *33.H*.

33.16 For example, the covering of Problem *33.I* is pq-fold. In many examples, the number of sheets is infinite (countable).

33.17 All even positive integers and only they. The first assertion is obvious (cf. *33.4*), but at the moment we actually cannot prove the second one. The argument below involves methods and results presented in subsequent sections (cf. *39.3*). Consider the homomorphism $p_* : \pi_1(S^1 \times I) \to \pi_1(M)$, which is a monomorphism. It is known that $\pi_1(S^1 \times I) \cong \mathbb{Z} \cong \pi_1(M)$, and, furthermore, the generator of $\pi_1(S^1 \times I)$ is taken to the $2k$-fold generator of $\pi_1(M)$. Consequently, by *39.G* (or *39.H*), the covering has an even number of sheets.

33.18 All odd positive integers (cf. *33.5*) and only them (see *39.4*).

33.19 All even positive integers (cf. *33.6*) and only them (see *39.5*).

33.20 All positive integers (cf. *33.7*).

33.21 Consider the covering $T_1 = S^1 \times S^1 \to T_2 = S^1 \times S^1 : \ (z, w) \mapsto (z^d, w)$. Denote by S_2 the surface obtained from the torus T_2 by making $p-1$

holes. The preimage of S_2 under this covering is a surface S_1 homeomorphi to a torus with $d(p-1)$ holes. If we fill each of the holes (in S_1 and S_2) by a handle, then we attach $p-1$ handles to S_2, and as a result we obtain a surface M_2, which is a sphere with p handles, and we attach $d(p-1)$ handles to S_1 thus obtaining a surface M_1, which is a sphere with $d(p-1)+1$ handles. It is clear that the covering $S_1 \to S_2$ extends to a d-fold covering $M_1 \to M_2$.

33.22 Consider an arbitrary point $z \in Z$, let $q^{-1}(z) = \{y_1, y_2, \dots, y_d\}$. If a neighborhood V of z is trivially covered with respect to the projection q, and W_k are neighborhoods of the points y_k, $k = 1, 2, \dots, d$, trivially covered with respect to the projection p, then $U = \bigcap_{k=1}^{d} q\big(W_k \cap q^{-1}(V)\big)$ is a neighborhood of z trivially covered with respect to the projection $q \circ p$. Therefore, $q \circ p : X \to Z$ is a covering.

33.23 Let Z be the union of an infinite set of the circles determined by the equations $x^2 + y^2 = \frac{2x}{n}$, $n \in \mathbb{N}$, and let Y be the union of the y axis and the "twice" infinite family $x^2 + (y-k)^2 = \frac{2x}{n}$, where $n \in \mathbb{N},\ n > 1,\ k \in \mathbb{Z}$. The covering $q : Y \to Z$ has the following structure: the y axis coversthe outer circle of Z, while the restrictions of q to the other circles are parallel translations. Construct a covering $p : X \to Y$ whose composition with q is not a covering. Furthermore, the covering p can even be two-fold.

33.24 1) We observe that the topology in the fiber (induced from X) is discrete. Therefore, if X is compact, then the fiber $F = p^{-1}(b)$ is closed in X and, consequently, is compact. Therefore, the set F is finite, thus the covering is finite-sheeted. 2) Since B is compact and Hausdorff, it follows that B is regular, therefore, each point has a neighborhood U_x such that the compact closure $\operatorname{Cl} U_x$ lies in a certain trivially covered neighborhood. Since the base is compact, we have $B = \cup U_{x_i}$, $X = \cup p^{-1}(\operatorname{Cl} U_{x_i})$. Since the covering is finite-sheeted, X is thus covered by a finite number of compact sets, therefore, X is compact itself.

33.25 Let $U \cap V = G_0 \cup G_1$, where G_0 and G_1 are open subsets. Consider the product $X \times \mathbb{Z}$ and the subset

$$Y = \{(x,k) \mid x \in U,\ k \text{ even}\} \cup \{(x,k) \mid x \in V,\ k \text{ odd}\},$$

which is a disjoint union of countably many copies of U and V. We introduce in Y the following relation:

$$\begin{aligned} (x,k) &\sim (x,k+1) \quad \text{if } x \in G_1,\ k \text{ even},\\ (x,k) &\sim (x,k-1) \quad \text{if } x \in G_0,\ k \text{ odd}. \end{aligned}$$

Consider the partition of Y into the pairs of points equivalent to each other and into singletons in $\big(Y \smallsetminus (U \cap V)\big) \times \mathbb{Z}$. Denote by Z the quotient space by this partition. Let $p : Z \to X$ be the factorization of the restriction $\operatorname{pr}_X|_Y$, where $\mathrm{pr}_X : X \times \mathbb{Z} \to X$ is the standard projection. Verify that $p : Z \to X$ is an infinite-sheeted covering. Apply the described construction to the circle S^1, which is the union of two open arcs with disconnected intersection; what covering will result?

34.1 By assumption, we have $X = B \times F$, where F is a discrete space, and $p = \mathrm{pr}_B$. Let $y_0 \in F$ be the second coordinate of the point x_0. The correspondence $a \mapsto (f(a), y_0)$ determines a continuous lifting $\widetilde{f} : A \to X$ of f.

34.2 Let $x_0 = (b_0, y_0) \in B \times F = X$. Consider the map $g = \mathrm{pr}_F \circ \widetilde{f} : A \to F$. Since the set A is connected and the topology in F is discrete, it follows that g is a constant map. Therefore, $\tilde{f}(a) = (f(a), y_0)$, consequently, the lifting is unique.

34.3 Consider the coincidence set $G = \{a \in A \mid f(a) = g(a)\}$ of f and g; by assumption, $G \neq \varnothing$. For each point $a \in A$, take a connected neighborhood $V_a \subset \varphi^{-1}(U_b)$, where U_b is a certain trivially covered neighborhood of $b = \varphi(a)$. If $V_a \cap G \neq \varnothing$, then $V_a \subset G$ by *34.2*. In particular, if $a \in G$, then $V_a \subset G$, consequently, the set G is open. Similarly, if $a \notin G$, then $V_a \cap G = \varnothing$, i.e., $V_a \subset A \setminus G$, therefore, the set $A \setminus G$ is also open. By assumption, A is connected and $G \neq \varnothing$, whence $A = G$.

34.5 Show that if $b_0 = -1$, $x_0 = \frac{1}{2}$, then the path $u : \ t \mapsto e^{3\pi it}$ has no lifting.

34.6 We have: $\widetilde{u}(t) = \ln(2-t)$, $\widetilde{v}(t) = \ln(1+t) + 2\pi it$, $\widetilde{uv} = \widetilde{u}\,\widetilde{v}$, and $\widetilde{vu} = \widetilde{v}\,\widetilde{\widetilde{u}}$, where $\widetilde{\widetilde{u}} = \ln(2-t) + 2\pi i$.

34.F If the covering is nontrivial and the covering space is path-connected, then there exists a path s joining two distinct points $x_0, x_1 \in p^{-1}(b_0)$. By assertion *34.E*, the loop $p \circ s$ is not null-homotopic, therefore, B is not simply connected.

34.7 This follows from *34.F*.

34.8 For example, $\mathbb{R}P^2$ is not simply connected.

34.9 For example, generalize Theorem *34.C* to the case of maps $f : S^n \to B$ with $n > 1$ (cf. *39.Xx* and *39.Yx*).

35.1 This is the class α. Indeed, the path $\widetilde{s}(t) = t^2$ covering the loop ends at the point $1 \in \mathbb{R}$, therefore, $\widetilde{s}$ is homotopic to s_1.

35.2 If $[s] = \alpha^n$, then $s \sim s_n$, therefore, the paths $\widetilde{s}$ and $\widetilde{s}_n$ end at the same point.

35.3 The universal covering space for the n-dimensional torus is $\mathbb{R}^n$, the covering p is defined by the formula $p(x_1, \dots, x_n) = (e^{2\pi ix_1}, \dots, e^{2\pi ix_n})$. The map $\deg : \pi_1((S^1)^n, (1, 1, \dots, 1)) \to \mathbb{Z}^n$ is defined as follows. If u is a loop on the torus and $\widetilde{u}$ is the path covering u and starting at the origin, then $\deg([u]) = \widetilde{u}(1) \in \mathbb{Z}^n \subset \mathbb{R}^n$. Prove that this map is well defined and is an isomorphism.

35.4 This assumption was used where we used the fact that the n-sphere is simply connected, in other words, the covering $S^n \to \mathbb{R}P^2$ is universal only for $n \geq 2$.

31.7 Consider the following three cases, where X: 1) contains no open singletons (i.e., no "open points"); 2) contains a unique open singleton; 3) contains two open singletons.

35.7 For example, construct an infinite-sheeted covering (in a narrow sense) of X (see *7.V*).

35.8 Let us show that $\pi_1(X) \cong \mathbb{Z}$. The universal covering space of X is $\mathcal{Z} = (\mathbb{Z}, \Omega_4)$, where the topology Ω_4 is determined by the base consisting of singletons $\{2k\}$, $k \in \mathbb{Z}$, and 3-point sets $\{2k, 2k+1, 2k+2\}$, $k \in \mathbb{Z}$. The projection $p : \mathcal{Z} \to X$ is such that

$$p^{-1}(a) = \{4k \mid k \in \mathbb{Z}\}, \quad p^{-1}(b) = \{4k+1 \mid k \in \mathbb{Z}\},$$

$$p^{-1}(c) = \{4k+2 \mid k \in \mathbb{Z}\}, \quad p^{-1}(d) = \{4k+3 \mid k \in \mathbb{Z}\}.$$

As when calculating the fundamental group of the circle, it suffices to show that $\mathcal{Z}$ is simply connected. We can start, e.g., with the fact that the sets $U = \{0, 1, 2\}$ and $V = \{2, 3, 4\}$ are open in $U \cup V$ and simply connected, and their intersection $U \cup V$ is path connected. Therefore, their union $U \cup V$ is also simply connected (see *31.11*). After that, use induction. Here is another argument showing that $\mathcal{Z}$ is simply connected. Put $J_n = \{0, 1, \dots, 2n\}$ and define $H_n : J_n \times I \to J_n$ as follows:

$$H_n(x,t) = x \text{ for } x \in J_{n-1}, \quad H_n(2n-1,t) = \begin{cases} 2n-1 & \text{if } t = 0, \\ 2n-2 & \text{if } t \in (0,1], \end{cases}$$

$$H_n(2n,t) = \begin{cases} 2n & \text{if } t \in \left[0, \frac{1}{3}\right), \\ 2n-1 & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 2n-2 & \text{if } t \in \left(\frac{2}{3}, 1\right]. \end{cases}$$

Let u be a loop at 0 with image lying in J_n. Then the formula $h_n(s,t) = H_n\big(u(s), t\big)$ determines a homotopy between u and a loop with image lying in J_{n-1}. Using induction, we see that u is null-homotopic.

35.9 1) The results of Problems *31.7*, *35.6*, and *35.7* imply that $n_0 = 4$. 2) The computation presented in the solution of Problem *35.8* implies that $\mathbb{Z}$ is the fundamental group of a certain 4-point space. Show that is the only option.

35.10 1) Consider the 7-point space $Z = \{a, b, c, d, e, f, g\}$, where the topology is determined by the base $\{\{a\}, \{b\}, \{c\}, \{a, b, d\}, \{b, c, e\}, \{a, b, f\}, \{b, c, g\}\}$. To see that Z is not simply connected, observe that the universal covering of Z is constructed in the same way as that of the bouquet of two circles, with minor changes only. Instead of the "cross" K, use the space $\widetilde{K} = \{a, b_+, b_-, c_+, c_-, d, e, f, g\}$. 2) By *35.9*, at least five points are needed. Consider the 5-point space $Y = \{a, b, c, d, e\}$, where the topology is determined by the base $\{\{a\}, \{c\}, \{a, b, c\}, \{a, c, d\}, \{a, c, e\}\}$. Verify that the fundamental group of Y is a free group with two generators.

35.12 Consider a topological space

$$X = \{a_0, b_0, c_0, a_1, a_1', b_1, b_1', c_1, c_1', a_2, b_2, c_2, d_2\}$$

with topology determined by the base

$$\begin{array}{llll} \{a_0\}, & \{a_0, b_0, c_1\}, & \{a_0, b_0, c_1'\}, & \{a_0, b_0, c_0, a_1, b_1', c_1', a_2\}, \\ \{b_0\}, & \{a_0, b_1, c_0\}, & \{a_0, b_1', c_0\}, & \{a_0, b_0, c_0, a_1', b_1, c_1', b_2\}, \\ \{c_0\}, & \{a_1, b_0, c_0\}, & \{a_1', b_0, c_0\}, & \{a_0, b_0, c_0, a_1', b_1', c_1, c_2\}, \\ & & & \{a_0, b_0, c_0, a_1, b_1, c_1, d_2\}. \end{array}$$

36.1 First of all, we observe that, since the fundamental group of the punctured plane is Abelian, the operator of translation along any loop is the identity homomorphism. Consequently, two homotopic maps $f, g : \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0$ induce the same homomorphism on the level of fundamental groups. Let f be the map $z \mapsto z^3$. The generator of the group $\pi_1(\mathbb{C} \setminus 0, 1)$ is the class α of the loop $s(t) = e^{2\pi i t}$. The image of $f_*(\alpha)$ is the class of the loop $f_\#(u) = f \circ u$, therefore, $f_\#(u)(t) = e^{6\pi i t}$, whence $f_*(\alpha) = \alpha^3 \neq \alpha$. Consequently, $f_* \neq \mathrm{id}_{\pi_1(\mathbb{C} \setminus 0, 1)}$, whence it follows that f is not homotopic to the identity.

36.2 Denote by i the inclusion $X \to \mathbb{R}^n$. If the map f extends to $F : \mathbb{R}^n \to Y$, then $f = F \circ i$, whence $f_* = F_* \circ i_*$. However, since $\mathbb{R}^n$ is simply connected, it follows that the homomorphism F_* is trivial, consequently, so is the homomorphism f_*.

36.3.1 Denote by φ a homeomorphism of an open set $U \subset X$ onto $S^1 \times S^1 \setminus (1, 1)$. If $X = U$, then the assertion is obvious because the group $\pi_1(S^1 \times S^1 \setminus (1, 1))$ is a free group with two generators. Otherwise, we define $f : X \to S^1 \times S^1$ by letting

$$f(x) = \begin{cases} \varphi(x) & \text{for } x \in U, \\ (1, 1) & \text{for } x \notin U. \end{cases}$$

Verify that f is a continuous map. Now we take a point $x_0 \in U$ and consider the homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(S^1 \times S^1, f(x_0)).$$

We easily see that f_* is an epimorphism.

36.4 Let $f(z) = \operatorname{diag}\{z, 1, 1, \dots, 1\}$ for each point $z \in S^1$, and let $g(A) = \frac{\det(A)}{|\det(A)|}$ for each matrix $A \in GL(n, \mathbb{C})$. We have thus defined the maps $f : S^1 \to GL(n, \mathbb{C})$ and $g : GL(n, \mathbb{C}) \to S^1$ whose composition $g \circ f$ is the identity map. Since $g_* \circ f_* = (g \circ f)_* = \mathrm{id}_{\pi_1(S^1)}$, it follows that g_* is an is an epimorphism, consequently, the fundamental group of $GL(n, \mathbb{C})$ is infinite.

***36.1*x** This is assertion *36.D*x.

***36.2*x** By *36.1*x, it is sufficient to check that if $a \in \operatorname{Int} D^2$ and i is the standard embedding of the standard circle S^1 into $\mathbb{R}^2 \smallsetminus a$, then the circular loop i determines a nontrivial element in the group $\pi_1(\mathbb{R}^2 \smallsetminus a)$. Indeed, the formula $h(z, t) = z + ta$ determines a homotopy between i and a circular loop whose class obviously generates the fundamental group of $\mathbb{R}^2 \smallsetminus a$.

***36.3*x** Take an arbitrary point $a \in \mathbb{R}^2$, let $R > |a| + m$. Consider the circular loops $\varphi : S^1 \to \mathbb{R}^2 \smallsetminus a : z \mapsto f(Rz)$ and $i_R : S^1 \to \mathbb{R}^2 \smallsetminus a : z \mapsto Rz$. If $h(z, t) = t\varphi(z) + (1 - t)i_R(z)$, then

$$|h(z, t)| = |Rz + t(f(Rz) - Rz)| \geq R - |f(Rz) - rz| \geq R - m > |a|,$$

therefore, h determines a homotopy between φ and i_R in $\mathbb{R}^2 \smallsetminus a$. Since the loop i_R is not null-homotopic in $\mathbb{R}^2 \smallsetminus a$, it follows that φ is also not null-homotopic. By *36.1*x, $a = f(Rz)$, where $|z| < 1$, thus, the point a belongs to the image of f.

36.4*x*.1 The easiest way here would be to check that the corresponding circular loop is not null-homotopic in $\mathbb{R}^2 \smallsetminus 0$ and to use Theorem *36.1*x. Certainly, the latter theorem concerns a disk, and not a square, but the square is homeomorphi to a disk, so that from the topological point of view there is no difference between the pairs $(I^2, \operatorname{Fr} I^2)$ and (D^2, S^1). However, to help the reader better grasp the main idea of the proof of Theorem *36.1*x, we also present a solution making no use of the theorem. Assume that $w(x, y) \neq 0$ for all $(x, y) \in I^2$. Consider the following paths going along the sides of the square:

$$s_1(\tau) = (1, \tau);\ s_2(\tau) = (1 - \tau, 1);\ s_3(\tau) = (0, 1 - \tau);\ s_4(\tau) = (\tau, 0).$$

It is clear that the product $s = s_1 s_2 s_3 s_4$ is defined, which is a null-homotopic loop in the square I^2. Now we consider the loop $w \circ s$ and show that it is not null-homotopic in the punctured plane $\mathbb{R}^2 \smallsetminus 0$. Since $w(s_1(\tau)) = u(1) - v(\tau)$, the image of the path $w \circ s_1$ lies in the first quadrant. It starts at the point $u(1) - v(0) = (1, 0)$ and ends at the point $u(1) - v(1) = (0, 1)$. Since the first quadrant is a simply connected set, it follows that the path $w \circ s_1$ is homotopic there to any path joining the same points, for example, the paths $\varphi_1(t) = e^{\pi it/2}$. Similarly, the path $w \circ s_2$ lies in the second quadrant and is homotopic there to the path $\varphi_2(t) = e^{\pi i(t+1)/2}$. Thus, the path $w \circ s$ is homotopic in $\mathbb{R}^2 \smallsetminus 0$ to the path $\varphi = \varphi_1\varphi_2\varphi_3\varphi_4$ defined by the formula $\varphi(\tau) = e^{2\pi i\tau}$. Consequently, the class of the loop $w \circ s$ generates $\pi_1(\mathbb{R}^2 \smallsetminus (1, 0))$, in particular, this loop is not null-homotopic. On the other hand, the loop $w \circ s$ is null-homotopic in $\mathbb{R}^2 \smallsetminus 0$ by *36.G.4*. The contradiction obtained proves that $u(x) - v(y) = w(x, y) = 0$ for certain $x \in I$ and $y \in I$, i.e., the paths u and v intersect.

***36.5*x** For example, consider the sets

$$F = \{(1,1)\} \cup \big([0,1) \times 0\big) \cup \bigcup_{n=1}^{\infty} \big(\tfrac{2n-1}{2n} \times [0, \tfrac{2n-1}{2n}]\big)$$

$$G = \{(1,0)\} \cup \big([0,1) \times 1\big) \cup \bigcup_{n=1}^{\infty} \big(\tfrac{2n}{2n+1} \times [\tfrac{1}{2n+1}, 1]\big).$$

***36.6*x** No, we cannot. We argue by contradiction. Let $\varepsilon = \rho(F, G) > 0$. The result of Problem *13.17* implies that the points $(0, 0), (1, 1) \in F$ are joined by a path u with image in the $\varepsilon/2$-neighborhood of F, and the points $(0, 1), (1, 0) \in G$ are joined by a path v with image in the $\varepsilon/2$-neighborhood of G. Furthermore, $u(I) \cap v(I) = \varnothing$ by our choice of ε, which contradicts the assertion of Problem *36.4*x.
Now we also present another solution of this problem. The result of Problem *13.4*x implies that there exists a simple broken line joining $(0, 0)$ and $(1, 1)$ and disjoint with G. Consider the polygon $K_0 \dots K_n PQR$. One of the remaining vertices lies inside the polygon, while the other one lies outside, whence these points cannot belong to a connected set disjoint with the polygon.

***36.8*x** We prove that if x and y are joined by a path that does not intersect the set $u(S^1)$, then $\operatorname{ind}(u, x) = \operatorname{ind}(u, y)$. Indeed, if there exists such a path s, then the formula

$$h(z,t) = \varphi_{u,s(t)}(z) = \frac{u(z) - s(t)}{|u(z) - s(t)|}$$

determines a homotopy between $\varphi_{u,x}$ and $\varphi_{u,y}$; we proceed further as in the proof of *36.E*x. Thus, if $\operatorname{ind}(u, x) \neq \operatorname{ind}(u, y)$, then x and y cannot be joined by a path whose image not meet the set $u(S^1)$.

***36.9*x** Assume for the simplicity that the disk contains the origin. The formula

$$h(z,t) = \frac{(1-t)u(z) - x}{|(1-t)u(z) - x|}$$

shows that $\varphi_{u,x}$ is null-homotopic, whence $\operatorname{ind}(u,x) = 0$.

***36.10*x** (a) $\operatorname{ind}(u,x) = 1$ if $|x| < 1$, and $\operatorname{ind}(u,x) = 0$ if $|x| > 1$. (b) $\operatorname{ind}(u,x) = -1$ if $|x| < 1$, and $\operatorname{ind}(u,x) = 0$ if $|x| > 1$. (c) $\{\operatorname{ind}(u,x) \mid x \in \mathbb{R}^2 \smallsetminus u(S^1)\} = \{0, 1, -1\}$.

***36.11*x** The lemniscate L splits the plane in three components. The index of any loop with image L with respect to any point in the unbounded component is equal to zero. For each pair (k,l) of integers, there is a loop u such that the index of u with respect to points in one bounded component is equal to k, while the index of u with respect to points in the other bounded component is equal to l.

***36.12*x** See the solution of Problem*36.11*x.

***36.13*x** We can assume that x is the origin and the ray R is the positive half of the x axis. It is more convenient to consider the loop $u : I \to S^1$, $u(t) = \frac{f(e^{2\pi it})}{|f(e^{2\pi it})|}$. Assume that the set $f^{-1}(R)$ is finite and consists of n points. Consequently, $u^{-1}(1) = \{t_0, t_1, \dots, t_n\}$, and we have $t_0 = 0$ and $t_n = 1$. The loop u is homotopic to the product of loops u_i, $i = 1, 2, \dots, n$, each of which has the following property: $u_i(t) = 1$ only for $t = 0, 1$. Prove that $[u_i]$ is equal either to zero, or to a generator of $\pi_1(S^1)$. Therefore, if the integer k_i is the image of $[u_i]$ under the isomorphism $\pi_1(S^1) \to \mathbb{Z}$ and $k = \operatorname{ind}(f,x)$ is the image of $[u]$ under this isomorphism, then

$$|k| = |k_1 + k_2 + \dots k_n| \le |k_1| + |k_2| + \dots + |k_n| \le n$$

because each of the numbers k_i is 0 or ± 1.

***36.14*x** Apply the Borsuk–Ulam Theorem to the function taking each point on the surface of Earth to the pair of numbers (t,p), where t is the temperature at the point and p is the pressure.

37.1 If $\rho_1 : X \to A$ and $\rho_2 : A \to B$ are retractions, then $\rho_2 \circ \rho_1 : X \to B$ is also a retraction.

37.2 If $\rho_1 : X \to A$ and $\rho_2 : Y \to B$ are retractions, then so is $\rho_1 \times \rho_2 : X \times Y \to A \times B$.

37.3 Put $f(x) = a$ for $x \le a$, $f(x) = x$ for $x \in [a,b]$, $f(x) = b$ for $x \ge b$ (i.e., $f(x) = \max\{a, \min\{x, b\}\}$). Then $f : \mathbb{R} \to [a,b]$ is a retraction.

37.4 This follows from *37.6*, or, in a more customary way: if $f(x) = x$ for all $x \in (a,b)$, then the continuity of f implies that $f(b) = b$, thus, there exists no continuous function on $\mathbb{R}$ with image (a,b).

37.5 The properties that are transferred from topological spaces to their subspaces and (or) to continuous images. For example, the Hausdorff axiom, connectedness, compactness, etc.

37.6 This follows from *14.4*.

37.7 Since this space is not path-connected.

37.8 No, it is not. Indeed, the group $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ is finite, while the group $\pi_1(\mathbb{R}P^1) = \pi_1(S^1) \cong \mathbb{Z}$ is infinite, consequently, the former group admits no epimorphism onto the latter one (there also is no monomorphism in the opposite direction). Therefore, by assertion *37.F*, there exists no retraction $\mathbb{R}P^2 \to \mathbb{R}P^1$.

37.9 Let L be the boundary circle of a Möbius strip M. It is clear that $\pi_1(L) \cong \pi_1(M) \cong \mathbb{Z}$. However (cf. *33.4*), we easily see (verify this!), that the homomorphism i_* induced by the inclusion $i : L \to M$ takes the generator $\alpha \in \pi_1(L)$ to the element 2β, where β is the generator of $\pi_1(M) \cong \mathbb{Z}$. If there exists a retraction $\rho : M \to L$, then the composition $\rho_* \circ i_*$ takes the generator $\alpha \in \pi_1(L)$ to the element $2\rho_*(\beta) \neq \alpha$, contrary to the fact that this composition is the identical isomorphism of $\pi_1(L)$.

37.10 Let L be the boundary circle of a handle K. It is clear that $\pi_1(L) \cong \mathbb{Z}$, and $\pi_1(K)$ is a free group with two generators a and b. Furthermore, it can be checked (do it!), that the inclusion homomorphism $i_* : \pi_1(L) \to \pi_1(K)$ takes the generator $\alpha \in \pi_1(L)$ to the commutator $aba^{-1}b^{-1}$. Assume the contrary: let $\rho : K \to L$ be a retraction. Then the composition $\rho_* \circ i_*$ takes the generator $\alpha \in \pi_1(L)$ to the neutral element of $\pi_1(L)$ because the element

$$\rho_* \circ i_*(\alpha) = \rho_*(aba^{-1}b^{-1}) = \rho_*(a)\rho_*(b)\rho_*(a)^{-1}\rho_*(b)^{-1}$$

is neutral since the group $\mathbb{Z}$ is Abelian. On the other hand, this composition must coincide with $\mathrm{id}_{\pi_1(L)}$. A contradiction.

37.11 The assertion is obvious because each property stated in topological terms is topological. However, the following question is of interest. Let a space X have the fixed point property, and let $h : X \to Y$ be a homeomorphism. Thus, we know that each continuous map $f : X \to X$ has a fixed point. How, knowing this, can we prove that an arbitrary continuous map $g : Y \to Y$ also has a fixed point? Show that one of the fixed points of g is $h(x)$, where x is a fixed point of a certain map $X \to X$.

37.12 Consider a continuous function $f : [a, b] \to [a, b]$ and the auxiliary function $g(x) = f(x) - x$. Since $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$, there is a point $x \in [a, b]$ such that $g(x) = 0$. Thus, $f(x) = x$, i.e., x is a fixed point of f.

37.13 Let $\rho : X \to A$ be a retraction. Consider an arbitrary continuous map $f : A \to A$ and the composition $g = \mathrm{in} \circ f \circ \rho : X \to X$. Let x be a fixed point of g, whence $x = f(\rho(x))$. Since $\rho(x) \in A$, we also have $x \in A$, so that $\rho(x) = x$, whence $x = f(x)$.

37.14 Denote by ω the point of the bouquet which is the image of the pair $\{x_0, y_0\}$ under the factorization map. $\boxed{\Longrightarrow}$ This follows from *37.13*. $\boxed{\Longleftarrow}$ Consider an arbitrary continuous map $f : X \vee Y \to X \vee Y$. For the sake of definiteness, assume that $f(\omega) \in X$. Let $i : X \to X \vee Y$ be the standard inclusion, and let $\rho : X \vee Y \to X$ be a retraction mapping the entire Y to the point ω. By assumption, the map $\rho \circ f \circ i$ has a fixed point $x \in X$, $\rho(f(i(x))) = x$, so that $\rho(f(x)) = x$. If $f(x) \in Y$, then $\rho(f(x)) = \omega$, so that $x = \omega$. On the other hand, we assumed that $f(\omega) \in X$, consequently, $f(\omega) = \omega$ is a fixed point of f. Now we assume that $f(x) \in X$. In this case, we have

$$x = (\rho \circ f \circ i)(x) = \rho(f(x)) = f(x),$$

therefore, x is a fixed point of f.

37.15 Since the segment has the fixed point property (see *37.12*), hence, by *37.14*, reasoning by induction, we see that each finite tree has this property. An arbitrary infinite tree does not necessarily have this property; an example is the real line. However, try to state an additional assumption under which an infinite tree also has the fixed point property.

37.16 For example, a parallel translation has no fixed points.

37.17 For example, the antipodal map $x \mapsto -x$ has no fixed points.

37.18 Let $n = 2k - 1$. For example, the map

$$(x_1 : x_2 : \ldots : x_{2k-1} : x_{2k}) \mapsto (-x_2 : x_1 : \ldots : -x_{2k} : x_{2k-1})$$

has no fixed points.

37.19 Let $n = 2k - 1$. For example, the map

$$(z_1 : z_2 : \ldots : z_{2k-1} : z_{2k}) \mapsto (-\bar z_2 : \bar z_1 : \ldots : -\bar z_{2k} : \bar z_{2k-1})$$

has no fixed points.

38.1 The map $f : [0,1] \to \{0\}$ is a homotopy equivalence; the corresponding homotopically inverse map is, for example, the inclusion $i : \{0\} \to [0,1]$. The composition $i \circ f$ is homotopic to id_I because any two continuous maps $I \to I$ are homotopic, and the composition $f \circ i : \{0\} \to \{0\}$ is the identity map itself. Certainly, f is not a homeomorphism.

38.2 Let X and Y be two homotopy equivalent spaces and denote by $\pi_0(X)$ and $\pi_0(Y)$ the sets of path-connected components of X and Y, respectively. Let $f : X \to Y$ and $g : Y \to X$ be two mutually inverse homotopy equivalences. Since f is a continuous map, it maps path-connected sets to path-connected ones. Consequently, f and g induce maps $\widehat{f} : \pi_0(X) \to \pi_0(Y)$ and $\widehat{g} : \pi_0(Y) \to \pi_0(X)$. Since the composition $g \circ f$ is homotopic to id_X, it follows that each point $x \in X$ lies in the same path-connected component as the point $g(f(x))$. Consequently, the composition $\widehat{g} \circ \hat{f}$ is the identity map. Similarly, $\widehat{f} \circ \widehat{g}$ is also identical. Consequently, $\widehat{f}$ and $\widehat{g}$ are mutually inverse maps, in particular, the sets $\pi_0(X)$ and $\pi_0(Y)$ have equal cardinalities.

38.3 The proof is similar to that of *38.2*.

38.4 For example, consider: a point, a segment, a bouquet of n segments with $n \geq 3$.

38.5 We prove that the midline L of the Möbius strip M (i.e., the image of the segment $I \times \frac{1}{2}$ under factorization $I \times I \to M$) is a strong deformation retract of M. The geometric argument is obvious: we define h_t as the contraction of M towards L with ratio $1 - t$. Thus, h_0 is identical, while h_1 maps M to L. Now we present the corresponding formulas. Since M is a quotient space of the square, first, consider the homotopy

$$H : I \times I \times I \to I \times I : \ (u, v, t) \mapsto \big(u, (1 - t)v + \tfrac{t}{2}\big).$$

Furthermore, we have $H\big(u, \frac{1}{2}, t\big) = \big(u, \frac{1}{2}\big)$ for all $t \in I$. Since $(1 - t)v + \frac{t}{2} + (1 - t)(1 - v) + \frac{t}{2} = 1$, it follows that this homotopy is compatible with the factorization and thus induces a homotopy $h : M \times I \to M$. We have $H(u, v, 0) = (u, v)$, whence $h_0 = \mathrm{id}_M$ and $H_1(u, v) = \big(u, \frac{1}{2}\big)$.

38.6 The letters *E,F,G,H,I,J,K,L,M,N,S,T,U,V,W,X,Y,Z* are homotopy equivalent to a point; *A,O,P,Q,R* are homotopy equivalent to a circle; finally, *B* is homotopy equivalent to a bouquet of two circles.

38.7 This can be proved in various ways. For example, we can produce circles lying in the handle $\mathcal{H}$ whose union is a strong deformation retract of $\mathcal{H}$. For this purpose, we present the handle as a result of factorizing the annulus $A = \{z \mid \frac{1}{2} \leq |z| \leq 1\}$ by the following relation: $e^{i\varphi} \sim -e^{-i\varphi}$ for $\varphi \in \big[-\frac{\pi}{4}, \frac{\pi}{4}\big]$, and $e^{i\varphi} \sim e^{-i\varphi}$ for $\varphi \in \big[\frac{\pi}{4}, \frac{3\pi}{4}\big]$. The image of the standard unit circle under the factorization by the above equivalence relation is the required bouquet of two circles lying in of the handle. The formula $H(z, t) = (1 - t)z + t\frac{z}{|z|}$ determines a homotopy between the identity map of A and the map $z \mapsto \frac{z}{|z|}$ of A onto the outer rim of A, and $H(z, t) = z$ for all $z \in S^1$ and $t \in I$. The quotient map of H is the required homotopy.

38.8 This follows from *38.7* and *38.I*.

38.9 Embed each of these spaces in $\mathbb{R}^3 \smallsetminus S^1$ so that the image of the embedding be a deformation retract of $\mathbb{R}^3 \smallsetminus S^1$. Let us present one more space homotopy equivalent to our two spaces: the union X of S^2 with one of the diameters. This X can also be embedded in $\mathbb{R}^3 \setminus S^1$ as a deformation retract.

38.10 Put $A = \{(z_1, z_2) \mid 4z_2 = z_1^2\} \subset \mathbb{C}^2$. Consider the map $f : \mathbb{C} \times (\mathbb{C} \setminus 0) \to \mathbb{C}^2 \setminus A : (z_1, z_2) \mapsto \left(z_1, z_2 + \frac{z_1^2}{4}\right)$. Verify that f is a homeomorphism and $\mathbb{C}^2 \setminus A \simeq \mathbb{C} \times (\mathbb{C} \setminus 0) \simeq S^1$. Furthermore, the circle can be embedded in $\mathbb{C} \setminus A$ as a deformation retract.

38.11 We prove that $O(n)$ is a deformation retract of $GL(n, \mathbb{R})$. Let $(\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n)$ be the collection of columns of a matrix $A \in GL(n, \mathbb{R})$, each of which is regarded as an element of $\mathbb{R}^n$. Let $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ be a result of the Gram–Schmidt orthogonalization procedure. Thus the matrix with columns formed by the coordinates of these vectors is orthogonal. The vectors $\mathbf{e}_k$ are expressed via $\mathbf{f}_k$ by the formulas

$$\begin{aligned}
&\mathbf{e}_1 = \lambda_{11}\mathbf{f}_1,\\
&\mathbf{e}_2 = \lambda_{21}\mathbf{f}_1 + \lambda_{22}\mathbf{f}_2,\\
&\dots,\\
&\mathbf{e}_n = \lambda_{n1}\mathbf{f}_1 + \lambda_{n2}\mathbf{f}_2 + \dots + \lambda_{nn}\mathbf{f}_n,
\end{aligned}$$

where $\lambda_{kk} > 0$ for all $k = 1, 2, \dots, n$.
We introduce the vectors

$$\mathbf{w}_k(t) = t(\lambda_{n1}\mathbf{f}_1 + \lambda_{n2}\mathbf{f}_2 + \dots + \lambda_{kk-1}\mathbf{f}_{k-1}) + (t\lambda_{kk} + 1 - t)\mathbf{f}_k$$

and consider the matrix $h(A, t)$ with columns consisting of the coordinates of these vectors. It is clear that the correspondence $(A, t) \mapsto h(A, t)$ determines a continuous map $GL(n, \mathbb{R}) \times I \to GL(n, \mathbb{R})$. We easily see that $h(A, 0) = A$, $h(A, 1) \in O(n)$, and $h(B, t) = B$ for all $B \in O(n)$. Thus, the map $A \mapsto h(A, 1)$ is the required deformation retraction.

38.13 Use, e.g., *19.43*.

38.14 We need the notion of the cylinder Z_f of a continuous map $f : X \to Y$. By definition, Z_f is obtained by attaching the ordinary cylinder $X \times I$ to Y via the map $X \times 0 \to Y$, $(x, 0) \mapsto f(x)$. Hence, Z_f is a result of factorization of the disjoint union $(X \times I) \sqcup Y$, under which the point $(x, 0) \in X \times 0$ is identified with the point $f(x) \in Y$. We identify X and $X \times 1 \subset Z_f$, and it is also natural to assume that the space Y lies in the mapping cylinder. There is an obvious strong deformation retraction $p_Y : Z_f \to Y$, which leaves Y fixed and takes the point $(x, t) \in X \times (0, 1)$ to $f(x)$. It remains to prove that if f is a homotopy equivalence, then X is also a deformation retract of Z_f. Let $g : Y \to X$ be a homotopy equivalence inverse to f. Thus, there exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = g(f(x))$ and $H(x, 1) = x$. We define the retraction $\rho : Z_f \to X$ as a quotient map of the map $(X \times I) \sqcup Y \to X :\ (x, t) \mapsto h(x, t), y \mapsto g(y)$. It remains to prove that the map ρ is a deformation retraction, i.e., to verify that $\mathrm{in}_X \circ\rho$ is homotopic to id_{Z_f}. This follows from the following chain, where the $\sim$ sign denotes a homotopy between compositions of homotopic maps:

$$\mathrm{in}_X \circ\rho = \overline{\rho} = \overline{\rho} \circ \mathrm{id}_{Z_f} \sim \overline{\rho} \circ p_Y = g \circ p_Y = \mathrm{id}_{Z_f} \circ(g \circ p_Y) \sim$$
$$\sim p_Y \circ (g \circ p_Y) = (p_Y \circ g) \circ p_Y = (f \circ g) \circ p_Y \sim \mathrm{id}_Y \circ p_Y = p_Y \sim \mathrm{id}_{Z_f} .$$

38.15 Use the rectilinear homotopies.

38.16 Let $h : X \times I \to X$ be a homotopy between id_X and the constant map $x \mapsto x_0$. The formula $u_x(t) = h(x,t)$ determines a path joining (an arbitrary) point x in X with x_0. Consequently, X is path-connected.

38.17 Assertions (a)–(d) are obviously pairwise equivalent. We prove that they are also equivalent to assertions (e) and (f).
(a) $\Longrightarrow$ (e): Let $h : X \times I \to X$ be a homotopy between id_X and a constant map. For each continuous map $f : Y \to X$, the formula $H = h \circ (f \times \mathrm{id}_I)$ (or, in a different way: $H(y,t) = h(f(y),t)$) determines a homotopy between f and a constant map.
(e) $\Longrightarrow$ (a): Put $Y = X$ and $f = \mathrm{id}_X$.
(a) $\Longrightarrow$ (f): Let h be the same as before. The formula $H = f \circ h$ determines a homotopy between $f : X \to Y$ and a constant map.
(f) $\Longrightarrow$ (a): Put $Y = X$ and $f = \mathrm{id}_X$.

38.18 Assertion (b) is true; assertion (a) holds true iff Y is path-connected.

38.19 Each of the spaces (a)–(e) is contractible.

38.20 $\boxed{\Longrightarrow}$ Let H be a homotopy between $\mathrm{id}_{X\times Y}$ and a constant map $(x,y) \mapsto (x_0,y_0)$. Then $X \times I : (x,t) \mapsto \mathrm{pr}_X\big(H(x,y_0,t)\big)$ is a homotopy between id_X and the constant map $x \mapsto x_0$. The contractibility of Y is proved in a similar way.
$\boxed{\Longleftarrow}$ Assume that X and Y are contractible, h is a homotopy between id_X and the constant map $x \mapsto x_0$, and g is a homotopy between id_Y and the constant map $y \mapsto y_0$. The formula $H(x,y,t) = \big(h(x,t), g(y,t)\big)$ determines a homotopy between $\mathrm{id}_{X\times Y}$ and the constant map $(x,y) \mapsto (x_0),y_0)$.

38.21 (a) Since $X = \mathbb{R}^3 \smallsetminus \mathbb{R}^1 \cong (\mathbb{R}^2 \smallsetminus 0) \times \mathbb{R}^1 \simeq S^1$, we have $\pi_1(X) \cong \mathbb{Z}$.
(b) It is clear that $X = \mathbb{R}^N \smallsetminus \mathbb{R}^n \cong (\mathbb{R}^{N-n} \smallsetminus 0) \times \mathbb{R}^n \simeq S^{N-n-1}$. Consequently, if $N = n+1$, then $X \simeq S^0$; if $N = n+2$, then $X \simeq S^1$, whence $\pi_1(X) \cong \mathbb{Z}$; if $N > n+2$, then X is simply connected.
(c)Since $S^3 \smallsetminus S^1 \cong \mathbb{R}^2 \times S^1$, we have $\pi_1(S^3 \smallsetminus S^1) \cong \mathbb{Z}$.
(d) If $N = n+1$, then $X = \mathbb{R}^N \smallsetminus S^{N-1}$ has two components, one of which is an open N-ball, and hence is contractible, while the second one is homotopy equivalent to S^{N-1}. If $N > n+1$, then X is homotopy equivalent to the

bouquet $S^{N-1} \vee S^{N-n-1}$. Consequently, for $N = 2$ and $n = 0$ $\pi_1(X)$ is a free group with two generators; for $N > 2$ or $N = n + 2$, we obtain the group $\mathbb{Z}$; in all remaining cases, X is simply connected.
(e) $\mathbb{R}^3 \smallsetminus S^1$ admits a deformation retraction to a sphere with two points identified, which is homotopy equivalent to the bouquet $X = S^1 \vee S^2$ by *38.9*. The universal covering of X is the real line $\mathbb{R}^1$, to which at all of the integer points 2-spheres are attached (a "garland"). Therefore, $\pi_1(\mathbb{R}^3 \smallsetminus S^1) \cong \pi_1(X) \cong \mathbb{Z}$.
(f) If $N = k+1$, then $S^N \smallsetminus S^{N-1}$ is homeomorphic to the union of two open N-balls, so that each of its two components is simply connected. Certainly, this fact is a consequence from the following general result: $S^N \smallsetminus S^k \cong S^{N-k-1} \times \mathbb{R}^{k+1}$, whence $\pi_1(S^N \smallsetminus S^k) \cong \mathbb{Z}$ for $N = k + 2$ and this group is trivial in other cases.
(g) It can be shown that $\mathbb{R}P^3 \smallsetminus \mathbb{R}P^1 \cong \mathbb{R}^2 \times S^1$, but it is easier to show that this space admits a deformation retraction to S^1. In both cases, it is clear that $\pi_1(\mathbb{R}P^3 \smallsetminus \mathbb{R}P^1) \cong \mathbb{Z}$.
(h) Since a handle is homotopy equivalent to a bouquet of two circles, it has free fundamental group with two generators.
(i) The midline (the core circle) of the Möbius strip M is a deformation retract of M, therefore, the fundamental group of M is isomorphic to $\mathbb{Z}$.
(j) The sphere with s holes is homotopy equivalent to a bouquet of $s - 1$ circles and so has free fundamental group with $s - 1$ generators (which, certainly, is trivial for $s = 1$).
(k) The punctured Klein bottle is homotopy equivalent to a bouquet of two circles, and so has free fundamental group with two generators.
(l) the punctured Möbius strip is homotopy equivalent to the letter θ, which, in turn, is homotopy equivalent to a bouquet of two circles. The Möbius strip with s punctures is homotopy equivalent to a bouquet of $s + 1$ circles and thus has free fundamental group with $s + 1$ generators.

38.22 Let K be the boundary circle of a Möbius strip M, L the midline of M, and T a solid torus whose boundary contains K. Consider the embeddings $i : K \to T \smallsetminus S$ and $j : T \smallsetminus S \to \mathbb{R}^3 \smallsetminus S$. Since $T \smallsetminus S \cong (D^2 \smallsetminus 0) \times S^1$, we have $\pi_1(T \smallsetminus S) \cong \mathbb{Z} \oplus \mathbb{Z}$. Denote by a and b the generators of the group $\pi_1(T \smallsetminus S)$. Let α be the generator of $\pi_1 K \cong \mathbb{Z}$, then $i_*(\alpha) = a + 2b$. Furthermore, $j_*(a)$ is a generator of $\pi_1(\mathbb{R}^3 \smallsetminus S)$, and $j_*(b) = 0$. Therefore, $j_*(i_*(\alpha)) \neq 0$. If there existed a disk D spanning K and having no other common points with M, then we would have $D \subset \mathbb{R}^3 \smallsetminus S$. Consequently, K would determine a null-homotopic loop in $\mathbb{R}^3 \smallsetminus S$. However, $j_*(i_*(\alpha)) \neq 0$.

38.23 1) Using the notation introduced in *38.10*, consider the map

$$Q \to (\mathbb{C} \smallsetminus 0) \times (\mathbb{C}^2 \smallsetminus A) \simeq S^1 \times S^1 : (a, b, c) \mapsto \left(a, \tfrac{b}{a}, \tfrac{c}{a}\right).$$

This is a homeomorphism. Therefore, the fundamental group of Q is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

2) The result of Problem *38.10* implies that Q_1 is homotopy equivalent to the circle, and, consequently, has fundamental group isomorphic to $\mathbb{Z}$.

39.1 This follows from *39.H* since the group $p_*(\pi_1(X, x_0))$ of the universal covering is trivial, and therefore its index is equal to the order of the fundamental group $\pi_1(B, b_0)$ of the base of the covering.

39.2 This follows from *39.H* because a group having a subgroup of nonzero index is obviously nontrivial.

39.3 All even positive integers. It can be proved that each of the boundary circles of the cylinder is mapped onto the boundary S of the Möbius strip M. Let α be the generator of the group $\pi_1(S^1 \times I)$, then $p_*(1) = b^k$, where the element $b \in \pi_1(M)$ is the image of the generator of $\pi_1(S)$ under the embedding $S \to M$. It remains to observe that $b = a^2$, where a is the generator of the group $\pi_1(M) \cong \mathbb{Z}$. Thus, $p_*(\alpha) = a^{2k}$, consequently, the index of $p_*(\pi_1(S^1 \times I))$ is an even positive integer. We easily see that there are coverings with an arbitrary even number of sheets (see *33.4*).

39.4 All odd positive integers, see *39.10*x.

39.5 All even positive numbers, see *39.10*x.

39.6 All positive integers, see *39.10*x.

39.7 If the base of the covering is compact, while the covering space is not, then the covering is infinite-sheeted by *33.24*.

39.8 See the hint to Problem *39.7*.

39.9 The class of the identity map.

***39.1*x** For example, consider the union of the standard unit segments on the x and y axes and of the segments $I_n = \left\{\left(\frac{1}{n}, y\right) \mid y \in I\right\}$, $n \in \mathbb{N}$ (the "*hair comb*").

***39.4*x** This is obvious because the group $\pi_1(X, a)$ is trivial, and we can put $U = X$.

***39.5*x** Consider the circle.

***39.6*x** Let V be the smallest neighborhood of a. Therefore, the topology on V is indiscrete. Let $h_t(x) = x$ for $t < 1$, $h_1(x) = a$. Prove that $h : V \times I \to V$ is a homotopy.

***39.7*x** This is true because already the inclusion homomorphism $\pi_1(V, a) \to \pi_1(U, a)$ is trivial.

***39.8*x** For example, such a space is $D^2 \smallsetminus \left\{\left(\frac{1}{n}, 0\right) \mid n \in \mathbb{N}\right\}$ (consider the point $(0, 0)$).

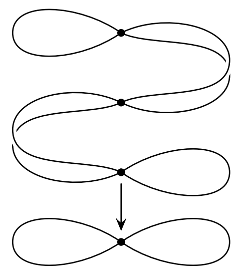
39.9x Consider the cone over the space of Problem *39.8x*.

39.10x By Theorem *39.Fx*, it suffices to describe the hierarchy of the classes of conjugate subgroups in the fundamental group of the base and present coverings with a given subgroup. In all examples except (e), the fundamental group of the space in question (the base) is Abelian. Therefore, it is sufficient to list all subgroups of the fundamental group and to determine their order with respect to the inclusion. In each case, all coverings are subordinate to the universal covering, and the trivial covering is subordinate to all coverings.
(a) The universal covering is the map $p : \mathbb{R} \to S^1$. The covering $p_k : S^1 \to S^1 :\ z \mapsto z^k$, where $k \in \mathbb{N}$, is subordinate to the covering p_l iff k divides l, and the subordination is the covering $p_{l/k}$.
(b) Since $\mathbb{R}^2 \smallsetminus 0 \cong S^1 \times \mathbb{R}$, the answer is similar to the preceding one.
(c) If M is a Möbius strip, then $\pi_1(M) \cong \mathbb{Z}$. Thus, as and the first example, all subgroups of the fundamental group of the base have the form $k\mathbb{Z}$. The difference is as follows: if k is odd, then the covering space is the Möbius strip, while if k is even, then the covering space is the cylinder $S^1 \times I$.
(d) The universal covering was constructed in the solution of Problem *35.7*. Since the fundamental group of this space is isomorphic to $\mathbb{Z}$, it is sufficient to present coverings with group $k\mathbb{Z} \subset \mathbb{Z}$. Construct them on your own. In contrast to example (a), the total spaces are not homeomorphi because each of them has its own number of points.
(e) The universal covering of the torus is the map $p : \mathbb{R}^1 \times \mathbb{R}^1 \to S^1 \times S^1 : (x, y) \mapsto \left(e^{2\pi ix}, e^{2\pi iy}\right)$. An example of a covering with group $k\mathbb{Z} \oplus l\mathbb{Z}$ is the following map of the torus to itself:

$$p_k \times p_l : S^1 \times S^1 \to S^1 \times S^1 :\ (z, w) \mapsto (z^k, w^l).$$

More generally, for each integer matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we can consider the covering $p_A : S^1 \times S^1 \to S^1 \times S^1 :\ (z, w) \mapsto (z^a w^b, z^c w^d)$, the group of which is the lattice $L \subset \mathbb{Z} \oplus \mathbb{Z}$ with basis vectors $\mathbf{a}(a, c)$ and $\mathbf{b}(b, d)$. The covering p_A is subordinate to the covering $p_{A'}$ determined by the matrix $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ if the lattice L' with basis vectors $\mathbf{a}'(a', c')$ and $\mathbf{b}'(b', d')$ is contained in the lattice L. In this case, the bases $\{\mathbf{a}, \mathbf{b}\}$ in L and $\{\mathbf{a}', \mathbf{b}'\}$ in L' can be chosen to be coordinated, i.e., so that $\mathbf{a}' = k\mathbf{a}$ and $\mathbf{b}' = l\mathbf{b}$ for certain $k, l \in \mathbb{N}$. The subordination here is the covering $p_k \times p_l$. Infinite-sheeted coverings are described up to equivalence by cyclic subgroups in $\mathbb{Z} \times \mathbb{Z}$, i.e., by the cyclic vectors $\mathbf{a}(a, c) \in \mathbb{Z}\times\mathbb{Z}$. Every such a vector determines the map $p_{\mathbf{a}} : S^1 \times \mathbb{R} \to S^1 \times S^1 :\ (z, t) \mapsto (z^a e^{2\pi it}, z^b)$. The covering $p_{\mathbf{a}}$ is subordinate to the covering $p_{\mathbf{b}}$ if $\mathbf{b} = k\mathbf{a}$, $k \in \mathbb{Z}$. In this case, the subordination has the form $S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} :\ (z,t) \mapsto (z^k, t)$. Description of subordinations between finite-sheeted and infinite-sheeted coverings is left to the reader as an exercise.

***39.11*x** See the figure.



***39.12*x** Indeed, any subgroup of an Abelian group is normal. We can also verify directly that for each loop $s : I \to B$ either each path in X covering s is a loop (independently of the starting point), or none of these paths is a loop.

***39.13*x** This is true because any subgroup of index two is normal.

***39.15*x** See the example constructed in the solution of Problem *39.11x*.

***39.16*x** This follows from assertion *39.Px*, (d).

40.3 The cellular partition of Z is obvious: if e^m is an open cell in X and e^n is an open cell in Y, then $e^m \times e^n$ is an open cell in Z because $B^m \times B^n \cong B^{m+n}$. Thus, the n-skeleton of Z is the union of pairwise products of all cells in X and Y whose of dimensions is at most n. Now we must describe the attaching maps of the corresponding closed cells. In order to construct the cellular space X, we start with a discrete topological space X_0, and then for each $m \in \mathbb{N}$ we construct the space X_m by attaching to X_{m-1} the disjoint union of m-disks $D^m_{X,\alpha}$ via an attaching map $\bigsqcup_\alpha S^{m-1}_{X,\alpha} \to X_{m-1}$. Clearly, X is a result of a simultaneous factorization of the disjoint union $\bigsqcup_{m,\alpha} D^m_{X,\alpha}$ by a certain single identification. The same is true for Y. Since in the present case the operations of factorization and multiplication of topological spaces commute (see *24.Tx*), the product $X \times Y$ is homeomorphi to a result of factorizing the disjoint union

$$\bigsqcup_{\substack{m,\alpha\\ n,\beta}} D^m_{X,\alpha} \times D^n_{Y,\beta}$$

of pairwise products of disks involved in the construction of X and Y. It remains to observe that this factorization, in turn, can be performed "by skeletons", starting with a discrete topological space $Z_0 = \bigsqcup \big(D^0_{X,\alpha} \times D^0_{Y,\beta}\big)$.

Attaching to Z_0 1-cells of the form $D^1_{X,\alpha} \times D^0_{Y,\beta}$ and $D^0_{X,\alpha} \times D^1_{Y,\beta}$, we obtain the 1-skeleton Z_1, etc. In dimensions grater than 1, Description of the attaching maps can cause difficulties. Consider a cell of the form $e^m \times e^n$. Its characteristic map $D^m \times D^n \to X \times Y$ is simply the product of the characteristic maps of the cells e^m and e^n, which maps the image of the boundary sphere of the "disk" $D^m \times D^n$ to the skeleton Z_{n+m-1}, which is already constructed. We have thus defined the attaching map $\omega : S^{n+m-1} \to Z_{n+m-1}$. Let us also give an explicit description of ω. To do this, we need the standard homeomorphism $\varkappa : D^{m+n} \to D^m \times D^n$ with $\varkappa(S^{m+n-1}) = (S^{m-1} \times D^n) \cup (D^m \times S^{n-1})$. Let $\varphi_1 : S^{m-1} \to X_{m-1}$ and $\varphi_2 : S^{n-1} \to Y_{n-1}$ be the attaching maps of the cells e^m and e^n. Then ω can be described as a composition

$$
\begin{aligned}
&S^{m+n-1} \to (D^m \times S^{n-1}) \cup (S^{m-1} \times D^n) \to \\
&\qquad \to [(X_{m-1} \cup_{\varphi_1} D^m) \times Y_{n-1}] \cup [X_{m-1} \times (Y_{n-1} \cup_{\varphi_2} D^n)] \hookrightarrow Z_{m+n-1},
\end{aligned}
$$

where the first map is a submap of the homeomorphism $\varkappa$, the second one is the obvious map defined on each part as the product of the characteristic and the attaching map, and the third one is an inclusion.

40.4 No, it does not. Show that the product topology on the product of two copies of the cellular space of Problem *40.9* is not cellular.

40.5 Actually, when solving Problem *40.H*, we used, firstly, the presentation $\mathbb{R}P^n = \bigcup_{k=0}^n \mathbb{R}P^k$, secondly, the fact that $\mathbb{R}P^k \smallsetminus \mathbb{R}P^{k-1}$ is an open k-cell. Use the presentation $\mathbb{C}P^n = \bigcup_{k=0}^n \mathbb{C}P^k$. Prove that for all integer $k \geq 0$ the difference $\mathbb{C}P^k \smallsetminus \mathbb{C}P^{k-1} \cong B^{2k}$. Furthermore, it is clear that the attaching map $S^{2k-1} \to \mathbb{C}P^{k-1}$ is the factorization map.

40.6 (a) Delete from the square a set homeomorphi to the open disk and bounded by a curve starting and ending at a certain vertex of the square I^2. The rest splits into 10 cells, and the quotient space of the complement splits into 5 cells and is homeomorphi to a handle.
(b) The Möbius strip is the quotient space of the square, which has a cellular partition consisting of 9 cells. After factorization, we obtain a partition of the Möbius strip consisting of 6 cells.
(c) As well as the space in the preceding item, $S^1 \times I$ is a quotient space of the square. Or, differently, see *40.3*.
(d)–(e) See *40.12*.

40.7 (a) 4 cells: present the Möbius strip as a result of factorization of a triangle under which all three vertices are identified into one. Show that one 1-cell is insufficient.
(b) $2p+2$ cells; (c) $q+2$ cells. See *40.12*. In order to show that this number

of cells is the smallest possible, use the computation of the fundamental groups of the above spaces, see 43°5.

40.8 We need at least three cells: a 0-cell, a 1-cell, and one more cell.

40.9 See *20.6*.

40.11 Notice that since any two points in $\mathbb{R}^\infty$ lie in a certain subspace $\mathbb{R}^N$, the distance between them is easy to define. Thus, we have a metric in $\mathbb{R}^\infty$, but it generates in $\mathbb{R}^\infty$ a wrong topology. To show that the topology in $\mathbb{R}^\infty$ is not generated by any metric, use the fact that $\mathbb{R}^\infty$ is not first countable (prove this).

40.12 We prove several assertions in this list.
(a) The word aa^{-1} describes the quotient space of D^2 by the partition into pairs of points of S^1 that are symmetric with respect to one of the diameters. This quotient space is homeomorphi to S^2. The cellular partition has two 0-cells, a 1-cell, and a 2-cell.
(b) The word aa describes the quotient space of D^2 by the partition into pairs of centrally symmetric points of the circle (and singletons formed by the remaining points). It is homeomorphi to the projective plane. The cellular partition consists of three cells: a 0-cell, a 1-cell, and a 2-cell.
(g) Consider the p-gon P with vertices at the common endpoints of the pairs of edges marked by a_1 and b_p^{-1}, a_2 and b_1^{-1}, …, a_p and b_{p-1}^{-1}, and cut the initial $4p$-gon along the sides of P. Factorizing P, we obtain a sphere with p holes. Factorizing the remaining pentagons, we obtain p handles.

40.13 For example, consider the so-called complete 5-graph K_5, i.e., the space with 5 vertices pairwise joined by edges. To prove that it cannot be embedded in $\mathbb{R}^2$, use the Euler Theorem *42.3*.

41.1x Let $\psi : D^n \to X$ be the characteristic map of the attached cell, let $i : A \to X$ be the inclusion. We can assume that $x = \psi(0)$, where 0 is the center of D^n. We introduce the map

$$g : X \smallsetminus x \to A : \ g(z) = \begin{cases} z & \text{if } z \in A, \\ \varphi(\psi^{-1}(z)/|\psi^{-1}(z)|) & \text{if } z \notin A. \end{cases}$$

We prove that the maps $\mathrm{id}_{X \smallsetminus x}$ and $i \circ g$ are A-homotopic. Consider the rectilinear homotopy $\widetilde{h} : (D^n \smallsetminus x) \times I \to D^n \smallsetminus x$ between the identity map and the projection $\rho : D^n \smallsetminus x \to D^n \smallsetminus x : z \mapsto \frac{z}{|z|}$. We define the homotopy

$$h : (A \sqcup (D^n \smallsetminus x)) \times I \to A \sqcup (D^n \smallsetminus x)$$

by letting

$$h(z,t) = \begin{cases} z & \text{if } z \in A, \\ \widetilde{h}(z,t) & \text{if } z \in D^n. \end{cases}$$

The quotient map $H : (X \smallsetminus x) \times I \to X \smallsetminus x$ of h is the required A-homotopy between $\mathrm{id}_{X \smallsetminus A}$ and $i \circ g$.

41.2x This follows from *41.1x* because closed n-cells together with X_{n-1} constitute a fundamental cover of X.

41.3x The assertion on $\mathbb{R}P^n$ follows from *41.1x* because $\mathbb{R}P^n$ is a result of attaching an n-cell to $\mathbb{R}P^{n-1}$, see *40.H*. The assertion about $\mathbb{C}P^n$ is proved in a similar way; see *40.5*. On the other hand, try to find explicit formulas for deformation retractions $\mathbb{R}P^n \smallsetminus \text{point} \to \mathbb{R}P^{n-1}$ and $\mathbb{C}P^n \smallsetminus \text{point} \to \mathbb{C}P^{n-1}$.

41.4x Consider a cellular partition of the solid torus that has one 3-cell and 2-skeleton homeomorphic to a torus with a disk attached along the meridian $S^1 \times 1$, and apply assertion *41.1x*.

41.5x Denote by $e_\varphi : D^{n+1} \to X_\varphi$ and $e_\psi : D^{n+1} \to X_\psi$ the characteristic maps of the $(n+1)$-cell attached to Y. Let $h : S^n \times I \to Y$ be a homotopy joining φ and ψ. Consider the maps $f' : Y \sqcup D^{n+1} \to X_\varphi$ and $g' : Y \sqcup D^{n+1} \to X_\psi$ that are the standard embeddings on Y, and are defined on the disks D^{n+1} by the formulas

$$f'(x) = \begin{cases} e_\psi(2x) & \text{for } |x| \le \frac{1}{2}, \\ h\left(\frac{x}{|x|}, 2(1-|x|)\right) & \text{for } \frac{1}{2} \le |x| \le 1, \end{cases}$$

$$g'(x) = \begin{cases} e_\varphi(2x) & \text{for } |x| \le \frac{1}{2}, \\ h\left(\frac{x}{|x|}, 2|x|-1\right) & \text{for } \frac{1}{2} \le |x| \le 1, \end{cases}$$

We easily see that the quotient maps $f : X_\varphi \to X_\psi$ and $g : X_\psi \to X_\varphi$ of f' and g' are defined. Show that f and g are mutually inverse homotopy equivalences.

41.6x Slightly modify the argument used in the solution of Problem *41.5x*.

41.7x Let A be the space obtained by attaching a disk to the circle via the map $\alpha : S^1 \to S^1$, $\alpha(z) = z^2$. Then $A \cong \mathbb{R}P^2$, whence $\pi_1(A) \cong \mathbb{Z}_2$. Consequently, the map $\varphi : S^1 \to A : \ z \mapsto z^3$ is homotopic to $\psi = \mathrm{id}_{S^1}$. By *41.5x*, X is homotopy equivalent to the space $A \cup_\psi D^2$, which coincides with $D^2 \cup_\alpha D^2$. Since the map $\alpha : S^1 \to D^2$ is null-homotopic, it follows (also by *41.5x*) that X is homotopy equivalent to the bouquet $D^2 \vee S^2$, which is homotopy equivalent to S^2:

$$X \simeq A \cup_\psi D^2 \simeq D^2 \cup_\alpha D^2 \simeq D^2 \vee S^2 \simeq S^2.$$

The sphere has a partition consisting of two cells, which, obviously, is the smallest possible number of cells.

41.9x The torus $S^1 \times S^1$ is obtained from the bouquet $S^1 \vee S^1$ by attaching a 2-cell via a certain map $\varphi : S^1 \to S^1 \vee S^1$. Denote by i the inclusion $S^1 \vee S^1 \to A = (1 \times S^1) \cup (D^2 \times 1)$ and show that the composition $i \circ \varphi : S^1 \to A$ is null-homotopic. Indeed, let α, β be the standard generators of $\pi_1(S^1 \vee S^1)$. Then $[\varphi] = \alpha\beta\alpha^{-1}\beta^{-1}$, and

$$[i\circ\varphi] = i_*([\varphi]) = i_*(\alpha\beta\alpha^{-1}\beta^{-1}) = i_*(\alpha)i_*(\beta)i_*(\alpha)^{-1}i_*(\beta)^{-1} = i_*(\alpha)i_*(\alpha)^{-1} = 1,$$

because $i_*(\beta) = 1 \in \pi_1(A)$. By Theorem *41.5x*,

$$A \cup_\varphi D^2 \simeq A \vee S^2 = S^1 \vee D^2 \vee S^2 \simeq S^1 \vee S^2.$$

41.10x Use the result of Problem *41.9x* and assertion *41.5x*.

41.11x Prove that $X \simeq S^1 \vee S^1 \vee S^2$, whence $\pi_1(X) \cong \mathcal{F}_2$, while $Y \simeq S^1 \times S^1$, so that $\pi_1(Y) \cong \mathbb{Z}^2$. Since $\pi_1(X) \not\cong \pi_1(Y)$, X and Y are not homotopy equivalent.

41.13x Consider a cellular partition of $\mathbb{C}P^2$ consisting of one 0-cell, one 1-cell, two 2-cells, and one 4-cell. Furthermore, we can assume that the 2-skeleton of the cellular space obtained is $\mathbb{C}P^1 \subset \mathbb{C}P^2$, while the 1-skeleton is the real part $RP^1 \subset \mathbb{C}P^1$. Let $\tau : \mathbb{C}P^2 \to \mathbb{C}P^2$ be the involution of complex conjugation, by which we factorize. Clearly, $\mathbb{C}P^1/_{[z \sim \tau(z)]} \cong D^2$. Consider the characteristic map $\psi : D^4 \to \mathbb{C}P^1$ of the 4-cell of the initial cellular partition. The quotient space $D^4/_{[z \sim \tau(z)]}$ is obviously homeomorphi to D^4. Therefore, the quotient map

$$D^4/_{[z \sim \tau(z)]} \to \mathbb{C}P^1/_{[z \sim \tau(z)]}$$

is the characteristic map for the 4-cell of X. Thus, X is a cellular space with 2-skeleton D^2. Therefore, by *41.Cx*, we have $X \simeq S^4$.

42.1 See *38.21*.

42.2 Let $X \cong S^2$. Denote by $v = c_0(X)$, $e = c_1(X)$, and $f = c_2(X)$ the number of 0-, 1-, and 2-cells in X, respectively. Deleting a point in each 2-cell of X, we obtain a space X' admitting a deformation retraction to its 1-skeleton. On the one hand, by *42.1*, $\pi_1(X')$ is a free group of rang $f - 1$. On the other hand, we have $\pi_1(X') \cong \pi_1(X_1)$, and the rang of the latter group is equal to $1 - \chi(X_1) = 1 - v + e$ by *42.B*. Thus, $f - 1 = 1 - e + v$, whence it follows that $\chi(X) = v - e + f = 2$.

42.3 This follows from *42.2*.

43.1 The fundamental group of S^n with $n > 1$ is trivial because there is a cellular partition of S^n with one-point 1-skeleton.

43.2 The group $\pi_1(\mathbb{C}P^n)$ is trivial for the same reason.

43.1x Take a point (x_0 and x_1) in each connected component of C so that we could join them in the 1-skeleton X_1 by two embedded segments $\overline{e}_A \subset A$ and $\overline{e}_B \subset B$, whose only common points are x_0 and x_1. The idea is to replace all the spaces by homotopy equivalent ones so that the 1-skeleton of X be the circle formed by the segments $\overline{e}_A$ and $\overline{e}_B$. For this purpose, we can use the techniques used in the solution of Problem *41.Fx*. As a result, we obtain a space having 1-skeleton with fundamental group isomorphic to $\mathbb{Z}$. It remains to observe that the image of the attaching map φ of a 2-cell cannot be the whole 1-skeleton since this cell lies either in A, or B, but not in both. Therefore, φ is null-homotopic, and, consequently, when we attach a 2-cell, no relations arise.

43.2x No, because in Theorem *43.Ax* the sets A and B are open in X, while in Theorem *43.2x* they are cellular subspaces, which are open only in exceptional cases. On the other hand, we can derive Theorem *43.Cx* from *43.Ax* if we construct neighborhoods of the cellular subspaces A, B, and C that admit deformation retractions to the spaces themselves.

43.3x Generally speaking, no, it may not (give an example).

43.4x Let us see how the fundamental group changes when we attach 2-cells to the 1-skeleton of X. We assume that the 0-skeleton is $\{x_0\}$. At the first step, we attach a 2-cell e to X_1, let $\varphi : S^1 \to X_1$ be the attaching map, and let $\chi : D^2 \to X_2$ be the characteristic map of e. Let $F \subset D^2$ be a closed disk (for example, of radius $\frac{1}{2}$), S the boundary of F, $A = \chi(D^2 \smallsetminus \operatorname{Int} F) \cup X_1$, $B = \chi(F)$, then $C = \chi(S) \cong S^1$. It is clear that X_1 is a (strong) deformation retract of the set A. Therefore, the group $\pi_1(A) \cong \pi_1(X_1)$ is a free group with generators α_i. On the other hand, we have $B \cong D^2$. Therefore, B is simply connected. The map $\chi|_S$ is homotopic to φ, consequently, the image of the generator of $\pi_1(C)$ is the class $\rho = [\varphi] \in \pi_1(X, x_0)$ of the attaching map of e. Consequently, in the fundamental group $\pi_1(X, x_0)$ there is a relation $\rho = 1$. When we attach cells of the highest dimension, no new relations on this group arise, because in this case the space $C \cong S^k$ is simply connected since $k > 1$. The Seifert–van Kampen theorem implies that the relations $[\varphi_i] = 1$ exhaust all relations between the standard generators of the fundamental group of the space.

43.5x If $m \neq 0$, then the fundamental group is a cyclic group of order $|m|$; if $m = 0$, then the fundamental group is isomorphic to $\mathbb{Z}$.

43.6x These spaces are homeomorphic to $S^2 \times S^1$ and S^3, respectively.

43.7x Instead of the complement of K, we consider the complement of a certain open neighborhood U of K homeomorphi to $\operatorname{Int} D^2 \times S^1$, for which K is the axial circle. It is more convenient to assume that all sets under consideration lie not in $\mathbb{R}^3$, but in S^3. Let $X = S^3 \smallsetminus U$. The torus T

splits S^3 into two solid tori $G = D^2 \times S^1$ and $F = S^1 \times D^2$. Put $A = G \smallsetminus U$ and $B = F \smallsetminus U$. Then $X = A \cup B$, and $C = A \cap B$ is the complement in T of the open strip, which is a neighborhood of the curve determined on T by the equation $pu = qv$, whence $\pi_1(C) \cong \pi_1(A) \cong \pi_1(B) \cong \mathbb{Z}$. By the Seifert–van Kampen Theorem, we have $\pi_1(X) = \langle \alpha, \beta \mid i_*(\gamma) = j_*(\gamma) \rangle$, where i and j are the inclusions $i : C \to A$ and $j : C \to B$. The loop in C representing the generator of $\pi_1(C)$ p times passes the torus along the parallel and q times along the meridian, whence $i_*(\gamma) = a^p$ and $j_*(\gamma) = b^q$. Therefore, $\pi_1(X) = \langle a, b \mid a^p = b^q \rangle$. Show that $H_1(X) \cong \mathbb{Z}$ (do not forget that p and q are co-prime).

***43.8*x** (a) This immediately follows from Theorem 43 (or Theorem *43.C*x). (b) Since the sets $A = X \vee V_{y_0}$ and $B = U_{x_0} \vee Y$ constitute an open cover of Z and their intersection $A \cap B = U_{x_0} \vee V_{y_0}$ is connected, we see that the fact that Z is simply connected follows from the result of Problem *31.11*.
(c)* Let $X \subset \mathbb{R}^3$ be the cone with vertex $(-1, 0, 1)$ over the union of the circles determined in the plane $\mathbb{R}^2$ by the equations $x^2 + \frac{2x}{n} + y^2 = 0$, $n \in \mathbb{N}$, and let Y be symmetric to X with respect to the z axis. Both X and Y are obviously contractible and, therefore, simply connected. Try to prove (this is not easy at all) that their union $X \cup Y$ is not simply connected.

***43.9*x** Yes, it is.

***43.10*x** The Klein bottle is a union of two Möbius strips pasted together along their the boundary circles.

***43.13*x** Verify that the class of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has order 2, and the class of $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ has order 3.

***43.14*x** Let us cut the torus (respectively, the Klein bottle) along a circle B so that as a result we obtain a cylinder, which will be our space C. Denote by β the generator of $\pi_1(B) \cong \mathbb{Z}$, and by α the generator of $\pi_1(C) \cong \mathbb{Z}$. In the case of torus, we have $\varphi_1 = \varphi_2 = \alpha$, while for the Klein bottle we have $\varphi_1 = \alpha = \varphi_2^{-1}$. Thus, by Theorem *43.F*x, we obtain a presentation of the fundamental group of the torus $\langle \alpha, \gamma \mid \gamma\alpha = \alpha\gamma \rangle$ and of the Klein bottle $\langle \alpha, \gamma \mid \gamma\alpha = \alpha\gamma^{-1} \rangle$.

***55.1*x** The construction from the proof of Theorem *55.D*x provides a covering with the required properties.

***55.2*x** Prove that for arbitrary covering of this sort there exist a splitting to a covering in the narrow sense of a handle and the trivial covering of the rest, see the proof of *55.D*x. Use such splittings to construct the homeomorphisms.

55.3x The simplest example is a pair of coverings $S^1 \times S^1 \to S^1 \times S^1$ defined by formulas $(z, w) \mapsto (z^4, w)$ and $(z, w) \mapsto (z^2, w^2)$ with automorphism groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectfully.

55.4x Yes, it covers a sphere with three crosscaps via the orientation covering. Another way to obtain the covering is to consider factorization by the action of symmetry with respect to a point. For this observe that the two handles can be attached to sphere in a symmetric way. Prove that the orbit space of the symmetry is non-orientable.