

Part 2

Algebraic Topology

This part of the book can be considered an introduction to algebraic topology. The latter is a part of topology which relates topological and algebraic problems. The relationship is used in both directions, but the reduction of topological problems to algebra is more useful at first stages because algebra is usually easier.

The relation is established according to the following scheme. One invents a construction that assigns to each topological space X under consideration an algebraic object $A(X)$. The latter may be a group, a ring, a space with a quadratic form, an algebra, etc. Another construction assigns to a continuous mapping $f : X \rightarrow Y$ a homomorphism $A(f) : A(X) \rightarrow A(Y)$. The constructions satisfy natural conditions (in particular, they form a functor), which make it possible to relate topological phenomena with their algebraic images obtained via the constructions.

There is an immense number of useful constructions of this kind. In this part we deal mostly with one of them which, historically, was the first one: the fundamental group of a topological space. It was invented by Henri Poincaré in the end of the XIXth century.

Fundamental Group

§29 Homotopy

§29°1 Continuous Deformations of Maps

29.A. Is it possible to deform continuously:

- (a) the identity map $\text{id} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to the constant map $\mathbb{R}^2 \rightarrow \mathbb{R}^2 : x \mapsto 0$,
- (b) the identity map $\text{id} : S^1 \rightarrow S^1$ to the symmetry $S^1 \rightarrow S^1 : x \mapsto -x$ (here x is considered a complex number because the circle S^1 is $\{x \in \mathbb{C} : |x| = 1\}$),
- (c) the identity map $\text{id} : S^1 \rightarrow S^1$ to the constant map $S^1 \rightarrow S^1 : x \mapsto 1$,
- (d) the identity map $\text{id} : S^1 \rightarrow S^1$ to the two-fold wrapping $S^1 \rightarrow S^1 : x \mapsto x^2$,
- (e) the inclusion $S^1 \rightarrow \mathbb{R}^2$ to a constant map,
- (f) the inclusion $S^1 \rightarrow \mathbb{R}^2 \setminus 0$ to a constant map?

29.B. Riddle. When you (tried to) solve the previous problem, what did you mean by “*deform continuously*”?

The present section is devoted to the notion of *homotopy* formalizing the naive idea of continuous deformation of a map.

§29°2 Homotopy as Map and Family of Maps

Let f and g be two continuous maps of a topological space X to a topological space Y , and $H : X \times I \rightarrow Y$ a continuous map such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for any $x \in X$. Then f and g are *homotopic*, and H is a *homotopy* between f and g .

For $x \in X$, $t \in I$ denote $H(x, t)$ by $h_t(x)$. This change of notation results in a change of the point of view of H . Indeed, for a fixed t the formula $x \mapsto h_t(x)$ determines a map $h_t : X \rightarrow Y$, and H becomes a family of maps h_t enumerated by $t \in I$.

29.C. Each h_t is continuous.

29.D. Does continuity of all h_t imply continuity of H ?

The conditions $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ in the above definition of a homotopy can be reformulated as follows: $h_0 = f$ and

$h_1 = g$. Thus a homotopy between f and g can be regarded as a family of continuous maps that connects f and g . Continuity of a homotopy allows us to say that it is a *continuous family of continuous maps* (see §29°10).

§29°3 Homotopy as Relation

29.E. *Homotopy of maps is an equivalence relation.*

29.E.1. If $f : X \rightarrow Y$ is a continuous map, then $H : X \times I \rightarrow Y : (x, t) \mapsto f(x)$ is a homotopy between f and f .

29.E.2. If H is a homotopy between f and g , then H' defined by $H'(x, t) = H(x, 1 - t)$ is a homotopy between g and f .

29.E.3. If H is a homotopy between f and f' and H' is a homotopy between f' and f'' , then H'' defined by

$$H''(x, t) = \begin{cases} H(x, 2t) & \text{if } t \in [0, \frac{1}{2}], \\ H'(x, 2t - 1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

is a homotopy between f and f'' .

Homotopy, being an equivalence relation by 29.E, splits the set $\mathcal{C}(X, Y)$ of all continuous mappings from a space X to a space Y into equivalence classes. The latter are *homotopy classes*. The set of homotopy classes of all continuous maps $X \rightarrow Y$ is denoted by $\pi(X, Y)$. Map homotopic to a constant map are said to be *null-homotopic*.

29.1. Prove that for any X , the set $\pi(X, I)$ has a single element.

29.2. Prove that two constant mappings $Z \rightarrow X$ are homotopic iff their images lie in one path-connected component of X .

29.3. Prove that the number of elements of $\pi(I, Y)$ is equal to the number of path connected components of Y .

§29°4 Rectilinear Homotopy

29.F. Any two continuous maps of the same space to \mathbb{R}^n are homotopic.

29.G. Solve the preceding problem by proving that for continuous maps $f, g : X \rightarrow \mathbb{R}^n$ formula $H(x, t) = (1 - t)f(x) + tg(x)$ determines a homotopy between f and g .

The homotopy defined in 29.G is a *rectilinear* homotopy.

29.H. Any two continuous maps of an arbitrary space to a convex subspace of \mathbb{R}^n are homotopic.

§29°5 Maps to Star-Shaped Sets

A set $A \subset \mathbb{R}^n$ is *star-shaped* if there exists a point $b \in A$ such that for any $x \in A$ the whole segment $[a, x]$ connecting x to a is contained in A . The point a is the *center* of the star. (Certainly, the center of the star is not uniquely determined.)

29.4. Prove that any two continuous maps of a space to a star-shaped subspace of \mathbb{R}^n are homotopic.

§29°6 Maps of Star-Shaped Sets

29.5. Prove that any continuous map of a star-shaped set $C \subset \mathbb{R}^n$ to any space is null-homotopic.

29.6. Under what conditions (formulated in terms of known topological properties of a space X) any two continuous maps of any star-shaped set to X are homotopic?

§29°7 Easy Homotopies

29.7. Prove that each non-surjective map of any topological space to S^n is null-homotopic.

29.8. Prove that any two maps of a one-point space to $\mathbb{R}^n \setminus 0$ with $n > 1$ are homotopic.

29.9. Find two nonhomotopic maps from a one-point space to $\mathbb{R} \setminus 0$.

29.10. For various m, n , and k , calculate the number of homotopy classes of maps $\{1, 2, \dots, m\} \rightarrow \mathbb{R}^n \setminus \{x_1, x_2, \dots, x_k\}$, where $\{1, 2, \dots, m\}$ is equipped with discrete topology.

29.11. Let f and g be two maps from a topological space X to $\mathbb{C} \setminus 0$. Prove that if $|f(x) - g(x)| < |f(x)|$ for any $x \in X$, then f and g are homotopic.

29.12. Prove that for any polynomials p and q over \mathbb{C} of the same degree in one variable there exists $r > 0$ such that for any $R > r$ formulas $z \mapsto p(z)$ and $z \mapsto q(z)$ determine maps of the circle $\{z \in \mathbb{C} : |z| = R\}$ to $\mathbb{C} \setminus 0$ and these maps are homotopic.

29.13. Let f, g be maps of an arbitrary topological space X to S^n . Prove that if $|f(a) - g(a)| < 2$ for each $a \in X$, then f is homotopic to g .

29.14. Let $f : S^n \rightarrow S^n$ be a continuous map. Prove that if it is fixed-point-free, i.e., $f(x) \neq x$ for every $x \in S^n$, then f is homotopic to the symmetry $x \mapsto -x$.

§29°8 Two Natural Properties of Homotopies

29.I. Let $f, f' : X \rightarrow Y$, $g : Y \rightarrow B$, $h : A \rightarrow X$ be continuous maps and $F : X \times I \rightarrow Y$ a homotopy between f and f' . Prove that then $g \circ F \circ (h \times \text{id}_I)$ is a homotopy between $g \circ f \circ h$ and $g \circ f' \circ h$.

29.J. Riddle. Under conditions of 29.I, define a natural mapping

$$\pi(X, Y) \rightarrow \pi(A, B).$$

How does it depend on g and h ? Write down all nice properties of this construction.

29.K. Prove that two maps $f_0, f_1 : X \rightarrow Y \times Z$ are homotopic iff $\text{pr}_Y \circ f_0$ is homotopic to $\text{pr}_Y \circ f_1$ and $\text{pr}_Z \circ f_0$ is homotopic to $\text{pr}_Z \circ f_1$.

§29°9 Stationary Homotopy

Let A be a subset of X . A homotopy $H : X \times I \rightarrow Y$ is *fixed* or *stationary* on A , or, briefly, an *A-homotopy* if $H(x, t) = H(x, 0)$ for all $x \in A, t \in I$. Two maps connected by an *A-homotopy* are *A-homotopic*.

Certainly, any two *A-homotopic* maps coincide on A . If we want to emphasize that a homotopy is not assumed to be fixed, then we say that it is *free*. If we want to emphasize the opposite (that the homotopy is fixed), then we say that it is *relative*.¹

29.L. Prove that, like free homotopy, *A-homotopy* is an equivalence relation.

The classes into which *A-homotopy* splits the set of continuous maps $X \rightarrow Y$ that agree on A with a map $f : A \rightarrow Y$ are *A-homotopy classes of continuous extensions of f to X* .

29.M. For what A is a rectilinear homotopy fixed on A ?

§29°10 Homotopies and Paths

Recall that a *path* in a space X is a continuous mapping from the segment I to X . (See Section §13.)

29.N. Riddle. In what sense is any path a homotopy?

29.O. Riddle. In what sense does any homotopy consist of paths?

29.P. Riddle. In what sense is any homotopy a path?

Recall that the *compact-open topology* in $\mathcal{C}(X, Y)$ is the topology generated by the sets $\{\varphi \in \mathcal{C}(X, Y) \mid \varphi(A) \subset B\}$ for compact $A \subset X$ and open $B \subset Y$.

29.15. Prove that any homotopy $h_t : X \rightarrow Y$ determines (see §29°2) a path in $\mathcal{C}(X, Y)$ with compact-open topology.

29.16. Prove that if X is locally compact and regular, then any path in $\mathcal{C}(X, Y)$ with compact-open topology determines a homotopy.

¹Warning: there is a similar, but different kind of homotopy, which is also called relative.

§29°11 Homotopy of Paths

29.Q. Prove that two paths in a space X are freely homotopic iff their images belong to the same path-connected component of X .

This shows that the notion of free homotopy in the case of paths is not interesting. On the other hand, there is a sort of relative homotopy playing a very important role. This is $(0 \cup 1)$ -homotopy. This causes the following commonly accepted deviation from the terminology introduced above: homotopy of paths always means not a free homotopy, but a homotopy fixed on the endpoints of I (i.e., on $0 \cup 1$).

Notation: a homotopy class of a path s is denoted by $[s]$.

§30 Homotopy Properties of Path Multiplication

§30°1 Multiplication of Homotopy Classes of Paths

Recall (see Section §13) that two paths u and v in a space X can be multiplied, provided the initial point $v(0)$ of v is the final point $u(1)$ of u . The product uv is defined by

$$uv(t) = \begin{cases} u(2t) & \text{if } t \in [0, \frac{1}{2}], \\ v(2t - 1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

30.A. *If a path u is homotopic to u' , a path v is homotopic to v' , and there exists the product uv , then $u'v'$ exists and is homotopic to uv .*

Define the product of homotopy classes of paths u and v as the homotopy class of uv . So, $[u][v]$ is defined as $[uv]$, provided uv is defined. This is a definition requiring a proof.

30.B. *The product of homotopy classes of paths is well defined.*²

§30°2 Associativity

30.C. Is multiplication of paths associative?

Certainly, this question might be formulated in more detail as follows.

30.D. Let u , v , and w be paths in a certain space such that products uv and vw are defined (i.e., $u(1) = v(0)$ and $v(1) = w(0)$). Is it true that $(uv)w = u(vw)$?

30.1. Prove that for paths in a metric space $(uv)w = u(vw)$ implies that u , v , and w are constant maps.

30.2. Riddle. Find nonconstant paths u , v , and w in an indiscrete space such that $(uv)w = u(vw)$.

30.E. *Multiplication of homotopy classes of paths is associative.*

30.E.1. Reformulate Theorem 30.E in terms of paths and their homotopies.

30.E.2. Find a map $\varphi : I \rightarrow I$ such that if u , v , and w are paths with $u(1) = v(0)$ and $v(1) = w(0)$, then $((uv)w) \circ \varphi = u(vw)$.

30.E.3. Any path in I starting at 0 and ending at 1 is homotopic to $\text{id} : I \rightarrow I$.

30.E.4. Let u , v and w be paths in a space such that products uv and vw are defined (thus, $u(1) = v(0)$ and $v(1) = w(0)$). Then $(uv)w$ is homotopic to $u(vw)$.

²Of course, when the initial point of paths in the first class is the final point of paths in the second class.

If you want to understand the essence of 30.E, then observe that the paths $(uv)w$ and $u(vw)$ have the same trajectories and differ only by the time spent in different fragments of the path. Therefore, in order to find a homotopy between them, we must find a continuous way to change one schedule to the other. The lemmas above suggest a formal way of such a change, but the same effect can be achieved in many other ways.

30.3. Present explicit formulas for the homotopy H between the paths $(uv)w$ and $u(vw)$.

§30°3 Unit

Let a be a point of a space X . Denote by e_a the path $I \rightarrow X : t \mapsto a$.

30.F. Is e_a a unit for multiplication of paths?

The same question in more detailed form:

30.G. For a path u with $u(0) = a$ is $e_a u = u$? For a path v with $v(1) = a$ is $ve_a = v$?

30.4. Prove that if $e_a u = u$ and the space satisfies the first separation axiom, then $u = e_a$.

30.H. *The homotopy class of e_a is a unit for multiplication of homotopy classes of paths.*

§30°4 Inverse

Recall that for a path u there is the inverse path $u^{-1} : t \mapsto u(1-t)$ (see Section §13).

30.I. Is the inverse path inverse with respect to multiplication of paths?

In other words:

30.J. For a path u beginning in a and finishing in b , is it true that $uu^{-1} = e_a$ and $u^{-1}u = e_b$?

30.5. Prove that for a path u with $u(0) = a$ equality $uu^{-1} = e_a$ implies $u = e_a$.

30.K. *For any path u , the homotopy class of the path u^{-1} is inverse to the homotopy class of u .*

30.K.1. Find a map $\varphi : I \rightarrow I$ such that $uu^{-1} = u \circ \varphi$ for any path u .

30.K.2. Any path in I that starts and finishes at 0 is homotopic to the constant path $e_0 : I \rightarrow I$.

We see that from the algebraic point of view multiplication of paths is terrible, but it determines multiplication of homotopy classes of paths, which has nice algebraic properties. The only unfortunate property is that the multiplication of homotopy classes of paths is defined not for any two classes.

30.L. Riddle. How to select a subset of the set of homotopy classes of paths to obtain a group?

§31 Fundamental Group

§31°1 Definition of Fundamental Group

Let X be a topological space, x_0 its point. A path in X which starts and ends at x_0 is a *loop* in X at x_0 . Denote by $\Omega_1(X, x_0)$ the set of loops in X at x_0 . Denote by $\pi_1(X, x_0)$ the set of homotopy classes of loops in X at x_0 .

Both $\Omega_1(X, x_0)$ and $\pi_1(X, x_0)$ are equipped with a multiplication.

31.A. For any topological space X and a point $x_0 \in X$ the set $\pi_1(X, x_0)$ of homotopy classes of loops at x_0 with multiplication defined above in Section §30 is a group.

$\pi_1(X, x_0)$ is the *fundamental group* of the space X with base point x_0 . It was introduced by Poincaré, and this is why it is also called the *Poincaré group*. The letter π in this notation is also due to Poincaré.

§31°2 Why Index 1?

The index 1 in the notation $\pi_1(X, x_0)$ appeared later than the letter π . It is related to one more name of the fundamental group: the first (or one-dimensional) homotopy group. There is an infinite series of groups $\pi_r(X, x_0)$ with $r = 1, 2, 3, \dots$ the fundamental group being one of them. The higher-dimensional homotopy groups were defined by Witold Hurewicz in 1935, thirty years after the fundamental group was defined. Roughly speaking, the general definition of $\pi_r(X, x_0)$ is obtained from the definition of $\pi_1(X, x_0)$ by replacing I with the cube I^r .

31.B. Riddle. How to generalize problems of this section in such a way that in each of them I would be replaced by I^r ?

There is even a “zero-dimensional homotopy group” $\pi_0(X, x_0)$, but it is not a group, as a rule. It is the set of path-connected components of X . Although there is no natural multiplication in $\pi_0(X, x_0)$, unless X is equipped with some special additional structures, there is a natural unit in $\pi_0(X, x_0)$. This is the component containing x_0 .

§31°3 Circular loops

Let X be a topological space, x_0 its point. A continuous map $l : S^1 \rightarrow X$ such that³ $l(1) = x_0$ is a (*circular*) *loop* at x_0 . Assign to each circular loop l the composition of l with the exponential map $I \rightarrow S^1 : t \mapsto e^{2\pi it}$. This is a usual loop at the same point.

31.C. Prove that any loop can be obtained in this way from a circular loop.

³Recall that S^1 is regarded as a subset of the plane R^2 , and the latter is identified in a canonical way with \mathbb{C} . Hence, $1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Two circular loops l_1 and l_2 are *homotopic* if they are 1-homotopic. A homotopy of a circular loop not fixed at x_0 is a *free* homotopy.

31.D. Prove that two circular loops are homotopic iff the corresponding ordinary loops are homotopic.

31.1. What kind of homotopy of loops corresponds to free homotopy of circular loops?

31.2. Describe the operation with circular loops corresponding to the multiplication of paths.

31.3. Let U and V be the circular loops with common base point $U(1) = V(1)$ corresponding to the loops u and v . Prove that the circular loop

$$z \mapsto \begin{cases} U(z^2) & \text{if } \operatorname{Im}(z) \geq 0, \\ V(z^2) & \text{if } \operatorname{Im}(z) \leq 0 \end{cases}$$

corresponds to the product of u and v .

31.4. Outline a construction of fundamental group using circular loops.

§31°4 The Very First Calculations

31.E. Prove that $\pi_1(\mathbb{R}^n, 0)$ is a trivial group (i.e., consists of one element).

31.F. Generalize 31.E to the situations suggested by 29.H and 29.4.

31.5. Calculate the fundamental group of an indiscrete space.

31.6. Calculate the fundamental group of the quotient space of disk D^2 obtained by identification of each $x \in D^2$ with $-x$.

31.7. Prove that if a two-point space X is path-connected, then X is simply connected.

31.G. Prove that $\pi_1(S^n, (1, 0, \dots, 0))$ with $n \geq 2$ is a trivial group.

Whether you have solved 31.G or not, we recommend you to consider problems 31.G.1, 31.G.2, 31.G.4, 31.G.5, and 31.G.6 designed to give an approach to 31.G, warn about a natural mistake and prepare an important tool for further calculations of fundamental groups.

31.G.1. Prove that any loop $s : I \rightarrow S^n$ that does not fill the entire S^n (i.e., $s(I) \neq S^n$) is null-homotopic, provided $n \geq 2$. (Cf. Problem 29.7.)

Warning: for any n there exists a loop filling S^n . See 9x:O.

31.G.2. Can a loop filling S^2 be null-homotopic?

31.G.3 Corollary of Lebesgue Lemma 16.W. Let $s : I \rightarrow X$ be a path, and Γ be an open cover of a topological space X . There exists a sequence of points $a_1, \dots, a_N \in I$ with $0 = a_1 < a_2 < \dots < a_{N-1} < a_N = 1$ such that $s([a_i, a_{i+1}])$ is contained in an element of Γ for each i .

31.G.4. Prove that if $n \geq 2$, then for any path $s : I \rightarrow S^n$ there exists a subdivision of I into a finite number of subintervals such that the restriction of s to each of the subintervals is homotopic to a map with nowhere-dense image via a homotopy fixed on the endpoints of the subinterval.

31.G.5. Prove that if $n \geq 2$, then any loop in S^n is homotopic to a non-surjective loop.

31.G.6. 1) Deduce *31.G* from *31.G.1* and *31.G.5*. 2) Find all points of the proof of *31.G* obtained in this way, where the condition $n \geq 2$ is used.

§31°5 Fundamental Group of Product

31.H. *The fundamental group of the product of topological spaces is canonically isomorphic to the product of the fundamental groups of the factors:*

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

31.8. Consider a loop $u : I \rightarrow X$ at x_0 , a loop $v : I \rightarrow Y$ at y_0 , and the loop $w = u \times v : I \rightarrow X \times Y$. We introduce the loops $u' : I \rightarrow X \times Y : t \mapsto (u(t), y_0)$ and $v' : I \rightarrow X \times Y : t \mapsto (x_0, v(t))$. Prove that $u'v' \sim w \sim v'u'$.

31.9. Prove that $\pi_1(\mathbb{R}^n \setminus 0, (1, 0, \dots, 0))$ is trivial if $n \geq 3$.

§31°6 Simply-Connectedness

A nonempty topological space X is *simply connected* (or *one-connected*) if X is path-connected and every loop in X is null-homotopic.

31.I. *For a path-connected topological space X , the following statements are equivalent:*

- (a) X is simply connected,
- (b) each continuous map $f : S^1 \rightarrow X$ is (freely) null-homotopic,
- (c) each continuous map $f : S^1 \rightarrow X$ extends to a continuous map $D^2 \rightarrow X$,
- (d) any two paths $s_1, s_2 : I \rightarrow X$ connecting the same points x_0 and x_1 are homotopic.

Theorem *31.I* is closely related to Theorem *31.J* below. Notice that since Theorem *31.J* concerns not all loops, but an individual loop, it is applicable in a broader range of situations.

31.J. *Let X be a topological space and $s : S^1 \rightarrow X$ be a circular loop. Then the following statements are equivalent:*

- (a) s is null-homotopic,
- (b) s is freely null-homotopic,
- (c) s extends to a continuous map $D^2 \rightarrow X$,
- (d) the paths $s_+, s_- : I \rightarrow X$ defined by formula $s_{\pm}(t) = s(e^{\pm\pi it})$ are homotopic.

31.J.1. Riddle. To prove that 4 statements are equivalent, we must prove at least 4 implications. What implications would you choose for the easiest proof of Theorem 31.J?

31.J.2. Does homotopy of circular loops imply that these circular loops are free homotopic?

31.J.3. A homotopy between a map of the circle and a constant map possesses a quotient map whose source space is homeomorphic to disk D^2 .

31.J.4. Represent the problem of constructing of a homotopy between paths s_+ and s_- as a problem of extension of a certain continuous map of the boundary of a square to a continuous of the whole square.

31.J.5. When we solve the extension problem obtained as a result of Problem 31.J.4, does it help to know that the circular loop $S^1 \rightarrow X : t \mapsto s(e^{2\pi it})$ extends to a continuous map of a disk?

31.10. Which of the following spaces are simply connected:

- | | | |
|----------------------------------|--------------------------|----------------------|
| (a) a discrete space; | (b) an indiscrete space; | (c) \mathbb{R}^n ; |
| (d) a convex set; | (e) a star-shaped set; | (f) S^n ; |
| (g) $\mathbb{R}^n \setminus 0$? | | |

31.11. Prove that if a topological space X is the union of two open simply connected sets U and V with path-connected intersection $U \cap V$, then X is simply connected.

31.12. Show that the assumption in 31.11 that U and V are open is necessary.

31.13*. Let X be a topological space, U and V its open sets. Prove that if $U \cup V$ and $U \cap V$ are simply connected, then so are U and V .

§31x°7 Fundamental Group of a Topological Group

Let G be a topological group. Given loops $u, v : I \rightarrow G$ starting at the unity $1 \in G$, let us define a loop $u \odot v : I \rightarrow G$ by the formula $u \odot v(t) = u(t) \cdot v(t)$, where \cdot denotes the group operation in G .

31x:A. Prove that the set $\Omega(G, 1)$ of all loops in G starting at 1 equipped with the operation \odot is a group.

31x:B. Prove that the operation \odot on $\Omega(G, 1)$ determines a group operation on $\pi_1(G, 1)$, which coincides with the standard group operation (determined by multiplication of paths).

31x:B.1. For loops $u, v \rightarrow G$ starting at 1, find $(ue_1) \odot (e_1v)$.

31x:C. The fundamental group of a topological group is Abelian.

31x:D. Formulate and prove analogs of Problems 31x:A and 31x:B for higher homotopy groups and $\pi_0(G, 1)$.

§31x°8 High Homotopy Groups

Let X be a topological space and x_0 its point. A continuous map $I^r \rightarrow X$ mapping the boundary ∂I^r of I^r to x_0 is a *spheroid of dimension* r of X at x_0 , or just an *r -spheroid*. Two r -spheroids are *homotopic* if they are ∂I^r -homotopic. For two r -spheroids u and v of X at x_0 , $r \geq 1$, define the product uv by the formula

$$uv(t_1, t_2, \dots, t_r) = \begin{cases} u(2t_1, t_2, \dots, t_r) & \text{if } t_1 \in [0, \frac{1}{2}], \\ v(2t_1 - 1, t_2, \dots, t_r) & \text{if } t_1 \in [\frac{1}{2}, 1]. \end{cases}$$

The set of homotopy classes of r -spheroids of a space X at x_0 is the r th (or r -dimensional) homotopy group $\pi_r(X, x_0)$ of X at x_0 . Thus,

$$\pi_r(X, x_0) = \pi(I^r, \partial I^r; X, x_0).$$

Multiplication of spheroids induces multiplication in $\pi_r(X, x_0)$, which makes $\pi_r(X, x_0)$ a group.

31x:E. Find $\pi_r(\mathbb{R}^n, 0)$.

31x:F. For any X and x_0 the group $\pi_r(X, x_0)$ with $r \geq 2$ is Abelian.

Similar to §31°3, higher-dimensional homotopy groups can be constructed not out of homotopy classes of maps $(I^r, \partial I^r) \rightarrow (X, x_0)$, but as

$$\pi(S^r, (1, 0, \dots, 0); X, x_0).$$

Another, also quite a popular way, is to define $\pi_r(X, x_0)$ as

$$\pi(D^r, \partial D^r; X, x_0).$$

31x:G. Construct natural bijections

$$\pi(I^r, \partial I^r; X, x_0) \rightarrow \pi(D^r, \partial D^r; X, x_0) \rightarrow \pi(S^r, (1, 0, \dots, 0); X, x_0)$$

31x:H. Riddle. For any X, x_0 and $r \geq 2$ present group $\pi_r(X, x_0)$ as the fundamental group of some space.

31x:I. Prove the following generalization of 31.H:

$$\pi_r(X \times Y, (x_0, y_0)) = \pi_r(X, x_0) \times \pi_r(Y, y_0).$$

§32 The Role of Base Point

§32°1 Overview of the Role of Base Point

Sometimes the choice of the base point does not matter, sometimes it is obviously crucial, sometimes this is a delicate question. In this section, we have to clarify all subtleties related to the base point. We start with preliminary formulations describing the subject in its entirety, but without some necessary details.

The role of the base point may be roughly described as follows:

- As the base point changes within the same path-connected component, the fundamental group remains in the same class of isomorphic groups.
- However, if the group is non-Abelian, it is impossible to find a natural isomorphism between the fundamental groups at different base points even in the same path-connected component.
- Fundamental groups of a space at base points belonging to different path-connected components have nothing to do to each other.

In this section these will be demonstrated. The proof involves useful constructions, whose importance extends far outside of the frameworks of our initial question on the role of base point.

§32°2 Definition of Translation Maps

Let x_0 and x_1 be two points of a topological space X , and let s be a path connecting x_0 with x_1 . Denote by σ the homotopy class $[s]$ of s . Define a map $T_s : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ by the formula $T_s(\alpha) = \sigma^{-1}\alpha\sigma$.

32.1. Prove that for any loop $a : I \rightarrow X$ representing $\alpha \in \pi_1(X, x_0)$ and any path $s : I \rightarrow X$ with $s(0) = x_0$ there exists a free homotopy $H : I \times I \rightarrow X$ between a and a loop representing $T_s(\alpha)$ such that $H(0, t) = H(1, t) = s(t)$ for $t \in I$.

32.2. Let $a, b : I \rightarrow X$ be loops homotopic via a homotopy $H : I \times I \rightarrow X$ such that $H(0, t) = H(1, t)$ (i.e., H is a free homotopy of loops: at each moment $t \in I$, it keeps the endpoints of the path coinciding). Set $s(t) = H(0, t)$ (hence, s is the path run through by the initial point of the loop under the homotopy). Prove that the homotopy class of b is the image of the homotopy class of a under $T_s : \pi_1(X, s(0)) \rightarrow \pi_1(X, s(1))$.

§32°3 Properties of T_s

32.A. T_s is a (group) homomorphism. ⁴

⁴Recall that this means that $T_s(\alpha\beta) = T_s(\alpha)T_s(\beta)$.

32.B. If u is a path connecting x_0 to x_1 and v is a path connecting x_1 with x_2 , then $T_{uv} = T_v \circ T_u$. In other words, the diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{T_u} & \pi_1(X, x_1) \\ & T_{uv} \searrow & \downarrow T_v \\ & & \pi_1(X, x_2) \end{array}$$

is commutative.

32.C. If paths u and v are homotopic, then $T_u = T_v$.

32.D. $T_{e_a} = \text{id} : \pi_1(X, a) \rightarrow \pi_1(X, a)$

32.E. $T_{s^{-1}} = T_s^{-1}$.

32.F. T_s is an isomorphism for any path s .

32.G. For any points x_0 and x_1 lying in the same path-connected component of X groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

In spite of the result of Theorem 32.G, we cannot write $\pi_1(X)$ even if the topological space X is path-connected. The reason is that although the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic, there may be no canonical isomorphism between them (see 32.J below).

32.H. The space X is simply connected iff X is path-connected and the group $\pi_1(X, x_0)$ is trivial for a certain point $x_0 \in X$.

§32°4 Role of Path

32.I. If a loop s represents an element σ of the fundamental group $\pi_1(X, x_0)$, then T_s is the inner automorphism of $\pi_1(X, x_0)$ defined by $\alpha \mapsto \sigma^{-1}\alpha\sigma$.

32.J. Let x_0 and x_1 be points of a topological space X belonging to the same path-connected component. The isomorphisms $T_s : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ do not depend on s iff $\pi_1(X, x_0)$ is an Abelian group.

Theorem 32.J implies that if the fundamental group of a topological space X is Abelian, we may simply write $\pi_1(X)$.

§32x°5 In Topological Group

In a topological group G there is another way to relate $\pi_1(G, x_0)$ with $\pi_1(G, x_1)$: there are homeomorphisms $L_g : G \rightarrow G : x \mapsto xg$ and $R_g : G \rightarrow G : x \mapsto gx$, so that there are the induced isomorphisms $(L_{x_0^{-1}x_1})_* : \pi_1(G, x_0) \rightarrow \pi_1(G, x_1)$ and $(R_{x_1x_0^{-1}})_* : \pi_1(G, x_0) \rightarrow \pi_1(G, x_1)$.

32x:A. Let G be a topological group, $s : I \rightarrow G$ be a path. Prove that

$$T_s = (L_{s(0)^{-1}s(1)})_* = (R_{s(1)s(0)^{-1}})_* : \pi_1(G, s(0)) \rightarrow \pi_1(G, s(1)).$$

32x:B. Deduce from 32x:A that the fundamental group of a topological group is Abelian (cf. 31x:C).

32x:1. Prove that the following spaces have Abelian fundamental groups:

- (a) the space of nondegenerate real $n \times n$ matrices $GL(n, \mathbb{R}) = \{A \mid \det A \neq 0\}$;
- (b) the space of orthogonal real $n \times n$ matrices $O(n, \mathbb{R}) = \{A \mid A \cdot ({}^t A) = \mathbb{E}\}$;
- (c) the space of special unitary complex $n \times n$ matrices $SU(n) = \{A \mid A \cdot ({}^t \bar{A}) = 1, \det A = 1\}$.

§32x°6 In High Homotopy Groups

32x:C. Riddle. Guess how T_s is generalized to $\pi_r(X, x_0)$ with any r .

Here is another form of the same question. We put it because its statement contains a greater piece of an answer.

32x:D. Riddle. Given a path $s : I \rightarrow X$ with $s(0) = x_0$ and a spheroid $f : I^r \rightarrow X$ at x_0 , how to cook up a spheroid at $x_1 = s(1)$ out of these?

32x:E. Let $s : I \rightarrow X$ be a path, $f : I^r \rightarrow X$ a spheroid with $f(\text{Fr } I^r) = s(0)$. Prove that there exists a homotopy $H : I^r \times I \rightarrow X$ of f such that $H(\text{Fr } I^r \times t) = s(t)$ for any $t \in I$. Furthermore, the spheroid obtained by such a homotopy is unique up to homotopy and determines an element of $\pi_r(X, s(1))$, which is uniquely determined by the homotopy class of s and the element of $\pi_r(X, s(0))$ represented by f .

Certainly, a solution of 32x:E gives an answer to 32x:D and 32x:C. The map $\pi_r(X, s(0)) \rightarrow \pi_r(X, s(1))$ defined by 32x:E is denoted by T_s . By 32.2, this T_s generalizes T_s defined in the beginning of the section for the case $r = 1$.

32x:F. Prove that the properties of T_s formulated in Problems 32.A – 32.F hold true in all dimensions.

32x:G. Riddle. What are the counterparts of 32x:A and 32x:B for higher homotopy groups?

Proofs and Comments

29.A (a), (b), (e): yes; (c), (d), (f): no. See 29.B.

29.B See §29°2.

29.C The mapping h_t is continuous as the restriction of the homotopy H to the fiber $X \times t \subset X \times I$.

29.D Certainly, no, it does not.

29.E See 29.E.1, 29.E.2, and 29.E.3.

29.E.1 The mapping H is continuous as the composition of the projection $p : X \times I \rightarrow X$ and the mapping f , and, furthermore, $H(x, 0) = f(x) = H(x, 1)$. Consequently, H is a homotopy.

29.E.2 The mapping H' is continuous as the composition of the homeomorphism $X \times I \rightarrow X \times I : (x, t) \mapsto (x, 1-t)$ and the homotopy H , and, furthermore, $H'(x, 0) = H(x, 1) = g(x)$ and $H'(x, 1) = H(x, 0) = f(x)$. Therefore, H' is a homotopy.

29.E.3 Indeed, $H''(x, 0) = f(x)$ and $H''(x, 1) = H'(x, 1) = f''(x)$. H'' is continuous since the restriction of the mapping H'' to each of the sets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ is continuous and these sets constitute a fundamental cover of $X \times I$.

Below we do not prove that the homotopies are continuous because this always follows from explicit formulas.

29.F Each of them is homotopic to the constant map mapping the entire space to the origin, for example, if $H(x, t) = (1-t)f(x)$, then $H : X \times I \rightarrow \mathbb{R}^n$ is a homotopy between f and the constant mapping $x \mapsto 0$. (There is a more convenient homotopy between arbitrary mappings to \mathbb{R}^n , see 29.G.)

29.G Indeed, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The mapping H is obviously continuous. For example, this follows from the inequality

$$|H(x, t) - H(x', t')| \leq |f(x) - f(x')| + |g(x) - g(x')| + (|f(x)| + |g(x)|)|t - t'|.$$

29.H Let K be a convex subset of \mathbb{R}^n , $f, g : X \rightarrow K$ two continuous mappings, and H the rectilinear homotopy between f and g . Then $H(x, t) \in K$ for all $(x, t) \in X \times I$, and we obtain a homotopy $H : X \times I \rightarrow K$.

29.I The mapping $H = g \circ F \circ (h \times \text{id}_I) : A \times I \rightarrow B$ is continuous, $H(a, 0) = g(F(h(a), 0)) = g(f(h(a)))$, and $H(a, 1) = g(F(h(a), 1)) = g(f'(h(a)))$. Consequently, H is a homotopy.

29.J Take the mapping $f : X \rightarrow Y$ to the composition $g \circ f \circ h : A \rightarrow B$. Assertion 29.I shows that this correspondence preserves the

homotopy relation, and, hence, it can be transferred to homotopy classes of mappings. Thus, a mapping $\pi(X, Y) \rightarrow \pi(A, B)$ is defined.

29.K Any mapping $f : X \rightarrow Y \times Z$ is uniquely determined by its components $\text{pr}_X \circ f$ and $\text{pr}_Y \circ f$. \Leftrightarrow If H is a homotopy between f and g , then $\text{pr}_Y \circ H$ is a homotopy between $\text{pr}_Y \circ f$ and $\text{pr}_Y \circ g$, and $\text{pr}_Z \circ H$ is a homotopy between $\text{pr}_Z \circ f$ and $\text{pr}_Z \circ g$.

\Leftarrow If H_Y is a homotopy between $\text{pr}_Y \circ f$ and $\text{pr}_Y \circ g$ and H_Z is a homotopy between $\text{pr}_Z \circ f$ and $\text{pr}_Z \circ g$, then the formula $H(x, t) = (H_Y(x, t), H_Z(x, t))$ determines a homotopy between f and g .

29.L The proof does not differ from that of assertion *29.E*.

29.M For the sets A such that $f|_A = g|_A$ (i.e., for the sets contained in the coincidence set of f and g).

29.N A path is a homotopy of a mapping of a point, cf. *29.8*.

29.O For each point $x \in X$, the mapping $u_x : I \rightarrow X : t \mapsto h(x, t)$ is a path.

29.P If H is a homotopy, then for each $t \in I$ the formula $h_t = H(x, t)$ determines a continuous mapping $X \rightarrow Y$. Thus, we obtain a mapping $\mathcal{H} : I \rightarrow \mathcal{C}(X, Y)$ of the segment to the set of all continuous mappings $X \rightarrow Y$. After that, see *29.15* and *29.16*.

29.15 This follows from *24x:V*.

29.16 This follows from *24x:W*.

29.Q This follows from the solution of Problem *29.3*.

30.A 1) We start with a visual description of the required homotopy. Let $u_t : I \rightarrow X$ be a homotopy joining u and u' , and $v_t : I \rightarrow X$ a homotopy joining v and v' . Then the paths $u_t v_t$ with $t \in [0, 1]$ form a homotopy between uv and $u'v'$.

2) Now we present a more formal argument. Since the product uv is defined, we have $u(1) = v(0)$. Since $u \sim u'$, we have $u(1) = u'(1)$, we similarly have $v(0) = v'(0)$. Therefore, the product $u'v'$ is defined. The homotopy between uv and $u'v'$ is the mapping

$$H : I \times I \rightarrow X : (s, t) \mapsto \begin{cases} H'(2s, t) & \text{if } s \in [0, \frac{1}{2}], \\ H''(2s - 1, t) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

(H is continuous because the sets $[0, \frac{1}{2}] \times I$ and $[\frac{1}{2}, 1] \times I$ constitute a fundamental cover of the square $I \times I$, and the restriction of H to each of these sets is continuous.)

30.B This is a straight-forward reformulation of *30.A*.

30.C No; see *30.D*, cf. *30.1*.

30.D No, this is almost always wrong (see 30.1 and 30.2). Here is the simplest example. Let $u(s) = 0$ and $w(s) = 1$ for all $s \in [0, 1]$ and $v(s) = s$. Then $(uv)w(s) = 0$ only for $s \in [0, \frac{1}{4}]$, and $u(vw)(s) = 0$ for $s \in [0, \frac{1}{2}]$.

30.E.1 Reformulation: for any three paths u , v , and w such that the products uv and vw are defined, the paths $(uv)w$ and $u(vw)$ are homotopic.

30.E.2 Let

$$\varphi(s) = \begin{cases} \frac{s}{2} & \text{if } s \in [0, \frac{1}{2}], \\ s - \frac{1}{4} & \text{if } s \in [\frac{1}{2}, \frac{3}{4}], \\ 2s - 1 & \text{if } s \in [\frac{3}{4}, 1]. \end{cases}$$

Verify that φ is the required function, i.e., $((uv)w)(\varphi(s)) = u(vw)(s)$.

30.E.3 Consider the rectilinear homotopy, which is in addition fixed on $\{0, 1\}$.

30.E.4 This follows from 29.I, 30.E.2, and 30.E.3.

30.F See 30.G.

30.G Generally speaking, no; see 30.4.

30.H Let

$$\varphi(s) = \begin{cases} 0 & \text{if } s \in [0, \frac{1}{2}], \\ 2s - 1 & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

Verify that $e_a u = u \circ \varphi$. Since $\varphi \sim \text{id}_I$, we have $u \circ \varphi \sim u$, whence

$$[e_a][u] = [e_a u] = [u \circ \varphi] = [u].$$

30.I See 30.J.

30.J Certainly not.

30.K.1 Consider the mapping

$$\varphi(s) = \begin{cases} 2s & \text{if } s \in [0, \frac{1}{2}], \\ 2 - 2s & \text{if } s \in [\frac{1}{2}, 1], \end{cases}$$

30.K.2 Consider the rectilinear homotopy.

30.L Groups are the sets of classes of paths u with $u(0) = u(1) = x_0$, where x_0 is a certain marked point of X , as well as their subgroups.

31.A This immediately follows from assertions 30.B, 30.E, 30.H, and 30.K.

31.B See §31x°8.

31.C If $u : I \rightarrow X$ is a loop, then there exists a quotient mapping $\tilde{u} : I/\{0, 1\} \rightarrow X$. It remains to observe that $I/\{0, 1\} \cong S^1$.

31.D \Leftrightarrow If $H : S^1 \times I \rightarrow X$ is a homotopy of circular loops, then the formula $H'(s, t) = H(e^{2\pi is}, t)$ determines a homotopy H' between ordinary loops.

\Leftrightarrow Homotopies of circular loops are quotient maps of homotopies of ordinary loops by the partition of the square induced by the relation $(0, t) \sim (1, t)$.

31.E This is true because there is a rectilinear homotopy between any loop in \mathbb{R}^n at the origin and a constant loop.

31.F Here is a possible generalization: for each convex (and even star-shaped) set $V \subset \mathbb{R}^n$ and any point $x_0 \in V$, the fundamental group $\pi_1(V, x_0)$ is trivial.

31.G.1 Let $p \in S^n \setminus u(I)$. Consider the stereographic projection $\tau : S^n \setminus p \rightarrow \mathbb{R}^n$. The loop $v = \tau \circ u$ is null-homotopic, let h be the corresponding homotopy. Then $H = \tau^{-1} \circ h$ is a homotopy joining the loop u and a constant loop on the sphere.

31.G.2 Such loops certainly exist. Indeed, if a loop u fills the entire sphere, then so does the loop uu^{-1} , which, however, is null-homotopic.

31.G.4 Let x be an arbitrary point of the sphere. We cover the sphere by two open sets $U = S^n \setminus x$ and $V = S^n \setminus \{-x\}$. By Lemma *31.G.3*, there is a sequence of points $a_1, \dots, a_N \in I$, where $0 = a_1 < a_2 < \dots < a_{N-1} < a_N = 1$, such that for each i the image $u([a_i, a_{i+1}])$ is entirely contained in U or in V . Since each of these sets is homeomorphi to \mathbb{R}^n , where any two paths with the same starting and ending points are homotopic, it follows that each of the restrictions $u|_{[a_i, a_{i+1}]}$ is homotopic to a path the image of which is, e.g., an “arc of a great circle” of S^n . Thus, the path u is homotopic to a path the image of which does not fill the sphere, and even is nowhere dense.

31.G.5 This immediately follows from Lemma *31.G.4*.

31.G.6 1) This is immediate. 2) The assumption $n \geq 2$ was used only in Lemma *31.G.4*.

31.H Take a loop $u : I \rightarrow X \times Y$ at the point (x_0, y_0) to the pair of loops in X and Y that are the components of u : $u_1 = \text{pr}_X \circ u$ and $u_2 = \text{pr}_Y \circ u$. By assertion *29.I*, the loops u and v are homotopic iff $u_1 \sim v_1$ and $u_2 \sim v_2$. Consequently, taking the class of the loop u to the pair $([u_1], [u_2])$, we obtain a bijection between the fundamental group $\pi_1(X \times Y, (x_0, y_0))$ of the product of the spaces and the product $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ of the fundamental groups of the factors. It remains

to verify that the bijection constructed is a homomorphism, which is also obvious because $\text{pr}_X \circ (uv) = (\text{pr}_X \circ u)(\text{pr}_X \circ v)$.

31.I (a) \implies (b): The space X is simply connected \implies each loop in X is null-homotopic \implies each circular loop in X is relatively null-homotopic \implies each circular loop in X is freely null-homotopic.

(b) \implies (c): By assumption, for an arbitrary mapping $f : S^1 \rightarrow X$ there is a homotopy $h : S^1 \times I \rightarrow X$ such that $h(p, 0) = f(p)$ and $h(p, 1) = x_0$. Consequently, there is a continuous mapping $h' : S^1 \times I / (S^1 \times 1) \rightarrow X$ such that $h = h' \circ \text{pr}$. It remains to observe that $S^1 \times I / (S^1 \times 1) \cong D^2$.

(c) \implies (d): Put $g(t, 0) = u_1(t)$, $g(t, 1) = u_2(t)$, $g(0, t) = x_0$, and $g(1, t) = x_1$ for $t \in I$. Thus, we mapped the boundary of the square $I \times I$ to X . Since the square is homeomorphi to a disk and its boundary is homeomorphi to a circle, it follows that the mapping extends from the boundary to the entire square. The extension obtained is a homotopy between u_1 and u_2 .

(d) \implies (a): This is obvious.

31.J.1 It is reasonable to consider the following implications: (a) \implies (b) \implies (c) \implies (d) \implies (a).

31.J.2 It certainly does. Furthermore, since the loop s is null-homotopic, it follows that the circular loop f is also null-homotopic, and the homotopy is even fixed at the point $1 \in S^1$. Thus, (a) \implies (b).

31.J.3 The assertion suggests the main idea of the proof of the implication (b) \implies (c). A null-homotopy of a certain circular loop f is a mapping $H : S^1 \times I \rightarrow X$ constant on the upper base of the cylinder. Consequently, there is a quotient mapping $S^1 \times I / S^1 \times 1 \rightarrow X$. It remains to observe that the quotient space of the cylinder by the upper base is homeomorphi to a disk.

31.J.4 By the definition of a homotopy $H : I \times I \rightarrow X$ between two paths, the restriction of H to the contour of the square is given. Consequently, the problem of constructing a homotopy between two paths is the problem of extending a map from the contour of the square to the entire square.

31.J.5 All that remains to observe for the proof of the implication (c) \implies (d), is the following fact: if $F : D^2 \rightarrow X$ is an extension of the circular loop f , then the formula $H(t, \tau) = F(\cos \pi t, (2\tau - 1) \sin \pi t)$ determines a homotopy between s_+ and s_- .

31.J In order to prove the theorem, it remains to prove the implication (d) \implies (a). Let us state this assertion without using the notion of circular loop. Let $s : I \rightarrow X$ be a loop. Put $s_+(t) = s(2t)$ and $s_-(t) = s(1 - 2t)$. Thus, we must prove that if the paths s_+ and s_- are

homotopic, then the loop s is null-homotopic. Try to prove this on your own.

31x:A The associativity of \odot follows from that of the multiplication in G ; the unity in the set $\Omega(G, 1)$ of all loops is the constant loop at the unity of the group; the element inverse to the loop u is the path v , where $v(s) = (u(s))^{-1}$.

31x:B.1 Verify that $(ue_1) \odot (e_1v) = uv$.

31x:B We prove that if $u \sim u_1$, then $u \odot v \sim u_1 \odot v$. For this purpose it suffices to check that if h is a homotopy between u and u_1 , then the formula $H(s, t) = h(s, t)v(s)$ determines a homotopy between $u \odot v$ and $u_1 \odot v$. Further, since $ue_1 \sim u$ and $e_1v \sim v$, we have $uv = (ue_1) \odot (e_1v) \sim u \odot v$, therefore, the paths uv and $u \odot v$ lie in one homotopy class. Consequently, the operation \odot induces the standard group operation in the set of homotopy classes of paths.

31x:C It is sufficient to prove that $uv \sim vu$, which fact follows from the following chain:

$$uv = (ue_1) \odot (e_1v) \sim u \odot v \sim (e_1u) \odot (ve_1) = vu.$$

31x:E This group is also trivial. The proof is similar to that of assertion 31.E.

32.A Indeed, if $\alpha = [u]$ and $\beta = [v]$, then

$$T_s(\alpha\beta) = \sigma^{-1}\alpha\beta\sigma = \sigma^{-1}\alpha\sigma\sigma^{-1}\beta\sigma = T_s(\alpha)T_s(\beta).$$

32.B Indeed,

$$T_{uv}(\alpha) = [uv]^{-1}\alpha[uv] = [v]^{-1}[u]^{-1}\alpha[u][v] = T_v(T_u(\alpha)).$$

32.C By the definition of translation along a path, the homomorphism T_s depends only on the homotopy class of s .

32.D This is so because $T_{e_a}([u]) = [e_aue_a] = [u]$.

32.E Since $s^{-1}s \sim e_{x_1}$, 32.B–32.D imply that

$$T_{s^{-1}} \circ T_s = T_{s^{-1}s} = T_{e_{x_1}} = \text{id}_{\pi_1(X, x_1)}.$$

Similarly, we have $T_s \circ T_{s^{-1}} = \text{id}_{\pi_1(X, x_0)}$, whence $T_{s^{-1}} = T_s^{-1}$.

32.F By 32.E, the homomorphism T_s has an inverse and, consequently, is an isomorphism.

32.G If x_0 and x_1 lie in one path-connected component, then they are joined by a path s . By 32.F, $T_s : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is an isomorphism.

32.H This immediately follows from Theorem 32.G.

32.I This directly follows from the definition of T_s .

32.J \Leftrightarrow Assume that the translation isomorphism does not depend on the path. In particular, the isomorphism of translation along any loop at x_0 is trivial. Consider an arbitrary element $\beta \in \pi_1(X, x_0)$ and a loop s in the homotopy class β . By assumption, $\beta^{-1}\alpha\beta = T_s(\alpha) = \alpha$ for each $\alpha \in \pi_1(X, x_0)$. Therefore, $\alpha\beta = \beta\alpha$ for any elements $\alpha, \beta \in \pi_1(X, x_0)$, which precisely means that the group $\pi_1(X, x_0)$ is Abelian.

\Leftarrow Consider two paths s_1 and s_2 joining x_0 and x_1 . Since $T_{s_1s_2^{-1}} = T_{s_2}^{-1} \circ T_{s_1}$, it follows that $T_{s_1} = T_{s_2}$ iff $T_{s_1s_2^{-1}} = \text{id}_{\pi_1(X, x_0)}$. Let $\beta \in \pi_1(X, x_0)$ be the class of the loop $s_1s_2^{-1}$. If the group $\pi_1(X, x_0)$ is Abelian, then $T_{s_1s_2^{-1}}(\alpha) = \beta^{-1}\alpha\beta = \alpha$, whence $T_{s_1s_2^{-1}} = \text{id}$, and so $T_{s_1} = T_{s_2}$.

32x:A Let u be a loop at $s(0)$. The formula $H(\tau, t) = u(\tau)s(0)^{-1}s(1)$ determines a free homotopy between u and the loop $L_{s(0)^{-1}s(1)}(u)$ such that $H(0, t) = H(1, t) = s(t)$. Therefore, by 32.2, the loops $L_{s(0)^{-1}s(1)}(u)$ and $s^{-1}us$ are homotopic, whence $T_s = (L_{s(0)^{-1}s(1)})_*$. The equality for $R_{s(0)^{-1}s(1)}$ is proved in a similar way.

32x:B By 32x:A, we have $T_s = (L_e)_* = \text{id}_{\pi_1(X, x_0)}$ for each loop s at x_0 . Therefore, if β is the class of the loop s , then $T_s(\alpha) = \beta^{-1}\alpha\beta = \alpha$, whence $\alpha\beta = \beta\alpha$.

Covering Spaces and Calculation of Fundamental Groups

§ 33 Covering Spaces

§ 33°1 Definition of Covering

Let X, B topological spaces, $p : X \rightarrow B$ a continuous map. Assume that p is surjective and each point of B possesses a neighborhood U such that the preimage $p^{-1}(U)$ of U is a disjoint union of open sets V_α and p maps each V_α homeomorphically onto U . Then $p : X \rightarrow B$ is a *covering* (of B), the space B is the *base* of this covering, X is the *covering space* for B and the *total space* of the covering. Neighborhoods like U are said to be *trivially covered*. The map p is a *covering map* or *covering projection*.

33.A. Let B be a topological space and F be a discrete space. Prove that the projection $\text{pr}_B : B \times F \rightarrow B$ is a covering.

33.1. If $U' \subset U \subset B$ and the neighborhood U is trivially covered, then the neighborhood U' is also trivially covered.

The following statement shows that in a certain sense any covering locally is organized as the covering of 33.A.

33.B. A continuous surjective map $p : X \rightarrow B$ is a covering iff for each point a of B the preimage $p^{-1}(a)$ is discrete and there exist a neighborhood U of a and a homeomorphism $h : p^{-1}(U) \rightarrow U \times p^{-1}(a)$ such that $p|_{p^{-1}(U)} = \text{pr}_U \circ h$. Here, as usual, $\text{pr}_U : U \times p^{-1}(a) \rightarrow U$.

However, the coverings of 33.A are not interesting. They are said to be *trivial*. Here is the first really interesting example.

33.C. Prove that $\mathbb{R} \rightarrow S^1 : x \mapsto e^{2\pi ix}$ is a covering.

To distinguish the most interesting examples, a covering with a connected total space is called a covering in a *narrow sense*. Of course, the covering of 33.C is a covering in a narrow sense.

§ 33°2 More Examples

33.D. $\mathbb{R}^2 \rightarrow S^1 \times \mathbb{R} : (x, y) \mapsto (e^{2\pi ix}, y)$ is a covering.

33.E. Prove that if $p : X \rightarrow B$ and $p' : X' \rightarrow B'$ are coverings, then so is $p \times p' : X \times X' \rightarrow B \times B'$.

If $p : X \rightarrow B$ and $p' : X' \rightarrow B'$ are two coverings, then $p \times p' : X \times X' \rightarrow B \times B'$ is the *product of the coverings* p and p' . The first example of the product of coverings is presented in 33.D.

33.F. $\mathbb{C} \rightarrow \mathbb{C} \setminus 0 : z \mapsto e^z$ is a covering.

33.2. Riddle. In what sense the coverings of 33.D and 33.F are the same? Define an appropriate equivalence relation for coverings.

33.G. $\mathbb{R}^2 \rightarrow S^1 \times S^1 : (x, y) \mapsto (e^{2\pi ix}, e^{2\pi iy})$ is a covering.

33.H. For any positive integer n , the map $S^1 \rightarrow S^1 : z \mapsto z^n$ is a covering.

33.3. Prove that for each positive integer n the map $\mathbb{C} \setminus 0 \rightarrow \mathbb{C} \setminus 0 : z \mapsto z^n$ is a covering.

33.I. For any positive integers p and q , the map $S^1 \times S^1 \rightarrow S^1 \times S^1 : (z, w) \mapsto (z^p, w^q)$ is a covering.

33.J. The natural projection $S^n \rightarrow \mathbb{R}P^n$ is a covering.

33.K. Is $(0, 3) \rightarrow S^1 : x \mapsto e^{2\pi ix}$ a covering? (Cf. 33.14.)

33.L. Is the projection $\mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto x$ a covering? Indeed, why is not an open interval $(a, b) \subset \mathbb{R}$ a trivially covered neighborhood: its preimage $(a, b) \times \mathbb{R}$ is the union of open intervals $(a, b) \times \{y\}$, which are homeomorphically projected onto (a, b) by the projection $(x, y) \mapsto x$?

33.4. Find coverings of the Möbius strip by a cylinder.

33.5. Find nontrivial coverings of Möbius strip by itself.

33.6. Find a covering of the Klein bottle by a torus. Cf. Problem 21.14.

33.7. Find coverings of the Klein bottle by the plane \mathbb{R}^2 and the cylinder $S^1 \times \mathbb{R}$, and a nontrivial covering of the Klein bottle by itself.

33.8. Describe explicitly the partition of \mathbb{R}^2 into preimages of points under this covering.

33.9. Find a covering of a sphere with any number of crosscaps by a sphere with handles.

§33°3 Local Homeomorphisms versus Coverings

33.10. Any covering is an open map.¹

A map $f : X \rightarrow Y$ is *locally homeomorphic* if each point of X has a neighborhood U such that the image $f(U)$ is open in Y and the submap $\text{ab}(f) : U \rightarrow f(U)$ is a homeomorphism.

¹We remind that a map is *open* if the image of any open set is open.

33.11. Any covering is a locally homeomorphic map.

33.12. Find a locally homeomorphic map which is not a covering.

33.13. Prove that the restriction of a locally homeomorphic map to an open set is locally homeomorphic.

33.14. For which subsets of \mathbb{R} is the restriction of the map of Problem 33.C a covering?

33.15. Find a nontrivial covering $X \rightarrow B$ with X homeomorphic to B and prove that it satisfies the definition of a covering.

§33°4 Number of Sheets

Let $p : X \rightarrow B$ be a covering. The cardinality (i.e., the number of points) of the preimage $p^{-1}(a)$ of a point $a \in B$ is the *multiplicity* of the covering at a or the *number of sheets of the covering over a* .

33.M. *If the base of a covering is connected, then the multiplicity of the covering at a point does not depend on the point.*

In the case of covering with connected base, the multiplicity is called the *number of sheets* of the covering. If the number of sheets is n , then the covering is *n -sheeted*, and we talk about an *n -fold* covering. Of course, unless the covering is trivial, it is impossible to distinguish the sheets of it, but this does not prevent us from speaking about the number of sheets. On the other hand, we adopt the following agreement. By definition, the preimage $p^{-1}(U)$ of any trivially covered neighborhood $U \subset B$ splits into open subsets: $p^{-1}(U) = \cup V_\alpha$, such that the restriction $p|_{V_\alpha} : V_\alpha \rightarrow U$ is a homeomorphism. Each of the subsets V_α is a *sheet* over U .

33.16. What are the numbers of sheets for the coverings from Section §33°2?

33.17. What numbers can you realize as the number of sheets of a covering of the Möbius strip by the cylinder $S^1 \times I$?

33.18. What numbers can you realize as the number of sheets of a covering of the Möbius strip by itself?

33.19. What numbers can you realize as the number of sheets of a covering of the Klein bottle by a torus?

33.20. What numbers can you realize as the number of sheets of a covering of the Klein bottle by itself?

In problems 33.17–33.19 we did not assume that you would rigorously justify your answers. This will be done below, see problems 39.3–39.6.

33.21. Construct a d -fold covering of a sphere with p handles by a sphere with $1 + d(p - 1)$ handles.

33.22. Let $p : X \rightarrow Y$ and $q : Y \rightarrow Z$ be coverings. Prove that if q has finitely many sheets, then $q \circ p : X \rightarrow Z$ is a covering.

33.23*. Is the hypothesis of finiteness of the number of sheets in Problem 33.22 necessary?

33.24. Let $p : X \rightarrow B$ be a covering with compact base B . 1) Prove that if X is compact, then the covering is finite-sheeted. 2) If B is Hausdorff and the covering is finite-sheeted, then X is compact.

33.25. Let X be a topological space presentable as the union of two open connected sets U and V . Prove that if the intersection $U \cap V$ is disconnected, then X has a connected infinite-sheeted covering.

§33°5 Universal Coverings

A covering $p : X \rightarrow B$ is *universal* if X is simply connected. The appearance of the word *universal* in this context is explained below in Section §39.

33.N. Which coverings of the problems stated above in this section are universal?

§ 34 Theorems on Path Lifting

§ 34°1 Lifting

Let $p : X \rightarrow B$ and $f : A \rightarrow B$ be arbitrary maps. A map $g : A \rightarrow X$ such that $p \circ g = f$ is said to *cover* f or be a *lifting* of f . Various topological problems can be phrased in terms of finding a continuous lifting of some continuous map. Problems of this sort are called *lifting problems*. They may involve additional requirements. For example, the desired lifting must coincide with a lifting already given on some subspace.

34.A. The identity map $S^1 \rightarrow S^1$ does not admit a continuous lifting with respect to the covering $\mathbb{R} \rightarrow S^1 : x \mapsto e^{2\pi i x}$. (In other words, there exists no continuous map $g : S^1 \rightarrow \mathbb{R}$ such that $e^{2\pi i g(x)} = x$ for $x \in S^1$.)

§ 34°2 Path Lifting

34.B Path Lifting Theorem. *Let $p : X \rightarrow B$ be a covering, $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Then for any path $s : I \rightarrow B$ starting at b_0 there exists a unique path $\tilde{s} : I \rightarrow X$ starting at x_0 and being a lifting of s . (In other words, there exists a unique path $\tilde{s} : I \rightarrow X$ with $\tilde{s}(0) = x_0$ and $p \circ \tilde{s} = s$.)*

We can also prove a more general assertion than Theorem 34.B: see Problems 34.1–34.3.

34.1. Let $p : X \rightarrow B$ be a trivial covering. Then for any continuous map f of any space A to B there exists a continuous lifting $\tilde{f} : A \rightarrow X$.

34.2. Let $p : X \rightarrow B$ be a trivial covering and $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Then for any continuous map f of a space A to B mapping a point a_0 to b_0 , a continuous lifting $\tilde{f} : A \rightarrow X$ with $\tilde{f}(a_0) = x_0$ is unique.

34.3. Let $p : X \rightarrow B$ be a covering, A a connected and locally connected space. If $f, g : A \rightarrow X$ are two continuous maps coinciding at some point and $p \circ f = p \circ g$, then $f = g$.

34.4. If we replace x_0 , b_0 , and a_0 in Problem 34.2 by pairs of points, then the lifting problem may happen to have no solution \tilde{f} with $\tilde{f}(a_0) = x_0$. Formulate a condition necessary and sufficient for existence of such a solution.

34.5. What goes wrong with the Path Lifting Theorem 34.B for the local homeomorphism of Problem 33.K?

34.6. Consider the covering $\mathbb{C} \rightarrow \mathbb{C} \setminus 0 : z \mapsto e^z$. Find liftings of the paths $u(t) = 2 - t$ and $v(t) = (1 + t)e^{2\pi i t}$ and their products uv and vu .

§ 34°3 Homotopy Lifting

34.C Path Homotopy Lifting Theorem. *Let $p : X \rightarrow B$ be a covering, $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Let $u, v : I \rightarrow B$ be paths starting at b_0 and $\tilde{u}, \tilde{v} : I \rightarrow X$ be the lifting paths for u, v starting*

at x_0 . If the paths u and v are homotopic, then the covering paths \tilde{u} and \tilde{v} are homotopic.

34.D Corollary. Under the assumptions of Theorem 34.C, the covering paths \tilde{u} and \tilde{v} have the same final point (i.e., $\tilde{u}(1) = \tilde{v}(1)$).

Notice that the paths in 34.C and 34.D are assumed to share the initial point x_0 . In the statement of 34.D, we emphasize that then they also share the final point.

34.E Corollary of 34.D. Let $p : X \rightarrow B$ be a covering and $s : I \rightarrow B$ be a loop. If there exists a lifting $\tilde{s} : I \rightarrow X$ of s with $\tilde{s}(0) \neq \tilde{s}(1)$ (i.e., there exists a covering path which is not a loop), then s is not null-homotopic.

34.F. Prove that if a path-connected space B has a nontrivial path-connected covering space, then the fundamental group of B is nontrivial.

34.7. Prove that any covering $p : X \rightarrow B$ with simply connected B and path connected X is a homeomorphism.

34.8. What corollaries can you deduce from 34.F and the examples of coverings presented above in Section §33?

34.9. Riddle. Is it really important in the hypothesis of Theorem 34.C that u and v are paths? To what class of maps can you generalize this theorem?

§35 Calculation of Fundamental Groups Using Universal Coverings

§35°1 Fundamental Group of Circle

For an integer n , denote by s_n the loop in S^1 defined by the formula $s_n(t) = e^{2\pi int}$. The initial point of this loop is 1. Denote the homotopy class of s_1 by α . Thus, $\alpha \in \pi_1(S^1, 1)$.

35.A. The loop s_n represents $\alpha^n \in \pi_1(S^1, 1)$.

35.B. Find the paths in \mathbb{R} starting at $0 \in \mathbb{R}$ and covering the loops s_n with respect to the universal covering $\mathbb{R} \rightarrow S^1$.

35.C. The homomorphism $\mathbb{Z} \rightarrow \pi_1(S^1, 1) : n \mapsto \alpha^n$ is an isomorphism.

35.C.1. The formula $n \mapsto \alpha^n$ determines a homomorphism $\mathbb{Z} \rightarrow \pi_1(S^1, 1)$.

35.C.2. Prove that a loop $s : I \rightarrow S^1$ starting at 1 is homotopic to s_n if the path $\tilde{s} : I \rightarrow \mathbb{R}$ covering s and starting at $0 \in \mathbb{R}$ ends at $n \in \mathbb{R}$ (i.e., $\tilde{s}(1) = n$).

35.C.3. Prove that if the loop s_n is null-homotopic, then $n = 0$.

35.1. Find the image of the homotopy class of the loop $t \mapsto e^{2\pi it^2}$ under the isomorphism of Theorem 35.C.

Denote by \deg the isomorphism inverse to the isomorphism of Theorem 35.C.

35.2. For any loop $s : I \rightarrow S^1$ starting at $1 \in S^1$, the integer $\deg([s])$ is the final point of the path starting at $0 \in \mathbb{R}$ and covering s .

35.D Corollary of Theorem 35.C. *The fundamental group of $(S^1)^n$ is a free Abelian group of rank n (i.e., isomorphic to \mathbb{Z}^n).*

35.E. On torus $S^1 \times S^1$ find two loops whose homotopy classes generate the fundamental group of the torus.

35.F Corollary of Theorem 35.C. *The fundamental group of punctured plane $\mathbb{R}^2 \setminus 0$ is an infinite cyclic group.*

35.3. Solve Problems 35.D – 35.F without reference to Theorems 35.C and 31.H, but using explicit constructions of the corresponding universal coverings.

§35°2 Fundamental Group of Projective Space

The fundamental group of the projective line is an infinite cyclic group. It is calculated in the previous subsection since the projective line is a circle. The zero-dimensional projective space is a point, hence its

fundamental group is trivial. Now we calculate the fundamental groups of projective spaces of all other dimensions.

Let $n \geq 2$, and let $l : I \rightarrow \mathbb{R}P^n$ be a loop covered by a path $\tilde{l} : I \rightarrow S^n$ which connects two antipodal points of S^n , say the poles $P_+ = (1, 0, \dots, 0)$ and $P_- = (-1, 0, \dots, 0)$. Denote by λ the homotopy class of l . It is an element of $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$.

35.G. For any $n \geq 2$ group $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$ is a cyclic group of order 2. It consists of two elements: λ and 1.

35.G.1 Lemma. Any loop in $\mathbb{R}P^n$ at $(1 : 0 : \dots : 0)$ is homotopic either to l or constant. This depends on whether the covering path of the loop connects the poles P_+ and P_- , or is a loop.

35.4. Where did we use the assumption $n \geq 2$ in the proofs of Theorem 35.G and Lemma 35.G.1 ?

§35°3 Fundamental Group of Bouquet of Circles

Consider a family of topological spaces $\{X_\alpha\}$. In each of the spaces, let a point x_α be marked. Take the disjoint sum $\bigsqcup_\alpha X_\alpha$ and identify all marked points. The resulting quotient space $\bigvee_\alpha X_\alpha$ is the *bouquet* of $\{X_\alpha\}$. Hence a *bouquet of q circles* is a space which is a union of q copies of circle. The copies meet at a single common point, and this is the only common point for any two of them. The common point is the *center* of the bouquet.

Denote the bouquet of q circles by B_q and its center by c . Let u_1, \dots, u_q be loops in B_q starting at c and parameterizing the q copies of circle comprising B_q . Denote by α_i the homotopy class of u_i .

35.H. $\pi_1(B_q, c)$ is a free group freely generated by $\alpha_1, \dots, \alpha_q$.

§35°4 Algebraic Digression: Free Groups

Recall that a group G is a free group freely generated by its elements a_1, \dots, a_q if:

- each element $x \in G$ is a product of powers (with positive or negative integer exponents) of a_1, \dots, a_q , i.e.,

$$x = a_{i_1}^{e_1} a_{i_2}^{e_2} \dots a_{i_n}^{e_n}$$

and

- this expression is unique up to the following trivial ambiguity: we can insert or delete factors $a_i a_i^{-1}$ and $a_i^{-1} a_i$ or replace a_i^m by $a_i^r a_i^s$ with $r + s = m$.

35.I. A free group is determined up to isomorphism by the number of its free generators.

The number of free generators is the *rank* of the free group. For a standard representative of the isomorphism class of free groups of rank q , we can take the group of words in an alphabet of q letters a_1, \dots, a_q and their inverses $a_1^{-1}, \dots, a_q^{-1}$. Two words represent the same element of the group iff they can be obtained from each other by a sequence of insertions or deletions of fragments $a_i a_i^{-1}$ and $a_i^{-1} a_i$. This group is denoted by $\mathbb{F}(a_1, \dots, a_q)$, or just \mathbb{F}_q , when the notation for the generators is not to be emphasized.

35.J. *Each element of $\mathbb{F}(a_1, \dots, a_q)$ has a unique shortest representative. This is a word without fragments that could have been deleted.*

The number $l(x)$ of letters in the shortest representative of $x \in \mathbb{F}(a_1, \dots, a_q)$ is the *length* of x . Certainly, this number is not well defined unless the generators are fixed.

35.5. Show that an automorphism of \mathbb{F}_q can map $x \in \mathbb{F}_q$ to an element with different length. For what value of q does such an example not exist? Is it possible to change the length in this way arbitrarily?

35.K. *A group G is a free group freely generated by its elements a_1, \dots, a_q iff every map of the set $\{a_1, \dots, a_q\}$ to any group X extends to a unique homomorphism $G \rightarrow X$.*

Theorem 35.K is sometimes taken as a definition of a free group. (Definitions of this sort emphasize relations among different groups, rather than the internal structure of a single group. Of course, relations among groups can tell everything about “internal affairs” of each group.)

Now we can reformulate Theorem 35.H as follows:

35.L. *The homomorphism*

$$\mathbb{F}(a_1, \dots, a_q) \rightarrow \pi_1(B_q, c)$$

taking a_i to α_i for $i = 1, \dots, q$ is an isomorphism.

First, for the sake of simplicity we restrict ourselves to the case where $q = 2$. This will allow us to avoid superfluous complications in notation and pictures. This is the simplest case, which really represents the general situation. The case $q = 1$ is too special.

To take advantages of this, let us change the notation. Put $B = B_2$, $u = u_1$, $v = u_2$, $\alpha = \alpha_1$, and $\beta = \alpha_2$.

Now Theorem 35.L looks as follows:

The homomorphism $\mathbb{F}(a, b) \rightarrow \pi(B, c)$ taking a to α and b to β is an isomorphism.

This theorem can be proved like Theorems 35.C and 35.G, provided the universal covering of B is known.

§35°5 Universal Covering for Bouquet of Circles

Denote by U and V the points antipodal to c on the circles of B . Cut B at these points, removing U and V and putting instead each of them two new points. Whatever this operation is, its result is a cross K , which is the union of four closed segments with a common endpoint c . There appears a natural map $P : K \rightarrow B$ that takes the center c of the cross to the center c of B and homeomorphically maps the rays of the cross onto half-circles of B . Since the circles of B are parameterized by loops u and v , the halves of each of the circles are ordered: the corresponding loop passes first one of the halves and then the other one. Denote by U^+ the point of $P^{-1}(U)$ belonging to the ray mapped by P onto the second half of the circle, and by U^- the other point of $P^{-1}(U)$. We similarly denote points of $P^{-1}(V)$ by V^+ and V^- .

The restriction of P to $K \setminus \{U^+, U^-, V^+, V^-\}$ maps this set homeomorphically onto $B \setminus \{U, V\}$. Therefore P provides a covering of $B \setminus \{U, V\}$. However, it fails to be a covering at U and V : none of these points has a trivially covered neighborhood. Furthermore, the preimage of each of these points consists of 2 points (the endpoints of the cross), where P is not even a local homeomorphism. To eliminate this defect, we can attach a copy of K at each of the 4 endpoints of K and extend P in a natural way to the result. But then 12 new endpoints appear at which the map is not a local homeomorphism. Well, we repeat the trick and recover the property of being a local homeomorphism at each of the 12 new endpoints. Then we do this at each of the 36 new points, etc. But if we repeat this infinitely many times, all bad points become nice ones.²

35.M. Formalize the construction of a covering for B described above.

Consider $\mathbb{F}(a, b)$ as a discrete topological space. Take $K \times \mathbb{F}(a, b)$. It can be thought of as a collection of copies of K enumerated by elements of $\mathbb{F}(a, b)$. Topologically this is a disjoint sum of the copies because $\mathbb{F}(a, b)$ is equipped with discrete topology. In $K \times \mathbb{F}(a, b)$, we identify points

²This sounds like a story about a battle with Hydra, but the happy ending demonstrates that modern mathematicians have a magic power of the sort that the heroes of myths and tales could not even dream of. Indeed, we meet a Hydra K with 4 heads, chop off all the heads, but, according to the old tradition of the genre, 3 new heads appear in place of each of the original heads. We chop them off, and the story repeats. We do not even try to prevent this multiplication of heads. We just chop them off. But contrary to the real heroes of tales, we act outside of Time and hence have no time limitations. Thus after infinite repetitions of the exercise with an exponentially growing number of heads we succeed! No heads left!

This is a typical success story about an infinite construction in mathematics. Sometimes, as in our case, such a construction can be replaced by a finite one, but dealing with infinite objects. However, there are important constructions in which an infinite fragment is unavoidable.

(U^-, g) with (U^+, ga) and (V^-, g) with (V^+, gb) for each $g \in \mathbb{F}(a, b)$. Denote the resulting quotient space by X .

35.N. The composition of the projection $K \times \mathbb{F}(a, b) \rightarrow K$ and $P : K \rightarrow B$ determines a continuous quotient map $p : X \rightarrow B$.

35.O. $p : X \rightarrow B$ is a covering.

35.P. X is path-connected. For any $g \in \mathbb{F}(a, b)$, there exists a path connecting $(c, 1)$ with (c, g) and covering the loop obtained from g by replacing a with u and b with v .

35.Q. X is simply connected.

§35°6 Fundamental Groups of Finite Topological Spaces

35.6. Prove that if a three-point space X is path-connected, then X is simply connected (cf. 31.7).

35.7. Consider a topological space $X = \{a, b, c, d\}$ with topology determined by the base $\{\{a\}, \{c\}, \{a, b, c\}, \{c, d, a\}\}$. Prove that X is path-connected, but not simply connected.

35.8. Calculate $\pi_1(X)$.

35.9. Let X be a finite topological space with nontrivial fundamental group. Let n_0 be the least possible cardinality of X . 1) Find n_0 . 2) What nontrivial groups arise as fundamental groups of n_0 -point spaces?

35.10. 1) Find a finite topological space with non-Abelian fundamental group. 2) What is the least possible cardinality of such a space?

35.11*. Let a topological space X be the union of two open path-connected sets U and V . Prove that if $U \cap V$ has at least three connected components, then the fundamental group of X is non-Abelian and, moreover, admits an epimorphism onto a free group of rank 2.

35.12*. Find a finite topological space with fundamental group isomorphic to \mathbb{Z}_2 .

Proofs and Comments

33.A Let us show that the set B itself is trivially covered. Indeed, $(\text{pr}_B)^{-1}(B) = X = \bigcup_{y \in F} (B \times y)$, and since the topology in F is discrete, it follows that each of the sets $B \times y$ is open in the total space of the covering, and the restriction of pr_B to each of them is a homeomorphism.

33.B \Leftrightarrow We construct a homeomorphism $h : p^{-1}(U) \rightarrow U \times p^{-1}(a)$ for an arbitrary trivially covered neighborhood $U \subset B$ of a . By the definition of a trivially covered neighborhood, we have $p^{-1}(U) = \bigcup U_\alpha$. Let $x \in p^{-1}(U)$, consider an open sets U_α containing x and take x to the pair $(p(x), c)$, where $\{c\} = p^{-1}(a) \cap U_\alpha$. It is clear that the correspondence $x \mapsto (p(x), c)$ determines a homeomorphism $h : p^{-1}(U) \rightarrow U \times p^{-1}(a)$.

\Leftarrow By assertion 33.1, U is a trivially covered neighborhood, hence, $p : X \rightarrow B$ is a covering.

33.C For each point $z \in S^1$, the set $U_z = S^1 \setminus \{-z\}$ is a trivially covered neighborhood of z . Indeed, let $z = e^{2\pi i x}$. Then the preimage of U_z is the union $\bigcup_{k \in \mathbb{Z}} (x + k - \frac{1}{2}, x + k + \frac{1}{2})$, and the restriction of the covering to each of the above intervals is a homeomorphism.

33.D The product $(S^1 \setminus \{-z\}) \times \mathbb{R}$ is a trivially covered neighborhood of a point $(z, y) \in S^1 \times \mathbb{R}$; cf. 33.E.

33.E Verify that the product of trivially covered neighborhoods of points $b \in B$ and $b' \in B'$ is a trivially covered neighborhood of the point $(b, b') \in B \times B'$.

33.F Consider the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{h} & \mathbb{C} \\ q \downarrow & & \downarrow p \\ S^1 \times \mathbb{R} & \xrightarrow{g} & \mathbb{C} \setminus 0, \end{array}$$

where $g(z, x) = ze^x$, $h(x, y) = y + 2\pi i x$, and $q(x, y) = (e^{2\pi i x}, y)$. The equality $g(q(x, y)) = e^{2\pi i x} \cdot e^y = e^{y+2\pi i x} = p(h(x, y))$ implies that the diagram is commutative. Clearly, g and h are homeomorphisms. Since q is a covering by 33.D, p is also a covering.

33.G By 33.E, this assertion follows from 33.C. Certainly, it is not difficult to prove it directly. The product $(S^1 \setminus z) \times (S^1 \setminus z')$ is a trivially covered neighborhood of the point $(z, z') \in S^1 \times S^1$.

33.H Let $z \in S^1$. The preimage $-z$ under the projection consists of n points, which partition the covering space into n arcs, and the restriction of the projection to each of them determines a homeomorphism of this arc onto the neighborhood $S^1 \setminus \{-z\}$ of z .

33.I By 33.E, this assertion follows from 33.H.

33.J The preimage of a point $y \in \mathbb{R}P^n$ is a pair $\{x, -x\} \subset S^n$ of antipodal points. The plane passing through the center of the sphere and orthogonal to the vector x splits the sphere into two open hemispheres, each of which is homeomorphically projected to a neighborhood (homeomorphi to \mathbb{R}^n) of the point $y \in \mathbb{R}P^n$.

33.K No, it is not, because the point $1 \in S^1$ has no trivially covered neighborhood.

33.L The open intervals mentioned in the statement are not open subsets of the plane. Furthermore, since the preimage of any interval is a connected set, it cannot be split into disjoint open subsets at all.

33.M Prove that the definition of a covering implies that the set of the points in the base with preimage of prescribed cardinality is open and use the fact that the base of the covering is connected.

33.N Those coverings where the covering space is $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^n \setminus 0$ with $n \geq 3$, and S^n with $n \geq 2$, i.e., a simply connected space.

34.A Assume that there exists a lifting g of the identity map $S^1 \rightarrow S^1$; this is a continuous injection $S^1 \rightarrow \mathbb{R}$. We show that there are no such injections. Let $g(S^1) = [a, b]$. The Intermediate Value Theorem implies that each point $x \in (a, b)$ is the image of at least two points of the circle. Consequently, g is not an injection.

34.B Cover the base by trivially covered neighborhoods and partition the segment $[0, 1]$ by points $0 = a_0 < a_1 < \dots < a_n = 1$, such that the image $s([a_i, a_{i+1}])$ is entirely contained in one of the trivially covered neighborhoods; $s([a_i, a_{i+1}]) \subset U_i, i = 0, 1, \dots, n-1$. Since the restriction of the covering to $p^{-1}(U_0)$ is a trivial covering and $f([a_0, a_1]) \subset U_0$, there exists a lifting of the mapping $s|_{[a_0, a_1]}$ such that $\tilde{s}(a_0) = x_0$, let $x_1 = \tilde{s}(a_1)$. Similarly, there exists a unique lifting $\tilde{s}|_{[a_1, a_2]}$ such that $\tilde{s}(a_1) = x_1$; let $x_2 = \tilde{s}(a_2)$, and so on. Thus, there exists a lifting $\tilde{s} : I \rightarrow X$. Its uniqueness is obvious. If you do not agree, use induction.

34.C Let $h : I \times I \rightarrow B$ be a homotopy between the paths u and v , thus, $h(\tau, 0) = u(\tau), h(\tau, 1) = v(\tau), h(0, t) = b_0$, and $h(1, t) = b_1 \in B$. We show that there exists a mapping $\tilde{h} : I \times I \rightarrow X$ covering h and such that $h(0, 0) = x_0$. The proof of the existence of the covering homotopy is similar to that of the Path Lifting Theorem. We subdivide the square $I \times I$ into smaller squares such that the h -image of each of them is contained in a certain trivially covered neighborhood in B . The restriction $h_{k,l}$ of the homotopy h to each of the “little” squares $I_{k,l}$ is covered by the corresponding mapping $\tilde{h}_{k,l}$. In order to obtain a homotopy covering h , we must only ensure that these mappings coincide

on the intersections of these squares. By 34.3, it suffices to require that these mappings coincide at least at one point. Let us make the first step: let $h(I_{0,0}) \subset U_{b_0}$ and let $\tilde{h}_{0,0} : I_{0,0} \rightarrow X$ be a covering mapping such that $\tilde{h}_{0,0}(a_0, c_0) = x_0$. Now we put $b_1 = h(a_1, c_0)$ and $x_1 = \tilde{h}(a_1, c_0)$. There is a mapping $\tilde{h}_{1,0} : I_{1,0} \rightarrow X$ covering $h|_{I_{1,0}}$ such that $\tilde{h}_{1,0}(a_1, c_0) = x_1$. Proceeding in this way, we obtain a mapping \tilde{h} defined on the entire square. It remains to verify that \tilde{h} is a homotopy of paths. Consider the covering path $\tilde{u} : t \mapsto \tilde{h}(0, t)$. Since $p \circ \tilde{u}$ is a constant path, the path \tilde{u} must also be constant, whence $\tilde{h}(0, t) = x_0$. Similarly, $\tilde{h}(1, t) = x_1$ is a marked point of the covering space. Therefore, \tilde{h} is a homotopy of paths. In conclusion, we observe that the uniqueness of this homotopy follows, once more, from Lemma 34.3.

34.D Formally speaking, this is indeed a corollary, but actually we already proved this when proving Theorem 34.C.

34.E A constant path is covered by a constant path. By 34.D, each null-homotopic loop is covered by a loop.

35.A Consider the paths $\tilde{s}_n : I \rightarrow \mathbb{R} : t \mapsto nt$, $\tilde{s}_{n-1} : I \rightarrow \mathbb{R} : t \mapsto (n-1)t$, and $\tilde{s}_1 : I \rightarrow \mathbb{R} : t \mapsto n-1+t$ covering the paths s_n , s_{n-1} , and s_1 , respectively. Since the product $\tilde{s}_{n-1}\tilde{s}_1$ is defined and has the same starting and ending points as the path \tilde{s}_n , we have $\tilde{s}_n \sim \tilde{s}_{n-1}\tilde{s}_1$, whence $s_n \sim s_{n-1}s_1$. Therefore, $[s_n] = [s_{n-1}]\alpha$. Reasoning by induction, we obtain the required equality $[s_n] = \alpha^n$.

35.B See the proof of assertion 35.A: this is the path defined by the formula $\tilde{s}_n(t) = nt$.

35.C By 35.C.1, the map in question is indeed a well-defined homomorphism. By 35.C.2, it is an epimorphism, and by 35.C.3 it is a monomorphism. Therefore, it is an isomorphism.

35.C.1 If $n \mapsto \alpha^n$ and $k \mapsto \alpha^k$, then $n+k \mapsto \alpha^{n+k} = \alpha^n \cdot \alpha^k$.

35.C.2 Since \mathbb{R} is simply connected, the paths \tilde{s} and \tilde{s}_n are homotopic, therefore, the paths s and s_n are also homotopic, whence $[s] = [s_n] = \alpha^n$.

35.C.3 If $n \neq 0$, then the path \tilde{s}_n ends at the point n , hence, it is not a loop. Consequently, the loop s_n is not null-homotopic.

35.D This follows from the above computation of the fundamental group of the circle and assertion 31.H:

$$\pi_1(\underbrace{S^1 \times \dots \times S^1}_{n \text{ factors}}, (1, 1, \dots, 1)) \cong \underbrace{\pi_1(S^1, 1) \times \dots \times \pi_1(S^1, 1)}_{n \text{ factors}} \cong \mathbb{Z}^n.$$

35.E Let $S^1 \times S^1 = \{(z, w) : |z| = 1, |w| = 1\} \subset \mathbb{C} \times \mathbb{C}$. The generators of $\pi_1(S^1 \times S^1, (1, 1))$ are the loops $s_1 : t \mapsto (e^{2\pi it}, 1)$ and $s_2 : t \mapsto (1, e^{2\pi it})$.

35.F Since $\mathbb{R}^2 \setminus 0 \cong S^1 \times \mathbb{R}$, we have $\pi_1(\mathbb{R}^2 \setminus 0, (1, 0)) \cong \pi_1(S^1, 1) \times \pi_1(\mathbb{R}, 1) \cong \mathbb{Z}$.

35.G.1 Let u be a loop in $\mathbb{R}P^n$, and let \tilde{u} be the covering u the path in S^n . For $n \geq 2$, the sphere S^n is simply connected, and if \tilde{u} is a loop, then \tilde{u} and hence also u are null-homotopic. Now if \tilde{u} is not a loop, then, once more since S^n is simply connected, we have $\tilde{u} \sim \tilde{l}$, whence $u \sim l$.

35.G By **35.G.1**, the fundamental group consists of two elements, therefore, it is a cyclic group of order two.

35.H See §35°5.

35.M See the paragraph following the present assertion.

35.N This obviously follows from the definition of the mapping P .

35.O This obviously follows from the definition of p .

35.P Use induction.

35.Q Use the fact that the image of any loop, as a compact set, intersects only a finite number of the segments constituting the covering space X , and use induction on the number of such segments.

Fundamental Group and Mappings

§36 Induced Homomorphisms and Their First Applications

§36°1 Homomorphisms Induced by a Continuous Map

Let $f : X \rightarrow Y$ be a continuous map of a topological space X to a topological space Y . Let $x_0 \in X$ and $y_0 \in Y$ be points such that $f(x_0) = y_0$. The latter property of f is expressed by saying that f maps pair (X, x_0) to pair (Y, y_0) and writing $f : (X, x_0) \rightarrow (Y, y_0)$.

Consider the map $f_{\#} : \Omega(X, x_0) \rightarrow \Omega(Y, y_0) : s \mapsto f \circ s$. This map assigns to a loop its composition with f .

36.A. $f_{\#}$ maps homotopic loops to homotopic loops.

Therefore, $f_{\#}$ induces a map $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$.

36.B. $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is a homomorphism for any continuous map $f : (X, x_0) \rightarrow (Y, y_0)$.

$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is the *homomorphism induced by f* .

36.C. Let $f : (X, x_0) \rightarrow (Y, y_0)$ and $g : (Y, y_0) \rightarrow (Z, z_0)$ be (continuous) maps. Then

$$(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \rightarrow \pi_1(Z, z_0).$$

36.D. Let $f, g : (X, x_0) \rightarrow (Y, y_0)$ be continuous maps homotopic via a homotopy fixed at x_0 . Then $f_* = g_*$.

36.E. Riddle. How can we generalize Theorem 36.D to the case of freely homotopic f and g ?

36.F. Let $f : X \rightarrow Y$ be a continuous map, x_0 and x_1 points of X connected by a path $s : I \rightarrow X$. Denote $f(x_0)$ by y_0 and $f(x_1)$ by y_1 . Then the diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ T_s \downarrow & & \downarrow T_{f \circ s} \\ \pi_1(X, x_1) & \xrightarrow{f_*} & \pi_1(Y, y_1) \end{array}$$

is commutative, i.e., $T_{f \circ s} \circ f_* = f_* \circ T_s$.

36.1. Prove that the map $\mathbb{C} \setminus 0 \rightarrow \mathbb{C} \setminus 0 : z \mapsto z^3$ is not homotopic to the identity map $\mathbb{C} \setminus 0 \rightarrow \mathbb{C} \setminus 0 : z \mapsto z$.

36.2. Let X be a subset of \mathbb{R}^n . Prove that if a continuous map $f : X \rightarrow Y$ extends to a continuous map $\mathbb{R}^n \rightarrow Y$, then $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$ is a trivial homomorphism (i.e., maps everything to unit) for any $x_0 \in X$.

36.3. Prove that if a Hausdorff space X contains an open set homeomorphic to $S^1 \times S^1 \setminus (1, 1)$, then X has infinite noncyclic fundamental group.

36.3.1. Prove that a space X satisfying the conditions of 36.3 can be continuously mapped to a space with infinite noncyclic fundamental group in such a way that the map would induce an epimorphism of $\pi_1(X)$ onto this infinite group.

36.4. Prove that the fundamental group of the space $GL(n, \mathbb{C})$ of complex $n \times n$ -matrices with nonzero determinant is infinite.

§36°2 Fundamental Theorem of Algebra

Our goal here is to prove the following theorem, which at first glance has no relation to fundamental group.

36.G Fundamental Theorem of Algebra. *Every polynomial of positive degree in one variable with complex coefficients has a complex root.*

In more detail:

Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial of degree $n > 0$ in z with complex coefficients. Then there exists a complex number w such that $p(w) = 0$.

Although it is formulated in an algebraic way and called “The Fundamental Theorem of Algebra,” it has no simple algebraic proof. Its proofs usually involve topological arguments or use complex analysis. This is so because the field \mathbb{C} of complex numbers as well as the field \mathbb{R} of reals is extremely difficult to describe in purely algebraic terms: all customary constructive descriptions involve a sort of completion construction, cf. Section §17.

36.G.1 Reduction to Problem on a Map. Deduce Theorem 36.G from the following statement:

For any complex polynomial $p(z)$ of a positive degree, the zero belongs to the image of the map $\mathbb{C} \rightarrow \mathbb{C} : z \mapsto p(z)$. In other words, the formula $z \mapsto p(z)$ does not determine a map $\mathbb{C} \rightarrow \mathbb{C} \setminus 0$.

36.G.2 Estimate of Remainder. Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a complex polynomial, $q(z) = z^n$, and $r(z) = p(z) - q(z)$. Then there exists a positive real R such that $|r(z)| < |q(z)| = R^n$ for any z with $|z| = R$.

36.G.3 Lemma on Lady with Doggy. (Cf. 29.11.) A lady $q(z)$ and her dog $p(z)$ walk on the punctured plane $\mathbb{C} \setminus 0$ periodically (i.e., say, with $z \in S^1$). Prove that if the lady does not let the dog to run further than by

$|q(z)|$ from her, then the doggy's loop $S^1 \rightarrow \mathbb{C} \setminus 0 : z \mapsto p(z)$ is homotopic to the lady's loop $S^1 \rightarrow \mathbb{C} \setminus 0 : z \mapsto q(z)$.

36.G.4 Lemma for Dummies. (Cf. 29.12.) If $f : X \rightarrow Y$ is a continuous map and $s : S^1 \rightarrow X$ is a null-homotopic loop, then $f \circ s : S^1 \rightarrow Y$ is also null-homotopic.

§36x°3 Generalization of Intermediate Value Theorem

36x:A. Riddle. How to generalize Intermediate Value Theorem 12.A to the case of maps $f : D^n \rightarrow \mathbb{R}^n$?

36x:B. Find out whether Intermediate Value Theorem 12.A is equivalent to the following statement:

Let $f : D^1 \rightarrow \mathbb{R}^1$ be a continuous map. If $0 \notin f(S^0)$ and the submap $f|_{S^0} : S^0 \rightarrow \mathbb{R}^1 \setminus 0$ of f induces a nonconstant map $\pi_0(S^0) \rightarrow \pi_0(\mathbb{R}^1 \setminus 0)$, then there exists a point $x \in D^1$ such that $f(x) = 0$.

36x:C. Riddle. Suggest a generalization of Intermediate Value Theorem to maps $D^n \rightarrow \mathbb{R}^n$ which would generalize its reformulation 36x:B. To do it, you must give a definition of the induced homomorphism for homotopy groups.

36x:D. Let $f : D^n \rightarrow \mathbb{R}^n$ be a continuous map. If $f(S^{n-1})$ does not contain $0 \in \mathbb{R}^n$ and the submap $f|_{S^{n-1}} : S^{n-1} \rightarrow \mathbb{R}^n \setminus 0$ of f induces a nonconstant map

$$\pi_{n-1}(S^{n-1}) \rightarrow \pi_{n-1}(\mathbb{R}^n \setminus 0),$$

then there exists a point $x \in D^1$ such that $f(x) = 0$.

Usability of Theorem 36x:D is impeded by a condition which is difficult to check if $n > 0$. For $n = 1$, this is still possible in the frameworks of the theory developed above.

36x:1. Let $f : D^2 \rightarrow \mathbb{R}^2$ be a continuous map. If $f(S^1)$ does not contain $a \in \mathbb{R}^2$ and the circular loop $f|_{S^1} : S^1 \rightarrow \mathbb{R}^2 \setminus a$ determines a nontrivial element of $\pi_1(\mathbb{R}^2 \setminus a)$, then there exists $x \in D^2$ such that $f(x) = a$.

36x:2. Let $f : D^2 \rightarrow \mathbb{R}^2$ be a continuous map that leaves fixed each point of the boundary circle S^1 . Then $f(D^2) \supset D^2$.

36x:3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuous map and there exists a real number m such that $|f(x) - x| \leq m$ for any $x \in \mathbb{R}^2$. Prove that f is a surjection.

36x:4. Let $u, v : I \rightarrow I \times I$ be two paths such that $u(0) = (0, 0)$, $u(1) = (1, 1)$ and $v(0) = (0, 1)$, $v(1) = (1, 0)$. Prove that $u(I) \cap v(I) \neq \emptyset$.

36x:4.1. Let u, v be as in 36x:4. Prove that $0 \in \mathbb{R}^2$ is a value of the map $w : I^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto u(x) - v(y)$.

36x:5. Prove that there exist connected disjoint sets $F, G \subset I^2$ such that $(0, 0), (1, 1) \in F$ and $(0, 1), (1, 0) \in G$.

36x:6. Can we require in addition that the sets F and G satisfying the assumptions of Problem 36x:5 be closed?

36x:7. Let C be a smooth simple closed curve on the plane with two inflection points. Prove that there is a line intersecting C in four points $a, b, c,$ and d with segments $[a, b], [b, c]$ and $[c, d]$ of the same length.

§36x°4 Winding Number

As we know (see 35.F), the fundamental group of the punctured plane $\mathbb{R}^2 \setminus 0$ is isomorphic to \mathbb{Z} . There are two isomorphisms, which differ by multiplication by -1 . We choose that taking the homotopy class of the loop $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ to $1 \in \mathbb{Z}$. In terms of circular loops, the isomorphism means that to any loop $f : S^1 \rightarrow \mathbb{R}^2 \setminus 0$ we assign an integer. Roughly speaking, it is the number of times the loop goes around 0 (with account of direction).

Now we change the viewpoint in this consideration, and fix the loop, but vary the point. Let $f : S^1 \rightarrow \mathbb{R}^2$ be a circular loop and let $x \in \mathbb{R}^2 \setminus f(S^1)$. Then f determines an element in $\pi_1(\mathbb{R}^2 \setminus x) = \mathbb{Z}$ (here we choose basically the same identification of $\pi_1(\mathbb{R}^2 \setminus x)$ with \mathbb{Z} that takes 1 to the homotopy class of $t \mapsto x + (\cos 2\pi t, \sin 2\pi t)$). This number is denoted by $\text{ind}(f, x)$ and called the *winding number* or *index* of x with respect to f .

It is also convenient to characterize the number $\text{ind}(u, x)$ as follows. Along with the circular loop $u : S^1 \rightarrow \mathbb{R}^2 \setminus x$, consider the mapping $\varphi_{u,x} : S^1 \rightarrow S^1 : z \mapsto \frac{u(z)-x}{|u(z)-x|}$. The homomorphism $(\varphi_{u,x})_* : \pi_1(S^1) \rightarrow \pi_1(S^1)$ takes the generator α of the fundamental group of the circle to the element $k\alpha$, where $k = \text{ind}(u, x)$.

36x:E. The formula $x \mapsto \text{ind}(u, x)$ defines a locally constant function on $\mathbb{R}^2 \setminus u(S^1)$.

36x:8. Let $f : S^1 \rightarrow \mathbb{R}^2$ be a loop and $x, y \in \mathbb{R}^2 \setminus f(S^1)$. Prove that if $\text{ind}(f, x) \neq \text{ind}(f, y)$, then any path connecting x and y in \mathbb{R}^2 meets $f(S^1)$.

36x:9. Prove that if $u(S^1)$ is contained in a disk, while a point x is not, then $\text{ind}(u, x) = 0$.

36x:10. Find the set of values of function $\text{ind} : \mathbb{R}^2 \setminus u(S^1) \rightarrow \mathbb{Z}$ for the following loops u :

a) $u(z) = z$; b) $u(z) = \bar{z}$; c) $u(z) = z^2$; d) $u(z) = z + z^{-1} + z^2 - z^{-2}$
(here $z \in S^1 \subset \mathbb{C}$).

36x:11. Choose several loops $u : S^1 \rightarrow \mathbb{R}^2$ such that $u(S^1)$ is a bouquet of two circles (a “lemniscate”). Find the winding number with respect to these loops for various points.

36x:12. Find a loop $f : S^1 \rightarrow \mathbb{R}^2$ such that there exist points $x, y \in \mathbb{R}^2 \setminus f(S^1)$ with $\text{ind}(f, x) = \text{ind}(f, y)$, but belonging to different connected components of $\mathbb{R}^2 \setminus f(S^1)$.

36x:13. Prove that any ray R radiating from x meets $f(S^1)$ at least at $|\text{ind}(f, x)|$ points (i.e., the number of points in $f^{-1}(R)$ is not less than $|\text{ind}(f, x)|$).

36x:F. If $u : S^1 \rightarrow \mathbb{R}^2$ is a restriction of a continuous map $F : D^2 \rightarrow \mathbb{R}^2$ and $\text{ind}(u, x) \neq 0$, then $x \in F(D^2)$.

36x:G. If u and v are two circular loops in \mathbb{R}^2 with common base point (i. e., $u(1) = v(1)$) and uv is their product, then $\text{ind}(uv, x) = \text{ind}(u, x) + \text{ind}(v, x)$ for each $x \in \mathbb{R}^2 \setminus uv(S^1)$.

36x:H. Let u and v be circular loops in \mathbb{R}^2 , and $x \in \mathbb{R}^2 \setminus (u(S^1) \cup v(S^1))$. If there exists a (free) homotopy u_t , $t \in I$ connecting u and v such that $x \in \mathbb{R}^2 \setminus u_t(S^1)$ for each $t \in I$, then $\text{ind}(u, x) = \text{ind}(v, x)$.

36x:I. Let $u : S^1 \rightarrow \mathbb{C}$ be a circular loop and $a \in \mathbb{C}^2 \setminus u(S^1)$. Then

$$\text{ind}(u, a) = \frac{1}{2\pi i} \int_{S^1} \frac{|u(z) - a|}{u(z) - a} dz.$$

36x:J. Let $p(z)$ be a polynomial with complex coefficients, $R > 0$, and let $z_0 \in \mathbb{C}$. Consider the circular loop $u : S^1 \rightarrow \mathbb{C} : z \mapsto p(Rz)$. If $z_0 \in \mathbb{C} \setminus u(S^1)$, then the polynomial $p(z) - z_0$ has (counting the multiplicities) precisely $\text{ind}(u, z_0)$ roots in the open disk $B_R^2 = \{z : |z| < R\}$.

36x:K. Riddle. By what can we replace the circular loop u , the domain B_R , and the polynomial $p(z)$ so that the assertion remain valid?

§ 36x°5 Borsuk–Ulam Theorem

36x:L One-Dimensional Borsuk–Ulam. For each continuous map $f : S^1 \rightarrow \mathbb{R}^1$ there exists $x \in S^1$ such that $f(x) = f(-x)$.

36x:M Two-Dimensional Borsuk–Ulam. For each continuous map $f : S^2 \rightarrow \mathbb{R}^2$ there exists $x \in S^2$ such that $f(x) = f(-x)$.

36x:M.1 Lemma. If there exists a continuous map $f : S^2 \rightarrow \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for each $x \in S^2$, then there exists a continuous map $\varphi : \mathbb{R}P^2 \rightarrow \mathbb{R}P^1$ inducing a nonzero homomorphism $\pi_1(\mathbb{R}P^2) \rightarrow \pi_1(\mathbb{R}P^1)$.

36x:14. Prove that at each instant of time, there is a pair of antipodal points on the earth's surface where the pressures and also the temperatures are equal.

Theorems *36x:L* and *36x:M* are special cases of the following general theorem. We do not assume the reader to be ready to prove Theorem *36x:N* in the full generality, but is there another easy special case?

36x:N Borsuk–Ulam Theorem. For each continuous map $f : S^n \rightarrow \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = f(-x)$.

§ 37 Retractions and Fixed Points

§ 37°1 Retractions and Retracts

A continuous map of a topological space onto a subspace is a *retraction* if the restriction of the map to the subspace is the identity mapping. In other words, if X is a topological space and $A \subset X$, then $\rho : X \rightarrow A$ is a retraction if ρ is continuous and $\rho|_A = \text{id}_A$.

37.A. Let ρ be a continuous map of a space X onto its subspace A . Then the following statements are equivalent:

- (a) ρ is a retraction,
- (b) $\rho(a) = a$ for any $a \in A$,
- (c) $\rho \circ \text{in} = \text{id}_A$,
- (d) $\rho : X \rightarrow A$ is an extension of the identity mapping $A \rightarrow A$.

A subspace A of a space X is a *retract* of X if there exists a retraction $X \rightarrow A$.

37.B. Any one-point subset is a retract.

Two-point set may be a non-retract.

37.C. Any subset of \mathbb{R} consisting of two points is not a retract of \mathbb{R} .

37.1. If A is a retract of X and B is a retract of A , then B is a retract of X .

37.2. If A is a retract of X and B is a retract of Y , then $A \times B$ is a retract of $X \times Y$.

37.3. A closed interval $[a, b]$ is a retract of \mathbb{R} .

37.4. An open interval (a, b) is not a retract of \mathbb{R} .

37.5. What topological properties of ambient space are inherited by a retract?

37.6. Prove that a retract of a Hausdorff space is closed.

37.7. Prove that the union of Y -axis and the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin \frac{1}{x}\}$ is not a retract of \mathbb{R}^2 and, moreover, is not a retract of any of its neighborhoods.

37.D. S^0 is not a retract of D^1 .

The role of the notion of retract is clarified by the following theorem.

37.E. A subset A of a topological space X is a retract of X iff for each space Y each continuous map $A \rightarrow Y$ extends to a continuous map $X \rightarrow Y$.

§ 37°2 Fundamental Group and Retractions

37.F. If $\rho : X \rightarrow A$ is a retraction, $i : A \rightarrow X$ is the inclusion, and $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \rightarrow \pi_1(A, x_0)$ is an epimorphism and $i_* : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ is a monomorphism.

37.G. Riddle. Which of the two statements of Theorem 37.F (about ρ_* or i_*) is easier to use for proving that a set $A \subset X$ is not a retract of X ?

37.H Borsuk Theorem in Dimension 2. S^1 is not a retract of D^2 .

37.8. Is the projective line a retract of the projective plane?

The following problem is more difficult than 37.H in the sense that its solution is not a straightforward consequence of Theorem 37.F, but rather demands to reexamine the arguments used in proof of 37.F.

37.9. Prove that the boundary circle of Möbius band is not a retract of Möbius band.

37.10. Prove that the boundary circle of a handle is not a retract of the handle.

The Borsuk Theorem in its whole generality cannot be deduced like Theorem 37.H from Theorem 37.F. However, it can be proven using a generalization of 37.F to higher homotopy groups. Although we do not assume that you can successfully prove it now relying only on the tools provided above, we formulate it here.

37.I Borsuk Theorem. *The $(n - 1)$ -sphere S^{n-1} is not a retract of the n -disk D^n .*

At first glance this theorem seems to be useless. Why could it be interesting to know that a map with a very special property of being a retraction does not exist in this situation? However, in mathematics nonexistence theorems are often closely related to theorems that may seem to be more attractive. For instance, the Borsuk Theorem implies the Brouwer Theorem discussed below. But prior to this we must introduce an important notion related to the Brouwer Theorem.

§37°3 Fixed-Point Property.

Let $f : X \rightarrow X$ be a continuous map. A point $a \in X$ is a *fixed point* of f if $f(a) = a$. A space X has the *fixed-point property* if every continuous map $X \rightarrow X$ has a fixed point. The fixed point property implies solvability of a wide class of equations.

37.11. Prove that the fixed point property is a topological property.

37.12. A closed interval $[a, b]$ has the fixed point property.

37.13. Prove that if a topological space has the fixed point property, then so does each of its retracts.

37.14. Let X and Y be two topological spaces, $x_0 \in X$ and $y_0 \in Y$. Prove that X and Y have the fixed point property iff so does their bouquet $X \vee Y = X \sqcup Y / [x_0 \sim y_0]$.

37.15. Prove that any finite tree (i.e., a connected space obtained from a finite collection of closed intervals by some identifying of their endpoints such that deleting of an internal point of each of the segments makes the space disconnected, see §42x°4) has the fixed-point property. Is this statement true for infinite trees?

37.16. Prove that \mathbb{R}^n with $n > 0$ does not have the fixed point property.

37.17. Prove that S^n does not have the fixed point property.

37.18. Prove that $\mathbb{R}P^n$ with odd n does not have the fixed point property.

37.19*. Prove that $\mathbb{C}P^n$ with odd n does not have the fixed point property.

Information. $\mathbb{R}P^n$ and $\mathbb{C}P^n$ with any even n have the fixed point property.

37.J Brouwer Theorem. D^n has the fixed point property.

37.J.1. Deduce from Borsuk Theorem in dimension n (i.e., from the statement that S^{n-1} is not a retract of D^n) Brouwer Theorem in dimension n (i.e., the statement that any continuous map $D^n \rightarrow D^n$ has a fixed point).

37.K. Derive the Borsuk Theorem from the Brouwer Theorem.

The existence of fixed points can follow not only from topological arguments.

37.20. Prove that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a periodic affine transformation (i.e., $\underbrace{f \circ \cdots \circ f}_p \text{ times} = \text{id}_{\mathbb{R}^n}$ for a certain p), then f has a fixed point.

§38 Homotopy Equivalences

§38°1 Homotopy Equivalence as Map

Let X and Y be two topological spaces, $f : X \rightarrow Y$ and $g : Y \rightarrow X$ continuous maps. Consider the compositions $f \circ g : Y \rightarrow Y$ and $g \circ f : X \rightarrow X$. They would be equal to the corresponding identity maps if f and g were mutually inverse homeomorphisms. If $f \circ g$ and $g \circ f$ are only homotopic to the identity maps, then f and g are said to be *homotopy inverse* to each other. If a continuous map f possesses a homotopy inverse map, then f is a *homotopy invertible map* or a *homotopy equivalence*.

38.A. Prove the following properties of homotopy equivalences:

- (a) any homeomorphism is a homotopy equivalence,
- (b) a map homotopy inverse to a homotopy equivalence is a homotopy equivalence,
- (c) the composition of two homotopy equivalences is a homotopy equivalence.

38.1. Find a homotopy equivalence that is not a homeomorphism.

§38°2 Homotopy Equivalence as Relation

Two topological spaces X and Y are *homotopy equivalent* if there exists a homotopy equivalence $X \rightarrow Y$.

38.B. Homotopy equivalence of topological spaces is an equivalence relation.

The classes of homotopy equivalent spaces are *homotopy types*. Thus homotopy equivalent spaces are said to be of the same homotopy type.

38.2. Prove that homotopy equivalent spaces have the same number of path-connected components.

38.3. Prove that homotopy equivalent spaces have the same number of connected components.

38.4. Find an infinite series of topological spaces that belong to the same homotopy type, but are pairwise not homeomorphic.

§38°3 Deformation Retraction

A retraction $\rho : X \rightarrow A$ is a *deformation retraction* if its composition in $\circ \rho$ with the inclusion $\text{in} : A \rightarrow X$ is homotopic to the identity id_X . If $\text{in} \circ \rho$ is A -homotopic to id_X , then ρ is a *strong deformation retraction*. If X admits a (strong) deformation retraction onto A , then A is a (*strong*) *deformation retract* of X .

38.C. Each deformation retraction is a homotopy equivalence.

38.D. If A is a deformation retract of X , then A and X are homotopy equivalent.

38.E. Any two deformation retracts of one and the same space are homotopy equivalent.

38.F. If A is a deformation retract of X and B is a deformation retract of Y , then $A \times B$ is a deformation retract of $X \times Y$.

§38°4 Examples

38.G. Circle S^1 is a deformation retract of $\mathbb{R}^2 \setminus 0$.

38.5. Prove that the Möbius strip is homotopy equivalent to a circle.

38.6. Classify letters of Latin alphabet up to homotopy equivalence.

38.H. Prove that a plane with s punctures is homotopy equivalent to a union of s circles intersecting in a single point.

38.I. Prove that the union of a diagonal of a square and the contour of the same square is homotopy equivalent to a union of two circles intersecting in a single point.

38.7. Prove that a handle is homotopy equivalent to a bouquet of two circles. (E.g., construct a deformation retraction of the handle to a union of two circles intersecting in a single point.)

This can be proved in various ways. For example, we can produce circles lying in the handle \mathcal{H} whose union is a strong deformation retract of \mathcal{H} . For this purpose, we present the handle as a result of factorizing the annulus $A = \{z \mid \frac{1}{2} \leq |z| \leq 1\}$ by the following relation: $e^{i\varphi} \sim -e^{-i\varphi}$ for $\varphi \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, and $e^{i\varphi} \sim e^{-i\varphi}$ for $\varphi \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. The image of the standard unit circle under the factorization by the above equivalence relation is the required bouquet of two circles lying in of the handle. The formula $H(z, t) = (1-t)z + t\frac{z}{|z|}$ determines a homotopy between the identical mapping of A and the mapping $z \mapsto \frac{z}{|z|}$ of A onto the outer rim of A , and $H(z, t) = z$ for all $z \in S^1$ and $t \in I$. The quotient mapping of H is the required homotopy.

38.8. Prove that a handle is homotopy equivalent to a union of three arcs with common endpoints (i.e., letter θ).

38.9. Prove that the space obtained from S^2 by identification of a two (distinct) points is homotopy equivalent to the union of a two-sphere and a circle intersecting in a single point.

38.10. Prove that the space $\{(p, q) \in \mathbb{C} : z^2 + pz + q \text{ has two distinct roots}\}$ of quadratic complex polynomials with distinct roots is homotopy equivalent to the circle.

38.11. Prove that the space $GL(n, \mathbb{R})$ of invertible $n \times n$ real matrices is homotopy equivalent to the subspace $O(n)$ consisting of orthogonal matrices.

38.12. Riddle. Is there any relation between a solution of the preceding problem and the Gram–Schmidt orthogonalization? Can the Gram–Schmidt orthogonalization algorithm be considered a deformation retraction?

38.13. Construct the following deformation retractions: (a) $\mathbb{R}^3 \setminus \mathbb{R}^1 \rightarrow S^1$; (b) $\mathbb{R}^n \setminus \mathbb{R}^m \rightarrow S^{n-m-1}$; (c) $S^3 \setminus S^1 \rightarrow S^1$; (d) $S^n \setminus S^m \rightarrow S^{n-m-1}$ (e) $\mathbb{R}P^n \setminus \mathbb{R}P^m \rightarrow \mathbb{R}P^{n-m-1}$.

§38°5 Deformation Retraction versus Homotopy Equivalence

38.J. Spaces of Problem 38.I cannot be embedded one to another. On the other hand, they can be embedded as deformation retracts in the plane with two punctures.

Deformation retractions comprise a special type of homotopy equivalences. For example, they are easier to visualize. However, as follows from 38.J, it may happen that two spaces are homotopy equivalent, but none of them can be embedded in the other one, and so none of them is homeomorphic to a deformation retract of the other one. Therefore, deformation retractions seem to be insufficient for establishing homotopy equivalences.

However, this is not the case:

38.14*. Prove that any two homotopy equivalent spaces can be embedded as deformation retracts in the same topological space.

§38°6 Contractible Spaces

A topological space X is *contractible* if the identity map $\text{id} : X \rightarrow X$ is null-homotopic.

38.15. Show that \mathbb{R} and I are contractible.

38.16. Prove that any contractible space is path-connected.

38.17. Prove that the following three statements about a topological space X are equivalent:

- (a) X is contractible,
- (b) X is homotopy equivalent to a point,
- (c) there exists a deformation retraction of X onto a point,
- (d) any point a of X is a deformation retract of X ,
- (e) any continuous map of any topological space Y to X is null-homotopic,
- (f) any continuous map of X to any topological space Y is null-homotopic.

38.18. Is it true that if X is a contractible space, then for any topological space Y

- (a) any two continuous maps $X \rightarrow Y$ are homotopic?
- (b) any two continuous maps $Y \rightarrow X$ are homotopic?

38.19. Find out if the spaces on the following list are contractible:

- (a) \mathbb{R}^n ,
- (b) a convex subset of \mathbb{R}^n ,
- (c) a star-shaped subset of \mathbb{R}^n ,

- (d) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \leq 1\}$,
 (e) a finite tree (i.e., a connected space obtained from a finite collection of closed intervals by some identifying of their endpoints such that deleting of an internal point of each of the segments makes the space disconnected, see §42x°4.)

38.20. Prove that $X \times Y$ is contractible iff both X and Y are contractible.

§38°7 Fundamental Group and Homotopy Equivalences

38.K. Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be homotopy inverse maps, and let $x_0 \in X$ and $y_0 \in Y$ be two points such that $f(x_0) = y_0$ and $g(y_0) = x_0$ and, moreover, the homotopies relating $f \circ g$ to id_Y and $g \circ f$ to id_X are fixed at y_0 and x_0 , respectively. Then f_* and g_* are inverse to each other isomorphisms between groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

38.L Corollary. If $\rho : X \rightarrow A$ is a strong deformation retraction, $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \rightarrow \pi_1(A, x_0)$ and $\text{in}_* : \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ are mutually inverse isomorphisms.

38.21. Calculate the fundamental group of the following spaces:

- (a) $\mathbb{R}^3 \setminus \mathbb{R}^1$, (b) $\mathbb{R}^N \setminus \mathbb{R}^n$, (c) $\mathbb{R}^3 \setminus S^1$, (d) $\mathbb{R}^N \setminus S^n$,
 (e) $S^3 \setminus S^1$, (f) $S^N \setminus S^k$, (g) $\mathbb{R}P^3 \setminus \mathbb{R}P^1$, (h) handle,
 (i) Möbius band, (j) sphere with s holes,
 (k) Klein bottle with a point re- (l) Möbius band with s holes.
 moved,

38.22. Prove that the boundary circle of the Möbius band standardly embedded in \mathbb{R}^3 (see 21.18) could not be the boundary of a disk embedded in \mathbb{R}^3 in such a way that its interior does not intersect the band.

38.23. 1) Calculate the fundamental group of the space Q of all complex polynomials $ax^2 + bx + c$ with distinct roots. 2) Calculate the fundamental group of the subspace Q_1 of Q consisting of polynomials with $a = 1$ (unital polynomials).

38.24. Riddle. Can you solve 38.23 along the lines of deriving the customary formula for the roots of a quadratic trinomial?

38.M. Suppose that the assumptions of Theorem 38.K are weakened as follows: $g(y_0) \neq x_0$ and/or the homotopies relating $f \circ g$ to id_Y and $g \circ f$ to id_X are *not* fixed at y_0 and x_0 , respectively. How would f_* and g_* be related? Would $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ be isomorphic?

§39 Covering Spaces via Fundamental Groups

§39°1 Homomorphisms Induced by Covering Projections

39.A. Let $p : X \rightarrow B$ be a covering, $x_0 \in X$, $b_0 = p(x_0)$. Then $p_* : \pi_1(X, x_0) \rightarrow \pi_1(B, b_0)$ is a monomorphism. Cf. 34.C.

The image of the monomorphism $p_* : \pi_1(X, x_0) \rightarrow \pi_1(B, b_0)$ induced by the covering projection $p : X \rightarrow B$ is the *group of the covering p with base point x_0* .

39.B. Riddle. Is the group of covering determined by the covering?

39.C Group of Covering versus Lifting of Loops. Describe loops in the base space of a covering, whose homotopy classes belong to the group of the covering, in terms provided by Path Lifting Theorem 34.B.

39.D. Let $p : X \rightarrow B$ be a covering, let $x_0, x_1 \in X$ belong to the same path-component of X , and $b_0 = p(x_0) = p(x_1)$. Then $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ are conjugate subgroups of $\pi_1(B, b_0)$ (i.e., there exists an $\alpha \in \pi_1(B, b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1}p_*(\pi_1(X, x_0))\alpha$).

39.E. Let $p : X \rightarrow B$ be a covering, $x_0 \in X$, $b_0 = p(x_0)$. For each $\alpha \in \pi_1(B, b_0)$, there exists an $x_1 \in p^{-1}(b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1}p_*(\pi_1(X, x_0))\alpha$.

39.F. Let $p : X \rightarrow B$ be a covering in a narrow sense, $G \subset \pi_1(B, b_0)$ the group of this covering with a base point x_0 . A subgroup $H \subset \pi_1(B, b_0)$ is a group of the same covering iff H is conjugate to G .

§39°2 Number of Sheets

39.G Number of Sheets and Index of Subgroup. Let $p : X \rightarrow B$ be a covering in a narrow sense with finite number of sheets. Then the number of sheets is equal to the index of the group of this covering.

39.H Sheets and Right Cosets. Let $p : X \rightarrow B$ be a covering in a narrow sense, $b_0 \in B$, and $x_0 \in p^{-1}(b_0)$. Construct a natural bijection of $p^{-1}(b_0)$ and the set $p_*(\pi_1(X, x_0)) \backslash \pi_1(B, b_0)$ of right cosets of the group of the covering in the fundamental group of the base space.

39.1 Number of Sheets in Universal Covering. The number of sheets of a universal covering equals the order of the fundamental group of the base space.

39.2 Nontrivial Covering Means Nontrivial π_1 . Any topological space that has a nontrivial path-connected covering space has a nontrivial fundamental group.

39.3. What numbers can appear as the number of sheets of a covering of the Möbius strip by the cylinder $S^1 \times I$?

39.4. What numbers can appear as the number of sheets of a covering of the Möbius strip by itself?

39.5. What numbers can appear as the number of sheets of a covering of the Klein bottle by torus?

39.6. What numbers can appear as the number of sheets of a covering of the Klein bottle by itself?

39.7. What numbers can appear as the numbers of sheets for a covering of the Klein bottle by plane \mathbb{R}^2 ?

39.8. What numbers can appear as the numbers of sheets for a covering of the Klein bottle by $S^1 \times \mathbb{R}$?

§39°3 Hierarchy of Coverings

Let $p : X \rightarrow B$ and $q : Y \rightarrow B$ be two coverings, $x_0 \in X$, $y_0 \in Y$, and $p(x_0) = q(y_0) = b_0$. The covering q with base point y_0 is *subordinate* to p with base point x_0 if there exists a map $\varphi : X \rightarrow Y$ such that $q \circ \varphi = p$ and $\varphi(x_0) = y_0$. In this case, the map φ is a *subordination*.

39.I. A subordination is a covering map.

39.J. If a subordination exists, then it is unique. Cf. 34.B.

Two coverings $p : X \rightarrow B$ and $q : Y \rightarrow B$ are *equivalent* if there exists a homeomorphism $h : X \rightarrow Y$ such that $p = q \circ h$. In this case, h and h^{-1} are *equivalences*.

39.K. If two coverings are mutually subordinate, then the corresponding subordinations are equivalences.

39.L. The equivalence of coverings is, indeed, an equivalence relation in the set of coverings with a given base space.

39.M. Subordination determines a nonstrict partial order in the set of equivalence classes of coverings with a given base.

39.9. What equivalence class of coverings is minimal (i.e., subordinate to all other classes)?

39.N. Let $p : X \rightarrow B$ and $q : Y \rightarrow B$ be coverings, $x_0 \in X$, $y_0 \in Y$ and $p(x_0) = q(y_0) = b_0$. If q with base point y_0 is subordinate to p with base point x_0 , then the group of covering p is contained in the group of covering q , i.e., $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$.

§39x°4 Existence of Subordinations

A topological space X is *locally path-connected* if for each point $a \in X$ and each neighborhood U of a the point a has a path-connected neighborhood $V \subset U$.

39x:1. Find a path connected, but not locally path connected topological space.

39x:A. Let B be a locally path-connected space, $p : X \rightarrow B$ and $q : Y \rightarrow B$ be coverings in a narrow sense, $x_0 \in X$, $y_0 \in Y$ and $p(x_0) = q(y_0) = b_0$. If $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$, then q is subordinate to p .

39x:A.1. Under the conditions of **39x:A**, if two paths $u, v : I \rightarrow X$ have the same initial point x_0 and a common final point, then the paths that cover $p \circ u$ and $p \circ v$ and have the same initial point y_0 also have the same final point.

39x:A.2. Under the conditions of **39x:A**, the map $X \rightarrow Y$ defined by **39x:A.1** (guess, what this map is!) is continuous.

39x:2. Construct an example proving that the hypothesis of local path connectedness in **39x:A.2** and **39x:A** is necessary.

39x:B. Two coverings $p : X \rightarrow B$ and $q : Y \rightarrow B$ with a common locally path-connected base are equivalent iff for some $x_0 \in X$ and $y_0 \in Y$ with $p(x_0) = q(y_0) = b_0$ the groups $p_*(\pi_1(X, x_0))$ and $q_*(\pi_1(Y, y_0))$ are conjugate in $\pi_1(B, b_0)$.

39x:3. Construct an example proving that the assumption of local path connectedness of the base in **39x:B** is necessary.

§39x°5 Micro Simply Connected Spaces

A topological space X is *micro simply connected* if each point $a \in X$ has a neighborhood U such that the inclusion homomorphism $\pi_1(U, a) \rightarrow \pi_1(X, a)$ is trivial.

39x:4. Any simply connected space is micro simply connected.

39x:5. Find a micro simply connected, but not simply connected space.

A topological space is *locally contractible at point a* if each neighborhood U of a contains a neighborhood V of a such that the inclusion $V \rightarrow U$ is null-homotopic. A topological space is *locally contractible* if it is locally contractible at each of its points.

39x:6. Any finite topological space is locally contractible.

39x:7. Any locally contractible space is micro simply connected.

39x:8. Find a space which is not micro simply connected.

In the literature, the micro simply connectedness is also called *weak local simply connectedness*, while a strong local simply connectedness is the following property: any neighborhood U of any point x contains a neighborhood V such that any loop at x in V is null-homotopic in U .

39x:9. Find a micro simply connected space which is not strong locally simply connected.

§39x°6 Existence of Coverings

39x:C. *A space having a universal covering space is micro simply connected.*

39x:D Existence of Covering With a Given Group. *If a topological space B is path connected, locally path connected, and micro simply connected, then for any $b_0 \in B$ and any subgroup π of $\pi_1(B, b_0)$ there exists a covering $p : X \rightarrow B$ and a point $x_0 \in X$ such that $p(x_0) = b_0$ and $p_*(\pi_1(X, x_0)) = \pi$.*

39x:D.1. Suppose that in the assumptions of Theorem 39x:D there exists a covering $p : X \rightarrow B$ satisfying all requirements of this theorem. For each $x \in X$, describe all paths in B that are p -images of paths connecting x_0 to x in X .

39x:D.2. Does the solution of Problem 39x:D.1 determine an equivalence relation in the set of all paths in B starting at b_0 , so that we obtain a one-to-one correspondence between the set X and the set of equivalence classes?

39x:D.3. Describe a topology in the set of equivalence classes from 39x:D.2 such that the natural bijection between X and this set be a homeomorphism.

39x:D.4. Prove that the reconstruction of X and $p : X \rightarrow B$ provided by problems 39x:D.1–39x:D.4 under the assumptions of Theorem 39x:D determine a covering whose existence is claimed by Theorem 39x:D.

Essentially, assertions 39x:D.1–39x:D.3 imply the uniqueness of the covering with a given group. More precisely, the following assertion holds true.

39x:E Uniqueness of the Covering With a Given Group. *Assume that B is path-connected, locally path-connected, and micro simply connected. Let $p : X \rightarrow B$ and $q : Y \rightarrow B$ be two coverings, and let $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0))$. Then the coverings p and q are equivalent, i.e., there exists a homeomorphism $f : X \rightarrow Y$ such that $f(x_0) = y_0$ and $p \circ f = q$.*

39x:F Classification of Coverings Over a Good Space. *There is a one-to-one correspondence between classes of equivalent coverings (in a narrow sense) over a path-connected, locally path-connected, and micro simply connected space B with base point b_0 , on the one hand, and conjugacy classes of subgroups of $\pi_1(B, b_0)$, on the other hand. This correspondence identifies the hierarchy of coverings (ordered by subordination) with the hierarchy of subgroups (ordered by inclusion).*

Under the correspondence of Theorem 39x:F, the trivial subgroup corresponds to a covering with simply connected covering space. Since this covering subordinates any other covering with the same base space, it is said to be *universal*.

39x:10. Describe all coverings of the following spaces up to equivalence and subordination:

- (a) circle S^1 ;
- (b) punctured plane $\mathbb{R}^2 \setminus 0$;
- (c) Möbius strip;
- (d) four point digital circle (the space formed by 4 points, a, b, c, d ; with the base of open sets formed by $\{a\}$, $\{c\}$, $\{a, b, c\}$ and $\{c, d, a\}$)
- (e) torus $S^1 \times S^1$;

§39x°7 Action of Fundamental Group on Fiber

39x:G Action of π_1 on Fiber. Let $p : X \rightarrow B$ be a covering, $b_0 \in B$. Construct a natural right action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$.

39x:H. When the action in 39x:G is transitive?

§39x°8 Automorphisms of Covering

A homeomorphism $\varphi : X \rightarrow X$ is an *automorphism* of a covering $p : X \rightarrow B$ if $p \circ \varphi = p$.

39x:I. Automorphisms of a covering form a group.

Denote the group of automorphisms of a covering $p : X \rightarrow B$ by $\text{Aut}(p)$.

39x:J. An automorphism $\varphi : X \rightarrow X$ of covering $p : X \rightarrow B$ is recovered from the image $\varphi(x_0)$ of any $x_0 \in X$. Cf. 39.J.

39x:K. Any two-fold covering has a nontrivial automorphism.

39x:11. Find a three-fold covering without nontrivial automorphisms.

Let G be a group and H its subgroup. Recall that the *normalizer* $Nr(H)$ of H is the subset of G consisting of $g \in G$ such that $g^{-1}Hg = H$. This is a subgroup of G , which contains H as a normal subgroup. So, $Nr(H)/H$ is a group.

39x:L. Let $p : X \rightarrow B$ be a covering, $x_0 \in X$ and $b_0 = p(x_0)$. Construct a map $\pi_1(B, b_0) \rightarrow p^{-1}(b_0)$ inducing a bijection of the set $p_*(\pi_1(X, x_0)) \setminus \pi_1(B, b_0)$ of right cosets onto $p^{-1}(b_0)$.

39x:M. Show that the bijection $p_*(\pi_1(X, x_0)) \setminus \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$ from 39x:L maps the set of images of a point x_0 under all automorphisms of a covering $p : X \rightarrow B$ to the group $Nr(p_*(\pi_1(X, x_0)))/p_*(\pi_1(X, x_0))$.

39x:N. For any covering $p : X \rightarrow B$ in a narrow sense, there is a natural injective map $\text{Aut}(p)$ to the group $Nr(p_*(\pi_1(X, x_0)))/p_*(\pi_1(X, x_0))$. This map is an *antihomomorphism*.¹

¹Recall that a map $\varphi : G \rightarrow H$ from a group G to a group H is an *antihomomorphism* if $\varphi(ab) = \varphi(b)\varphi(a)$ for any $a, b \in G$.

39x:O. Under assumptions of Theorem 39x:N, if the base space B is locally path connected, then the antihomomorphism $\text{Aut}(p) \rightarrow \text{Nr}(p_*(\pi_1(X, x_0)))/p_*(\pi_1(X, x_0))$ is bijective.

§39x°9 Regular Coverings

39x:P Regularity of Covering. Let $p : X \rightarrow B$ be a covering in a narrow sense, $b_0 \in B$, $x_0 \in p^{-1}(b_0)$. The following conditions are equivalent:

- (a) $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(B, b_0)$;
- (b) $p_*(\pi_1(X, x))$ is a normal subgroup of $\pi_1(B, p(x))$ for each $x \in X$;
- (c) all groups $p_*\pi_1(X, x)$ for $x \in p^{-1}(b)$ are the same;
- (d) for any loop $s : I \rightarrow B$ either every path in X covering s is a loop (independent on the its initial point) or none of them is a loop;
- (e) the automorphism group acts transitively on $p^{-1}(b_0)$.

A covering satisfying to (any of) the equivalent conditions of Theorem 39x:P is said to be *regular*.

39x:12. The coverings $\mathbb{R} \rightarrow S^1 : x \mapsto e^{2\pi ix}$ and $S^1 \rightarrow S^1 : z \mapsto z^n$ for integer $n > 0$ are regular.

39x:Q. The automorphism group of a regular covering $p : X \rightarrow B$ is naturally anti-isomorphic to the quotient group $\pi_1(B, b_0)/p_*\pi_1(X, x_0)$ of the group $\pi_1(B, b_0)$ by the group of the covering for any $x_0 \in p^{-1}(b_0)$.

39x:R Classification of Regular Coverings Over a Good Base. There is a one-to-one correspondence between classes of equivalent coverings (in a narrow sense) over a path connected, locally path connected, and micro simply connected space B with a base point b_0 , on one hand, and anti-epimorphisms $\pi_1(B, b_0) \rightarrow G$, on the other hand.

Algebraic properties of the automorphism group of a regular covering are often referred to as if they were properties of the covering itself. For instance, a *cyclic covering* is a regular covering with cyclic automorphism group, an *Abelian covering* is a regular covering with Abelian automorphism group, etc.

39x:13. Any two-fold covering is regular.

39x:14. Which coverings considered in Problems of Section §33 are regular? Is out there any nonregular covering?

39x:15. Find a three-fold nonregular covering of a bouquet of two circles.

39x:16. Let $p : X \rightarrow B$ be a regular covering, $Y \subset X$, $C \subset B$, and let $q : Y \rightarrow C$ be a submap of p . Prove that if q is a covering, then this covering is regular.

§39x°10 Lifting and Covering Maps

39x:S. Riddle. Let $p : X \rightarrow B$ and $f : Y \rightarrow B$ be continuous maps. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $p(x_0) = f(y_0)$. Formulate in terms of homomorphisms $p_* : \pi_1(X, x_0) \rightarrow \pi_1(B, p(x_0))$ and $f_* : \pi_1(Y, y_0) \rightarrow \pi_1(B, f(y_0))$ a necessary condition for existence of a lifting $\tilde{f} : Y \rightarrow X$ of f such that $\tilde{f}(y_0) = x_0$. Find an example where this condition is not sufficient. What additional assumptions can make it sufficient?

39x:T Theorem on Lifting a Map. *Let $p : X \rightarrow B$ be a covering in a narrow sense and $f : Y \rightarrow B$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $p(x_0) = f(y_0)$. If Y is a locally path-connected space and $f_*\pi_1(Y, y_0) \subset p_*\pi_1(X, x_0)$, then there exists a unique continuous map $\tilde{f} : Y \rightarrow X$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = x_0$.*

39x:U. *Let $p : X \rightarrow B$ and $q : Y \rightarrow C$ be coverings in a narrow sense and $f : B \rightarrow C$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $fp(x_0) = q(y_0)$. If there exists a continuous map $F : X \rightarrow Y$ such that $fp = qF$ and $F(x_0) = y_0$, then $f_*p_*\pi_1(X, x_0) \subset q_*\pi_1(Y, y_0)$.*

39x:V Theorem on Covering of a Map. *Let $p : X \rightarrow B$ and $q : Y \rightarrow C$ be coverings in a narrow sense and $f : B \rightarrow C$ be a continuous map. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $fp(x_0) = q(y_0)$. If Y is locally path connected and $f_*p_*\pi_1(X, x_0) \subset q_*\pi_1(Y, y_0)$, then there exists a unique continuous map $F : X \rightarrow Y$ such that $fp = qF$ and $F(x_0) = y_0$.*

§39x°11 Induced Coverings

39x:W. *Let $p : X \rightarrow B$ be a covering and $f : A \rightarrow B$ a continuous map. Denote by W a subspace of $A \times X$ consisting of points (a, x) such that $f(a) = p(x)$. Let $q : W \rightarrow A$ be a restriction of $A \times X \rightarrow A$. Then $q : W \rightarrow A$ is a covering with the same number of sheets as p .*

A covering $q : W \rightarrow A$ obtained as in Theorem 39x:W is said to be *induced* from $p : X \rightarrow B$ by $f : A \rightarrow B$.

39x:17. Represent coverings from problems 33.D and 33.F as induced from $\mathbb{R} \rightarrow S^1 : x \mapsto e^{2\pi ix}$.

39x:18. Which of the coverings considered above can be induced from the covering of Problem 35.7?

§39x°12 High-Dimensional Homotopy Groups of Covering Space

39x:X. *Let $p : X \rightarrow B$ be a covering. Then for any continuous map $s : I^n \rightarrow B$ and a lifting $u : I^{n-1} \rightarrow X$ of the restriction $s|_{I^{n-1}}$ there exists a unique lifting of s extending u .*

39x:Y. For any covering $p : X \rightarrow B$ and points $x_0 \in X$, $b_0 \in B$ such that $p(x_0) = b_0$ the homotopy groups $\pi_r(X, x_0)$ and $\pi_r(B, b_0)$ with $r > 1$ are canonically isomorphic.

39x:Z. Prove that homotopy groups of dimensions greater than 1 of circle, torus, Klein bottle and Möbius strip are trivial.

Proofs and Comments

36.A This follows from 29.I.

36.B Let $[u], [v] \in \pi_1(X, x_0)$. Since $f \circ (uv) = (f \circ u)(f \circ v)$, we have $f_{\#}(uv) = f_{\#}(u)f_{\#}(v)$ and

$$\begin{aligned} f_*([u][v]) &= f_*([uv]) = [f_{\#}(uv)] = [f_{\#}(u)f_{\#}(v)] = \\ &= [f_{\#}(u)][f_{\#}(v)] = f_*([u])f_*([v]). \end{aligned}$$

36.C Let $[u] \in \pi_1(X, x_0)$. Since $(g \circ f)_{\#}(u) = g \circ f \circ u = g_{\#}(f_{\#}(u))$, consequently,

$$(g \circ f)_*([u]) = [(g \circ f)_{\#}(u)] = [g_{\#}(f_{\#}(u))] = g_*([f_{\#}(u)]) = g_*(f_*([u])),$$

thus, $(g \circ f)_* = g_* \circ f_*$.

36.D Let $H : X \times I \rightarrow Y$ be a homotopy between f and g , and let $H(x_0, t) = y_0$ for all $t \in I$; u is a certain loop in X . Consider a mapping $h = H \circ (u \times \text{id}_I)$, thus, $h : (\tau, t) \mapsto H(u(\tau), t)$. Then $h(\tau, 0) = H(u(\tau), 0) = f(u(\tau))$ and $h(\tau, 1) = H(u(\tau), 1) = g(u(\tau))$, so that h is a homotopy between the loops $f \circ u$ and $g \circ u$. Furthermore, $h(0, t) = H(u(0), t) = H(x_0, t) = y_0$, and we similarly have $h(1, t) = y_0$, therefore, h is a homotopy between the loops $f_{\#}(u)$ and $g_{\#}(u)$, whence

$$f_*([u]) = [f_{\#}(u)] = [g_{\#}(u)] = g_*([u]).$$

36.E Let H be a homotopy between the mappings f and g and the loop s is defined by the formula $s(t) = H(x_0, t)$. By assertion 32.2, $g_* = T_s \circ f_*$.

36.F This obviously follows from the equality

$$f_{\#}(s^{-1}us) = (f \circ s)^{-1}f_{\#}(u)(f \circ s).$$

36.G.1 This is the assertion of Theorem 36.G.

36.G.2 For example, it is sufficient to take R such that

$$R > \max\{1, |a_1| + |a_2| + \dots + |a_n|\}.$$

36.G.3 Use the rectilinear homotopy $h(z, t) = tp(z) + (1-t)q(z)$. It remains to verify that $h(z, t) \neq 0$ for all z and t . Indeed, since $|p(z) - q(z)| < q(z)$ by assumption, we have

$$|h(z, t)| \geq |q(z)| - t|p(z) - q(z)| \geq |q(z)| - |p(z) - q(z)| > 0.$$

36.G.4 Indeed, this is a quite obvious lemma; see 36.A.

36.G Take a number R satisfying the assumptions of assertion 36.G.2 and consider the loop $u : u(t) = Re^{2\pi it}$. The loop u , certainly, is null-homotopic in \mathbb{C} . Now we assume that $p(z) \neq 0$ for all z with $|z| \leq R$.

Then the loop $p \circ u$ is null-homotopic in $\mathbb{C} \setminus 0$, by 36.G.3, and the loop $q \circ u$ is null-homotopic in $\mathbb{C} \setminus 0$. However, $(q \circ u)(t) = R^n e^{2\pi i n t}$, therefore, this loop is not null-homotopic. A contradiction.

36x:A See 36x:D.

36x:B Yes, it is.

36x:C See 36x:D.

36x:D Let $i : S^{n-1} \rightarrow D^n$ be the inclusion. Assume that $f(x) \neq 0$ for all $x \in D^n$. We preserve the designation f for the submapping $D^n \rightarrow \mathbb{R}^n \setminus 0$ and consider the inclusion homomorphisms $i_* : \pi_{n-1}(S^{n-1}) \rightarrow \pi_{n-1}(D^n)$ and $f_* : \pi_{n-1}(D^n) \rightarrow \pi_{n-1}(\mathbb{R}^n \setminus 0)$. Since all homotopy groups of D^n are trivial, the composition $(f \circ i)_* = f_* \circ i_*$ is a zero homomorphism. However, the composition $f \circ i$ is the mapping f_0 , which, by assumption, induces a nonzero homomorphism $\pi_{n-1}(S^{n-1}) \rightarrow \pi_{n-1}(\mathbb{R}^n \setminus 0)$.

36x:E Consider a circular neighborhood U of x disjoint with the image $u(S^1)$ of the circular loop under consideration and let $y \in U$. Join x and y by a rectilinear path $s : t \mapsto ty + (1 - t)x$. Then

$$h(z, t) = \varphi_{u, s(t)}(z) = \frac{u(z) - s(t)}{|u(z) - s(t)|}$$

determines a homotopy between the mappings $\varphi_{u, x}$ and $\varphi_{u, y}$, whence $(\varphi_{u, x})_* = (\varphi_{u, y})_*$, whence it follows that $\text{ind}(u, y) = \text{ind}(u, x)$ for any point $y \in U$. Consequently, the function $\text{ind} : x \mapsto \text{ind}(u, x)$ is constant on U .

36x:13 We can assume that x is the origin and the ray R is the positive half of the x axis. It is more convenient to consider the loop $u : I \rightarrow S^1$, $u(t) = \frac{f(e^{2\pi i t})}{|f(e^{2\pi i t})|}$. Assume that the set $f^{-1}(R)$ is finite and consists of n points. Consequently, $u^{-1}(1) = \{t_0, t_1, \dots, t_n\}$, and we have $t_0 = 0$ and $t_n = 1$. The loop u is homotopic to the product of loops u_i , $i = 1, 2, \dots, n$, each of which has the following property: $u_i(t) = 1$ only for $t = 0, 1$. Prove that $[u_i]$ is equal either to zero, or to a generator of $\pi_1(S^1)$. Therefore, if the integer k_i is the image of $[u_i]$ under the isomorphism $\pi_1(S^1) \rightarrow \mathbb{Z}$ and $k = \text{ind}(f, x)$ is the image of $[u]$ under this isomorphism, then

$$|k| = |k_1 + k_2 + \dots + k_n| \leq |k_1| + |k_2| + \dots + |k_n| \leq n$$

because each of the numbers k_i is 0 or ± 1 .

36x:F If $x \notin F(D^2)$, then the circular loop u is null-homotopic in $\mathbb{R}^2 \setminus x$ because $u = F \circ i$, where i is the standard embedding $S^1 \rightarrow D^2$, and i is null-homotopic in D^2 .

36x:G This is true because we have $[uv] = [u][v]$ and $\pi_1(\mathbb{R}^2 \setminus x) \rightarrow \mathbb{Z}$ is a homomorphism.

36x:H The formula

$$h(z, t) = \varphi_{u_t, x}(z) = \frac{u_t(z) - x}{|u_t(z) - x|}$$

determines a homotopy between the mappings $\varphi_{u, x}$ and $\varphi_{v, x}$, whence $\text{ind}(u, x) = \text{ind}(v, x)$; cf. **36x:E**.

36x:L We define a mapping $\varphi : S^1 \rightarrow \mathbb{R} : x \mapsto f(x) - f(-x)$. Then

$$\varphi(-x) = f(-x) - f(x) = -(f(x) - f(-x)) = -\varphi(x),$$

thus φ is an odd mapping. Consequently, if, for example, $\varphi(1) \neq 0$, then the image $\varphi(S^1)$ contains values with distinct signs. Since the circle is connected, there is a point $x \in S^1$ such that $f(x) - f(-x) = \varphi(x) = 0$.

36x:M.1 Assume that $f(x) \neq f(-x)$ for all $x \in S^2$. In this case, the formula $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ determines a mapping $g : S^2 \rightarrow S^1$.

Since $g(-x) = -g(x)$, it follows that g takes antipodal points of S^2 to antipodal points of S^1 . The quotient mapping of g is a continuous mapping $\varphi : \mathbb{R}P^2 \rightarrow \mathbb{R}P^1$. We show that the induced homomorphism $\varphi_* : \pi_1(\mathbb{R}P^2) \rightarrow \pi_1(\mathbb{R}P^1)$ is nontrivial. The generator λ of the group $\pi_1(\mathbb{R}P^2)$ is the class of the loop l covered by the path \tilde{l} joining two opposite points of S^2 . The path $g \circ \tilde{l}$ also joins two opposite points lying on the circle, consequently, the loop $\varphi \circ l$ covered by $g \circ \tilde{l}$ is not null-homotopic. Thus, $\varphi_*(\lambda)$ is a nontrivial element of $\pi_1(\mathbb{R}P^1)$.

36x:M To prove the Borsuk–Ulam Theorem, it only remains to observe that there are no nontrivial homomorphisms $\pi_1(\mathbb{R}P^2) \rightarrow \pi_1(\mathbb{R}P^1)$ because the first of these groups is isomorphic to \mathbb{Z}_2 , while the second one is isomorphic to \mathbb{Z} .

37.A Prove this assertion on your own.

37.B Since any mapping to a singleton is continuous, the mapping $\rho : X \rightarrow \{x_0\}$ a retraction.

37.C The line is connected. Therefore, its retract (being its continuous image) is connected, too. However, a pair of points in the line is not connected.

37.D See the proof of assertion **37.C**.

37.E \Leftrightarrow Let $\rho : X \rightarrow A$ be a retraction. and let $f : A \rightarrow Y$ be a continuous mapping. Then the composition $F = f \circ \rho : X \rightarrow Y$ extends f .

⊔ Consider the identical mapping $\text{id} : A \rightarrow A$. Its continuous extension to X is the required retraction $\rho : X \rightarrow A$.

37.F Since $\rho_* \circ i_* = (\rho \circ i)_* = (\text{id}_A)_* = \text{id}_{\pi_1(A, x_0)}$, it follows that the homomorphism ρ_* is an epimorphism, and the homomorphism i_* is a monomorphism.

37.G About i_* ; for example, see the proof of the following assertion.

37.H Since the group $\pi_1(D^2)$ is trivial, while $\pi_1(S^1)$ is not, it follows that $i_* : \pi_1(S^1, 1) \rightarrow \pi_1(D^2, 1)$ cannot be a monomorphism. Consequently, by assertion 37.F, the disk D^2 cannot be retracted to its boundary S^1 .

37.I The proof word by word repeats that of Theorem 37.H, only instead of fundamental groups we must use $(n-1)$ -dimensional homotopy groups. The reason for this is that the group $\pi_{n-1}(D^n)$ is trivial, while $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$ (i.e., this group is nontrivial).

37.J Assume that a mapping $f : D^n \rightarrow D^n$ has no fixed points. For each $x \in D^n$, consider the ray starting at $f(x) \in D^n$ and passing through x , and denote by $\rho(x)$ the point of its intersection with the boundary sphere S^{n-1} . It is clear that $\rho(x) = x$ for $x \in S^{n-1}$. Prove that the mapping ρ is continuous. Therefore, $\rho : D^n \rightarrow S^{n-1}$ is a retraction. However, this contradicts the Borsuk Theorem.

38.A Prove this assertion on your own.

38.B This immediately follows from assertion 38.A.

38.C Since ρ is a retraction, it follows that one of the conditions in the definition of homotopically inverse mappings is automatically fulfilled: $\rho \circ \text{id}_A = \text{id}_A$. The second requirement: $\text{id}_X \circ \rho$ is homotopic to id_X , is fulfilled by assumption.

38.D This immediately follows from assertion 38.C.

38.E This follows from 38.D and 38.B.

38.F Let $\rho_1 : X \rightarrow A$ and $\rho_2 : Y \rightarrow B$ be deformation retractions. Prove that $\rho_1 \times \rho_2$ is a deformation retraction.

38.G Let the mapping $\rho : \mathbb{R}^2 \setminus 0 \rightarrow S^1$ be defined by the formula $\rho(x) = \frac{x}{|x|}$. The formula $h(x, t) = (1-t)x + t\frac{x}{|x|}$ determines a rectilinear homotopy between the identical mapping of $\mathbb{R}^2 \setminus 0$ and the composition $\rho \circ i$, where i is the standard inclusion $S^1 \rightarrow \mathbb{R}^2 \setminus 0$.

38.H The topological type of $\mathbb{R}^2 \setminus \{x_1, x_2, \dots, x_s\}$ does not depend on the position of the points x_1, x_2, \dots, x_s in the plane. We put them on the unit circle: for example, let them be roots of unity of degree s . Consider s simple closed curves on the plane each of which encloses exactly one of the points and passes through the origin, and which have no other

common points except the origin. Instead of curves, maybe it is simpler to take, e.g., rhombi with centers at our points. It remains to prove that the union of the curves (or rhombi) is a deformation retract of the plane with s punctures. Clearly, it makes little sense to write down explicit formulas, although this is possible. Consider an individual rhombus R and its center c . The central projection maps $R \setminus c$ to the boundary of R , and there is a rectilinear homotopy between the projection and the identical map of $R \setminus c$. It remains to show that the part of the plane lying outside the union of the rhombi also admits a deformation retraction to the union of their boundaries. What can we do in order to make the argument look more like a proof? First consider the polygon P whose vertices are the vertices of the rhombi opposite to the origin. We easily see that P is a strong deformation retract of the plane (as well as the disk is). It remains to show that the union of the rhombi is a deformation retract of P , which is obvious, is not it?

38.I We subdivide the square into four parts by two midlines and consider the set K formed by the contour, the midlines, and the two quarters of the square containing one of the diagonals. Show that each of the following sets is a deformation retract of K : the union of the contour and the mentioned diagonal of the square; the union of the contours of the “empty” quarters of this square.

38.J 1) None of these spaces can be embedded in another. Prove this on your own, using the following lemma. Let J_n be the union of n segments with a common endpoint. Then J_n cannot be embedded in J_k for any $n > k \geq 2$. 2) The second question is answered in the affirmative; see the proof of assertion 38.I.

38.14 We need the notion of the cylinder Z_f of a continuous mapping $f : X \rightarrow Y$. By definition, Z_f is obtained by attaching the ordinary cylinder $X \times I$ to Y via the mapping $X \times 0 \rightarrow Y$, $(x, 0) \mapsto f(x)$. Hence, Z_f is a result of factorization of the disjoint union $(X \times I) \sqcup Y$, under which the point $(x, 0) \in X \times 0$ is identified with the point $f(x) \in Y$. We identify X and $X \times 1 \subset Z_f$, and it is also natural to assume that the space Y lies in the mapping cylinder. There is an obvious strong deformation retraction $p_Y : Z_f \rightarrow Y$, which leaves Y fixed and takes the point $(x, t) \in X \times (0, 1)$ to $f(x)$. It remains to prove that if f is a homotopy equivalence, then X is also a deformation retract of Z_f . Let $g : Y \rightarrow X$ be a homotopy equivalence inverse to f . Thus, there exists a homotopy $H : X \times I \rightarrow X$ such that $H(x, 0) = g(f(x))$ and $H(x, 1) = x$. We define the retraction $\rho : Z_f \rightarrow X$ as a quotient map of the mapping $(X \times I) \sqcup Y \rightarrow X : (x, t) \mapsto h(x, t), y \mapsto g(y)$. It remains to prove that the mapping ρ is a deformation retraction, i.e., to verify that $\text{id}_X \circ \rho$ is homotopic to id_{Z_f} . This follows from the following chain, where the \sim

sign denotes a homotopy between compositions of homotopic mappings:

$$\begin{aligned} \text{in}_X \circ \rho &= \bar{\rho} = \bar{\rho} \circ \text{id}_{Z_f} \sim \bar{\rho} \circ p_Y = g \circ p_Y = \text{id}_{Z_f} \circ (g \circ p_Y) \sim \\ &\sim p_Y \circ (g \circ p_Y) = (p_Y \circ g) \circ p_Y = (f \circ g) \circ p_Y \sim \text{id}_Y \circ p_Y = p_Y \sim \text{id}_{Z_f}. \end{aligned}$$

38.K Since the composition $g \circ f$ is x_0 -null-homotopic, we have $g_* \circ f_* = (g \circ f)_* = \text{id}_{\pi_1(X, x_0)}$. Similarly, $f_* \circ g_* = \text{id}_{\pi_1(Y, y_0)}$. Thus, f_* and g_* are mutually inverse homomorphisms.

38.L Indeed, this immediately follows from Theorem 38.K.

38.M Let $x_1 = g(x_0)$. For any homotopy h between id_X and $g \circ f$, the formula $s(t) = h(x_0, t)$ determines a path at x_0 . By the answer to Riddle 36.E, the composition $g_* \circ f_* = T_s$ is an isomorphism. Similarly, the composition $f_* \circ g_*$ is an isomorphism. Therefore, f_* and g_* are isomorphisms.

39.A If u is a loop in X such that the loop $p \circ u$ in B is null-homotopic, then by the Path Homotopy Lifting Theorem 34.C the loop u is also null-homotopic. Thus, if $p_*([u]) = [p \circ u] = 0$, then $[u] = 0$, which precisely means that p_* is a monomorphism.

39.B No, it is not. If $p(x_0) = p(x_1) = b_0$, $x_0 \neq x_1$, and the group $\pi_1(B, b_0)$ is non-Abelian, then the subgroups $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ can easily be distinct (see 39.D).

39.C The group $p_*(\pi_1(X, x_0))$ of the covering consists of the homotopy classes of those loops at b_0 whose covering path starting at x_0 is a loop.

39.D Let s be a path in X joining x_0 and x_1 . Denote by α the class of the loop $p \circ s$ and consider the inner automorphism $\varphi : \pi_1(B, b_0) \rightarrow \pi_1(B, b_0) : \beta \mapsto \alpha^{-1}\beta\alpha$. We prove that the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{T_s} & \pi_1(X, x_1) \\ p_* \downarrow & & \downarrow p_* \\ \pi_1(B, b_0) & \xrightarrow{\varphi} & \pi_1(B, b_0). \end{array}$$

Indeed, since $T_s([u]) = [s^{-1}us]$, we have

$$p_*(T_s([u])) = [p \circ (s^{-1}us)] = [(p \circ s^{-1})(p \circ u)(p \circ s)] = \alpha^{-1}p_*([u])\alpha.$$

Since the diagram is commutative and T_s is an isomorphism, it follows that

$$p_*(\pi_1(X, x_1)) = \varphi(p_*(\pi_1(X, x_0))) = \alpha^{-1}p_*(\pi_1(X, x_0))\alpha,$$

thus, the groups $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ are conjugate.

39.E Let s be a loop in X representing the class $\alpha \in \pi_1(B, b_0)$. Let the path \tilde{s} cover s and start at x_0 . If we put $x_1 = \tilde{s}(1)$, then, as it follows from the proof of assertion 39.D, we have $p_*(\pi_1(X, x_1)) = \alpha^{-1}p_*(\pi_1(X, x_0))\alpha$.

39.F This follows from 39.D and 39.E.

39.G See 39.H.

39.H For brevity, put $H = p_*(\pi_1(X, x_0))$. Consider an arbitrary point $x_1 \in p^{-1}(b_0)$; let s be the path starting at x_0 and ending at x_1 , and $\alpha = [p \circ s]$. Take x_1 to the right coset $H\alpha \subset \pi_1(B, b_0)$. Let us verify that this definition is correct. Let s_1 be another path from x_0 to x_1 , $\alpha_1 = [p \circ s_1]$. The path ss_1^{-1} is a loop, so that $\alpha\alpha_1^{-1} \in H$, whence $H\alpha = H\alpha_1$. Now we prove that the described correspondence is a surjection. Let $H\alpha$ be a coset. Consider a loop u representing the class α , let \tilde{u} be the path covering u and starting at x_0 , and $x_1 = \tilde{u}(1) \in p^{-1}(b_0)$. By construction, x_1 is taken to the coset $H\alpha$, therefore, the above correspondence is surjective. Finally, let us prove that it is injective. Let $x_1, x_2 \in p^{-1}(b_0)$, and let s_1 and s_2 be two paths joining x_0 with x_1 and x_2 , respectively; let $\alpha_i = [p \circ s_i]$, $i = 1, 2$. Assume that $H\alpha_1 = H\alpha_2$ and show that then $x_1 = x_2$. Consider a loop $u = (p \circ s_1)(p \circ s_2^{-1})$ and the path \tilde{u} covering u , which is a loop because $\alpha_1\alpha_2^{-1} \in H$. It remains to observe that the paths s'_1 and s'_2 , where $s'_1(t) = u(\frac{t}{2})$ and $s'_2(t) = u(1 - \frac{t}{2})$, start at x_0 and cover the paths $p \circ s_1$ and $p \circ s_2$, respectively. Therefore, $s_1 = s'_1$ and $s_2 = s'_2$, thus,

$$x_1 = s_1(1) = s'_1(1) = \tilde{u}(\frac{1}{2}) = s'_2(1) = s_2(1) = x_2.$$

39.I Consider an arbitrary point $y \in Y$, let $b = q(y)$, and let U_b be a neighborhood of b that is trivially covered for both p and q . Further, let V be the sheet over U_b containing y , and let $\{W_\alpha\}$ be the collection of sheets over U_b the union of which is $\varphi^{-1}(V)$. Clearly, the mapping $\varphi|_{W_\alpha} = (q|_V)^{-1} \circ p|_{W_\alpha}$ is a homeomorphism.

39.J Let p and q be two coverings. Consider an arbitrary point $x \in X$ and a path s joining the marked point x_0 with x . Let $u = p \circ s$. By assertion 34.B, there exists a unique path $\tilde{u} : I \rightarrow Y$ covering u and starting at y_0 . Therefore, $\tilde{u} = \varphi \circ s$, consequently, the point $\varphi(x) = \varphi(s(1)) = \tilde{u}(1)$ is uniquely determined.

39.K Let $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ be subordinations, and let $\varphi(x_0) = y_0$ and $\psi(y_0) = x_0$. Clearly, the composition $\psi \circ \varphi$ is a subordination of the covering $p : X \rightarrow B$ to itself. Consequently, by the uniqueness of a subordination (see 39.J), we have $\psi \circ \varphi = \text{id}_X$. Similarly, $\varphi \circ \psi = \text{id}_Y$, which precisely means that the subordinations φ and ψ are mutually inverse equivalences.

39.L This relation is obviously symmetric, reflexive, and transitive.

39.M It is clear that if two coverings p and p' are equivalent and q is subordinate to p , then q is also subordinate to p' , therefore, the subordination relation is transferred from coverings to their equivalence classes. This relation is obviously reflexive and transitive, and it is proved in **39.K** that two coverings subordinate to each other are equivalent, therefore this relation is antisymmetric.

39.N Since $p_* = (q \circ \varphi)_* = q_* \circ \varphi_*$, we have

$$p_*(\pi_1(X, x_0)) = q_*(\varphi_*(\pi_1(X, x_0))) \subset q_*(\pi_1(Y, y_0)).$$

39x:A.1 Denote by $\tilde{u}, \tilde{v} : I \rightarrow Y$ the paths starting at y_0 and covering the paths $p \circ u$ and $p \circ v$, respectively. Consider the path uv^{-1} , which is a loop at x_0 by assumption, the loop $(p \circ u)(p \circ v)^{-1} = p \circ (uv^{-1})$, and its class $\alpha \in p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. Thus, $\alpha \in q_*(\pi_1(Y, y_0))$, therefore, the path starting at y_0 and covering the loop $(p \circ u)(p \circ v)^{-1}$ is also a loop. Consequently, the paths covering $p \circ u$ and $p \circ v$ and starting at y_0 end at one and the same point. It remains to observe that they are the paths \tilde{u} and \tilde{v} .

39x:A.2 We define the mapping $\varphi : X \rightarrow Y$ as follows. Let $x \in X$, u – a path joining x_0 and x . Then $\varphi(x) = y$, where y is the endpoint of the path $\tilde{u} : I \rightarrow Y$ covering the path $p \circ u$. By assertion **39x:A.1**, the mapping φ is well defined. We prove that $\varphi : X \rightarrow Y$ is continuous. Let $x_1 \in X$, $b_1 = p(x_1)$ and $y_1 = \varphi(x_1)$; by construction, we have $q(y_1) = b_1$. Consider an arbitrary neighborhood V of y_1 . We can assume that V is a sheet over a trivially covered path-connected neighborhood U of b_1 . Let W be the sheet over U containing x_1 , thus, the neighborhood W is also path-connected. Consider an arbitrary point $x \in W$. Let a path $v : I \rightarrow W$ join x_1 and x . It is clear that the image of the path \tilde{v} starting at y_1 and covering the path $p \circ v$ is contained in the neighborhood V , whence $\varphi(x) \in V$. Thus, $\varphi(W) \subset V$, consequently, φ is continuous at x .

39x:B This follows from **39.E**, **39x:A**, and **39.K**.

39x:C Let $X \rightarrow B$ be a universal covering, U a trivially covered neighborhood of a point $a \in B$, and V one of the “sheets” over U . Then the inclusion $i : U \rightarrow B$ is the composition $p \circ j \circ (p|_V)^{-1}$, where j is the inclusion $V \rightarrow X$. Since the group $\pi_1(X)$ is trivial, the inclusion homomorphism $i_* : \pi_1(U, a) \rightarrow \pi_1(B, a)$ is also trivial.

39x:D.1 Let two paths u_1 and u_2 join b_0 and b . The paths covering them and starting at x_0 end at one and the same point x iff the class of the loop $u_1u_2^{-1}$ lies in the subgroup π .

39x:D.2 Yes, it does. Consider the set of all paths in B starting at b_0 , endow it with the following equivalence relation: $u_1 \sim u_2$ if $[u_1u_2^{-1}] \in \pi$,

and let \tilde{X} be the quotient set by this relation. A natural bijection between X and \tilde{X} is constructed as follows. For each point $x \in X$, we consider a path u joining the marked point x_0 with of a point x . The class of the path $p \circ u$ in \tilde{X} is the image of x . The described correspondence is obviously a bijection $f : X \rightarrow \tilde{X}$. The mapping $g : \tilde{X} \rightarrow X$ inverse to f has the following structure. Let $u : I \rightarrow B$ represent a class $y \in \tilde{X}$. Consider the path $v : I \rightarrow X$ covering u and starting at x_0 . Then $g(y) = v(1)$.

39x:D.3 We define a base for the topology in \tilde{X} . For each pair (U, x) , where U is an open set in B and $x \in \tilde{X}$, the set U_x consists of the classes of all possible paths uv , where u is a path in the class x , and v is a path in U starting at $u(1)$. It is not difficult to prove that for each point $y \in U_x$ we have the identity $U_y = U_x$, whence it follows that the collection of the sets of the form U_x is a base for the topology in \tilde{X} . In order to prove that f and g are homeomorphisms, it is sufficient to verify that each of them maps each set in a certain base for the topology to an open set. Consider the base consisting of trivially covered neighborhoods $U \subset B$, each of which, firstly, is path-connected, and, secondly, each loop in which is null-homotopic in B .

39x:D.4 The space \tilde{X} is defined in **39x:D.2**. The projection $p : \tilde{X} \rightarrow B$ is defined as follows: $p(y) = u(1)$, where u is a path in the class $y \in \tilde{X}$. The mapping p is continuous without any assumptions on the properties of B . Prove that if a set U in B is open and path-connected and each loop in U is null-homotopic in B , then U is a trivially covered neighborhood.

39x:F Consider the subgroups $\pi \subset \pi_0 \subset \pi_1(B, b_0)$ and let $p : \tilde{Y} \rightarrow B$ and $q : \tilde{Y} \rightarrow B$ be the coverings constructed by π and π_0 , respectively. The construction of the covering implies that there exists a mapping $f : \tilde{X} \rightarrow \tilde{Y}$. Show that f is the required subordination.

39x:G We say that the group G acts from the right on a set F if each element $\alpha \in G$ determines a mapping $\varphi_\alpha : F \rightarrow F$ so that: 1) $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta$; 2) if e is the unity of the group G , then $\varphi_e = \text{id}_F$. Put $F = p^{-1}(b_0)$. For each $\alpha \in \pi_1(B, b_0)$, we define a mapping $\varphi_\alpha : F \rightarrow F$ as follows. Let $x \in F$. Consider a loop u at b_0 , such that $[u] = \alpha$. Let the path \tilde{u} cover u and start at x . Put $\varphi_\alpha(x) = \tilde{u}(1)$. The Path Homotopy Lifting Theorem implies that the mapping φ_α depends only on the homotopy class of u , therefore, the definition is correct. If $[u] = e$, i.e., the loop u is null-homotopic, then the path \tilde{u} is also a

loop, whence $\tilde{u}(1) = x$, thus, $\varphi_e = \text{id}_F$. Verify that the first property in the definition of an action of a group on a set is also fulfilled.

39x:H See 39x:P.

39x:I The group operation in the set of all automorphisms is their composition.

39x:J This follows from 39.J.

39x:K Show that the mapping transposing the two points in the preimage of each point in the base, is a homeomorphism.

39x:L This is assertion 39.H.

39x:12 Indeed, any subgroup of an Abelian group is normal. We can also verify directly that for each loop $s : I \rightarrow B$ either each path in X covering s is a loop (independently of the starting point), or none of these paths is a loop.

39x:Q This follows from 39x:N and 39x:P.

CHAPTER 9

Cellular Techniques

§40 Cellular Spaces

§40°1 Definition of Cellular Spaces

In this section, we study a class of topological spaces that play a very important role in algebraic topology. Their role in the context of this book is more restricted: this is the class of spaces for which we learn how to calculate the fundamental group.¹

A *zero-dimensional cellular space* is just a discrete space. Points of a 0-dimensional cellular space are also called (*zero-dimensional*) *cells*, or *0-cells*.

A *one-dimensional cellular space* is a space that can be obtained as follows. Take any 0-dimensional cellular space X_0 . Take a family of maps $\varphi_\alpha : S^0 \rightarrow X_0$. Attach to X_0 via φ_α the sum of a family of copies of D^1 (indexed by the same indices α as the maps φ_α):

$$X_0 \cup_{\sqcup \varphi_\alpha} \left(\bigsqcup_{\alpha} D^1 \right).$$

The images of the interior parts of copies of D^1 are called (*open*) *1-dimensional cells*, *1-cells*, *one-cells*, or *edges*. The subsets obtained from D^1 are *closed 1-cells*. The cells of X_0 (i.e., points of X_0) are also called *vertices*. Open 1-cells and 0-cells constitute a partition of a one-dimensional cellular space. This partition is included in the notion of cellular space, i.e., a one-dimensional cellular space is a topological space equipped with a partition that can be obtained in this way.²

¹This class of spaces was introduced by J. H. C. Whitehead. He called these spaces *CW-complexes*, and they are known under this name. However, it is not a good name for plenty of reasons. With very rare exceptions (one of which is *CW-complex*, the other is *simplicial complex*), the word *complex* is used nowadays for various algebraic notions, but not for spaces. We have decided to use the term *cellular space* instead of *CW-complex*, following D. B. Fuchs and V. A. Rokhlin, *Beginner's Course in Topology: Geometric Chapters*. Berlin; New York: Springer-Verlag, 1984. [?].

²One-dimensional cellular spaces are also associated with the word *graph*. However, rather often this word is used for objects of other classes. For example, one can call in this way one-dimensional cellular spaces in which attaching maps of different

A *two-dimensional cellular space* is a space that can be obtained as follows. Take any cellular space X_1 of dimension 0 or 1. Take a family of continuous³ maps $\varphi_\alpha : S^1 \rightarrow X_1$. Attach the sum of a family of copies of D^2 to X_1 via φ_α :

$$X_1 \cup_{\sqcup \varphi_\alpha} \left(\bigsqcup_{\alpha} D^2 \right).$$

The images of the interior parts of copies of D^2 are (*open*) *2-dimensional cells*, *2-cells*, *two-cells*, or *faces*. The cells of X_1 are also regarded as cells of the 2-dimensional cellular space. Open cells of both kinds constitute a partition of a 2-dimensional cellular space. This partition is included in the notion of cellular space, i.e., a two-dimensional cellular space is a topological space equipped with a partition that can be obtained in the way described above. The set obtained out of a copy of the whole D^2 is a *closed 2-cell*.

A *cellular space of dimension n* is defined in a similar way: This is a space equipped with a partition. It is obtained from a cellular space X_{n-1} of dimension less than n by attaching a family of copies of the n -disk D^n via by a family of continuous maps of their boundary spheres:

$$X_{n-1} \cup_{\sqcup \varphi_\alpha} \left(\bigsqcup_{\alpha} D^n \right).$$

The images of the interiors of the attached n -disks are (*open*) *n -dimensional cells* or simply *n -cells*. The images of the entire n -disks are *closed n -cells*. Cells of X_{n-1} are also regarded as cells of the n -dimensional cellular space. The mappings φ_α are the *attaching mappings*, and the restrictions of the factorization map to the n -disks D^n are the *characteristic mappings*.

A *cellular space* is obtained as a union of increasing sequence of cellular spaces $X_0 \subset X_1 \subset \cdots \subset X_n \subset \dots$ obtained in this way from each other.

one-cells are not allowed to coincide, or the boundary of a one-cell is prohibited to consist of a single vertex. When one-dimensional cellular spaces are to be considered anyway, despite of this terminological disregard, they are called *multigraphs* or *pseudographs*. Furthermore, sometimes one includes into the notion of graph an additional structure. Say, a choice of orientation on each edge. Certainly, all these variations contradict a general tendency in mathematical terminology to call in a simpler way decent objects of a more general nature, passing to more complicated terms along with adding structures and imposing restrictions. However, in this specific situation there is no hope to implement that tendency. Any attempt to fix a meaning for the word *graph* apparently only contributes to this chaos, and we just keep this word away from important formulations, using it as a short informal synonym for more formal term of one-dimensional cellular space. (Other overused common words, like *curve* and *surface*, also deserve this sort of caution.)

³In the above definition of a 1-dimensional cellular space, the attaching maps φ_α also were continuous, although their continuity was not required since any map of S^0 to any space is continuous.

The sequence may be finite or infinite. In the latter case, the topological structure is introduced by saying that the cover of the union by X_n 's is fundamental, i.e., a set $U \subset \bigcup_{n=0}^{\infty} X_n$ is open iff its intersection $U \cap X_n$ with each X_n is open in X_n .

The partition of a cellular space into its open cells is a *cellular decomposition*. The union of all cells of dimension less than or equal to n of a cellular space X is the *n -dimensional skeleton* of X . This term may be misleading since the n -dimensional skeleton may contain no n -cells, and so it may coincide with the $(n-1)$ -dimensional skeleton. Thus, the n -dimensional skeleton may have dimension less than n . For this reason, it is better to speak about the *n th skeleton* or *n -skeleton*.

40.1. In a cellular space, skeletons are closed.

A cellular space is *finite* if it contains a finite number of cells. A cellular space is *countable* if it contains a countable number of cells. A cellular space is *locally finite* if each of its points has a neighborhood intersecting finitely many cells.

Let X be a cellular space. A subspace $A \subset X$ is a *cellular subspace* of X if A is a union of open cells and together with each cell e contains the closed cell \bar{e} . This definition admits various equivalent reformulations. For instance, $A \subset X$ is a *cellular subspace* of X iff A is both a union of closed cells and a union of open cells. Another option: together with each point $x \in A$ the subspace A contains the closed cell $\bar{e} \in x$. Certainly, A is equipped with a partition into the open cells of X contained in A . Obviously, the k -skeleton of a cellular space X is a cellular subspace of X .

40.2. Prove that the union and intersection of any collection of cellular subspaces are cellular subspaces.

40.A. Prove that a cellular subspace of a cellular space is a cellular space. (Probably, your proof will involve assertion 40x:G.)

40.A.1. Let X be a topological space, and let $X_1 \subset X_2 \subset \dots$ be an increasing sequence of subsets constituting a fundamental cover of X . Let $A \subset X$ be a subspace, put $A_i = A \cap X_i$. Let one of the following conditions be fulfilled:

- 1) X_i are open in X ;
- 2) A_i are open in X ;
- 3) A_i are closed in X .

Then $\{A_i\}$ is a fundamental cover of A .

§40°2 First Examples

40.B. A cellular space consisting of two cells, one of which is a 0-cell and the other one is an n -cell, is homeomorphic to S^n .

40.C. Represent D^n with $n > 0$ as a cellular space made of three cells.

40.D. A cellular space consisting of a single 0-cell and q one-cells is a bouquet of q circles.

40.E. Represent torus $S^1 \times S^1$ as a cellular space with one 0-cell, two 1-cells, and one 2-cell.

40.F. How to obtain a presentation of torus $S^1 \times S^1$ as a cellular space with 4 cells from a presentation of S^1 as a cellular space with 2 cells?

40.3. Prove that if X and Y are finite cellular spaces, then $X \times Y$ has a natural structure of a finite cellular space.

40.4*. Does the statement of 40.3 remain true if we skip the finiteness condition in it? If yes, prove this; if no, find an example where the product is not a cellular space.

40.G. Represent sphere S^n as a cellular space such that spheres $S^0 \subset S^1 \subset S^2 \subset \dots \subset S^{n-1}$ are its skeletons.

40.H. Represent $\mathbb{R}P^n$ as a cellular space with $n + 1$ cells. Describe the attaching maps of the cells.

40.5. Represent $\mathbb{C}P^n$ as a cellular space with $n + 1$ cells. Describe the attaching maps of its cells.

40.6. Represent the following topological spaces as cellular ones

- | | | |
|------------------------------|--------------------------------|----------------------|
| (a) handle; | (b) Möbius strip; | (c) $S^1 \times I$, |
| (d) sphere with p handles; | (e) sphere with p crosscaps. | |

40.7. What is the minimal number of cells in a cellular space homeomorphic to

- | | | |
|-------------------|------------------------------|--------------------------------|
| (a) Möbius strip; | (b) sphere with p handles; | (c) sphere with p crosscaps? |
|-------------------|------------------------------|--------------------------------|

40.8. Find a cellular space where the closure of a cell is not equal to a union of other cells. What is the minimal number of cells in a space containing a cell of this sort?

40.9. Consider the disjoint sum of a countable collection of copies of closed interval I and identify the copies of 0 in all of them. Represent the result (which is the bouquet of the countable family of intervals) as a countable cellular space. Prove that this space is not first countable.

40.I. Represent \mathbb{R}^1 as a cellular space.

40.10. Prove that for any two cellular spaces homeomorphic to \mathbb{R}^1 there exists a homeomorphism between them homeomorphically mapping each cell of one of them onto a cell of the other one.

40.J. Represent \mathbb{R}^n as a cellular space.

Denote by \mathbb{R}^∞ the union of the sequence of Euclidean spaces $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \dots \subset \mathbb{R}^n \subset$ canonically included to each other: $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Equip \mathbb{R}^∞ with the topological structure for which the spaces \mathbb{R}^n constitute a fundamental cover.

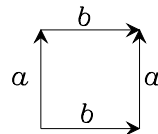
40.K. Represent \mathbb{R}^∞ as a cellular space.

40.11. Show that \mathbb{R}^∞ is not metrizable.

§40°3 Further Two-Dimensional Examples

Let us consider a class of 2-dimensional cellular spaces that admit a simple combinatorial description. Each space in this class is a quotient space of a finite family of convex polygons by identification of sides via affine homeomorphisms. The identification of vertices is determined by the identification of the sides. The quotient space has a natural decomposition into 0-cells, which are the images of vertices, 1-cells, which are the images of sides, and faces, the images of the interior parts of the polygons.

To describe such a space, we need, first, to show, what sides are identified. Usually this is indicated by writing the same letters at the sides to be identified. There are only two affine homeomorphisms between two closed intervals. To specify one of them, it suffices to show the orientations of the intervals that are identified by the homeomorphism. Usually this is done by drawing arrows on the sides. Here is a description of this sort for the standard presentation of torus $S^1 \times S^1$ as the quotient space of square:



We can replace a picture by a combinatorial description. To do this, put letters on *all* sides of polygon, go around the polygons counterclockwise and write down the letters that stay at the sides of polygon along the contour. The letters corresponding to the sides whose orientation is opposite to the counterclockwise direction are put with exponent -1 . This yields a collection of words, which contains sufficient information about the family of polygons and the partition. For instance, the presentation of the torus shown above is encoded by the word $ab^{-1}a^{-1}b$.

40.12. Prove that:

- (a) the word $a^{-1}a$ describes a cellular space homeomorphic to S^2 ,
- (b) the word aa describes a cellular space homeomorphic to $\mathbb{R}P^2$,
- (c) the word $aba^{-1}b^{-1}c$ describes a handle,
- (d) the word $abcb^{-1}$ describes cylinder $S^1 \times I$,
- (e) each of the words aab and $abac$ describe Möbius strip,
- (f) the word $abab$ describes a cellular space homeomorphic to $\mathbb{R}P^2$,
- (g) each of the words $aabb$ and $ab^{-1}ab$ describe Klein bottle,
- (h) the word

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \dots a_gb_ga_g^{-1}b_g^{-1}.$$

describes sphere with g handles,

- (i) the word $a_1a_1a_2a_2\dots a_ga_g$ describes sphere with g crosscaps.

§40°4 Embedding to Euclidean Space

40.L. Any countable 0-dimensional cellular space can be embedded into \mathbb{R} .

40.M. Any countable locally finite 1-dimensional cellular space can be embedded into \mathbb{R}^3 .

40.13. Find a 1-dimensional cellular space which you cannot embed into \mathbb{R}^2 . (We do not ask you to prove rigorously that no embedding is possible.)

40.N. *Any finite dimensional countable locally finite cellular space can be embedded into Euclidean space of sufficiently high dimension.*

40.N.1. Let X and Y be topological spaces such that X can be embedded into \mathbb{R}^p and Y can be embedded into \mathbb{R}^q , and both embeddings are proper mappings (see §18x°3; in particular, their images are closed in \mathbb{R}^p and \mathbb{R}^q , respectively). Let A be a closed subset of Y . Assume that A has a neighborhood U in Y such that there exists a homeomorphism $h : ClU \rightarrow A \times I$ mapping A to $A \times 0$. Let $\varphi : A \rightarrow X$ be a proper continuous map. Then the initial embedding $X \rightarrow \mathbb{R}^p$ extends to an embedding $X \cup_\varphi Y \rightarrow \mathbb{R}^{p+q+1}$.

40.N.2. Let X be a locally finite countable k -dimensional cellular space and A be the $(k-1)$ -skeleton of X . Prove that if A can be embedded to \mathbb{R}^p , then X can be embedded into \mathbb{R}^{p+k+1} .

40.O. Any countable locally finite cellular space can be embedded into \mathbb{R}^∞ .

40.P. Any finite cellular space is metrizable.

40.Q. Any finite cellular space is normal.

40.R. Any countable cellular space can be embedded into \mathbb{R}^∞ .

40.S. Any cellular space is normal.

40.T. Any locally finite cellular space is metrizable.

§40x°5 Simplicial Spaces

Recall that in §23x°3 we introduced a class of topological spaces: simplicial spaces. Each simplicial space is equipped with a partition into subsets, called open simplices, which are indeed homeomorphic to open simplices of Euclidean space.

40x:A. Any simplicial space is cellular, and its partition into open simplices is the corresponding partition into open cells.

§40x°6 Topological Properties of Cellular Spaces

The present section contains assertions of mixed character. For example, we study conditions ensuring that a cellular space is compact ($40x:K$) or separable ($40x:O$). We also prove that a cellular space X is connected, iff X is path-connected ($40x:S$), iff the 1-skeleton of X is path-connected ($40x:V$). On the other hand, we study the cellular topological structure as such. For example, any cellular space is Hausdorff ($40x:B$). Further, it is not obvious at all from the definition of a cellular space that a closed cell is the closure of the corresponding open cell (or that closed cells are closed at all). In this connection, the present section includes assertions of technical character. (We do not formulate them as lemmas to individual theorems because often they are lemmas for several assertions.) For example: closed cells constitute a fundamental cover of a cellular space ($40x:D$).

We notice that, say, in the textbook [FR], a cellular space is defined as a Hausdorff topological space equipped by a cellular partition with two properties: (C) each closed cell intersects only a finite number of (open) cells; (W) closed cells constitute a fundamental cover of the space. The results of assertions $40x:B$, $40x:C$, and $40x:F$ imply that cellular spaces in the sense of the above definition are cellular spaces in the sense of Rokhlin–Fuchs’ textbook (i.e., in the standard sense), the possibility of inductive construction for which is proved in [RF]. Thus, both definitions of a cellular space are equivalent.

An advice to the reader: first try to prove the above assertions for finite cellular spaces.

$40x:B$. Each cellular space is a Hausdorff topological space.

$40x:C$. In a cellular space, the closure of any cell e is the closed cell \bar{e} .

$40x:D$. Closed cells constitute a fundamental cover of a cellular space.

$40x:E$. *Each cover of a cellular space by cellular subspaces is fundamental.*

$40x:F$. In a cellular space, any closed cell intersects only a finite number of open cells.

$40x:G$. If A is cellular subspace of a cellular space X , then A is closed in X .

$40x:H$. The space obtained as a result of pasting two cellular subspaces together along their common subspace, is cellular.

$40x:I$. If a subset A of a cellular space X intersects each open cell along a finite set, then A is closed. Furthermore, the induced topology on A is discrete.

40x:J. Prove that any compact subset of a cellular space intersects a finite number of cells.

40x:K Corollary. *A cellular space is compact iff it is finite.*

40x:L. Any cell of a cellular space is contained in a finite cellular subspace of this space.

40x:M. Any compact subset of a cellular space is contained in a finite cellular subspace.

40x:N. *A subset of a cellular space is compact iff it is closed and intersects only a finite number of open cells.*

40x:O. A cellular space is separable iff it is countable.

40x:P. Any path-connected component of a cellular space is a cellular subspace.

40x:Q. A cellular space is locally path-connected.

40x:R. Any path-connected component of a cellular space is both open and closed. It is a connected component.

40x:S. A cellular space is connected iff it is path connected.

40x:T. A locally finite cellular space is countable iff it has countable 0-skeleton.

40x:U. Any connected locally finite cellular space is countable.

40x:V. *A cellular space is connected iff its 1-skeleton is connected.*

§41 Cellular Constructions

§41°1 Euler Characteristic

Let X be a finite cellular space. Let $c_i(X)$ denote the number of its cells of dimension i . The *Euler characteristic* of X is the alternating sum of $c_i(X)$:

$$\chi(X) = c_0(X) - c_1(X) + c_2(X) - \cdots + (-1)^i c_i(X) + \cdots$$

41.A. Prove that Euler characteristic is additive in the following sense: for any cellular space X and its finite cellular subspaces A and B we have

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

41.B. Prove that Euler characteristic is multiplicative in the following sense: for any finite cellular spaces X and Y the Euler characteristic of their product $X \times Y$ is $\chi(X)\chi(Y)$.

§41°2 Collapse and Generalized Collapse

Let X be a cellular space, e and f its open cells of dimensions n and $n - 1$, respectively. Suppose:

- the attaching map $\varphi_e : S^{n-1} \rightarrow X_{n-1}$ of e determines a homeomorphism of the open upper hemisphere S_+^{n-1} onto f ,
- f does not meet images of attaching maps of cells, distinct from e ,
- the cell e is disjoint from the image of attaching map of any cell.

41.C. $X \setminus (e \cup f)$ is a cellular subspace of X .

41.D. $X \setminus (e \cup f)$ is a deformation retract of X .

We say that $X \setminus (e \cup f)$ is obtained from X by an *elementary collapse*, and we write $X \searrow X \setminus (e \cup f)$.

If a cellular subspace A of a cellular space X is obtained from X by a sequence of elementary collapses, then we say that X *is collapsed* onto A and also write $X \searrow A$.

41.E. *Collapsing does not change the Euler characteristic: if X is a finite cellular space and $X \searrow A$, then $\chi(A) = \chi(X)$.*

As above, let X be a cellular space, let e and f be its open cells of dimensions n and $n - 1$, respectively, and let the attaching map $\varphi_e : S^{n-1} \rightarrow X_{n-1}$ of e determine a homeomorphism S_+^{n-1} on f . Unlike the preceding situation, here we assume neither that f is disjoint from the images of attaching maps of cells different from e , nor that e is disjoint from the images of attaching maps of whatever cells. Let $\chi_e : D^n \rightarrow X$ be a characteristic map of e . Furthermore, let $\psi : D^n \rightarrow S^{n-1} \setminus \varphi_e^{-1}(f) = S^{n-1} \setminus S_+^{n-1}$ be a deformation retraction.

41.F. Under these conditions, the quotient space $X/[\chi_e(x) \sim \varphi_e(\psi(x))]$ of X is a cellular space where the cells are the images under the natural projections of all cells of X except e and f .

Cellular space $X/[\chi_e(x) \sim \varphi_e(\psi(x))]$ is said to be obtained by *cancelation of cells* e and f .

41.G. The projection $X \rightarrow X/[\chi_e(x) \sim \varphi_e(\psi(x))]$ is a homotopy equivalence.

41.G.1. Find a cellular subspace Y of a cellular space X such that the projection $Y \rightarrow Y/[\chi_e(x) \sim \varphi_e(\psi(x))]$ would be a homotopy equivalence by Theorem 41.D.

41.G.2. Extend the mapping $Y \rightarrow Y \setminus (e \cup f)$ to a mapping $X \rightarrow X'$, which is a homotopy equivalence by 41x:6.

§41x°3 Homotopy Equivalences of Cellular Spaces

41x:1. Let $X = A \cup_{\varphi} D^n$ be the space obtained by attaching an n -disk to a topological space A via a continuous mapping $\varphi : S^{n-1} \rightarrow A$. Prove that the complement $X \setminus x$ of any point $x \in X \setminus A$ admits a (strong) deformation retraction to A .

41x:2. Let X be an n -dimensional cellular space, and let K be a set intersecting each of the open n -cells of X at a single point. Prove that the $(n - 1)$ -skeleton X_{n-1} of X is a deformation retract of $X \setminus K$.

41x:3. Prove that the complement $\mathbb{R}P^n \setminus \text{point}$ is homotopy equivalent to $\mathbb{R}P^{n-1}$; the complement $\mathbb{C}P^n \setminus \text{point}$ is homotopy equivalent to $\mathbb{C}P^{n-1}$.

41x:4. Prove that the punctured solid torus $D^2 \times S^1 \setminus \text{point}$, where point is an arbitrary interior point, is homotopy equivalent to a torus with a disk attached along the meridian $S^1 \times 1$.

41x:5. Let A be cellular space of dimension n , let $\varphi : S^n \rightarrow A$ and $\psi : S^n \rightarrow A$ be continuous maps. Prove that if φ and ψ are homotopic, then the spaces $X_{\varphi} = A \cup_{\varphi} D^{n+1}$ and $X_{\psi} = A \cup_{\psi} D^{n+1}$ are homotopy equivalent.

Below we need a more general fact.

41x:6. Let $f : X \rightarrow Y$ be a homotopy equivalence, $\varphi : S^{n-1} \rightarrow X$ and $\varphi' : S^{n-1} \rightarrow Y$ continuous mappings. Prove that if $f \circ \varphi \sim \varphi'$, then $X \cup_{\varphi} D^n \simeq Y \cup_{\varphi'} D^n$.

41x:7. Let X be a space obtained from a circle by attaching of two copies of disk by maps $S^1 \rightarrow S^1 : z \mapsto z^2$ and $S^1 \rightarrow S^1 : z \mapsto z^3$, respectively. Find a cellular space homotopy equivalent to X with smallest possible number of cells.

41x:8. Riddle. Generalize the result of Problem 41x:7.

41x:9. Prove that if we attach a disk to the torus $S^1 \times S^1$ along the parallel $S^1 \times 1$, then the space K obtained is homotopy equivalent to the bouquet $S^2 \vee S^1$.

41x:10. Prove that the torus $S^1 \times S^1$ with two disks attached along the meridian $\{1\} \times S^1$ and parallel $S^1 \times 1$, respectively, is homotopy equivalent to S^2 .

41x:11. Consider three circles in \mathbb{R}^3 : $S_1 = \{x^2 + y^2 = 1, z = 0\}$, $S_2 = \{x^2 + y^2 = 1, z = 1\}$, and $S_3 = \{z^2 + (y - 1)^2 = 1, x = 0\}$. Since $\mathbb{R}^3 \cong S^3 \setminus \text{point}$, we can assume that S_1 , S_2 , and S_3 lie in S^3 . Prove that the space $X = S^3 \setminus (S_1 \cup S_2)$ is not homotopy equivalent to the space $Y = S^3 \setminus (S_1 \cup S_3)$.

41x:A. Let X be a cellular space, $A \subset X$ a cellular subspace. Then the union $(X \times 0) \cup (A \times I)$ is a retract of the cylinder $X \times I$.

41x:B. Let X be a cellular space, $A \subset X$ a cellular subspace. Assume that we are given a mapping $F : X \rightarrow Y$ and a homotopy $h : A \times I \rightarrow Y$ of the restriction $f = F|_A$. Then the homotopy h extends to a homotopy $H : X \times I \rightarrow Y$ of F .

41x:C. Let X be a cellular space, $A \subset X$ a contractible cellular subspace. Then the projection $\text{pr} : X \rightarrow X/A$ is a homotopy equivalence.

Problem 41x:C implies the following assertions.

41x:D. If a cellular space X contains a closed 1-cell e homeomorphic to I , then X is homotopy equivalent to the cellular space X/e obtained by contraction of e .

41x:E. Any connected cellular space is homotopy equivalent to a cellular space with one-point 0-skeleton.

41x:F. A simply connected finite 2-dimensional cellular space is homotopy equivalent to a cellular space with one-point 1-skeleton.

41x:12. Solve Problem 41x:9 with the help of Theorem 41x:C.

41x:13. Prove that the quotient space

$$\mathbb{C}P^2 / [(z_0 : z_1 : z_2) \sim (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)]$$

of the complex projective plane $\mathbb{C}P^2$ is homotopy equivalent to S^4 .

Information. We have $\mathbb{C}P^2 / [z \sim \tau(z)] \cong S^4$.

41x:G. Let X be a cellular space, and let A be a cellular subspace of X such that the inclusion $\text{in} : A \rightarrow X$ is a homotopy equivalence. Then A is a deformation retract of X .

§42 One-Dimensional Cellular Spaces

§42°1 Homotopy Classification

42.A. Any connected finite 1-dimensional cellular space is homotopy equivalent to a bouquet of circles.

42.A.1 Lemma. Let X be a 1-dimensional cellular space, e a 1-cell of X attached by an injective map $S^0 \rightarrow X_0$ (i.e., e has two distinct endpoints). Prove that the projection $X \rightarrow X/e$ is a homotopy equivalence. Describe the homotopy inverse map explicitly.

42.B. A finite connected cellular space X of dimension one is homotopy equivalent to the bouquet of $1 - \chi(X)$ circles, and its fundamental group is a free group of rank $1 - \chi(X)$.

42.C Corollary. The Euler characteristic of a finite connected one-dimensional cellular space is invariant under homotopy equivalence. It is not greater than one. It equals one iff the space is homotopy equivalent to point.

42.D Corollary. The Euler characteristic of a finite one-dimensional cellular space is not greater than the number of its connected components. It is equal to this number iff each of its connected components is homotopy equivalent to a point.

42.E Homotopy Classification of Finite 1-Dimensional Cellular Spaces. Finite connected one-dimensional cellular spaces are homotopy equivalent, iff their fundamental groups are isomorphic, iff their Euler characteristics are equal.

42.1. The fundamental group of a 2-sphere punctured at n points is a free group of rank $n - 1$.

42.2. Prove that the Euler characteristic of a cellular space homeomorphic to S^2 is equal to 2.

42.3 The Euler Theorem. For any convex polyhedron in \mathbb{R}^3 , the sum of the number of its vertices and the number of its faces equals the number of its edges plus two.

42.4. Prove the Euler Theorem without using fundamental groups.

42.5. Prove that the Euler characteristic of any cellular space homeomorphic to the torus is equal to 0.

Information. The Euler characteristic is homotopy invariant, but the usual proof of this fact involves the machinery of singular homology theory, which lies far beyond the scope of our book.

§42°2 Spanning Trees

A one-dimensional cellular space is a *tree* if it is connected, while the complement of each of its (open) 1-cells is disconnected. A cellular

subspace A of a cellular space X is a *spanning tree* of X if A is a tree and is not contained in any other cellular subspace $B \subset X$ which is a tree.

42.F. Any finite connected one-dimensional cellular space contains a spanning tree.

42.G. Prove that a cellular subspace A of a cellular space X is a spanning tree iff A is a tree and contains all vertices of X .

Theorem 42.G explains the term *spanning tree*.

42.H. Prove that a cellular subspace A of a cellular space X is a spanning tree iff it is a tree and the quotient space X/A is a bouquet of circles.

42.I. Let X be a one-dimensional cellular space and A its cellular subspace. Prove that if A is a tree, then the projection $X \rightarrow X/A$ is a homotopy equivalence.

Problems 42.F, 42.I, and 42.H provide one more proof of Theorem 42.A.

§42x°3 Dividing Cells

42x:A. *In a one-dimensional connected cellular space each connected component of the complement of an edge meets the closure of the edge. The complement has at most two connected component.*

A complete local characterization of a vertex in a one-dimensional cellular space is its *valency*. This is the total number of points in the preimages of the vertex under attaching maps of all one-cells of the space. It is more traditional to define the degree of a vertex v as the number of edges incident to v , counting with multiplicity 2 the edges that are incident only to v .

42x:B. *1) Each connected component of the complement of a vertex in a connected one-dimensional cellular space contains an edge with boundary containing the vertex. 2) The complement of a vertex of valency m has at most m connected components.*

§42x°4 Trees and Forests

A one-dimensional cellular space is a *tree* if it is connected, while the complement of each of its (open) 1-cells is disconnected. A one-dimensional cellular space is a *forest* if each of its connected components is a tree.

42x:C. Any cellular subspace of a forest is a forest. In particular, any connected cellular subspace of a tree is a tree.

42x:D. In a tree the complement of an edge consists of two connected components.

42x:E. In a tree, the complement of a vertex of valency m has consists of m connected components.

42x:F. A finite tree has there exists a vertex of valency one.

42x:G. Any finite tree collapses to a point and has Euler characteristic one.

42x:H. Prove that any point of a tree is its deformation retract.

42x:I. Any finite one-dimensional cellular space that can be collapsed to a point is a tree.

42x:J. In any finite one-dimensional cellular space the sum of valencies of all vertices is equal to the number of edges multiplied by two.

42x:K. A finite connected one-dimensional cellular space with Euler characteristic one has a vertex of valency one.

42x:L. A finite connected one-dimensional cellular space with Euler characteristic one collapses to a point.

§42x°5 Simple Paths

Let X be a one-dimensional cellular space. A *simple path of length n* in X is a finite sequence $(v_1, e_1, v_2, e_2, \dots, e_n, v_{n+1})$, formed by vertices v_i and edges e_i of X such that each term appears in it only once and the boundary of every edge e_i consists of the preceding and subsequent vertices v_i and v_{i+1} . The vertex v_1 is the *initial* vertex, and v_{n+1} is the *final* one. The simple path *connects* these vertices. They are connected by a path $I \rightarrow X$, which is a topological embedding with image contained in the union of all cells involved in the simple path. The union of these cells is a cellular subspace of X . It is called a *simple broken line*.

42x:M. *In a connected one-dimensional cellular space, any two vertices are connected by a simple path.*

42x:N Corollary. *In a connected one-dimensional cellular space X , any two points are connected by a path $I \rightarrow X$ which is a topological embedding.*

42x:1. Can a path-connected space contain two distinct points that cannot be connected by a path which is a topological embedding?

42x:2. Can you find a Hausdorff space with this property?

42x:O. *A connected one-dimensional cellular space X is a tree iff there exists no topological embedding $S^1 \rightarrow X$.*

42x:P. *In a one-dimensional cellular space X there exists a non-null-homotopic loop $S^1 \rightarrow X$ iff there exists a topological embedding $S^1 \rightarrow X$.*

42x:Q. *A one-dimensional cellular space is a tree iff any two distinct vertices are connected in it by a unique simple path.*

42x:3. Prove that any finite tree has fixed point property.

Cf. 37.12, 37.13, and 37.14.

42x:4. Is this true for any tree; for any finite connected one-dimensional cellular space?

§43 Fundamental Group of a Cellular Space

§43°1 One-Dimensional Cellular Spaces

43.A. *The fundamental group of a connected finite one-dimensional cellular space X is a free group of rank $1 - \chi(X)$.*

43.B. Let X be a finite connected one-dimensional cellular space, T a spanning tree of X , and $x_0 \in T$. For each 1-cell $e \subset X \setminus T$, choose a loop s_e that starts at x_0 , goes inside T to e , then goes once along e , and then returns to x_0 in T . Prove that $\pi_1(X, x_0)$ is freely generated by the homotopy classes of s_e .

§43°2 Generators

43.C. Let A be a topological space, $x_0 \in A$. Let $\varphi : S^{k-1} \rightarrow A$ be a continuous map, $X = A \cup_{\varphi} D^k$. If $k > 1$, then the inclusion homomorphism $\pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ is surjective. Cf. 43.G.4 and 43.G.5.

43.D. *Let X be a cellular space, x_0 its 0-cell and X_1 the 1-skeleton of X . Then the inclusion homomorphism*

$$\pi_1(X_1, x_0) \rightarrow \pi_1(X, x_0)$$

is surjective.

43.E. Let X be a finite cellular space, T a spanning tree of X_1 , and $x_0 \in T$. For each cell $e \subset X_1 \setminus T$, choose a loop s_e that starts at x_0 , goes inside T to e , then goes once along e , and finally returns to x_0 in T . Prove that $\pi_1(X, x_0)$ is generated by the homotopy classes of s_e .

43.1. Deduce Theorem 31.G from Theorem 43.D.

43.2. Find $\pi_1(\mathbb{C}P^n)$.

§43°3 Relations

Let X be a cellular space, x_0 its 0-cell. Denote by X_n the n -skeleton of X . Recall that X_2 is obtained from X_1 by attaching copies of the disk D^2 via continuous maps $\varphi_{\alpha} : S^1 \rightarrow X_1$. The attaching maps are circular loops in X_1 . For each α , choose a path $s_{\alpha} : I \rightarrow X_1$ connecting $\varphi_{\alpha}(1)$ with x_0 . Denote by N the normal subgroup of $\pi_1(X, x_0)$ generated (as a normal subgroup⁴) by the elements

$$T_{s_{\alpha}}[\varphi_{\alpha}] \in \pi_1(X_1, x_0).$$

43.F. N does not depend on the choice of the paths s_{α} .

⁴Recall that a subgroup N is *normal* if N coincides with all conjugate subgroups of N . The normal subgroup N generated by a set A is the minimal normal subgroup containing A . As a subgroup, N is generated by elements of A and elements conjugate to them. This means that each element of N is a product of elements conjugate to elements of A .

43.G. The normal subgroup N is the kernel of the inclusion homomorphism $\text{in}_* : \pi_1(X_1, x_0) \rightarrow \pi_1(X, x_0)$.

Theorem 43.G can be proved in various ways. For example, we can derive it from the Seifert–van Kampen Theorem (see 43x:4). Here we prove Theorem 43.G by constructing a “rightful” covering space. The inclusion $N \subset \text{Ker in}_*$ is rather obvious (see 43.G.1). The proof of the converse inclusion involves the existence of a covering $p : Y \rightarrow X$, whose submapping over the 1-skeleton of X is a covering $p_1 : Y_1 \rightarrow X_1$ with group N , and the fact that Ker in_* is contained in the group of each covering over X_1 that extends to a covering over the entire X . The scheme of argument suggested in Lemmas 1–7 can also be modified. The thing is that the inclusion $X_2 \rightarrow X$ induces an isomorphism of fundamental groups. It is not difficult to prove this, but the techniques involved, though quite general and natural, nevertheless lie beyond the scope of our book. Here we just want to emphasize that this result replaces Lemmas 4 and 5.

43.G.1 **Lemma 1.** $N \subset \text{Ker } i_*$, cf. 31.J (c).

43.G.2 **Lemma 2.** Let $p_1 : Y_1 \rightarrow X_1$ be a covering with covering group N . Then for any α and a point $y \in p_1^{-1}(\varphi_\alpha(1))$ there exists a lifting $\tilde{\varphi}_\alpha : S^1 \rightarrow Y_1$ of φ_α with $\tilde{\varphi}_\alpha(1) = y$.

43.G.3 **Lemma 3.** Let Y_2 be a cellular space obtained by attaching copies of disk to Y_1 by all liftings of attaching maps φ_α . Then there exists a map $p_2 : Y_2 \rightarrow X_2$ extending p_1 which is a covering.

43.G.4 **Lemma 4.** Attaching maps of n -cells with $n \geq 3$ are lift to any covering space. Cf. 39x:X and 39x:Y.

43.G.5 **Lemma 5.** Covering $p_2 : Y_2 \rightarrow X_2$ extends to a covering of the whole X .

43.G.6 **Lemma 6.** Any loop $s : I \rightarrow X_1$ realizing an element of $\text{Ker } i_*$ (i.e., null-homotopic in X) is covered by a loop of Y . The covering loop is contained in Y_1 .

43.G.7 **Lemma 7.** $N = \text{Ker in}_*$.

43.H. The inclusion $\text{in}_2 : X_2 \rightarrow X$ induces an isomorphism between the fundamental groups of a cellular space and its 2-skeleton.

43.3. Check that the covering over the cellular space X constructed in the proof of Theorem 43.G is universal.

§43°4 Writing Down Generators and Relations

Theorems 43.E and 43.G imply the following recipe for writing down a presentation for the fundamental group of a finite dimensional cellular space by generators and relations:

Let X be a finite cellular space, x_0 a 0-cell of X . Let T a spanning tree of the 1-skeleton of X . For each 1-cell $e \notin T$ of X , choose a loop s_e that starts at x_0 , goes inside T to e , goes once along e , and then returns to x_0 in T . Let g_1, \dots, g_m be the homotopy classes of these loops. Let $\varphi_1, \dots, \varphi_n : S^1 \rightarrow X_1$ be the attaching maps of 2-cells of X . For each φ_i choose a path s_i connecting $\varphi_i(1)$ with x_0 in the 1-skeleton of X . Express the homotopy class of the loop $s_i^{-1}\varphi_i s_i$ as a product of powers of generators g_j . Let r_1, \dots, r_n are the words in letters g_1, \dots, g_m obtained in this way. The fundamental group of X is generated by g_1, \dots, g_m , which satisfy the defining relations $r_1 = 1, \dots, r_n = 1$.

43.I. Check that this rule gives correct answers in the cases of $\mathbb{R}P^n$ and $S^1 \times S^1$ for the cellular presentations of these spaces provided in Problems 40.H and 40.E.

In assertion 41x:F proved above we assumed that the cellular space is 2-dimensional. The reason for this was that at that moment we did not know that the inclusion $X_2 \rightarrow X$ induces an isomorphism of fundamental groups.

43.J. Each finite simply connected cellular space is homotopy equivalent to a cellular space with one-point 1-skeleton.

§43°5 Fundamental Groups of Basic Surfaces

43.K. *The fundamental group of a sphere with g handles admits presentation*

$$\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$

43.L. *The fundamental group of a sphere with g crosscaps admits the following presentation*

$$\langle a_1, a_2, \dots, a_g \mid a_1^2 a_2^2 \dots a_g^2 = 1 \rangle.$$

43.M. *Fundamental groups of spheres with different numbers of handles are not isomorphic.*

When we want to prove that two finitely presented groups are not isomorphic, one of the first natural moves is to abelianize the groups. (Recall that to *abelianize* a group G means to quotient it out by the commutator subgroup. The commutator subgroup $[G, G]$ is the normal subgroup generated by the commutators $a^{-1}b^{-1}ab$ for all $a, b \in G$. Abelianization means adding relations that $ab = ba$ for any $a, b \in G$.)

Abelian finitely generated groups are well known. Any finitely generated Abelian group is isomorphic to a product of a finite number of cyclic groups. If the abelianized groups are not isomorphic, then the original groups are not isomorphic as well.

43.M.1. The abelianized fundamental group of a sphere with g handles is a free Abelian group of rank $2g$ (i.e., is isomorphic to \mathbb{Z}^{2g}).

43.N. *Fundamental groups of spheres with different numbers of crosscaps are not isomorphic.*

43.N.1. The abelianized fundamental group of a sphere with g crosscaps is isomorphic to $\mathbb{Z}^{g-1} \times \mathbb{Z}_2$.

43.O. *Spheres with different numbers of handles are not homotopy equivalent.*

43.P. *Spheres with different numbers of crosscaps are not homotopy equivalent.*

43.Q. *A sphere with handles is not homotopy equivalent to a sphere with crosscaps.*

If X is a path-connected space, then the abelianized fundamental group of X is the *1-dimensional* (or *first*) *homology group* of X and denoted by $H_1(X)$. If X is not path-connected, then $H_1(X)$ is the direct sum of the first homology groups of all path-connected components of X . Thus *43.M.1* can be rephrased as follows: if F_g is a sphere with g handles, then $H_1(F_g) = \mathbb{Z}^{2g}$.

§43x°6 Seifert–van Kampen Theorem

To calculate fundamental group, one often uses the Seifert–van Kampen Theorem, instead of the cellular techniques presented above.

43x:A *Seifert–van Kampen Theorem. Let X be a path-connected topological space, A and B be its open path-connected subspaces covering X , and let $C = A \cap B$ be also path-connected. Then $\pi_1(X)$ can be presented as amalgamated product of $\pi_1(A)$ and $\pi_1(B)$ with identified subgroup $\pi_1(C)$. In other words, if $x_0 \in C$,*

$$\pi_1(A, x_0) = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$

$$\pi_1(B, x_0) = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1 \rangle,$$

$\pi_1(C, x_0)$ is generated by its elements $\gamma_1, \dots, \gamma_t$, and $\text{in}_A : C \rightarrow A$ and $\text{in}_B : C \rightarrow B$ are inclusions, then $\pi_1(X, x_0)$ can be presented as

$$\begin{aligned} &\langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \\ &\quad \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \\ &\quad \text{in}_{A^*}(\gamma_1) = \text{in}_{B^*}(\gamma_1), \dots, \text{in}_{A^*}(\gamma_t) = \text{in}_{B^*}(\gamma_t) \rangle. \end{aligned}$$

Now we consider the situation where the space X and its subsets A and B are cellular.

43x:B. Assume that X is a connected finite cellular space, and A and B are two cellular subspaces of X covering X . Denote $A \cap B$ by C . How are the fundamental groups of X , A , B , and C related to each other?

43x:C Seifert–van Kampen Theorem. *Let X be a connected finite cellular space, A and B – connected cellular subspaces covering X , $C = A \cap B$. Assume that C is also connected. Let $x_0 \in C$ be a 0-cell,*

$$\pi_1(A, x_0) = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$

$$\pi_1(B, x_0) = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1 \rangle,$$

and let the group $\pi_1(C, x_0)$ be generated by the elements $\gamma_1, \dots, \gamma_t$. Denote by $\xi_i(\alpha_1, \dots, \alpha_p)$ and $\eta_i(\beta_1, \dots, \beta_q)$ the images of the elements γ_i (more precisely, their expression via the generators) under the inclusion homomorphisms

$$\pi_1(C, x_0) \rightarrow \pi_1(A, x_0) \text{ and, respectively, } \pi_1(C, x_0) \rightarrow \pi_1(B, x_0).$$

Then

$$\begin{aligned} \pi_1(X, x_0) = \langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \\ \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \\ \xi_1 = \eta_1, \dots, \xi_t = \eta_t \rangle. \end{aligned}$$

43x:1. Let X , A , B , and C be as above. Assume that A and B are simply connected and C consists of two connected components. Prove that $\pi_1(X)$ is isomorphic to \mathbb{Z} .

43x:2. Is Theorem 43x:C a special case of Theorem 43x:A?

43x:3. May the assumption of openness of A and B in 43x:A be omitted?

43x:4. Deduce Theorem 43.G from the Seifert–van Kampen Theorem 43x:A.

43x:5. Compute the fundamental group of the lens space, which is obtained by pasting together two solid tori via the homeomorphism $S^1 \times S^1 \rightarrow S^1 \times S^1 : (u, v) \mapsto (u^k v^l, u^m v^n)$, where $kn - lm = 1$.

43x:6. Determine the homotopy and the topological type of the lens space for $m = 0, 1$.

43x:7. Find a presentation for the fundamental group of the complement in \mathbb{R}^3 of a torus knot K of type (p, q) , where p and q are relatively prime positive integers. This knot lies on the revolution torus T , which is described by parametric equations

$$\begin{cases} x = (2 + \cos 2\pi u) \cos 2\pi v \\ y = (2 + \cos 2\pi u) \sin 2\pi v \\ z = \sin 2\pi u, \end{cases}$$

and K is described on T by equation $pu = qv$.

43x:8. Let (X, x_0) and (Y, y_0) be two simply connected topological spaces with marked points, and let $Z = X \vee Y$ be their bouquet.

(a) Prove that if X and Y are cellular spaces, then Z is simply connected.

- (b) Prove that if x_0 and y_0 have neighborhoods $U_{x_0} \subset X$ and $V_{y_0} \subset Y$ that admit strong deformation retractions to x_0 and y_0 , respectively, then Z is simply connected.
- (c) Construct two simply connected topological spaces X and Y with a non-simply connected bouquet.

**§43x°7 Group-Theoretic Digression:
Amalgamated Product of Groups**

At first glance, description of the fundamental group of X given above in the statement of Seifert - van Kampen Theorem is far from being invariant: it depends on the choice of generators and relations of other groups involved. However, this is actually a detailed description of a group - theoretic construction in terms of generators and relations. By solving the next problem, you will get a more complete picture of the subject.

43x:D. Let A and B be groups,

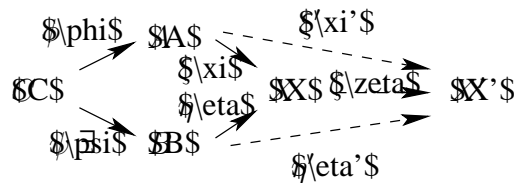
$$A = \langle \alpha_1, \dots, \alpha_p \mid \rho_1 = \dots = \rho_r = 1 \rangle,$$

$$B = \langle \beta_1, \dots, \beta_q \mid \sigma_1 = \dots = \sigma_s = 1 \rangle,$$

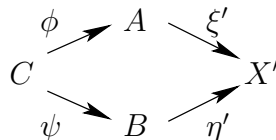
and C be a group generated by $\gamma_1, \dots, \gamma_t$. Let $\xi : C \rightarrow A$ and $\eta : C \rightarrow B$ be arbitrary homomorphisms. Then

$$X = \langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \mid \rho_1 = \dots = \rho_r = \sigma_1 = \dots = \sigma_s = 1, \xi(\gamma_1) = \eta(\gamma_1), \dots, \xi(\gamma_t) = \eta(\gamma_t) \rangle.$$

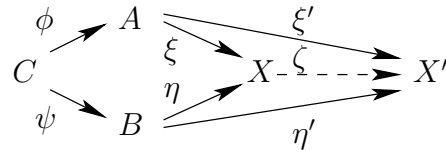
and homomorphisms $\phi : A \rightarrow X : \alpha_i \mapsto \alpha_i, i = 1, \dots, p$ and $\psi : B \rightarrow X : \beta_j \mapsto \beta_j, j = 1, \dots, q$ take part in commutative diagram



and for each group X' and homomorphisms $\phi' : A \rightarrow X'$ and $\psi' : B \rightarrow X'$ involved in commutative diagram



there exists a unique homomorphism $\zeta : X \rightarrow X'$ such that diagram



is commutative. The latter determines the group X up to isomorphism.

The group X described in 43x:D is a *free product of A and B with amalgamated subgroup C* , it is denoted by $A *_C B$. Notice that the name is not quite precise, as it ignores the role of the homomorphisms ϕ and ψ and the possibility that they may be not injective.

If the group C is trivial, then $A *_C B$ is denoted by $A * B$ and called the *free product* of A and B .

43x:9. Is a free group of rank n a free product of n copies of \mathbb{Z} ?

43x:10. Represent the fundamental group of Klein bottle as $\mathbb{Z} *_Z \mathbb{Z}$. Does this decomposition correspond to a decomposition of Klein bottle?

43x:11. Riddle. Define a free product as a set of equivalence classes of words in which the letters are elements of the factors.

43x:12. Investigate algebraic properties of free multiplication of groups: is it associative, commutative and, if it is, then in what sense? Do homomorphisms of the factors determine a homomorphism of the product?

43x:13*. Find decomposition of modular group

$$Mod = SL(2, \mathbb{Z}) / \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

as free product $\mathbb{Z}_2 * \mathbb{Z}_3$.

§43x°8 Addendum to Seifert–van Kampen Theorem

Seifert-van Kampen Theorem appeared and used mainly as a tool for calculation of fundamental groups. However, it helps not in any situation. For example, it does not work under assumptions of the following theorem.

43x:E. Let X be a topological space, A and B open sets covering X and $C = A \cap B$. Assume that A and B are simply connected and C consists of two connected components. Then $\pi_1(X)$ is isomorphic to \mathbb{Z} .

Theorem 43x:E also holds true if we assume that C consists of two path-connected components. The difference seems to be immaterial, but the proof becomes incomparably more technical.

Seifert and van Kampen needed more universal tool for calculation of fundamental group, and theorems published by them were much more general than 43x:A. Theorem 43x:A is all that could penetrate from there original papers to textbooks. Theorem 43x:1 is another special case of

their results. The most general formulation is cumbersome, and we restrict ourselves to one more special case, which was distinguished by van Kampen. Together with 43x:A, it allows one to calculate fundamental groups in all situations that are available with the most general formulations by van Kampen, although not that fast. We formulate the original version of this theorem, but recommend, first, to restrict to a cellular version, in which the results presented in the beginning of this section allow one to obtain a complete answer about calculation of fundamental groups, and only after that to consider the general situation.

First, let us describe the situation common for both formulations. Let A be a topological space, B its closed subset and U a neighborhood of B in A such that $U \setminus B$ is a union of two disjoint sets, M_1 and M_2 , open in A . Put $N_i := B \cup M_i$. Let C be a topological space that can be represented as $(A \setminus U) \cup (N_1 \sqcup N_2)$ and in which the sets $(A \setminus U) \cup N_1$ and $(A \setminus U) \cup N_2$ with the topology induced from A form a fundamental cover. There are two copies of B in C , which come from N_1 and N_2 . The space A can be identified with the quotient space of C obtained by identification of the two copies of B via the natural homeomorphism. However, our description begins with A , since this is the space whose fundamental group we want to calculate, while the space B is auxiliary constructed out of A . See Figure 1.

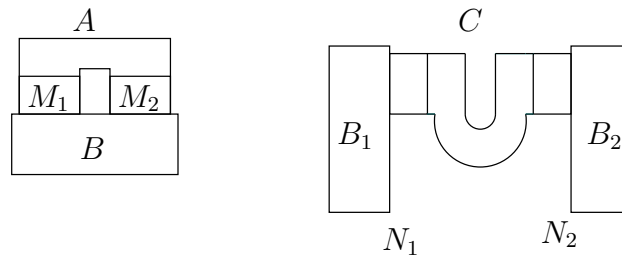


FIGURE 1.

In the cellular version of the statement formulated below, space A is supposed to be cellular, and B its cellular subspace. Then C is also equipped with a natural cellular structure such that the natural map $C \rightarrow A$ is cellular.

43x:F. *Let in the situation described above C is path-connected and $x_0 \in C \setminus (B_1 \cup B_2)$. Let $\pi_1(C, x_0)$ is presented by generators $\alpha_1, \dots, \alpha_n$ and relations $\psi_1 = 1, \dots, \psi_m = 1$. Assume that base points $y_i \in B_i$ are mapped to the same point y under the mapping $C \rightarrow A$, and σ_i is a homotopy class of a path connecting x_0 with y_i in C . Let β_1, \dots, β_p be generators of $\pi_1(B, y)$, and $\beta_{1i}, \dots, \beta_{pi}$ the corresponding elements of $\pi_1(B_i, y_i)$. Denote by φ_{li} a word representing $\sigma_i \beta_{li} \sigma_i^{-1}$ in terms of $\alpha_1,$*

\dots, α_n . Then $\pi_1(A, x_0)$ has the following presentation:

$$\langle \alpha_1, \dots, \alpha_n, \gamma \mid \psi_1 = \dots = \psi_m = 1, \gamma\varphi_{11} = \varphi_{12}\gamma, \dots, \gamma\varphi_{p1} = \varphi_{p2}\gamma \rangle.$$

43x:14. Using 43x:F, calculate fundamental groups of torus and Klein bottle.

43x:15. Using 43x:F, calculate the fundamental groups of basic surfaces.

43x:16. Deduce Theorem 43x:1 from 43x:A and 43x:F.

43x:17. Riddle. Develop an algebraic theory of group-theoretic construction contained in Theorem 43x:F.

Proofs and Comments

40.A Let A be a cellular subspace of a cellular space X . For $n = 0, 1, \dots$, we see that $A \cap X_{n+1}$ is obtained from $A \cap X_n$ by attaching the $(n + 1)$ -cells contained in A . Therefore, if A is contained in a certain skeleton, then A certainly is a cellular space and the intersections $A_n = A \cap X_n$, $n = 0, 1, \dots$, are the skeletons of A . In the general case, we must verify that the cover of A by the sets A_n is fundamental, which follows from assertion 3 of Lemma 40.A.1 below, Problem 40.1, and assertion 40x:G.

40.A.1 We prove only assertion 3 because it is needed for the proof of the theorem. Assume that a subset $F \subset A$ intersects each of the sets A_i along a set closed in A_i . Since $F \cap X_i = F \cap A_i$ is closed in A_i , it follows that this set is closed in X_i . Therefore, F is closed in X since the cover $\{X_i\}$ is fundamental. Consequently, F is also closed in A , which proves that the cover $\{A_i\}$ is fundamental.

40.B This is true because attaching D^n to a point along the boundary sphere we obtain the quotient space $D^n/S^{n-1} \cong S^n$.

40.C These (open) cells are: a point, the $(n-1)$ -sphere S^{n-1} without this point, the n -ball B^n bounded by S^{n-1} : $e^0 = x \in S^{n-1} \subset D^n$, $e^{n-1} = S^n \setminus x$, $e^n = B^n$.

40.D Indeed, factorizing the disjoint union of segments by the set of all of their endpoints, we obtain a bouquet of circles.

40.E We present the product $I \times I$ as a cellular space consisting of 9 cells: four 0-cells – the vertices of the square, four 1-cells – the sides of the square, and a 2-cell – the interior of the square. After the standard factorization under which the square becomes a torus, from the four 0-cells we obtain one 0-cell, and from the four 1-cells we obtain two 1-cells.

40.F Each open cell of the product is a product of open cells of the factors, see Problem 40.3.

40.G Let $S^k = S^n \cap \mathbb{R}^{k+1}$, where

$$\mathbb{R}^{k+1} = \{(x_1, x_2, \dots, x_{k+1}, 0, \dots, 0)\} \subset \mathbb{R}^{n+1}.$$

If we present S^n as the union of the constructed spheres of smaller dimensions: $S^n = \bigcup_{k=0}^n S^k$, then for each $k \in \{1, \dots, n\}$ the difference $S^k \setminus S^{k-1}$ consists of exactly two k -cells: open hemispheres.

40.H Consider the cellular partition of S^n described in the solution of Problem 40.G. Then the factorization $S^n \rightarrow \mathbb{R}P^n$ identifies both cells in each dimension into one. Each of the attaching maps is the projection $D^k \rightarrow \mathbb{R}P^k$ mapping the boundary sphere S^{k-1} onto $\mathbb{R}P^{k-1}$.

40.I 0-cells are all integer points, and 1-cells are the open intervals $(k, k + 1)$, $k \in \mathbb{Z}$.

40.J Since $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$ (n factors), the cellular structure of \mathbb{R}^n can be determined by those of the factors (see 40.3). Thus, the 0-cells are the points with integer coordinates. The 1-cells are open intervals with endpoints $(k_1, \dots, k_i, \dots, k_n)$ and $(k_1, \dots, k_i + 1, \dots, k_n)$, i.e., segments parallel to the coordinate axes. The 2-cells are squares parallel to the coordinate 2-planes, etc.

40.K See the solution of Problem 40.J.

40.L This is obvious: each infinite countable 0-dimensional space is homeomorphi to $\mathbb{N} \subset \mathbb{R}$.

40.M We map 0-cells to integer points $A_k(k, 0, 0)$ on the x axis. The embeddings of 1-cells will be piecewise linear and performed as follows. Take the n th 1-cell of X to the pair of points with coordinates $C_n(0, 2n - 1, 1)$ and $D_n(0, 2n, 1)$, $n \in \mathbb{N}$. If the endpoints of the 1-cell are mapped to A_k and A_l , then the image of the 1-cell is the three-link polyline $A_k C_n D_n A_l$ (possibly, closed). We easily see that the images of distinct open cells are disjoint (because their outer third parts lie on two skew lines). We have thus constructed an injection $f : X \rightarrow \mathbb{R}^3$, which is obviously continuous. The inverse mapping is continuous because it is continuous on each of the constructed polylines, which in addition constitute a closed locally-finite cover of $f(X)$, which is fundamental by 9.U.

40.N Use induction on skeletons and 40.N.2. The argument is simplified a great deal in the case where the cellular space is finite.

40.N.1 We assume that $X \subset \mathbb{R}^p \subset \mathbb{R}^{p+q+1}$, where \mathbb{R}^p is the coordinate space of the first p coordinate lines in \mathbb{R}^{p+q+1} , and $Y \subset \mathbb{R}^q \subset \mathbb{R}^{p+q+1}$, where \mathbb{R}^q is the coordinate space of the last q coordinate lines in \mathbb{R}^{p+q+1} . Now we define a mapping $f : X \sqcup Y \rightarrow \mathbb{R}^{p+q+1}$. Put $f(x) = x$ if $x \in X$, and $f(y) = (0, \dots, 0, 1, y)$ if $y \notin V = h^{-1}(A \times [0, \frac{1}{2}])$. Finally, if $y \in U$, $h(y) = (a, t)$, and $t \in [0, \frac{1}{2}]$, then we put

$$f(y) = ((1 - 2t)\varphi(a), 2t, 2ty).$$

We easily see that f is a proper mapping. The quotient mapping $\widehat{f} : X \cup_{\varphi} Y \rightarrow \mathbb{R}^{p+q+1}$ is a proper injection, therefore, \widehat{f} is an embedding by 18x:O (cf. 18x:P).

40.N.2 By the definition of a cellular space, X is obtained by attaching a disjoint union of closed k -disks to the $(k - 1)$ -skeleton of X . Let Y be a countable union of k -balls, A the union of their boundary spheres. (The assumptions of Lemma 40.N.1 is obviously fulfilled: let

the neighborhood U be the complement of the union of concentric disks with radius $\frac{1}{2}$.) Thus, Lemma 40.N.2 follows from 40.N.1.

40.O This follows from 40.N.2 by the definition of the cellular topology.

40.P This follows from 40.O and 40.N.

40.Q This follows from 40.P.

40.R Try to prove this assertion at least for 1-dimensional spaces.

40.S This can be proved by somewhat complicating the argument used in the proof of 40x.B.

40.T See, [FR, p. 93].

40x:A We easily see that the closure of any open simplex is canonically homeomorphic to the closed n -simplex. and, since any simplicial space Σ is Hausdorff, Σ is homeomorphic to the quotient space obtained from a disjoint union of several closed simplices by pasting them together along entire faces via affine homeomorphisms. Since each simplex Δ is a cellular space and the faces of Δ are cellular subspaces of Δ , it remains to use Problem 40x.H.

40x:B Let X be a cellular space, $x, y \in X$. Let n be the smallest number such that $x, y \in X_n$. We construct their disjoint neighborhoods U_n and V_n in X_n . Let, for example, $x \in e$, where e is an open n -cell. Then let U_n be a small ball centered at x , and let V_n be the complement (in X_n) of the closure of U_n . Now let a be the center of an $(n+1)$ -cell, $\varphi : S^n \rightarrow X_n$ the attaching mapping. Consider the open cones over $\varphi^{-1}(U_n)$ and $\varphi^{-1}(V_n)$ with vertex a . Let U_{n+1} and V_{n+1} be the unions of the images of such cones over all $(n+1)$ -cells of X . Clearly, they are disjoint neighborhoods of x and y in X_{n+1} . The sets $U = \bigcup_{k=n}^{\infty} U_k$ and $V = \bigcup_{k=n}^{\infty} V_k$ are disjoint neighborhoods of x and y in X .

40x:C Let X be a cellular space, $e \subset X$ a cell of X , $\psi : D^n \rightarrow X$ the characteristic mapping of e , $B = B^n \subset D^n$ the open unit ball. Since the mapping ψ is continuous, we have $\bar{e} = \psi(D^n) = \psi(\text{Cl } B) \subset \text{Cl}(\psi(B)) = \text{Cl}(e)$. On the other hand, $\psi(D^n)$ is a compact set, which is closed by 40x.B, whence $\bar{e} = \psi(D^n) \supset \text{Cl}(e)$.

40x:D Let X be a cellular space, X_n the n -skeleton of X , $n \in \mathbb{N}$. The definition of the quotient topology easily implies that X_{n-1} and closed n -cells of X form a fundamental cover of X_n . Starting with $n = 0$ and reasoning by induction, we prove that the cover of X_n by closed k -cells with $k \leq n$ is fundamental. And since the cover of X by the skeletons X_n is fundamental by the definition of the cellular topology, so is the cover of X by closed cells (see 9.31).

40x:E This follows from assertion 40x:D, the fact that, by the definition of a cellular subspace, each closed cell is contained in an element of the cover, and assertion 9.31.

40x:F Let X be a cellular space, X_k the k -skeleton of X . First, we prove that each compact set $K \subset X_k$ intersects only a finite number of open cells in X_k . We use induction on the dimension of the skeleton. Since the topology on the 0-skeleton is discrete, each compact set can contain only a finite number of 0-cells of X . Let us perform the step of induction. Consider a compact set $K \subset X_n$. For each n -cell e_α meeting K , take an open ball $U_\alpha \subset e_\alpha$ such that $K \cap U_\alpha \neq \emptyset$. Consider the cover $\Gamma = \{e_\alpha, X_n \setminus \cup \text{Cl}(U_\alpha)\}$. It is clear that Γ is an open cover of K . Since K is compact, Γ contains a finite subcovering. Therefore, K intersects finitely many n -cells. The intersection of K with the $(n - 1)$ -skeleton is closed, therefore, it is compact. By the inductive hypothesis, this set (i.e., $K \cap X_{n-1}$) intersects finitely many open cells. Therefore, the set K also intersects finitely many open cells.

Now let $\varphi : S^{n-1} \rightarrow X_{n-1}$ be the attaching mapping for the n -cell, $F = \varphi(S^{n-1}) \subset X_{n-1}$. Since F is compact, F can intersect only a finite number of open cells. Thus we see that each closed cell intersects only a finite number of open cells.

40x:G Let A be a cellular subspace of X . By 40x:D, it is sufficient to verify that $A \cap \bar{e}$ is closed for each cell e of X . Since a cellular subspace is a union of open (as well as of closed) cells, i.e., $A = \cup e_\alpha = \cup \bar{e}_\alpha$, it follows from 40x:F that we have

$$A \cap \bar{e} = (\cup e_\alpha) \cap \bar{e} = (\cup_{i=1}^n e_{\alpha_i}) \cap \bar{e} \subset (\cup_{i=1}^n \bar{e}_{\alpha_i}) \cap \bar{e} \subset A \cap \bar{e}$$

and, consequently, the inclusions in this chain are equalities. Consequently, by 40x:C, the set $A \cap \bar{e} = \cup_{i=1}^n (\bar{e}_{\alpha_i} \cap \bar{e})$ is closed as a union of a finite number of closed sets.

40x:I Since, by 40x:F, each closed cell intersects only a finite number of open cells, it follows that the intersection of any closed cell \bar{e} with A is finite and consequently (since cellular spaces are Hausdorff) closed, both in X , and *a fortiori* in \bar{e} . Since, by 40x:D, closed cells constitute a fundamental cover, the set A itself is also closed. Similarly, each subset of A is also closed in X and *a fortiori* in A . Thus, indeed, the induced topology in A is discrete.

40x:J Let $K \subset X$ be a compact subset. In each of the cells e_α meeting K , we take a point $x_\alpha \in e_\alpha \cap K$ and consider the set $A = \{x_\alpha\}$. By 40x:I, the set A is closed, and the topology on A is discrete. Since A is compact as a closed subset of a compact set, therefore, A is finite. Consequently, K intersects only a finite number of open cells.

40x:K \Leftrightarrow Use 40x:J. \Leftarrow A finite cellular space is compact as a union of a finite number of compact sets – closed cells.

40x:L We can use induction on the dimension of the cell because the closure of any cell intersects finitely many cells of smaller dimension. Notice that the closure itself is not necessarily a cellular subspace.

40x:M This follows from 40x:J, 40x:L, and 40.2.

40x:N \Rightarrow Let K be a compact subset of a cellular space. Then K is closed because each cellular space is Hausdorff. Assertion 40x:J implies that K meets only a finite number of open cells.

\Leftarrow If K intersects finitely many open cells, then by 40x:L K lies in a finite cellular subspace Y , which is compact by 40x:K, and K is a closed subset of Y .

40x:O Let X be a cellular space. \Rightarrow We argue by contradiction. Let X contain an uncountable set of n -cells e_α^n . Put $U_\alpha^n = e_\alpha^n$. Each of the sets U_α^n is open in the n -skeleton X_n of X . Now we construct an uncountable collection of disjoint open sets in X . Let a be the center of a certain $(n+1)$ -cell, $\varphi : S^n \rightarrow X_n$ the attaching mapping of the cell. We construct the cone over $\varphi^{-1}(U_\alpha^n)$ with vertex at a and denote by U_α^{n+1} the union of such cones over all $(n+1)$ -cells of X . It is clear that $\{U_\alpha^{n+1}\}$ is an uncountable collection of sets open in X_{n+1} . Then the sets $U_\alpha = \bigcup_{k=n}^{\infty} U_\alpha^k$ constitute an uncountable collection of disjoint sets that are open in the entire X . Therefore, X is not second countable and, therefore, nonseparable.

\Leftarrow If X has a countable set of cells, then, taking in each cell a countable everywhere dense set and uniting them, we obtain a countable set dense in the entire X (check this!). Thus, X is separable.

40x:P Indeed, any path-connected component Y of a cellular space together with each point $x \in Y$ entirely contains each closed cell containing x and, in particular, it contains the closure of the open cell containing x .

40x:R Cf. the argument used in the solution of Problem 40x:O.

40x:R This is so because a cellular space is locally path-connected, see 40x:Q.

40x:S This follows from 40x:R.

40x:T \Rightarrow Obvious. \Leftarrow We show by induction that the number of cells in each dimension is countable. For this purpose, it is sufficient to prove that each cell intersects finitely many closed cells. It is more convenient to prove a stronger assertion: any closed cell \bar{e} intersects finitely many closed cells. It is clear that any neighborhood meeting the closed cell also meets the cell itself. Consider the cover of \bar{e} by neighborhoods each of which intersects finitely many closed cells. It remains to use the fact that \bar{e} is compact.

40x:U By Problem 40x:T, the 1-skeleton of X is connected. The result of Problem 40x:T implies that it is sufficient to prove that the 0-skeleton of X is countable. Fix a 0-cell x_0 . Denote by A_1 the union of all closed 1-cells containing x_0 . Now we consider the set A_2 – the union of all closed 1-cells meeting A_1 . Since X is locally finite, each of the sets A_1 and A_2 contains a finite number of cells. Proceeding in a similar way, we obtain an increasing sequence of 1-dimensional cellular subspaces $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$, each of which is finite. Put $A = \bigcup_{k=1}^{\infty} A_k$. The set A contains countably many cells. The definition of the cellular topology implies that A is both open and closed in X_1 . Since X_1 is connected, we have $A = X_1$.

40x:V \Leftrightarrow Assume the contrary: let the 1-skeleton X_1 be disconnected. Then X_1 is the union of two closed sets: $X_1 = X'_1 \cup X''_1$. Each 2-cell is attached to one of these sets, whence $X_2 = X'_2 \cup X''_2$. A similar argument shows that for each positive integer n the n -skeleton is a union of its closed subsets. Put $X' = \bigcup_{n=0}^{\infty} X'_n$ and $X'' = \bigcup_{n=0}^{\infty} X''_n$. By the definition of the cellular topology, X' and X'' are closed, consequently, X is disconnected. \Leftarrow This is obvious.

41.A This immediately follows from the obvious equality $c_i(A \cup B) = c_i(A) + c_i(B) - c_i(A \cap B)$.

41.B Here we use the following artificial trick. We introduce the polynomial $\chi_A(t) = c_0(A) + c_1(A)t + \dots + c_i(A)t^i + \dots$. This is the *Poincaré polynomial*, and its most important property for us here is that $\chi(X) = \chi_X(-1)$. Since $c_k(X \times Y) = \sum_{i=0}^k c_i(X)c_{k-i}(Y)$, we have

$$\chi_{X \times Y}(t) = \chi_X(t) \cdot \chi_Y(t),$$

whence $\chi(X \times Y) = \chi_{X \times Y}(-1) = \chi_X(-1) \cdot \chi_Y(-1) = \chi(X) \cdot \chi(Y)$.

41.C Set $X' = X \setminus (e \cup f)$. It follows from the definition that the union of all open cells in X' coincides with the union of all closed cells in X' , consequently, X' is a cellular subspace of X .

41.D The deformation retraction of D^n to the lower closed hemisphere S_-^{n-1} determines a deformation retraction $X \rightarrow X \setminus (e \cup f)$.

41.E The assertion is obvious because each elementary combinatorial collapse decreases by one the number of cells in each of two neighboring dimensions.

41.F Let $p : X \rightarrow X'$ be the factorization map. The space X' has the same open cells as X except e and f . The attaching mapping for each of them is the composition of the initial attaching map and p .

41.G.1 Put $Y = X_{n-1} \cup_{\varphi_e} D^n$. Clearly, $Y' \cong Y \setminus (e \cup f)$, and so we identify these spaces. Then the projection $p' : Y \rightarrow Y'$ is a homotopy equivalence by 41.D.

41.G.2 Let $\{e_\alpha\}$ be a collection of n -cells of X distinct from the cell e , φ_α – the corresponding attaching mappings. Consider the mapping $p' : Y \rightarrow Y'$. Since

$$X_n = Y \cup_{(\bigsqcup_\alpha \varphi_\alpha)} \left(\bigsqcup_\alpha D_\alpha^n \right),$$

we have

$$X'_n = Y' \cup_{(\bigsqcup_\alpha p' \circ \varphi_\alpha)} \left(\bigsqcup_\alpha D_\alpha^n \right).$$

Since the mapping p' is a homotopy equivalence by *41.G.1*, the result of *41x:6* implies that p' extends to a homotopy equivalence $p_n : X_n \rightarrow X'_n$. Using induction on skeletons, we obtain the required assertion.

41x:A We use induction on the dimension. Clearly, we should consider only those cells which do not lie in A . If there is a retraction

$$\rho_{n-1} : (X_{n-1} \cup A) \times I \rightarrow (X_{n-1} \times 0) \cup (A \times I),$$

and we construct a retraction

$$\tilde{\rho}_n : (X_n \cup A) \times I \rightarrow (X_n \times 0) \cup ((X_{n-1} \cup A) \times I),$$

then it is obvious how, using their “composition”, we can obtain a retraction

$$\rho_n : (X_n \cup A) \times I \rightarrow (X_n \times 0) \cup (A \times I).$$

We need the standard retraction $\rho : D^n \times I \rightarrow (D^n \times 0) \cup (S^{n-1} \times I)$. (It is most easy to define ρ geometrically. Place the cylinder in a standard way in \mathbb{R}^{n+1} and consider a point p lying over the center of the upper base. For $z \in D^n \times I$, let $\rho(z)$ be the point of intersection of the ray starting at p and passing through z with the union of the base $D^n \times 0$ and the lateral area $S^{n-1} \times I$ of the cylinder.) The quotient mapping ρ is a mapping $\bar{e} \times I \rightarrow (X_n \times 0) \cup (X_{n-1} \times I)$. Extending it identically to $X_{n-1} \times I$, we obtain a mapping

$$\rho_e : (\bar{e} \times I) \cup (X_{n-1} \times I) \rightarrow (X_n \times 0) \cup (X_{n-1} \times I).$$

Since the closed cells constitute a fundamental cover of a cellular space, the retraction $\tilde{\rho}_n$ is thus defined.

41x:B The formulas $\tilde{H}(x, 0) = F(x)$ for $x \in X$ and $\tilde{H}(x, t) = h(x, t)$ for $(x, t) \in A \times I$ determine a mapping $\tilde{H} : (X \times 0) \cup (A \times I) \rightarrow Y$. By *41x:A*, there is a retraction $\rho : X \times I \rightarrow (X \times 0) \cup (A \times I)$. The composition $H = \tilde{H} \circ \rho$ is the required homotopy.

41x:C Denote by $h : A \times I \rightarrow A$ a homotopy between the identical mapping of A and the constant mapping $A \rightarrow A : a \mapsto x_0$. Consider the homotopy $\tilde{h} = i \circ h : A \times I \rightarrow X$. By Theorem *41x:B*, \tilde{h} extends to

a homotopy $H : X \times I \rightarrow X$ of the identical mapping of the entire X . Consider the mapping $f : X \rightarrow X$, $f(x) = H(x, 1)$. By the construction of the homotopy \tilde{h} , we have $f(A) = \{x_0\}$, consequently, the quotient map of f is a continuous mapping $g : X/A \rightarrow X$. We prove that pr and g are mutually inverse homotopy equivalences. To do this we must verify that $g \circ \text{pr} \sim \text{id}_X$ and $\text{pr} \circ g \sim \text{id}_{X/A}$.

1) We observe that $H(x, 1) = g(\text{pr}(x))$ by the definition of g . Since $H(x, 0) = x$ for all $x \in X$, it follows that H is a homotopy between id_X and the composition $g \circ \text{pr}$.

2) If we factorize each fiber $X \times t$ by $A \times t$, then, since $H(x, t) \in A$ for all $x \in A$ and $t \in I$, the homotopy H determines a homotopy $\tilde{H} : X/A \rightarrow X/A$ between $\text{id}_{X/A}$ and the composition $p \circ g$.

41x:F Let X be the space. By 41x:E, we can assume that X has one 0-cell, and therefore the 1-skeleton X_1 is a bouquet of circles. Consider the characteristic mapping $\psi : I \rightarrow X_1$ of a certain 1-cell. Instead of the loop ψ , it is more convenient to consider the circular loop $S^1 \rightarrow X_1$, which we denote by the same letter. Since X is simply connected, the loop ψ extends to a mapping $f : D^2 \rightarrow X$. Now consider the disk D^3 . To simplify the notation, we assume that f is defined on the lower hemisphere $S^2_- \subset D^3$. Put $Y = X \cup_f D^3 \simeq X$. The space Y is cellular and is obtained by adding two cells to X : a 2- and a 3-cell. The new 2-cell e , i.e., the image of the upper hemisphere in D^3 , is a contractible cellular space. Therefore, we have $Y/e \simeq Y$, and Y/e contains one 1-cell less than the initial space X . Proceeding in this way, we obtain a space with one-point 2-skeleton. Notice that our construction yielded a 3-dimensional cellular space. Actually, in our assumptions the space is homotopy equivalent to: a point, a 2-sphere, or a bouquet of 2-spheres, but the proof of this fact involves more sophisticated techniques (the homology).

41x:13 Consider a cellular partition of $\mathbb{C}P^2$ consisting of one 0-cell, one 1-cell, two 2-cells, and one 4-cell. Furthermore, we can assume that the 2-skeleton of the cellular space obtained is $\mathbb{C}P^1 \subset \mathbb{C}P^2$, while the 1-skeleton is the real part $RP^1 \subset \mathbb{C}P^1$. Let $\tau : \mathbb{C}P^2 \rightarrow \mathbb{C}P^2$ be the involution of complex conjugation, by which we factorize. Clearly, $\mathbb{C}P^1/[z \sim \tau(z)] \cong D^2$. Consider the characteristic mapping $\psi : D^4 \rightarrow \mathbb{C}P^1$ of the 4-cell of the initial cellular partition. The quotient space $D^4/[z \sim \tau(z)]$ is obviously homeomorphi to D^4 . Therefore, the quotient map

$$D^4/[z \sim \tau(z)] \rightarrow \mathbb{C}P^1/[z \sim \tau(z)]$$

is the characteristic mapping for the 4-cell of X . Thus, X is a cellular space with 2-skeleton D^2 . Therefore, by 41x:C, we have $X \simeq S^4$.

41x:G Let the mapping $f : X \rightarrow A$ be homotopically inverse to the inclusion in_A . By assumption, the restriction of f to the subspace A , i.e., the composition $f \circ \text{in}$, is homotopic to the identical mapping id_A . By Theorem 41x:B, this homotopy extends to a homotopy $H : X \times I \rightarrow A$ of f . Put $\rho(x) = H(x, 1)$; then $\rho(x, 1) = x$ for all $x \in A$. Consequently, ρ is a retraction. It remains to observe that, since ρ is homotopic to f , it follows that $\text{in} \circ \rho$ is homotopic to the composition $\text{in}_A \circ f$, which is homotopic to id_X because f and in are homotopically inverse by assumption.

42.A Prove this by induction, using Lemma 42.A.1.

42.A.1 Certainly, the fact that the projection is a homotopy equivalence is a special case of assertions 41x:D and 41.G. However, here we present an independent argument, which is more visual in the 1-dimensional case. All homotopies will be fixed outside a neighborhood of the 1-cell e of the initial cellular space X and outside a neighborhood of the 0-cell x_0 , which is the image of e in the quotient space $Y = X/e$. For this reason, we consider only the closures of such neighborhoods. Furthermore, to simplify the notation, we assume that the spaces under consideration coincide with these neighborhoods. In this case, X is the 1-cell e with the segments I_1, I_2, \dots, I_k (respectively, J_1, J_2, \dots, J_n) attached to the left endpoint, (respectively, to the right endpoint). The space Y is simply a bouquet of all these segments with a common point x_0 . The mapping $f : X \rightarrow Y$ has the following structure: each of the segments I_i and J_j is mapped onto itself identically, and the cell e is mapped to x_0 . The mapping $g : Y \rightarrow X$ takes x_0 to the midpoint of e and maps a half of each of the segments I_s and J_t to the left and to the right half of e , respectively. Finally, the remaining half of each of these segments is mapped (with double stretching) onto the entire segment. We prove that the described mappings are homotopically inverse. Here it is important that the homotopies be fixed on the free endpoints of I_s and J_t . The composition $f \circ g : Y \rightarrow Y$ has the following structure. The restriction of $f \circ g$ to each of the segments in the bouquet is, strictly speaking, the product of the identical path and the constant path, which is known to be homotopic to the identical path. Furthermore, the homotopy is fixed both on the free endpoints of the segments and on x_0 . The composition $g \circ f$ maps the entire cell e to the midpoint of e , while the halves of each of the segments I_s and J_t adjacent to e are mapped a half of e , and their remaining parts are doubly stretched and mapped onto the entire corresponding segment. Certainly, the mapping under consideration is homotopic to the identity.

42.B By 42.A.1, each connected 1-dimensional finite cellular space X is homotopy equivalent to a space X' , where the number of 0- and 1-cells is one less than in X , whence $\chi(X) = \chi(X')$. Reasoning by induction, we obtain as a result a space with a single 0-cell and with

Euler characteristic equal to $\chi(X)$ (cf. 41.E). Let k be the number of 1-cells in this space. Then $\chi(X) = 1 - k$, whence $k = 1 - \chi(X)$. It remains to observe that k is precisely the rang of $\pi_1(X)$.

42.C This follows from 42.B because the fundamental group of a space is invariant with respect to homotopy equivalences.

42.D This follows from 42.C.

42.E By 42.B, if two finite connected 1-dimensional cellular spaces have isomorphic fundamental groups (or equal Euler characteristics), then each of them is homotopy equivalent to a bouquet consisting of one and the same number of circles, therefore, the spaces are homotopy equivalent. If the spaces are homotopy equivalent, then, certainly, their fundamental groups are isomorphic, and, by 42.C, their Euler characteristics are also equal.

42x:A Let e be an open cell. If the image $\varphi_e(S^0)$ of the attaching mapping of e is one-point, then $X \setminus e$ is obviously connected. Assume that $\varphi_e(S^0) = \{x_0, x_1\}$. Prove that each connected component of $X \setminus e$ contains at least one of the points x_0 and x_1 .

42x:B 1) Let X be a connected 1-dimensional cellular space, $x \in X$ a vertex. If a connected component of $X \setminus x$ contains no edges whose closure contains x , then, since cellular spaces are locally connected, the component is both open and closed in the entire X , contrary to the connectedness of X . 2) This follows from the fact that a vertex of degree m lies in the closure of at most m distinct edges.

43.A See 42.B.

43.B This follows from 42.I (or 41x:C) because of 35.L.

43.C It is sufficient to prove that each loop $u : I \rightarrow X$ is homotopic to a loop v with $v(I) \subset A$. Let $U \subset D^k$ be the open ball with radius $\frac{2}{3}$, and let V be the complement in X of a closed disk with radius $\frac{1}{3}$. By the Lebesgue Lemma 16.W, the segment I can be subdivided segments I_1, \dots, I_N the image of each of which is entirely contained in one of the sets U or V . Assume that $u(I_l) \subset U$. Since in D^k any two paths with the same starting and ending points are homotopic, it follows that the restriction $u|_{I_l}$ is homotopic to a path that does not meet the center $a \in D^k$. Therefore, the loop u is homotopic to a loop u' whose image does not contain a . It remains to observe that the space A is a deformation retract of $X \setminus a$, therefore, u' is homotopic to a loop v with image lying in A .

43.D Let s be a loop at x_0 . Since the set $s(I)$ is compact, $s(I)$ is contained in a finite cellular subspace Y of X . It remains to apply assertion 43.C and use induction on the number of cells in Y .

43.E This follows from 43.D and 43.B.

43.F If we take another collection of paths s'_α , then the elements $T_{s_\alpha}[\varphi_\alpha]$ and $T_{s'_\alpha}[\varphi_\alpha]$ will be conjugate in $\pi_1(X_1, x_0)$, and since the subgroup N is normal, N contains the collection of elements $\{T_{s_\alpha}[\varphi_\alpha]\}$ iff N contains the collection $\{T_{s'_\alpha}[\varphi_\alpha]\}$.

43.G We can assume that the 0-skeleton of X is the singleton $\{x_0\}$, so that the 1-skeleton X_1 is a bouquet of circles. Consider a covering $p_1 : Y_1 \rightarrow X_1$ with group N . Its existence follows from the more general Theorem 39x:D on the existence of a covering with given group. In the case considered, the covering space is a 1-dimensional cellular space. Now the proof of the theorem consists of several steps, each of which is the proof of one of the following seven lemmas. It will also be convenient to assume that $\varphi_\alpha(1) = x_0$, so that $T_{s_\alpha}[\varphi_\alpha] = [\varphi_\alpha]$.

43.G.1 Since, clearly, $\text{in}_*([\varphi_\alpha]) = 1$ in $\pi_1(X, x_0)$, we have $\text{in}_*([\varphi_\alpha]) = 1$ in $\pi_1(X_1, x_0)$, therefore, each of the elements $[\varphi_\alpha] \in \text{Ker } i_*$. Since the subgroup $\text{Ker } i_*$ is normal, it contains N , which is the smallest subgroup generated by these elements.

43.G.2 This follows from 39x:P.

43.G.3 Let $F = p_1^{-1}(x_0)$ be the fiber over x_0 . The mapping p_2 is a quotient mapping

$$Y_1 \sqcup \left(\bigsqcup_{\alpha} \bigsqcup_{y \in F_\alpha} D_{\alpha, y}^2 \right) \rightarrow X_1 \sqcup \left(\bigsqcup_{\alpha} D_{\alpha}^2 \right),$$

whose submapping $Y_1 \rightarrow X_1$ is the mapping p_1 , and the mappings $\bigsqcup_{y \in F_\alpha} D_{\alpha}^2 \rightarrow D_{\alpha}^2$ are identities on each of the disks D_{α}^2 . It is clear that for each point $x \in \text{Int } D_{\alpha}^2 \subset X_2$ the entire interior of the disk is a trivially covered neighborhood. Now assume that for point $x \in X_1$ the set U_1 is a trivially covered neighborhood of x with respect to the covering p_1 . Put $U = U_1 \cup (\bigcup_{\alpha'} \psi_{\alpha'}(B_{\alpha'}))$, where $B_{\alpha'}$ is the open cone with vertex at the center of $D_{\alpha'}^2$ and base $\varphi_{\alpha'}^{-1}(U)$. The set U is a trivially covered neighborhood of x with respect to the mapping p_2 .

43.G.4 First, we prove this for $n = 3$. So, let $p : X \rightarrow B$ be an arbitrary covering, $\varphi : S^2 \rightarrow B$ an arbitrary mapping. Consider the subset $A = S^1 \times 0 \cup 1 \times I \cup S^1 \times 1$ of the cylinder $S^1 \times I$, and let $q : S^1 \times I \rightarrow S^1 \times I/A$ be the factorization map. We easily see that $S^1 \times I/A \cong S^2$. Therefore, we assume that $q : S^1 \times I \rightarrow S^2$. The composition $h = \varphi \circ q : S^1 \times I \rightarrow B$ is a homotopy between one and the same constant loop in the base of the covering. By the Path Homotopy Lifting Theorem 34.C, the homotopy h is covered by the mapping \tilde{h} , which also is a homotopy between two constant paths, therefore, the

quotient map of \tilde{h} is the mapping $\tilde{\varphi} : S^2 \rightarrow X$ covering φ . For $n > 3$, use 39x:Y.

43.G.5 The proof is similar to that of Lemma 3.

43.G.6 Since the loop in $\circ s : I \rightarrow X$ is null-homotopic, it is covered by a loop, the image of which automatically lies in Y_1 .

43.G.7 Let s be a loop in X_1 such that $[s] \in \text{Ker}(i_1)_*$. Lemma 6 implies that s is covered by a loop $\tilde{s} : I \rightarrow Y_1$, whence $[s] = (p_1)_*([\tilde{s}]) \in N$. Therefore, $\text{Ker in}_* \subset N$, whence $N = \text{Ker in}_*$ by Lemma 1.

43.I For example, $\mathbb{R}P^2$ is obtained by attaching D^2 to S^1 via the mapping $\varphi : S^1 \rightarrow S^1 : z \mapsto z^2$. The class of the loop φ in $\pi_1(S^1) = \mathbb{Z}$ is the doubled generator, whence $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, as it should have been expected. The torus $S^1 \times S^1$ is obtained by attaching D^2 to the bouquet $S^1 \vee S^1$ via a mapping φ representing the commutator of the generators of $\pi_1(S^1 \vee S^1)$. Therefore, as it should have been expected, the fundamental group of the torus is \mathbb{Z}^2 .

43.K See 40.12 (h).

43.L See 40.12 (i).

43.M.1 Indeed, the single relation in the fundamental group of the sphere with g handles means that the product of g commutators of the generators a_i and b_i equals 1, and so it “vanishes” after the abelianization.

43.N.1 Taking the elements a_1, \dots, a_{g-1} , and $b_n = a_1 a_2 \dots a_g$ as generators in the commuted group, we obtain an Abelian group with a single relation $b_n^2 = 1$.

43.O This follows from 43.M.1.

43.O This follows from 43.N.1.

43.Q This follows from 43.M.1 and 43.N.1.

43x:A We do not assume that you can prove this theorem on your own. The proof can be found, for example, in [Massey].

43x:B Draw a commutative diagram comprising all inclusion homomorphisms induced by all inclusions occurring in this situation.

43x:C Since, as we will see in Section §43x°7, the group presented as above, actually, up to canonical isomorphism does not depend on the choice of generators and relations in $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ and the choice of generators in $\pi_1(C, x_0)$, we can use the presentation which is most convenient for us. We derive the theorem from Theorems 43.D and 43.G. First of all, it is convenient to replace X, A, B , and C by homotopy equivalent spaces with one-point 0-skeletons. We do this with the help of the following construction. Let T_C be a spanning tree in the 1-skeleton of C . We complete T_C to a spanning tree $T_A \supset T_C$ in A , and

also complete T_C to a spanning tree $T_B \supset T_C$. The union $T = T_A \cup T_B$ is a spanning tree in X . It remains to replace each of the spaces under consideration with its quotient space by a spanning tree. Thus, the 1-skeleton of each of the spaces X , A , B , and C either coincides with the 0-cell x_0 , or is a bouquet of circles. Each of the circles of the bouquets determines a generator of the fundamental group of the corresponding space. The image of $\gamma_i \in \pi_1(C, x_0)$ under the inclusion homomorphism is one of the generators, let it be α_i (β_i) in $\pi_1(A, x_0)$ (respectively, in $\pi_1(B, x_0)$). Thus, $\xi_i = \alpha_i$ and $\eta_i = \beta_i$. The relations $\xi_i = \eta_i$, and, in this case, $\alpha_i = \beta_i$, $i = 1, \dots, t$ arise because each of the circles lying in C determines a generator of $\pi_1(X, x_0)$. All the remaining relations, as it follows from assertion 43.G, are determined by the attaching mappings of the 2-cells of X , each of which lies in at least one of the sets A or B , and hence is a relation between the generators of the fundamental groups of these spaces.

43x:D Let \mathcal{F} be a free group with generators $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$. By definition, the group X is the quotient group of F by the normal hull N of the elements

$$\{\rho_1, \dots, \rho_r, \sigma_1, \dots, \sigma_s, \xi(\gamma_1)\eta(\gamma_1)^{-1}, \dots, \xi(\gamma_t)\eta(\gamma_t)^{-1}\}.$$

Since the first diagram is commutative, it follows that the subgroup N lies in the kernel of the homomorphism $F \rightarrow X' : \alpha_i \mapsto \varphi'(\alpha_i), \beta_i \mapsto \psi'(\alpha_i)$, consequently, there is a homomorphism $\zeta : X \rightarrow X'$. Its uniqueness is obvious. Prove the last assertion of the theorem on your own.

43x:E Construct a universal covering of X .