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Topologies and Sheaves Appeared as Syntax and Semantics of Natural Language

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Abstract: In a sheaf-theoretic framework, we describe the process of interpretation of a text written in some unspecified natural language, say in English. We consider only texts written for human understanding, those we call *admissible*. A *meaning* of a part of a text is accepted as the communicative content grasped in a reading process following the reader's interpretive initiative formalized by the term *sense*. For the meaningfulness correlative with an idealized reader's linguistic competence, the set of all meaningful parts of an admissible text is stable under arbitrary unions and finite intersections, and hence it defines a topology that we call *phonocentric*. We interpret syntactic notions in terms of topology and order; it is a kind of *topological formal syntax*. The connectedness and the T_0-separability of such a phonocentric topology are *linguistic universals*. According to a particular sense of reading, we assign to each meaningful fragment of a given text the set of all its meanings those may be grasped in all possible readings in this sense. This way, to any sense of reading, we assign a sheaf of fragmentary meanings. All such sheaves constitute a category, in terms of which we develop a *sheaf-theoretic formal semantics*. It allows us to generalize Frege's compositionality and contextuality principles related with the *Frege duality* between the category of all sheaves of fragmentary meanings and the category of all bundles of contextual meanings. The acceptance of one of these principles implies the acceptance of the other. This Frege duality gives rise to a representation of fragmentary meanings by continuous functions. Finally, we develop a kind of *dynamic semantics* that describes how the interpretation proceeds as a stepwise extension of a meaning representation function from the initial meaningful fragment to the whole interpreted text.

Keywords: sense, meaning, phonocentric topology, linguistic universals, sheaf of fragmentary meanings, compositionality principle, contextuality principle, bundle of contextual meanings, Frege duality, dynamic semantics.

1. Introduction and informal outline

In this work, we apply rigorous mathematical methods in studying the process in which the understanding of a written text or an uttered discourse is reached. Our aim is to present a formal model for the understanding of a text or a discourse in a natural language communication process.

Any natural language serves as a means of communication between members of a community that shares this language. The life of a human society, primitive or developed, ancient or contemporary would be impossible without linguistic communication. When we communicate with each other, we are involved in the activity of exchange with two complementary sides, that is, the production and the understanding of language messages in oral or in written form. Any linguistic communication presupposes the emitting activity that produces a message and the receiving activity that produces an understanding. The message is an externalization of thoughts either by utterance or by writing. As a linguistic message unit, a single stand-alone sentence (or phrase) does not suffice to express the variety of thoughts and ideas that people need to communicate. The minimal exchange units that serve as messages in linguistic communications are written texts and uttered discourses. Linguistics is a discipline that studies the use of a language; for empirical objects, it has, therefore, texts and discourses as the units of human interaction, and not stand-alone words or phrases favoured by traditional grammars and the logic in the wake of Aristotelian tradition primarily concerned with questions of reference and truth.

The main parts of traditional grammars are syntax and semantics. A traditional syntax is a study of sentence structures in a given language, specifically in terms of word order. A semantics, of whatever kind, is the study of relationships between the linguistic expressions and their meanings. Traditional approaches are very restrictive or even inadequate to extend grammatical concepts and theories to the level of text or discourse in order to describe linguistic communication in all its forms.

The present work proposes a mathematical framework that generalizes syntax and semantics of a natural language from the traditional level of a stand-alone sentence or phrase to the level of written or spoken discourse. We propose a kind of a discourse analysis that describes the process of a natural language message interpretation in a uniform manner at all semantic levels.

The paper is organized as follows:

- In the next Sect. 2, we discuss in details our acceptance of basic semantic notions *meaning*, *sense*, and *reference*. We study the interpretation of a text in a certain unspecified natural language, say in English, considered as a means of linguistic communication (mostly in written form). We consider the class of minimal communicative units of a language as made up of texts, and thus it is broader than the class of all stand-alone sentences studied in traditional logical and grammatical theories.

From the set-theoretic point of view, any text is a sequence of its constituent sentences.[1] But from the theoretic point of view on linguistic communication, do we need to define somehow what is a genuine text? It seems useless to set some formal criteria of *textuality* those, likewise to formal criteria of *grammaticality*, would decide that a given sequence of sentences is a well-formed text. Although some particular sequence of words or sentences does not appear to be well-formed, nobody can guarantee the contrary for the future, because a natural language is always open for changes. However, the ethics of linguistic communication presupposes that a genuine text is written by its author(s) as a message intended to be understood by a reader. That is why, instead of adopting any criterion of textuality, we restrict the domain of our study to texts that we assume to be written 'with good grace' as messages intended for human understanding; those we call *admissible*. All sequences of words written in order to imitate some human writings are cast aside as irrelevant to the linguistic communication.

A *meaning* of a part of text is accepted as the communicative content grasped in a particular reading of this part following the idealized reader's attitude, presupposition and intention put together in the term *sense*. We adopt this acceptance of terms 'sense' and 'meaning' because it is close to the ordinary usage of these words in everyday English. The advantage of such a choice of terminology is that we can use words 'sense' and 'meaning' sometimes as linguistic terms, sometimes as ordinary words without specifying each time their mode of use. Otherwise, we were to accept in the use their definitions that we reject in the theory. Thus, we may ask, e.g., "What does this word

[1]It is clear that any such a sequence is made up of so-called 'sentence-tokens', not of so-called 'sentence-types'. Likewise, a sentence is a sequence of word-tokens, and a word is a sequence of morpheme-tokens. Nevertheless, in speaking further about a sequence of certain language units, we shall sometimes omit the word 'token', in order to not overload the terminology.

(or expression, sentence, text) mean in the literal (or metaphorical, allegorical, moral, Platonic, Fregean, narrow, wide, common, etc.) sense?" So, our acceptance of terms *sense* and *meaning* differs from *Sinn* and *Bedeutung* of Frege's famous paper of 1892. We discuss the difference further.

• In Sect. 3, we discuss topology and order structures underlying an admissible text considered as a means of communication. The linguistic communication may be adequately modelled by a formalism that takes as its object of study texts and discourses in their production and interpretation.

Whatever the human language is, the speaker produces an utterance when putting words one after another in an acoustic string. The listener is forced to interpret such a chain of sounds without the possibility of suspending its course with the purpose to return or to make a leap forward. Everyone knows this property empirically, owing to personal experience of speaker and listener; it should undoubtedly be taken into account by everyone who writes a text intended for a human understanding. We argue that such a fundamental feature of linguistic behaviour enables us to endow an admissible text X with the structure of a finite T_0 topological space where the set of opens $\mathfrak{O}(X)$ is the set of all meaningful parts of a given text X. We call *phonocentric* such a topology defined on the text X.

It is well known that the category **FinTOP**$_0$ of finite T_0 topological spaces with continuous maps is isomorphic to the category **FinORD** of finite partial ordered sets (*posets*, for short) with order preserving maps.[2] We consider two functors L and Q establishing such an isomorphism between these categories. It allows us to define on an admissible text topological and order structures, both of *deep* and *surface* kinds. The writing process consists in endowing the text with the surface structure of so-called linear 'word order' (and corresponding topology). The process of interpretation consists in a backward recovering of the deep structure of the specialization order (and corresponding phonocentric topology) on the text.

Thereafter, we define a phonocentric topology in a similar manner at each semantic level of an admissible text. The mathematical interpretation of different linguistic notions in terms of topology and order is a kind of *topological formal syntax*.

• In Sect. 4, we elaborate in mathematical details the aforesaid topological formal syntax. We argue that the T_0-*separability* and the *connectedness* of a

[2]See, for instance, [8, 23].

phonocentric topology are two *linguistic universals* of a topological nature.

• In Sect. 5, we study the process of understanding of an admissible text considered as a means of communication. To understand a text or a compound expression is to grasp what it means, i.e., what communicative content it conveys. Thus, the understanding of a text during its reading is a dynamic process that develops gradually as the reading progresses over the time.

On the other hand, a speaker (a writer) uses words as a preexisting means to express thoughts, and one combines them to convey thoughts one wants to communicate. So the meaning of a compound expression is determined by the meanings of its (meaningful) constituents, as well as the meaning of the whole text is determined by the meanings of its (meaningful) parts.

In the traditional hermeneutics, the relationship between the understanding of (meaningful) parts and the understanding of the whole text was conceived as a fundamental principle of text interpretation called the *hermeneutic circle*. As its counterpart in linguistic theories, there is a need for some principles those describe how the passage from the meanings of parts to the meaning of the whole and the passage in the reverse direction are proceeding. In logic, linguistics and philosophy of language, there exist such two complementary principles both traditionally ascribed to Frege, namely the *compositionality principle* and the *contextuality principle*, those manifest itself in different terms following a particular theoretical framework.

According to J. F. Pelletier [26, p. 89], R. Carnap was the first to attribute the compositionality principle explicitly to Frege in *Meaning and Necessity* [3], where he stated this principle in terms of a functional dependence. The majority of researchers followed him when formulating their definitions of Frege's compositionality principle in the mathematical paradigm of a function. To illustrate this, we cite a few definitions:

> [...] the meaning (semantical interpretation) of a complex expression is a function of the meanings (semantical interpretations) of its constituent expressions. (J. Hintikka [14, p. 31])

> Like Frege, we seek to do this [...] in such a way that [...] the assignment to a compound will be a function of the entities assigned to its components. (R. Montague [24, p. 217])

> [...] The meaning of a whole is a function of the meanings of the parts. (B. H. Partee [25, p. 313])

In many similar definitions, the meaning of a compound expression is set to be a function of the meanings of its parts, whereas what the meanings are differs substantially. Also, these definitions remain reticent about the explicit form of a function concerned. In sharpening her definition, B. H. Partee notices that "the Principle of Compositionality requires a notion of partwhole structure that is based on syntactic structure", and then she modifies the latter definition to the following one:

> The meaning of a whole is a function of the meanings of the parts and of the way they are syntactically combined. (B. H. Partee [25, p. 313])

Nevertheless, the modified definition of the compositionality principle remains implicit with regard to the function it refers to. In fact, the pages subsequent to definitions of compositionality principle in [25, p. 313] are devoted to the discussion of how one may explicitly define the input values (arguments) of such a function, and describe how this function acts on its arguments, and what it returns as output values. On this way, B. H. Partee leads the reader to the formal definitions given in the Montague's seminal paper [24].

To sum up our discussion, we have to note that in agreement with the tradition going back to Carnap, almost all generally accepted definitions of the compositionality principle convey the mathematical concept of a function in a set-theoretic paradigm.

In the contemporary mathematics, there are different formalizations of the concept of a function and functional dependence. In a prevailing set-theoretic paradigm, a function (map, mapping) is identified with its *graph*. Formally, a function $f\colon X \to Y$ is a set of ordered pairs $f \subseteq X \times Y$ (a *graph*) that satisfies the following two Claims:

$\mathbf{1}^\circ$ For *every* argument's value $x \in X$, *there exists* a function's value $y \in Y$ such that $\langle x, y \rangle \in f$;

$\mathbf{2}^\circ$ This function's value y is *unique* as such, that is, whenever $\langle x, y \rangle$ and $\langle x, z \rangle$ are members of f, then $y = z$. Thus, all functions are single-valued.

Intuitively, for an ordered pair $\langle x, y \rangle \in f$, a function f is a 'rule' that assigns the element y to the element x. This y is the value of f for the argument x, that is denoted usually as $y = f(x)$.

What is a function in the set-theoretic paradigm is understood in an unambiguous manner by all the scientific community, and the rigorous definition of a function is therefore imposed on any attempt to clarify a vague notion that bears in germ the idea of functional dependence. This is also true for the notion of compositionality in natural language semantics. Any attempt to define explicitly the principle of compositionality as a function $f\colon X \to Y$ in the set-theoretic paradigm meets with serious technical problems to explain what are these sets X, Y, and how is defined the functional graph $f \subseteq X \times Y$. This is a difficult task and even a trap for any attempt to translate literally the set-theoretic notion of a function into the linguistic notion of a compositionality.

The aim of an adequate semantic theory is to conceptualize how the understandings of parts are integrated during the process of reading to produce the understanding of the whole. However, any semantic theory that combines the compositionality defined as the functionality (meant in the 'function as graph' paradigm) with the non-postponed understanding (meant as a dynamic process that develops step by step while the reading progresses over the time) should be obviously inconsistent.

There are two main directions in which the solution of this apparent conflict might be sought:

- either one conserves the compositionality meant as a set-theoretic functionality but refuses to take into account the process of text understanding over the time, and then establishes a kind of *static semantics*;

- or otherwise, one renounces of compositionality meant as a set-theoretic functionality, or somehow redefines it, and then studies the process of text understanding over the time, in order to establish a kind of *dynamic semantics*.

If the semantic compositionality is taken to be the functionality in a set-theoretic paradigm, then it imposes the almost indubitable conclusion that Frege had never explicitly stated (in this way) the principle of semantic compositionality generally ascribed to him, whatever it were, the compositionality of *Sinn* or the compositionality of *Bedeutung*. In several papers, T. M. V. Janssen had carefully analyzed the development of Frege's views on such a semantic compositionality during his long scientific career, and then concluded, as a result, that Frege "would always be against compositionality" [15, p. 19]. Another point of view is expressed by F. J. Pelletier who writes in a solid historical research that "Frege may have believed the principle of semantic compositionality, although there is no straightforward evidence for it and in any case it does not play any central role in any writing of his [. . .]." [26, p. 111].

However, another theoretical view on the part-whole text structure without prejudice to define the compositionality as a kind of the set-theoretic functionality allows us to interpret Frege's views on the subject in a different way. We notice that in the unpublished work *Logic in Mathematics* of 1914, Frege writes:

> As a sentence is generally a complex sign, so the thought expressed by it is complex too: in fact it is put together in such a way that parts of the thought correspond to parts of the sentence. So as a general rule when a group of signs occurs in a sentence it will have a sense which is part of the thought expressed. (G. Frege [10, pp. 207–208])

In this translation, the expression 'will have a sense' concerning a group of signs should really mean 'will be understandable'. In fact, it is an implicit expression of the hermeneutic circle principle in the particular case of a stand-alone sentence. In a general case, this principle prescribes **'to understand a part in accordance with the understanding of the whole'**. It means that Frege believed the hermeneutic circle principle at the semantic level of a stand-alone sentence. As a logician, Frege was interested primarily in a particular case of sentences, that is, in judgements. It does not really matter whether Frege was familiar with the philological discipline of hermeneutics or not. The principle of hermeneutic circle reveals one of key cognitive operations involved in a natural language text (or discourse) understanding process, and so it is implicitly known by any competent language user. We argue that the hermeneutic circle principle carries in germ the mathematical concept of a *sheaf*, which expresses a passage from a local data to the global one, and which is very close to the idea of a functional dependence. From the sheaf-theoretic point of view, one can revise the aforesaid Frege's quotation like this: 'a family of compatible understandings of parts of the sentence are composable into the understanding of the whole sentence'. However, Frege considered words as being elementary units of a sentence, and he believed in the contextuality principle, bearing today his name, in accordance with which words have no meanings in isolation, "but only in the context of a sentence" [9]. We hypothesize that the reluctance to be got involved into the confusion between elements and parts of a whole (between "words [. . .] in isolation" and "parts of the sentence" in his formulations) prevented Frege from stating explicitly what would be called the compositionality principle. Surely, a meaningful sentence has some meaningful parts, the meanings of which are constitutive to the meaning of this sentence as a whole; but not every of word-tokens may be found among such meaningful parts. This is a kind of the type difference between an element and a subset of a given set.

For an adopted sense $\mathscr{F}$ of reading of a given text X, to each non-empty open (that is to say, meaningful) part $U \subseteq X$ we assign the set $\mathscr{F}(U)$ of all its meanings that may be grasped in all its possible readings in this sense. In fact, it assigns naturally a presheaf $\mathscr{F}$ of fragmentary meanings to the adopted sense of reading. In the beginning of Sect. 5, we argue that such a presheaf $\mathscr{F}$ should satisfy to both Claims **S** and **C** needed for a presheaf to be a sheaf. Thus, the presheaf $\mathscr{F}(U)$ of fragmentary meanings attached to a sense (mode of reading) of an admissible text is really a sheaf. This statement is our generalization of Frege's compositionality principle in the sheaf-theoretic framework. The issuing *sheaf-theoretic formal semantics* takes its departure from another formalization of a functional dependence that is based on the mathematical concept of a sheaf. We use this revised concept of functional dependence in order to define explicitly what is, or rather what should be the compositionality of fragmentary meanings. In this generalized concept of functionality, the arguments and their numbers are not given in advance (one takes for arguments any family of locally compatible *sheaf sections*); but due to the Claim **C**, *for every* such a family of arguments, *there exists* the global sheaf section that becomes their composition; and due to the Claim **S**, this composition is *unique* as such. In the Subsect. 5.1 we show that these Claims **C** and **S** are analogous to those Claims $\mathbf{1}^\circ$ and $\mathbf{2}^\circ$ in the aforesaid formal definition of a function in a set-theoretic paradigm.

• So far, we have considered only the meanings of open sets in the phonocentric topology that we have defined in Sect. 3 at any semantic level. Then, in Sect. 6, we describe how we have to define the meanings of points in the phonocentric topology at any semantic level. For this goal, we recast a famous Frege's contextuality principle in order to define the set of contextual meanings of any point x that belongs to the phonocentric topological space X of some semantic level, whatever this point x may be, a word, a sentence, a paragraph, etc., when considered as an element of a syntactic entity of the higher type. For any semantic level, it is the distinction between the notion of a contextual meaning of a primitive element (a point) at this level and the notion of a fragmentary meaning of a part (a subset) of the whole at this level, that is, of the whole space endowed with a phonocentric topology. The contextual meaning of a point x is defined to be the *inductive limit* of fragmentary meanings s of different open neighbourhoods $U \ni x$ those are got identified on some smaller common open neighbourhood of x. Finally, we generalize Frege's contextuality principle in the categorical terms of *bundles* of contextual meanings.

• In Sect. 7, we show that these generalized Frege's compositionality and contextuality principles are related by a *duality* that we formulate in terms of category theory, and that we name after Frege. This sheaf-theoretic duality sheds new light on the delicate relation between Frege's compositionality and contextuality principles, in revealing that the acceptance of one of them implies the acceptance of the other. It resolves Frege's embarrassing situation with the reconciliation of two principles those bear now his name. As two sides of the same coin, Frege's compositionality and contextuality principles express indeed two complementary parts of the hermeneutic circle principle. That is why they always come together in philosophy, linguistics, and logic. *Grosso modo*, the compositionality principle prescribes to understand a meaningful whole by means of understanding of its meaningful parts, whereas the contextuality principle prescribes to understand the meaning of an entity in accordance with the understanding of its meaningful neighbourhoods.

• Once explicitly stated, Frege duality gives rise to a functional representation of fragmentary meanings. In Sect. 8, this functional representation enables us to develop a kind of *compositional dynamic semantics* that describes how the interpretation proceeds over the time as the step-by-step extension of a meaning representation function, from the initial meaningful fragment to the whole interpreted text. Defined in the proposed sheaf-theoretic framework, such a dynamic semantics conceptualizes the compositionality in a uniform manner at each semantic level: word, clause, sentence, paragraph, section, chapter, text as a whole. Moreover, it treats the polysemy in a realistic manner as one of the essential features of a natural language. This sheaf-theoretic dynamic semantics provides the mathematical model of a text interpretation process, while rejecting attempts to codify interpretative practice as a kind of calculus. We call such a mathematical model of a natural language text interpretation process as *formal hermeneutics* (see, e.g., [29, 31, 32]).

• Then, in Sect. 9, we compare the compositional dynamic semantics proposed in our sheaf-theoretic framework with several algebraic compositional semantics. We notice that an algebraic semantic, of whatever kind, is always static because the meaning of the whole sentence is calculated just after the calculation of meanings of all its syntactic components was done. Algebraic semantic theories are appropriate to study the synonymy, but their irremovable drawback is the inability to describe the polysemy. Any kind of formal grammar that formalizes the compositionality as the functionality in a set-theoretic paradigm shares this fallacy with an algebraic semantics described by T. M. V. Janssen in [15] as “a homomorphism from syntax to semantics”.

By contrast, the proposed mathematical framework formalizes the compositionality of fragmentary meanings in a sheaf-theoretic paradigm of functional dependence. In this formal framework, the dynamic semantics describes how the interpretation is incrementally built up as a meaning representation function stepwise extension from the initial meaningful fragment to the whole text. Moreover, in this approach the process of a natural language text interpretation is modelled in a similar manner at all semantic levels.

• The present article culminates in the final Sect. 10 devoted to the statement of a *sheaf-theoretic formal hermeneutics* that describes a natural language in the category of *textual spaces* **Logos**. Appeared as syntax and semantics of a natural language, phonocentric topologies and sheaves of fragmentary meanings constitute together an adequate mathematical framework to formalize different linguistic phenomena in our works, such as linguistic universals of geometric nature in [29], as dynamic semantics in [34], as interpretations of one text by the others, as text summarization and abstracting, as well as many other aspects of intertextuality in [31].

2. Basic semantic concepts

Concerning the linguistic terminology to be used in this work, we have certain difficulties because the sciences of language do not have a unified terminology. According to F. Rastier [37], two traditions seem dominant in the sciences of language: (1) the grammatical tradition centered on the issue of the sign, that confines itself to the word and the sentence; (2) the rhetoric and hermeneutic tradition centered on the communication, that privileges the text and the discourse. Based on different conceptions, these two traditions differ in problematic and in terminology. When using the definition of a technical term proper to one doctrine, we have to privilege this doctrine compared with others, that would not be our goal. The aim of our work is to discern the mathematical structures underlying the process of reading, with the purpose to design a semantic theory that formalizes a natural language understanding process in a uniform manner at all semantic levels (word, sentence, text). We are therefore obliged to accept a terminology based on distinctions that are valid at all semantic levels of an admissible text. In this perspective, we have to study only those spoken or written language segments that are admissible as units of linguistic communication. Therefore, we keep to the hermeneutic tradition in the analysis of a text understanding process. We recognize that there are different scientific trends in discourse analysis; that is why we have to clarify basic semantic terms we use in the present paper. The technical acceptance of terms *meaning*, *sense*, and *reference* as these are used in the present paper may be explained as follows:

Meaning. The term *fragmentary meaning* of some fragment of a given text X is accepted as the communicative content grasped in some particular situation of reading. In this terminological acceptance, a *fragmentary meaning* is immanent not in a given fragment of a text, but in the interpretative process of its reading based on the linguistic competence, which is rooted in the social practice of communication with others through the medium of a language. Any reading is really an interpretative process where the historicity of the reader and the historicity of the text are involved. The understanding of meaning is based not only on the shared language but also on the shared experience as a common life-world, and it deals so with the reality. According to Gadamer, this being-with-each-other is a general building principle both in life and in language. The understanding of a natural language text results from being together in a common world. This understanding as a presumed agreement on 'what this fragment $U \subseteq X$ wants to say' becomes for the reader its fragmentary meaning s. In this acceptance, the meaning of an expression is the communicative content that a competent reader grasps when s/he understands it; and such an understanding can be reached regardless of the ontological status of its *reference*.

The process of coming to some fragmentary meaning s of a fragment $U \subseteq X$ demonstrates a human communicative ability in action. When we qualify some fragment as being meaningful, we state that an idealized competent reader can understand a communicative content that this fragment conveys; the understanding manifests itself as the ability of the reader to express at once this content in other words or in another language (e.g., if the reader is bilingual).

The fact of having such an understanding may be labelled with a certain abstract entity s called *fragmentary meaning* of U. When someone acknowledges the fact that a meaning of U has been understood, this situation may be described by saying that 'this fragment U has the fragmentary meaning s'; it presumes implicitly that the understanding of the meaning s of the fragment U is arrived at through some linguistic communication, direct or mediated. This meaning may be shared in a dialogue with another native speaker, and such a possibility describes the ontological status of the meaning s as being some abstract entity subtracted from the linguistic communication. This situation may be summed up by an external observer as 'the understanding of the fragmentary meaning s of a fragment U', where the 'meaning' may be perceived as a linguistic term in our technical acceptance, and also as an ordinary word of English language. So, our use of the term *fragmentary meaning* corresponds well to the common English usage.

We have noticed above that for any admissible text X, one should distinguish a fragmentary meaning of a meaningful part $U \subseteq X$ and a contextual meaning of an element (point) $x \in X$. It expresses the fact that clauses are parts of a sentence, but idioms and words are its indivisible elements. A fragmentary meaning s is assigned to the part $U \subseteq X$, and this s conveys some part of the communicative content of the whole X in a concrete situation of linguistic communication. This part U is a sequence of primitive elements (tokens) x those have contextual meanings in the context of U.

In the situation of linguistic communication, a unit that is proper to convey a communicative content may be some text or its fragment, some sentence or its clause, some elliptic expression, and yet a word or an exclamation in certain cases of communication. Thus, a meaning is related to the communicative content, regardless of its possible truth value, whatever it may be: true, false or indefinite.

However, the linguistic communication, either spoken or written, consists of the use of words in a conventional way. It is quite difficult to trace the history of how a single word enters the lexicon (vocabulary) of a language. Taken beyond the situation of linguistic communication, a single word is not a discourse nor a part of it, and this word says nothing to nobody. But this word had entered the lexicon in the process of repeated participation in a variety of situations of linguistic communication, with the result that native speakers of the language have a clear idea of the situations in which the use of a particular word is appropriate, and what it then means. These so-called literal meanings of words are recorded in the dictionaries and thesauruses. Generally, by means of examples, these dictionaries allow us to understand what meaning is associated with the use of each word in several standard situations of its use. In this way, dictionaries define the abstract objects those are called the literal meanings of words. Such definitions carry the entire history of the language and the experience of the numerous uses of the words in the specific situations of communication. The dictionaries thereby demonstrate that the relationship of each word with the set of its possible meanings in specific contexts had gained a normative value. This usage is normative for native speakers of a particular linguistic community, in a particular historic period. These descriptions are aimed to help for a competent reader to adjust better the orientation of his/her efforts to grasp a meaning. In this terminological acceptance, a word, a fragment, a text has a specific meaning only in the situation of linguistic communication, direct or mediated.

However, when using a particular expression in a particular situation of linguistic communication, each interlocutor establishes his/her own connection between this expression and its meaning, which is a mental concept (signified), grasped by means of this expression used in this particular situation of communication. This meaning is the mental concept concerning either some physical objects of the world, or some ideas, or some fictional entity, but this meaning is not itself a referred object in the world (in contrast to Frege's *Bedeutung*). As the mental concept, this meaning is apprehended as a being of intersubjective nature because it may be shared with native speakers of the same linguistic community. We equate the 'meaning' with the 'communicative content' because a message (in spoken or written form) is intended by its author as a carrier of a certain communicative content to be grasped by the addressee, that is, as a carrier of a certain meaning to be understood.

Let us take for example the word 'wolf'. A hunter, a scientist zoologist, an adult urban dweller who have never seen of living wolves, or a child who is familiar with them only by fairy tales, they all have different concepts conceived in connection with the word 'wolf'. The ostensive definition of the meaning of this word by pointing out wolves in a zoo, and its definition by dictionaries as a 'wild, flesh-eating animal of the dog family' are conveying different concepts. It implies certainly that an adequate semantic theory should take into account that a lexicon of a competent reader counts not only one but several literal meanings of the word 'wolf'. Every competent native speaker knows also about the use of this word in one of figurative senses, for example, in the moral sense of the proverb: "Who lives with the wolves should howl like a wolf".

It is, therefore, the intention of the reader that controls the choice of meanings during the reading. Which of possible meanings of a particular expression is grasped by the reader depends on the specific situation of reading guided by the reader's intention in the interpretative process, presuppositions and preferences, that we denominate by the term *sense* (or *mode of reading*).

Sense. In our acceptance, the term *sense* (or *mode of reading*) denotes a kind of semantic orientation in the interpretative process that relates to the whole text or its meaningful fragment, to some sentence or its syntagma, and involves the reader's subjective premises that what is to be understood constitutes a meaningful whole. Concerning a word-token of a phrase, one may ask a question "What does this word mean here in a literal sense?", and as we have argued above, an answer consists of the choice of only one meaning from the set of many possible ones. Likewise for a question, "What could it mean in a metaphoric sense?", as for many similar questions in a reading process. In such an acceptance, the term 'sense' is correlative to the intentionality of our interpretative efforts; that is, a sense is not immanent to the text we read, but in some way, it may even precede the reading process. For example, one may intend to read a fable in the moral sense yet in advance of its reading. But when the reading unfolds in time, one still controls own intentions following the current reading situation. These examples illustrate the acceptance of the term 'sense' as the reader's interpretative intention, and the acceptance of the term 'meaning' as the content actualized during the process of communication.

To some extent, our acceptance of the term 'sense' is close to the exegetic conception of four senses of the Holy Scripture. The traditional presentation of this conception of biblical hermeneutics is summarized by the famous distich of Augustine of Dacia: "Littera gesta docet, quid credas allegoria, moralis quid agas, quo tendas anagogia."[3]

According to the biblical hermeneutics, the readings of the Scripture in literal, allegorical, moral, and anagogical senses are coherent in each of its parts. Suppose we read the whole text of the Scripture by fragments, where each fragment was read in one of four senses: literal, allegorical, moral, or anagogical, but the choice of sense was not the same for all fragments. The composition of these four senses is a method of interpretation that gives rise

[3]Augustine of Dacia, *Rotulus pugillaris*, I: ed. A. Walz: Angelicum 6 (1929) p. 256. The distich is translated in English as: "The letter tells us what went down, the allegory what faith is sound, the moral how to act well – the anagogy where our course is bound."

to a large number of senses of the whole text. Indeed, the overall sense $\mathscr{F}$, as the integral intention in the reading process, is the result of all local intentions taken during these partial readings.

But what guides the subsequent choice of local intentions of an empirical reader? Following Fathers of Church, it is the presence of the Holy Spirit that guides the soul of the individual believer who reads the text of the Scripture. But for a secular text, how can we characterize in linguistic terms the possibility to join these partial senses? It is the presumed sincerity and a goodwill on the part of the author, whom we suppose to be of sound mind and perfect memory, while writing this text intended to communicate something to an alleged reader.

However, the local intentions those were taken in the writing process were got integrated into an overall intention of an empirical author; so, these partial writings are consistent to satisfy a certain *gluing condition* of the type that we discuss further in Sect. 5.4. Since the empirical author is almost always inaccessible for a dialogue, how can we understand what does the text mean by virtue of its textual coherence denoted by U. Eco as the *intentio operis*? According to U. Eco [7, p. 65], "it is possible to speak of the text's intention only as the result of a conjecture on the part of the reader. The initiative of the reader basically consists in making a conjecture about the text's intention." He asks further, "How to prove a conjecture about the *intentio operis*?", and he responds: "The only way is to check it upon the text as a coherent whole." He continues then that this idea comes from *De doctrina Christiana* of St. Augustine:

> [. . .] any interpretation given of a certain portion of a text can be accepted if it is confirmed by, and must be rejected if it is challenged by, another portion of the same text. [7, p. 65]

According to St. Augustine, the presumed textual coherence controls the partial interpretations that are made by an empirical reader. Therefore, in the process of reading, all these local intentions to understand a text have also to verify the *gluing condition* of the type that we discuss further in the Sect. 5.4.

In the process of actual communication, a mere consistency of the local interpretations would be insufficient. The inference on the speaker's intention is essential here for the understanding; the contact of interlocutors allows them to get into the coordination between the intention of the sender and the intention of the recipient.

With regard to a text produced not for a single recipient, but for a community of readers, the strategy of a *model author* is to lead his *model reader* to speculate about the text. Among these leading indexes, the central place is held by the *semantic isotopy* that A. J. Greimas defines as "a complex of plural semantic categories which makes possible the uniform reading of a story." [12, p. 188]. Concerning the notion of isotopy, U. Eco notices in [6, pp. 189–190] that "The category would then have the function of textual or transsentential disambiguation, but on various occasions Greimas furnishes examples dealing with sentences and outright noun phrases."

Following B. Pottier, the seme does not exist in isolation but as a part of a sememe, or as the set of coexisting semes.

> Le sémème, l'être de langue (en compétence), s'actualise dans le discours [. . .] . Le sémème donne le sens (l'orientation sémantique), et la mise en discours le transforme en signification.[4] [27, pp. 66, 67]

From this definition, we retain the acceptance of the term *sense* as the semantic orientation of the reader's intentions provoked by a sememe, and the fact that a meaning is actualized in the discourse. The reader's conjecture on the subject discussed in a text determines the first interpretive intention that will be clarified in the course of the reading when the recognition of a semantic isotopy becomes possible owing to the context that is more and more revealed. Following U. Eco,

> The first movement toward the recognition of a semantic isotopy is a conjecture about the topic of a given discourse: once this conjecture has been attempted, the recognition of a possible constant semantic isotopy is the textual proof of the 'aboutness' of the discourse in question. [7, p. 63]

In *Two Problems in Textual Interpretation* published in 1980, U. Eco describes the interpretative process as based on the reader's interpretive cooperation:

> Between the theory that the interpretation is wholly determined by the author's intention and the theory that it is wholly determined by the will of the interpreter there is undoubtedly a third way. Interpretive cooperation is an act in the course of which the reader of a text, through successive abductive inferences, proposes topics, ways of reading, and hypotheses of coherence, on the basis of suitable encyclopedic competence; but this interpretive initiative of his is, in a way, determined by the nature of the text. [2, pp. 43–44]

But later in 1992, in the analysis of so-called *superinterpretation*, U. Eco raises again the problem of a reader's conjectures about the empirical author's

[4]Our translation of this quotation is: "The sememe, the entity of language (in competence), is actualized in the discourse [. . .]. The sememe gives the sense (the semantic orientation), and the putting into discourse transforms it into meaning."

intention during the reading. His updated conception of the interpretation of texts “makes the notion of the intention of an empirical author radically unnecessary” [7, p. 60]. He defends this thesis with the support of his own experience as a writer who has discussed with his readers a few different interpretations of his novels.

To summarize now our acceptance of the term *sense* (or *mode of reading*), we have to say that it is close to the latter acceptance described by U. Eco. The term *sense* concerns the reader’s initiative in the interpretation of the text; it is wholly determined by the reader’s intention to understand possible meanings of the text. In Sect. 5, we identify a particular sense $\mathscr{F}$ (in our acceptance) with the assignment to each meaningful fragment U of a given text X the set of all its meanings $\mathscr{F}(U)$ that may be grasped in all possible readings of U in this sense $\mathscr{F}$. This way, to any *sense* (or *mode of reading*), we assign a *sheaf* of fragmentary meanings.

Remark. It should be noticed that our terminological acceptance of basic semantic notions of *sense* and *meaning* differs from their acceptance in the theories developed within the tradition that goes back to Carnap’s semantic theory, sometimes called the theory of “intension and extension”. In such theories, expressions of different syntactic kinds refer to entities of different kinds as their extensions, and also refer to entities of different kinds as their intensions. The terms *intension, intensional* are not to be confused with the terms *intention, intentional* we have discussed above. The notions ‘intension’, ‘intensional’ primarily concern the domain of logic, whereas ‘intention’, ‘intentional’ concern the philosophy of mind. According to A. R. Lacey, “Intuitively extensions can be thought of as the extents which certain kinds of terms range over and intensions as that in virtue of which they do so.” [18, p. 164], whereas the intentionality is “that feature of certain mental states by which they are directed at or about objects and states of affairs in the world” [18, p. 50].

Reference. Certainly, the referential function of a language is important in the linguistic communication, which concerns the world where the interlocutors live. A natural language has a huge arsenal of denoting expressions to designate real and imaginary objects during communication. The linguistic competence is characterized by the know-how in production and comprehension of natural language expressions realizing the referential relationship called *reference* or *denotation*. In the analytic philosophy of language, the study of denoting expressions plays a considerable role, because the reference to objects with an uncertain ontological status is responsible for some logical paradoxes.

In the present work, we assume a total referential competence of an idealized reader who knows the lexicon of a language and follows the rules of common usage. In short, we assume that the reader has a total language skill, combined with a general knowledge. Such a reader meets no problems to understand the meaning of denotative expressions and the ontological status of objects so defined.

3. Topologies appeared as syntax

The author of an admissible text doesn't suppose that the reader's understanding will be suspended until the end of reading because everybody knows that the words already read trigger intellectual mechanisms of interpretation based on the indissoluble links between the signifier and the signified. To be understood in linguistic communication, one must take it into account and organize one's writing in such a way that the reader's understanding at every moment may be arrived at on the basis of what has been already read. It seems that the primacy of speech over writing is a cause that implies in writing the subordination of graphic expressions to acoustic ones. A spoken utterance is a temporal series of sounds produced by a speaker using a human articulatory apparatus. When written, an acoustic signal is converted into a series of signs whose positions are linearly ordered following an adopted convention; in English, it is from left to right within the lines, and from top to bottom between them. Once a particular sign is taken as the initial, it allows us to specify the position of the following signs by enumeration. From the mathematical point of view, the whole segment may be considered as a finite sequence when the last sign is specified. Thus, we ought to consider a text X as a finite sequence $(x_1, x_2, x_3, \ldots, x_n)$ of its constituent sentences x_i, and so it is formally identified with the graph of a function $i \mapsto x_i$ defined on some interval of natural numbers. When reading a particular fragment of the text X, we delete mentally the other sentences but follow the induced order of remaining ones. Important is the induced order of their reading and not the concrete index numbers of their occupied places. Thus, any part of the text is a subsequence whose graph is a subset of the whole sequence graph. Likewise for a sentence considered as a finite sequence of its words.

While reading a text, the understanding is not postponed until the final sentence. So the text should have the meaningful parts, and the meanings of these parts determine the meaning of the whole as it is postulated by the hermeneutic circle principle. For the meaningfulness conveying an idealized reader's linguistic competence, a meaning of a meaningful part is the communicative content grasped in a particular reading of this part guided by the reader's presuppositions and preferences in the interpretative process, that is, guided by the sense (or mode) of reading.

Certainly, there are many meaningful fragments in the text. A simple example of a meaningful fragment is supplied by the interval including all sentences, from the first x_1 until the last x_n. Anybody reads the text as if it would be a written transcription of the story uttered by the author. When telling or writing a story, an author should take into account that the understanding can't be postponed, for "the texts never know the suspense of interpretation. It is compulsive and uncontrollable", as it is noticed by F. Rastier in [36]. If the author don't want to be misunderstood, s/he has to organize the text in such a way that any sentence x is preceded by certain sentences those provide a necessary context for the understanding of x. Thus, any meaningful part contains each sentence together with some its context, and this is characteristic of any part to be meaningful. It is clear that this property fails for a part including, e.g., all sentences x_i whose placehold number i is divisible by 100, and that is why this part is meaningless, and nobody try to read the text in such a manner. In [28, 31–33], we argue that in agreement with our linguistic intuition, the set of all meaningful parts of any admissible text should satisfy two properties:

(t_1) *The union of any set of meaningful parts is a meaningful part.*

(t_2) *The non-empty intersection of two meaningful parts is a meaningful part.*

The first property (t_1) is taken for granted, because it expresses the precept of generally accepted hermeneutic circle principle, which ensures us to understand the union of a given set of meaningful parts through the understanding of all its constitutive members. In the union of any set of meaningful parts, each part contains every its sentence together with some its context, whence the union itself is a part that has such a property. To be more accurate, we have to take into account that the meaning s of a meaningful part U isn't immanent to this part itself, but this meaning is grasped in the reading process following a sense (mode of reading) $\mathscr{F}$ guided by the reader's interpretative intentions. Thus, in the statement (t_1), some sense (or mode of reading) $\mathscr{F}$ is implicitly presumed to be the same for all members of the union. In the following Sect. 5–8, we discuss in details how the resulting meaning of the whole is obtained via the meanings of its constitutive parts.

The second property (t_2) expresses the contextuality of understanding. To understand a meaningful part U of the text X is to understand contextually all sentences $x \in U$, where the context of a particular sentence x is some meaningful part W such that $x \in W \subseteq U$. In the standard process of reading (i.e., from the beginning up to x), this part W should contain a subsequence of sentences those precede x and provide a necessary context for the understanding of x in the sense $\mathscr{F}$. For a particular sense $\mathscr{F}$, there should exist a smaller subsequence $(x_{i_1} \ldots, x_{i_m}) \subseteq W$ whose sentences have been understood during the reading, and then have been taken into account at the moment when the reader understands a meaning of x grasped in the sense $\mathscr{F}$. Let us denote $U_x = (x_{i_1}, \ldots, x_{i_m})$. The tokens x_{i_k} of U_x may be consecutive or dispersed among other tokens of W, it does not matter, but they should be read before the reading of x.

Consider first the case of one session process of reading of X in some sense $\mathscr{F}$. When the part U_x belongs to any meaningful part $W \subseteq X$ such that $x \in W$. Let U, V be two meaningful parts such that $x \in U \cap V$. According to our premises, $x \in U_x \subseteq U$ and $x \in U_x \subseteq V$; hence $x \in U_x \subseteq U \cap V$.

Consider now the case when $x \in U \cap V$, and parts U, V were read in two different sessions of reading, but in the same sense $\mathscr{F}$. This means that the reader is self-identical, and the reading is guided by the same intentionality. It implies that $U_x \subseteq U$ and $U_x \subseteq V$. Hence $x \in U_x \subseteq U \cap V$.

Thus in both cases, $U \cap V$ is meaningful because $U \cap V = \cup_{x \in U \cap V} U_x$ is the union of meaningful parts, due to (t_1).

Since an admissible text X is supposed to be meaningful as a whole by the very definition, it remains only to define formally the meaning of its empty part (for example, as a singleton) in order to satisfy the third property:

(t_3) *The whole admissible text and the empty part are meaningful.*

This enables us to endow an admissible text X with some *topology* in a strict mathematical sense, where the set $\mathfrak{O}(X)$ of open sets is defined to be the set of all meaningful parts. We call the topology so defined *phonocentric topology* to indicate in its name the subordination of graphic expressions to phonetic ones.

An admissible text X gives rise to a finite space; hence an arbitrary intersection of its open sets is open and so it is an *Alexandrov space*.

In general, a topology on a set X is defined by specifying the set $\mathfrak{O}(X)$ of open subsets of X satisfying axioms similar to ours (t_1), (t_2), and (t_3). But almost always it is impossible to enumerate all the open subsets. Instead, a topology is usually defined by specifying a smaller set of open subsets, called a *basis*, and then generating all the open subsets from this basis.

Likewise, when studying the process of interpretation of an admissible text X, many of linguistic concepts may be well expressed in terms of the phonocentric topology on X that is defined by specifying the set of open subsets $\mathfrak{O}(X)$ to be the set of all meaningful parts satisfying properties (t_1), (t_2), and (t_3). However, it will be more convenient and useful to develop the theory in more concrete, say even constructive, terms of empirically given meaningful parts those constitute a basis for a phonocentric topology.

Fortunately, the set of all meaningful parts $\mathfrak{O}(X)$ of a given text X may be described by specifying a class of fairly simple meaningful parts given as an empirical data related to a reading process. In the reading of a particular text X, the reader is practically concerned with a smaller class of meaningful parts $(U_x)_{x\in X}$, where each part U_x contains a sentence x and provides the smallest context that is necessary for a reader to grasp a particular meaning of x. Because the phonocentric topology $\mathfrak{O}(X)$ is finite, for each x, there exists such a smallest open neighbourhood U_x that is defined as the intersection of all open neighbourhoods of x.

For a given sentence x, the understanding of a whole U_x requires the grasping of meanings of all constitutive sentences of U_x; hence, for any sentence $y \in U_x$, its smallest context U_y should be a part of U_x. Suppose now that we are given two smallest meaningful parts U_x and U_y such that $U_x \cap U_y \neq \varnothing$. Then for each $z \in U_x \cap U_y$, we have $U_z \subseteq U_x$ and $U_z \subseteq U_y$; hence $U_z \subseteq U_x \cap U_y$. Therefore, the set $\mathfrak{B}(X) = \{U_x \colon x \in X\}$ is the set of meaningful parts of X satisfying two properties:

(b_1) For each $x \in X$, there exists $U_x \in \mathfrak{B}(X)$ such that $x \in U_x$.

(b_2) For every two $U_x, U_y \in \mathfrak{B}(X)$ such that $U_x \cap U_y \neq \varnothing$, and for each sentence $z \in U_x \cap U_y$, there exists $U_z \in \mathfrak{B}(X)$ such that $z \in U_z$ and $U_z \subseteq U_x \cap U_y$.

So, the set $\mathfrak{B}(X)$ is a basis for a phonocentric topology on X, because any meaningful part (i.e., open) $V \subseteq X$ is the union $V = \cup_{x\in V} U_x$ of the members of some subset of $\mathfrak{B}(X)$. Recall that a set $\mathfrak{B}$ of open sets of a topological space X is called a *basis* for its topology if and only if every open set U of X is the union of the members of a subset of $\mathfrak{B}$. Thus, the class of open sets $\mathfrak{O}(X)$ in a phonocentric topology on X is defined by the subclass $\mathfrak{B}(X)$ of all open sets of the type U_x, that is, a phonocentric topology on X is defined by the empirical data $\mathfrak{B}(X)$.

Any explicitly stated *concept of meaning* or a *criterion of meaningfulness* satisfying conditions (t_1), (t_2), and (t_3) allows us to define some type of *discursive topology* on texts, and then to interpret several problems of discourse analysis in topological terms [31]. In what follows, we consider only admissible texts endowed with a phonocentric topology that is a particular type of discursive topology corresponding to the criterion of meaningfulness conveying the linguistic competence of an idealized reader, meant as the ability to grasp a communicative content.

3.1. Phonocentric topology and partial order

In the ordinary process of reading, any sentence x of a text X should be understood on the basis of the part already read because the interpretation of a natural language text cannot be postponed, although it may be made more precise and corrected in further reading and rereading. In [36], F. Rastier describes this fundamental feature of a competent reader's linguistic behaviour as the following:

> Alors que le régime herméneutique des langages formels est celui du suspens, car leur interprétation peut se déployer après le calcul, les textes ne connaissent jamais le suspens de l'interprétation. Elle est compulsive et incoercible. Par exemple, les mots inconnus, les noms propres, voire les non-mots sont interprétés, validement ou non, peu importe.[5] [36, pp. 165, 166]

Thus, for every pair of distinct sentences x, y of X, there exists an open part U containing one of them (to be read first in the natural order $\leq$ of sentences reading) but not the other. This means explicitly that the phonocentric topology satisfies the *separation axiom* T_0 of Kolmogorov.

For a sentence $x \in X$, we have defined the open neighbourhood U_x to be the intersection of all the meaningful parts those contain x, that is the smallest open neighbourhood of x. The *specialization relation* $x \preceq y$ (read as 'x is

[5] Our translation of this quotation is: "While the hermeneutic regime of formal languages is that of suspense, because their interpretation can be deployed after the calculation, the texts never know the suspense of interpretation. It is compulsive and uncontrollable. For example, unknown words, proper names, even non-words are interpreted, valid or not, whatever."

more special than y') on a topological space X is defined by setting $x \preceq y$ if and only if $x \in U_y$ or, equivalently, $U_x \subseteq U_y$. It is clear that $x \in U_y$ if and only if $y \in \mathrm{cl}(\{x\})$, where $\mathrm{cl}(\{x\})$ denotes the topological closure of a one-point set $\{x\}$.

Key properties of these notions are summarized in the Propositions 1, 2 those are linguistic versions of general mathematical results concerning the interplay of topological and order structures defined on a finite set. The proofs may be found in many sources, as for example, in [23].

Proposition 1. *For an admissible text X, the set of all smallest opens $\{U_x : x \in X\}$ is a basis for a phonocentric topology on X. Since the phonocentric topology on X satisfies the separation axiom T_0, it defines a partial order $\preceq$ on X by means of the specialization relation. The initial phonocentric topology can be recovered from this partial order $\preceq$ in a unique way as the topology with the basis made up of all sets of the kind $U_x = \{z : z \preceq x\}$.*

Proposition 2. *Let X, Y be admissible texts endowed with phonocentric topologies. Then the following statements are equivalent:*
1. *The function $f : X \to Y$ is continuous.*
2. *For each $x \in X$, the function f maps a basis set into a basis set, that is $f(U_x) \subseteq U_{f(x)}$.*
3. *The function f preserves the specialization order, that is $x \preceq y$ implies $f(x) \preceq f(y)$.*

Example. A continuous function $f_1 : X_2 \to X_1$ arises in writing process when an author goes from a first plan X_1 of some future text to its more detailed plan X_2, where a sentence x_d of X_1 is substituted by some passage $(x_{d_1}, \ldots, x_{d_m})$. And so on, in going to more and more detailed texts $X_3, \ldots, X_n$, one gets a sequence of continuous functions

$$X_n \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1.$$

3.2. Deep structures and surface structures

Let $\mathbf{FinTOP}_0$ be the category of finite T_0-topological spaces and continuous maps, and let **FinORD** be the category of finite partially ordered sets (posets) and their monotone maps.

Given a finite partially ordered set $(X, \leq)$, one defines a T_0-topology τ on X by means of the basis for τ made up of all *low sets* $\{z : z \leq x\}$.

Thus, one obtains a functor $L\colon$ **FinORD** $\to$ $\mathbf{FinTOP}_0$ acting identically on the maps of underlying set. Conversely, one defines the specialization functor $Q\colon$ $\mathbf{FinTOP}_0$ $\to$ **FinORD**, assigning to each finite T_0-topological space (X, τ) a poset $(X, \preceq)$ with the specialization order $\preceq$, and acting identically on the maps of underlying set. Thus, the functors L and Q establish the isomorphism between the category $\mathbf{FinTOP}_0$ and the category **FinORD**. From the mathematical point of view, the study of one of these two categories is equivalent to the study of the other.

Now we generalize and summarize the considerations of the mathematical structures of topology and order underlying an admissible text:

The considerations in the beginning of Sect. 3 may be slightly modified in order to define a *phonocentric* topology at the semantic level of sentence and even word [31]. Thus, at each semantic level, there exist two topological structures:

(i) *the natural phonocentric topology at a considered semantic level;*

(ii) *the topology defined by applying the functor* L *to the linear order* $x \leq y$ *of reading.*

At an arbitrary semantic level (where the whole is a sequence of primitive elements), the difference between topologies can be summed up so that in the phonocentric topology the least neighbourhood U_x of a primitive element x contains only such primitive elements that precede x in the linear order of writing and provide the context necessary to understand the meaning of x in the adopted sense $\mathscr{F}$; whereas in the topology defined by the functor L applied to $(X, \leq)$, the least neighbourhood U_x of a primitive element x contains all primitive elements that precede x in the linear order of writing.

Note that the explicit definition of the phonocentric topology at the semantic level of sentence requires more delicate work in treatment of different grammatical types of sentences due to the lack of space, so to speak. Here there is a certain analogy with the topological classification of varieties that turns out to be more difficult in dimensions 3 and 4 than in lower and in higher dimensions.

On the other hand, at each semantic level, there exist two order structures:

(i′) *the specialization order* $x \preceq y$ *defined by applying the specialization functor* Q *to the natural phonocentric topology of a considered semantic level;*

(ii′) *the linear order* $x \leq y$ *of ordinary text reading.*

Similar to a generative grammar, we will qualify the equivalent structures of (i) and (i′) as *deep structures* compared to the equivalent structures of (ii) and (ii′) qualified as *surface structures*. We notice that this denomination has nothing to do with the acceptance of these terms in a generative grammar.

Remark. The relation $x \preceq y$ implies obviously the relation $x \leq y$, for all the primitive units x, y of the same semantic level. In particular, at the level of text, where the sentences are primitive units, the map $\mathrm{id}\colon L(X, \preceq) \to L(X, \leq)$, which acts as identity $x \mapsto x$ of the underlying set, is a continuous map of topological spaces. Thus, the necessary linearization during the writing process, that is the passage from $(X, \preceq)$ to $(X, \leq)$, results in weakening of the phonocentric topology by transition from $L(X, \preceq)$ to $L(X, \leq)$. The process of interpretation consists in a backward recovering of the phonocentric topology (or equally, of the specialization order) on the text.

3.3. Phonocentric topology at the level of text

There is a simple intuitive tool for graphical representation of a finite poset, called Hasse diagram. For a poset $(X, \preceq)$, the *cover relation* $x \prec y$ (read as 'x is covered by y') is defined by setting $x \prec y$ if and only if $x \preceq y$ and there is no other z such that $x \preceq z \preceq y$. For a given poset $(X, \preceq)$, its Hasse diagram is defined as the graph whose vertices are the elements of X and whose edges are those pairs $\langle x, y \rangle$ for which $x \prec y$. In the picture, the vertices of Hasse diagram are labeled by the elements of X and the edge $\langle x, y \rangle$ is drawn by an arrow going from x to y (or sometimes by an indirected line connecting x and y, but in this case the vertex y is displayed lower than the vertex x); moreover, the vertices are displayed in such a way that each line meets only two vertices.

The usage of some kind of Hasse diagram named *Leitfaden* is widely spread in the mathematical textbooks to facilitate the understanding of logical dependence of its chapters or paragraphs. Mostly, the poset is constituted of all chapters of the book. So, in *Local Fields* by J.-P. Serre [39] and in *A Mathematical Logic* by Yu. I. Manin [21], there are such diagrams.

These diagrams may surely be 'split' in order to draw the corresponding ones whose vertices are all the paragraphs, like it is done directly in *Differential Forms in Algebraic Topology* by R. Bott and L. W. Tu [1], where authors suppose indeed the linear reading of paragraphs 1-6, 8-11, 13-16 and 20-22, but it may be drawn explicitly. These three Hasse diagrams are shown in the Fig. 1.

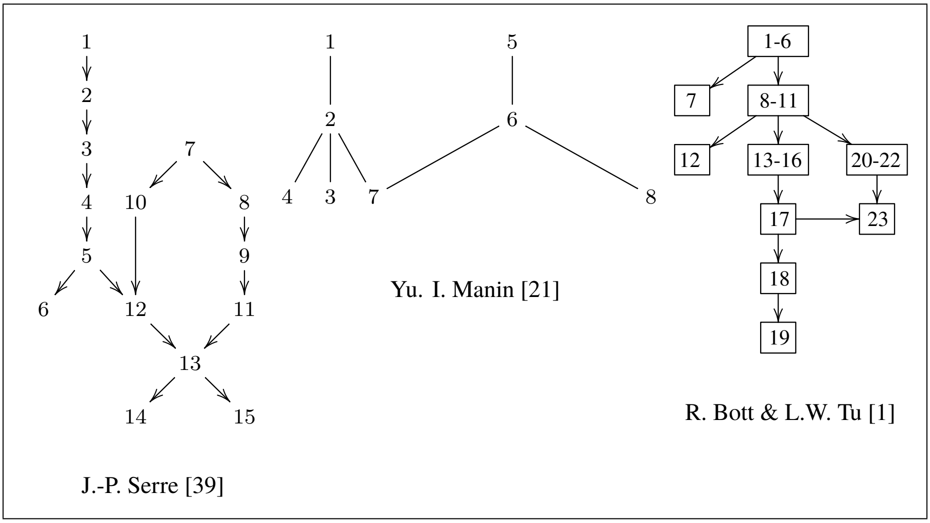

Figure 1: *Leitfiden* of J.-P. Serre [39], Yu. I. Manin [21], R. Bott & L. W. Tu [1].

This way, one may go further and do the next step. For every sentence x of a given admissible text X, one can find a basis open set of the kind U_x in order to define the phonocentric topology at the semantic level of text (where points are sentences), and then to draw the Hasse diagram of the corresponding poset.

In [31], we describe how one may interpret this way the most of diagrams from the *Rhetorical Structure Theory* (RST) conceived in the 1980s the by W. C. Mann and S. A. Thompson [22]. Since then, RST has seen a great development, especially in the computational linguistics, where it is often used for the automatic generation of coherent texts, as well as for the automatic analysis of the structure of texts. The RST aims to describe an arbitrary *coherent text*, which is not the random sequence of sentences. The textual coherence demands that for every part of a coherent text there exists a reason for its presence, which is obvious to a competent reader. It seems that RST notion of a coherent text is similar to our notion of an admissible text. In [31], we show that the RST analysis of contextual dependencies between sentences of certain small textual fragments represented as RST diagram may be redrawn as the Hasse diagram for the partial order structure of the corresponding specialization relationship. But the RST diagram may be drawn only for certain small textual fragments such that their sentences are *nucleus* and *satellite* in the sense of the RST. On the other hand, it is not the case when such a fragment is a part of a larger text. Then, according to the RST, there will be no link between a sentence x belonging to such a fragment and any other sentence y that is far enough in the text, because rhetorical relations can only bind adjacent segments. While in our approach, such a link is possible in the specialization relation (of deep order). This link is seen on the corresponding Hasse diagram as a direct edge $\langle x, y \rangle$ or as a sequence of edges that link these two sentences x and y. Thus, our approach is more general than this one of the RST.

3.4. Phonocentric topology at the level of sentence

In order to define a phonocentric topology at the semantic level of sentence, we must distinguish there the meaningful fragments that are similar to meaningful fragments at the level of text. Let x, y be any two word-tokens such that $x \preceq y$ in the specialization order at the level of sentence that is similar to the specialization order coming from the 'logical relations among the different chapters' in a text. This relation $x \preceq y$ means that the word-token x should necessary be an element of the set of word-tokens U_y required to understand the meaning of the word-token y in the interpreted sentence. So we have $x \leq y$ in the order of writing and there should be some syntactic dependence between them. It means that a grammar in which the notion of dependence between pairs of words plays an essential role will be closer to our topological theory than a grammar of Chomsky's type.

There are many formal grammars focused on links between words. The history of this stream of ideas is described by S. Kahane in a detailed review [16]. We think that the theoretical approach of the *special link grammar* of D. Sleator and D. Temperley is most appropriate to define a phonocentric topology at the level of sentence, because in whose formalism "[t]he grammar is distributed among the words" [40, p. 3], and "the links are not allowed to form cycles" [40, p. 13] comparing with *dependency grammars* that draw syntactic structure of sentence as a planar tree with one distinguished root word.

For a given sentence s, the link grammar assigns to it a syntactic structure (called *linkage diagram*) that consists of a set of labeled links connecting pairs of words. We use these diagrams to define all phonocentric topologies on this sentence s.

Example. To explain how to define phonocentric topologies on a particular sentence, let us borrow from [42] the following example of an ambiguous sentence:

(1) John saw the girl with a telescope.

We had yet considered this sentence in [29] by using Chomsky's generative grammar, and also in [31] by using link grammar. The analysis of this sentence by means of the *Link Parser 4.0* of D. Temperley, D. Sleator, and J. Lafferty [41] gives two linkage diagrams shown in the Fig. 2.

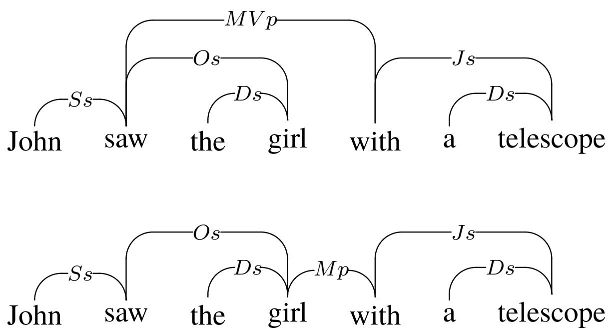

Figure 2: Two linkage diagrams with connector names.

These two diagrams rewritten with arrows that indicate the direction in which the connectors match (instead of connector name) have the appearance shown in the Fig. 3.

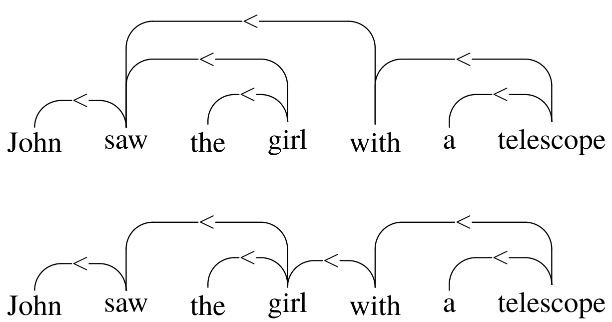

Figure 3: Two linkage diagrams with arrows instead of connector names.

It is clear that the transitive closure $x \preceq y$ of this relation $<$ between pairs of words defines two partial order structures on the sentence (1). By applying the functor L defined in Sect. 3.2, we can endow the sentence (1) with a phonocentric topology in two different ways. The Hasse diagrams of corresponding posets are shown in the Fig. 4.

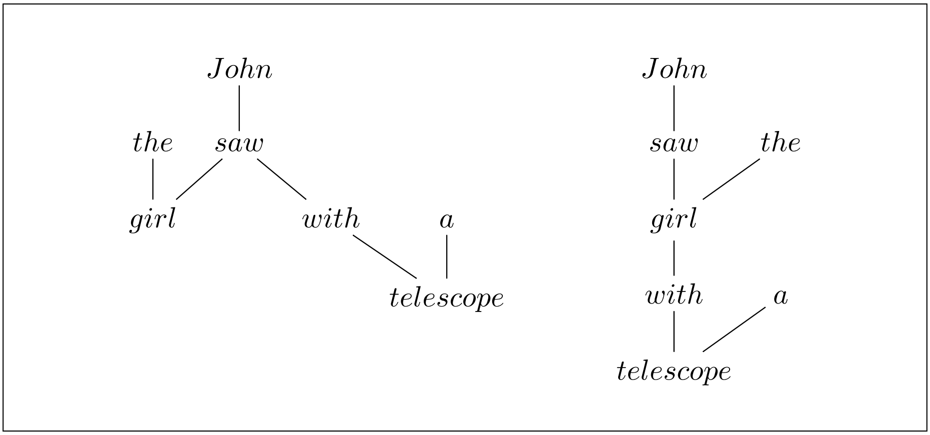

Figure 4: Two Hasse diagrams of the sentence (1) as displayed in [29, 31].

To understand the sentence (1), the reader has to do the *ambiguity resolution* when arriving to the word-token x =“with” by choosing only one of two possible basis sets:

$$U_x = \{\langle 1, \text{John}\rangle, \langle 2, \text{saw}\rangle, \langle 5, \text{with}\rangle\};$$
$$U_x = \{\langle 1, \text{John}\rangle, \langle 2, \text{saw}\rangle, \langle 3, \text{the}\rangle, \langle 4, \text{girl}\rangle, \langle 5, \text{with}\rangle\}.$$

In the general case, the step by step choice of an appropriate context $U_x = \{z \colon z \preceq x\}$ for each word x results in endowing the interpreted sentence with a particular phonocentric topology among many possible.

In [31], we have shown how to define a phonocentric topology at the level of word considered as a sequence of morphemes.

We summarize the results of our analysis presented in Sect. 3 as the following:

Slogan (Phonocentric Topologies as Syntax). Once the phonocentric topology and the corresponding specialization order are determined at a given semantic level, the systematic interpretation of linguistic phenomena in terms of topology and specialization order, and their mathematical study is a formal syntax at this level.

4. Linguistic universals of a topological nature

Throughout the history of scientific study of human languages, researchers are interested in discovering linguistic universals, that is, particular traits common to all languages. Because it is impossible to recognize everything about all languages, it is necessary to first decide where and how to look for linguistic universals. It appears that our sheaf-theoretic approach makes here a small contribution.

By its very origin, a human language is used for linguistic communication; for that reason, written texts and uttered discourses should be considered as communicative units. We must therefore look for linguistic universals, not only in terms of word as it is done by J. H. Greenberg [11] and his successors, but especially in terms of text. A true linguistic universals at the level of text (or discourse) must have a corresponding counterpart at the level of sentence.

By linguistic universals, we understand the characteristic properties of texts those are admissible as messages having communicative purposes, regardless of the language in which they are written. The question is, therefore, reduced to this: What criteria should we accept to be sure that a particular characteristic is truly shared by all admissible texts in any natural language? One can adopt a statistical criterion ensuring, to a certain extent, that if some property is shared by hundreds of natural languages, it is likely that it is shared by all. Such an approach is taken up in the classical works of J. H. Greenberg. But there are no guarantees that a particular trait of the languages already studied is also shared by the language of a lost Indian tribe that escaped the statistical body of research.

To our deep conviction, the way to avoid counter-examples is to adopt a criterion based not only on statistical considerations, but mainly on the analysis of the communicative function of languages. In our talk [30] at the 39th Annual Meeting of *SLE*, we argued that the properties of a phonocentric topology to satisfy the separation axiom T_0 of Kolmogorov and to be connected are linguistic universals. These properties should be required of the underlying phonocentric topology on any text written for the purpose to be understood in the linguistic communication.

A correct translation of an admissible text from one language into another is done by successive translation of each sentence in a manner to conserve their contextual relations. It results in a bijection between the original text and its translation, and also in a homeomorphism between corresponding topological spaces.

It is clear that a phonocentric topology on an admissible text written in one language (as well as the corresponding Hasse diagram) is invariant under translation into another language. Hence, a phonocentric topology on a text X and its properties and geometric invariants (say T_0-separability, connectedness, homology groups, etc.) are stable under translation from one language into another (i.e., under homeomorphism), and so they are formal invariants of the text X.

The properties those are shared by all texts in all natural languages are absolute linguistic universals. In [30–32], we argue that the T_0-*separability*, the *connectedness* of a phonocentric topology, and the *acyclicity* of corresponding Hasse diagram are features shared by the majority of languages.

4.1. Kolmogorov's axiom T_0 as a linguistic universal

One important example of a topological linguistic universal seems to be the *separation axiom* T_0 of Kolmogorov. In the Sect. 3, we argued for the relevance of the separation axiom T_0 to all semantic levels of an admissible text on the base of a lucid formulation by F. Rastier [36]. Anyway, there is an essential difference between the hermeneutic regime of formal languages and that one of natural languages; it is important for us that texts written in a natural language "never know the suspense of interpretation" [36, p. 166]. It's still the same idea that Origen expresses in the biblical hermeneutics regarding the non-understanding. According to Origen, yet for an *imbulatum*, there is a meaning as a sign of divine presence in the text.

Such an empirical truth that everyone knows from his/her own experience of reader still deserves a more nuanced discussion. Firstly, this property of understanding of texts in natural language is obviously taken into account by everyone who writes a text intended for human understanding, whether he/she is a professional writer or not; the rule is accepted as that one of a 'writing game', so to speak.

If we do not want to be misunderstood, we do not propose the reader to suspend understanding until the end of writing because we know that the words already read trigger intellectual interpretation mechanisms based on indissoluble links between signifier and signified. This is well expressed by the colourful Russian saying: *A word is not a sparrow; you can't catch it when it flies away!* In order to be understood, we must organize our writing in such a way that the reader's understanding would always be based on the part of text already read, in total ignorance of its future development.

The second reading (as all subsequent readings) is governed by the same rule, despite the fact that we already know the whole text. The repetitive reading respects the unpredictability of the future; while reading at the time being, we are being in the 'here and now', that leads us to identify the physical real time with the time of the narrative. What lies in the pages that follow makes no context for the understanding of what has been read. In particular, this rule is just applicable to scientific texts.

A question arises: What is the reason for this indisputable empirical phenomenon? It seems to us that it is the primacy of speech over writing, which causes the subordination of graphic expressions to phonetic ones.

Preliterate civilizations existed thousands of years before the advent of writing and even still exist somewhere else. Even today there are thousands of people who cannot read. The cultural history of the human species is repeated in the personal history of each individual because we learn to speak before we learn to read and write. But as a physical phenomenon, a phonetic expression exists in the dimension of time, and here the physiological properties of our speech organs are just involved.

In a conversation, the interlocutors have access only to whatever is already said, because the future remains unpredictable. Once said, the spoken word is flying away and the only chance to get by in such a situation is to understand on the spot all that is said by the others.

For anybody speaking, this attitude quickly becomes a habit and even a conditioned reflex on the situation of linguistic communication. As functional and even physiological in origin, this property of the oral communication is inherited by the written communication. So it becomes a linguistic universal because it is specific to understanding in linguistic communication, regardless of the natural language concerned. In our formalism, this linguistic universal is expressed by the statement that the topological space underlying any semantic level of an admissible text satisfies the separation axiom T_0 of Kolmogorov.

4.2. Topological connectedness as a linguistic universal

In Sect. 3, we have considered some examples of phonocentric topologies at various levels of semantic description of an admissible text. In all these examples, we see that their underlying topological spaces are connected. This shows empirically an important topological property of all genuine natural language texts, namely the connectedness, in the mathematical sense, of their phonocentric topology. The reasons for it aren't accidental, but it reveals a very important topological property of genuine natural language texts. At the conference [30], we presented arguments that the topological connectedness is one of the linguistic universals.

Any literary work has a property to be the communicative unity of meaning. So, for any two novels X and Y yet of the same kind, say historical, detective or biographical, their concatenation Z under one and the same cover doesn't constitute a new one. What does it mean, topologically speaking? We see that for any $x \in X$ there exists an open neighbourhood U of x that doesn't meet Y, and for any $y \in Y$ there exists an open neighbourhood V of y that doesn't meet X. Therefore, $Z = X \bigsqcup Y$ (i.e., Z is a disjoint union of two non-empty open subsets X and Y); hence, Z isn't connected. Thus, a property of a literary work to be the communicative unity of meaning may be expressed as a connectedness of a topological space related to text.

Recall that a space X is said to be *connected* if it is not the disjoint union of two non-empty open subsets. It is the same to say that X and $\varnothing$ are the only subsets opened and closed at a time. Such a property is called the connectedness of the space X. In any topological space X, a connected set is a subset U of X that is a connected space for the induced topology. It is clear that the union of connected parts having one point in common is also a connected part.

Define on a topological space X the relation $\sim$ by setting $x \sim y$ if and only if x and y belong to a connected subset of X. It is immediate that this relation is an equivalence; the equivalence class containing a point x is a connected part that is called *connected component* of x. It is clear that a topological space X is the disjoint union of its connected components, and any connected part is contained in exactly one component. If $f\colon X \to Y$ is a continuous mapping of topological spaces where the space X is connected, then $f(X)$ is a connected subset of Y.

Let X be an Alexandrov topological space. It is clear that for all $x \in X$, the smallest open U_x is connected. So, each open set U_x of the basis $\mathfrak{B}(X)$ of a phonocentric topology is connected.

For all $x, y \in X$ such that $x \neq y$, the subspace $\{x, y\}$ is connected if and only if $x \in U_y$ or $y \in U_x$; in terms of the specialization order, this amounts to saying that $x \preceq y$ or $y \preceq x$. The following well-known proposition (see, e.g., [23, p. 8]) characterizes connected Alexandrov topological spaces:

Proposition 3. *Let X be a connected Alexandrov topological space. Then for every pair of points x, y of X, there exists a finite sequence $(z_1, \dots, z_s)$ of points in X such that $z_1 = x$, $z_s = y$ and each $\{z_i, z_{i+1}\}$ is connected (i.e., $z_i \preceq z_{i+1}$ or $z_i \succeq z_{i+1}$) for all $i = 1, \dots, s-1$.*

Indeed, let Z be a set of points accessible by a finite sequence $(z_1, \dots, z_s)$ of points in X starting from $x = z_1$, such that each set $\{z_i, z_{i+1}\}$ is connected for $i = 1, \dots, s-1$. For each $z \in Z$, we have $U_z \subseteq Z$ because any element $y \in U_z$ is itself also accessible by a chain $(z_1, \dots, z, y)$. We have $Z \subseteq \bigcup_{z \in Z} U_z \subseteq Z$; hence Z is open. For each $z \in Z$, we have also $\mathrm{cl}(\{z\}) \subseteq Z$ because, for all $y \in \mathrm{cl}(\{z\})$, any neighbourhood of y, including U_y, contains z. This implies $z \preceq y$ and $y \in Z$. We have $Z \subseteq \bigcup_{z \in Z} \mathrm{cl}(\{z\}) \subseteq Z$; therefore Z is closed because X is an Alexandrov space. Now, the set Z is non-empty because $x \in Z$, opened and closed subset of the connected space X. Hence, $Z = X$.

It should be noticed that the formulation and the proof of the Proposition 3 are valid regardless of the (finite or infinite) number of points in the space X.

Since the relation $x \preceq y$ is transitive, we can, in the assertion of Proposition 3, exclude unnecessary elements of the finite sequence $(z_1, \dots, z_s)$. Namely, after excluding repetitive elements, we can reduce each subsequence $z_i \prec z_{i+1} \prec z_{i+2}$ to $z_i \prec z_{i+2}$ if any exists, and we can reduce each subsequence $z_j \succ z_{j+1} \succ z_{j+2}$ to subsequence $z_j \succ z_{j+2}$ if any exists.

After a finite number of such steps of reduction, we have a sequence $(z_1, \dots, z_r)$, such that in this sequence, the relations $\prec$ and $\succ$ follow one after the other, namely:

if $z_i \prec z_{i+1}$, then $z_{i-1} \succ z_i \prec z_{i+1}$ for all i such that $1 < i < s$;
if $z_i \succ z_{i+1}$, then $z_{i-1} \prec z_i \succ z_{i+1}$ for all i such that $1 < i < s$.

Example. In the Hasse diagram of the book [21], one immediately sees such a sequence $(4 \succ 2 \prec 7 \succ 6 \prec 8)$, which connects the Chapter 4 with the Chapter 8, that is shown in the Fig. 5.

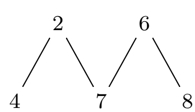

Figure 5: A Khalimsky arc traced in the Leitfaden of [21] shown in the Fig. 1.

The Hasse diagram of the type shown in the Fig. 5 is called *Khalimsky arc*.

We define now the *Khalimsky topology* by means of a structure that differs slightly from the original definition of [17]. Let us first define the partition $\mathbb{R} = \bigcup_{m\in\mathbb{Z}} P_m$ of Euclidean line of real numbers $\mathbb{R}$ by setting:

$P_m = [m - \frac{1}{2}, m + \frac{1}{2}]$, closed interval of real numbers $\{t\colon m - \frac{1}{2} \leqslant x \leqslant m + \frac{1}{2}\}$, for each even integer $m \in \mathbb{Z}$;

$P_m =]m - \frac{1}{2}, m + \frac{1}{2}[$, open interval of real numbers $\{t\colon m - \frac{1}{2} < x < m + \frac{1}{2}\}$, for each odd integer $m \in \mathbb{Z}$.

Recall the notion of a *quotient topology*. Let X be a topological space, and let P be an equivalence relation on X. The quotient topology on the quotient set $X/_\mathsf{P}$ is the finest topology making continuous the canonical projection $X \to X/_\mathsf{P}$ that associates to each element of X its equivalence class. That is, the set of equivalence classes of $X/_\mathsf{P}$ is open in the quotient topology if and only if its inverse image is open in X.

Let P be an equivalence relation on $\mathbb{R}$ associated with the partition $\mathbb{R} = \bigcup_{m\in\mathbb{Z}} P_m$. We then define a quotient topology on $X/_\mathsf{P}$. By identifying $P_m \in X/_\mathsf{P}$ with $m \in \mathbb{Z}$, we define the *Khalimsky topology* on $\mathbb{Z}$. The set of integers $\mathbb{Z}$ endowed with the Khalimsky topology is called the *Khalimsky line*. Since $\mathbb{R}$ is connected, the Khalimsky line is connected as well.

It is immediate that an even point is closed, and that an odd point is open. Concerning the smallest neighbourhoods, we have $U_m = \{m\}$ if m is odd, and we have $U_m = \{m-1, m, m+1\}$ if m is even. For integers $m \leqslant n$, we define a *Khalimsky interval* to be the interval $[m, n] \cap \mathbb{Z}$ with the topology induced from Khalimsky line, and we denote it by $[m, n]_\mathbb{Z}$. We call a *Khalimsky arc* any topological space that is homeomorphic to a Khalimsky interval $[m, n]_\mathbb{Z}$. We say that the points that are images of m and n are connected by a Khalimsky arc. Now it is clear that the Proposition 3 is equivalent to the following:

Proposition 4. *An Alexandrov topological space X is connected if and only if for every pair of points x, y of X, there exists a Khalimsky arc that connects them.*

In other words, for an Alexandrov space, the connectedness and the connectedness by a Khalimsky arc are equal.

It is obvious that all topological spaces whose Hasse diagrams are shown in the Fig. 1 are connected. It is difficult to imagine a book in which there is a single chapter that has no contextual links to other chapters. The same holds not only at the semantic level where primitive elements are chapters, but also at the semantic level where primitive elements are sentences of the text (such a level is called the semantic level of text). If at the end of the reading, we realize that a sentence x has nothing to do with the reminder of the text, we have a feeling that ‘a noise crept into the message’ because the reading of the text is finished, but the sentence x remains to be its completely strange ingredient.

On the contrary, if during the reading we meet a sentence that does not have direct contextual links with the sentences already read (like the item 7 in the Hasse diagram of the textbook [39] as shown in the Fig. 1), we have a feeling to be on a turning point in the narrative, and that the author prepares the reader for the future development, where the suspended sentence will be necessary for the understanding. For an admissible text, these considerations confirm that the connectedness of the underlying topological space expresses mathematically the necessary requirement of a textuality in the sense one understands this concept in the semiotics of text.

This explains why a basic unit that is pertinent as a message in the situation of linguistic communication should be an admissible text (or discourse) whose underlying topological space is connected! It is a connected unit because, after having communicated such a message, the transmitter (author, sender) may become silent to give the floor to its receptor (reader, receiver).

At the level of text, the connectedness of message is also a requirement specific to the kind of linguistic communication qualified as a dialogue, that is, to a bi-directional communication with others. If somebody produces, as a message, a series of phrases that disintegrates into pieces that have no links between, it reveals the disregard for the interlocutor, or the absence of the desire to communicate, or the use of a language for purely expressive purposes without a desire to communicate.

It is the same at the semantic level of sentence with regard to connectedness, although the formal definition of a phonocentric topology at the level of sentence needs more delicate work.

Remark. It should be noticed that for an admissible text, the corresponding Hasse diagram with directed edges is acyclic at any semantic level. It is clear that this property of a phonocentric topology is stable under homeomorphism. This means that the *acyclicity* of the Hasse diagram corresponding to the phonocentric topology is yet another linguistic universal of a topological nature.

5. Sheaves of meanings appeared as semantics

Let X be an admissible text endowed with a phonocentric topology, and let $\mathscr{F}$ be an adopted sense of reading. In a Platonic manner, for each non-empty open (that is meaningful) part $U \subseteq X$, we collect in the set $\mathscr{F}(U)$ all fragmentary meanings of this part U read in the sense $\mathscr{F}$; also we define $\mathscr{F}(\varnothing)$ to be a singleton pt. Thus, we are given a map

$$U \mapsto \mathscr{F}(U) \tag{1}$$

defined on the set $\mathfrak{O}(X)$ of all open sets in a phonocentric topology on X.

Following the precept of hermeneutic circle **'to understand a part in accordance with the understanding of the whole'**, for each inclusion $U \subseteq V$ of non-empty opens, the adopted sense of reading $\mathscr{F}$ gives rise to restriction map $\mathrm{res}_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$. We will consider the inclusion of sets $U \subseteq V$ as being the canonical injection map $U \overset{\mathrm{inj}}{\hookrightarrow} V$. Thus, we are also given a map

$$\{\, U \overset{\mathrm{inj}}{\hookrightarrow} V \,\} \mapsto \{\, \mathscr{F}(V) \xrightarrow{\mathrm{res}_{V,U}} \mathscr{F}(U) \,\} \tag{2}$$

with the properties:

(i) $\mathrm{id}_V \mapsto \mathrm{id}_{\mathscr{F}(V)}$ for all opens V of X;

(ii) $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ for all nested opens $U \subseteq V \subseteq W$ of X.

The first property means that the restriction $\mathrm{res}_{V,U}$ respects identity inclusions. The second property means that two consecutive restrictions may be done by one step.

As for the empty part $\varnothing$ of X, the restriction maps $\mathrm{res}_{\varnothing,\varnothing}$ and $\mathrm{res}_{V,\varnothing}$ with the same properties are obviously defined.

Let $(X, \mathfrak{O}(X))$ be a topological space. We can consider its topology $\mathfrak{O}(X)$ as the category $\mathbf{Open}_X$ whose objects are open sets of X, and where for two open sets $U, V \in \mathfrak{O}(X)$, the class of morphisms $\mathrm{Mor}(U, V)$ is empty if $U \subsetneq V$, and $\mathrm{Mor}(U, V)$ is the set reduced to the canonical injection $U \overset{\mathrm{inj}}{\hookrightarrow} V$ if $U \subseteq V$. The composition of morphisms is defined as the composition of canonical injections.

From the mathematical point of view, the assignments (1) and (2) give rise to a presheaf $\mathscr{F}$ defined as a contravariant functor from the category $\mathbf{Open}_X$

to the category $\mathbf{Set}$ of sets and maps

$$\mathscr{F}\colon \mathbf{Open}_X \to \mathbf{Set}, \tag{3}$$

acting on objects as defined by (1), and acting on morphisms as defined by (2).

In sheaf theory, an element $s \in \mathscr{F}(V)$ is called *section* (*over* V); sections over the whole space X are said to be *global*.

We consider the reading process of an open fragment U as its covering by some family of open subfragments $(U_j)_{j\in J}$ already read, that is $U = \bigcup_{j\in J} U_j$.

Following Quine, **"There is no entity without identity"** [35]. We argue that two fragmentary meanings should be equal globally if and only if they are equal locally. It motivates the following *identity criterion*:

Claim S (Separability). *Let X be an admissible text, and let U be an open fragment of X. Suppose that $s, t \in \mathscr{F}(U)$ are two fragmentary meanings of U and there is an open covering $U = \bigcup_{j\in J} U_j$ such that $\mathrm{res}_{U,\,U_j}(s) = \mathrm{res}_{U,\,U_j}(t)$ for all fragments U_j. Then $s = t$.*

According to the precept of hermeneutic circle, **'to understand the whole by means of understandings of its parts'**, a presheaf $\mathscr{F}$ of fragmentary meanings satisfies the following:

Claim C (Compositionality). *Let X be an admissible text, and let U be an open fragment of X. Suppose that $U = \bigcup_{j\in J} U_j$ is an open covering of U; suppose we are given a family $(s_j)_{j\in J}$ of fragmentary meanings, $s_j \in \mathscr{F}(U_j)$ for all fragments U_j, such that $\mathrm{res}_{U_i,\,U_i\cap U_j}(s_i) = \mathrm{res}_{U_j,\,U_i\cap U_j}(s_j)$. Then there exists some meaning s of the whole fragment U such that $\mathrm{res}_{U,\,U_j}(s) = s_j$ for all fragments U_j.*

Thus, any presheaf of fragmentary meanings defined as above should satisfy both Claims **S** and **C**, and so it is a *sheaf* by the very definition. This motivates the following definition:

Frege's Generalized Compositionality Principle. *A presheaf of fragmentary meanings naturally attached to any sense* (*mode of reading*) *of an admissible text is really a sheaf; its sections over a meaningful fragment of the text are its fragmentary meanings; its global sections are the meanings of the text as a whole.*

Traditionally attributed to Frege, the compositionality principle arises in logic, linguistics and philosophy of language in many different formulations, which all however convey the concept of functionality.

We note that the Claim **S** guarantees the meaning s (whose existence is stated by the Claim **C**) to be unique as such. It is not so hard to see that these two conditions **C** and **S** needed for a presheaf to be a sheaf are analogous to those two conditions $\mathbf{1}^\circ$ and $\mathbf{2}^\circ$ needed for a binary relation to be a function.

5.1. Sheaf-theoretic conception of a functional dependence

Formally, for a function f of n variables, it is set that: $\mathbf{1}^\circ$ *for any* family of variables' values $(s_1, \ldots, s_n)$, *there exists* a function's value $f(s_1, \ldots, s_n)$ being dependent on them, and $\mathbf{2}^\circ$ this function's value is *unique*. Likewise, for a sheaf $\mathscr{F}$, it is set that: (due to **C**) *for any* family of sections $(s_i)_{i \in I}$ those are locally compatible on an open U, *there exists* a section s being their composition dependent on them, and (due to **S**) this composition s is *unique* as such. In this generalized (sheaf-theoretic) conception of a functional dependence, the variables and their number are not fixed in advance (we consider an arbitrary family of pairwise compatible sections as variables), but for any such a family of variables, *there exists* the glued section considered as their composition (analogous to the function's value in a given family of variables) and such a section is *unique*. So the true formulation of Frege's compositionality principle does not demand a set-theoretic functionality, but demands its sheaf-theoretic generalization stating that any presheaf of fragmentary meanings naturally attached to an admissible text ought *de facto* to be a sheaf. The sheaves arise whenever some consistent local data glues into a global one.

5.2. Schleiermacher category of sheaves of fragmentary meanings

The reader should become at home with the senses treated as functors although we call them sometimes as 'modes of readings' instead of 'senses' not only to emphasize the character of intentionality of each actual process of reading but rather to avoid a possible confusion that may be caused by another technical acceptance of the term 'sense'. So one can think, for example, about the historical sense $\mathscr{F}$ and the moral sense $\mathscr{G}$ of some biographical text.

Let us consider now any two senses (modes of reading) $\mathscr{F}, \mathscr{G}$ of a given text X, and let $U \subseteq V$ be two arbitrary meaningful fragments of the text X. It seems to be very natural to consider that any meaning s of fragment V understood in the historical sense $\mathscr{F}$ gives a certain well-defined meaning $\phi(V)(s)$ of the same fragment V understood in the moral sense $\mathscr{G}$. Hence, for each $V \subseteq X$, we are given a map $\phi(V)\colon \mathscr{F}(V) \to \mathscr{G}(V)$. To transfer from the meaning s of V in the historical sense to its meaning $\phi(V)(s)$ in the moral sense and then to restrict the latter to a subfragment $U \subseteq V$ is the same operation as to make first the restriction from V to U of the meaning s in the historical sense, and to make then a change of the historical sense to the moral one. This kind of transfer from the understanding in one sense $\mathscr{F}$ to the understanding in another sense $\mathscr{G}$ is a usual matter of linguistic communication. In the Christian theology, the possibility of such a transfer from one of four senses of any biblical verse to some another its sense is considered as the cornerstone method of exegesis.

Formally, this idea is well expressed by the notion of *morphism* of the corresponding sheaves $\phi\colon \mathscr{F} \mapsto \mathscr{F}'$ defined as a family of maps $\phi(V)\colon \mathscr{F}(V) \to \mathscr{F}'(V)$ those commute with restrictions for all opens $U \subseteq V$, that is, $\mathrm{res}'_{V,\,U} \circ \phi(V) = \phi(U) \circ \mathrm{res}_{V,\,U}$. This can be expressed in a simple way by saying that the following diagram

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\mathrm{res}_{V,\,U}"'] & \mathscr{G}(V) \arrow[d, "\mathrm{res}'_{V,\,U}"] \\
\mathscr{F}(U) \arrow[r, "\phi(U)"'] & \mathscr{G}(U)
\end{tikzcd}$$

commutes for all opens $U \subseteq V$ of X.

This notion of morphism is very near to that of *incorporeal transformation* of G. Deleuze and F. Guattari illustrated by several examples, one of which we quote:

> In an airplane hijacking, the threat of a hijacker brandishing a revolver is obviously an action; so is the execution of the hostages, if it occurs. But the transformation of the passengers into hostages, and of the plane-body into a prison-body, is an instantaneous incorporeal transformation, a "mass media act" in the sense in which the English speak of "speech acts." [5, p. 102]

To adapt this example, we need only to transform it into some written story about a hijacking. Hence, the family of maps $(\phi(V))_{V \in \mathfrak{O}(X)}$ defines a change of mode of reading of a given text X, or simply a morphism $\phi\colon \mathscr{F} \mapsto \mathscr{G}$. It is obvious that a family of identical maps $\mathrm{id}_{\mathscr{F}(V)}\colon \mathscr{F}(V) \to \mathscr{F}(V)$ given for each open $V \subseteq X$ defines the identical morphism of the sheaf $\mathscr{F}$ that will be denoted as $\mathrm{id}_{\mathscr{F}}$. The composition of morphisms is defined in an obvious manner: For two arbitrary morphisms $\phi\colon \mathscr{F} \mapsto \mathscr{G}$, $\psi\colon \mathscr{G} \mapsto \mathscr{H}$, we define

$(\psi \circ \phi)(V) = \psi(V) \circ \phi(V)$. It is clear that this composition is associative every time it may be defined.

Thus, given an admissible text X, the data of all sheaves $\mathscr{F}$ of fragmentary meanings together with all its morphisms constitutes some *category* in a strict mathematical sense of the term. We name this category of particular sheaves describing the exegesis of the text X as *category of Schleiermacher* and denote it as $\mathbf{Schl}(X)$ because he is generally considered to be the author of the cornerstone principle of a natural language text understanding, called later by Dilthey as the *hermeneutic circle*. The parts are understood in terms of the whole, and the whole is understood in terms of the parts. This part-whole structure in the understanding, he claimed, is principal in the matter of interpretation of any text in natural language.

The theoretical principle of *hermeneutic circle* is a precursor to Frege's principles of *compositionality* and *contextuality* formulated later. The succeeded development of hermeneutics has confirmed the importance of Schleiermacher's concept of circularity in text understanding. From our point of view, the concept of part-whole structure expressed by Schleiermacher in 1829 as the hermeneutic circle principle reveals, in the linguistic form, the fundamental mathematical concept of a sheaf formulated by Leray in 1945, more than a hundred years later. This justifies us to name the particular category of sheaves $\mathbf{Schl}(X)$ after Schleiermacher.

5.3. Building a sheaf of fragmentary meanings from local data

An admissible text X is endowed with a phonocentric topology in such a way that the set $\mathfrak{O}(X)$ of all open sets of this topology is made up of all meaningful parts of X. The Hasse diagram presents a perfect visualization of this topological structure but its construction requires a lot of analytical work. It seems that the author has such a representation about his/her proper text, as well as the structure of text may be rebuild after philological considerations. But for a reader, how this topological structure is obtained during the reading? Obviously, the understanding is manifested in the reader's conscience as an empirical fact of having grasped the meaning of a sentence read in the present moment. Thus, the meaningful parts that are most clearly manifested during the reading process are the opens U_x of the phonocentric topology basis $\mathfrak{B}(X)$ that is defined in the Sect. 3. These meaningful parts U_x provide the set of contexts for the understanding of the whole text.

The Proposition 1 states that the set of all these fragments U_x constitutes the minimal basis for a phonocentric topology. Formally, this means that any arbitrary open set is the union of a family of these basis sets U_x. The liberty in choice of basis sets whose union gives an open set $U \subseteq X$ makes us doubtful whether it would be too strong to impose the satisfaction of Claims **S** and **C** for all opens of the topology $\mathfrak{O}(X)$. Would it be more convenient and more useful to develop the very theory in more concrete terms, say even in constructive terms of opens U_x of the minimal basis $\mathfrak{B}(X)$ of a phonocentric topology on X? The answer is plain and simple: From the psychological point of view, yes, perhaps; but from the mathematical point of view, this approach will be formally equivalent but less technically convenient! Moreover, a general truth is sometimes more understandable that a mass of concrete data. In what follows, we will present formal arguments to justify this point of view.

A topological space $(X, \mathfrak{O}(X))$ may be considered as the category $\mathbf{Open}_X$ with open sets $U \in \mathfrak{O}(X)$ as objects, and injection maps $U \xhookrightarrow{\text{inj}} V$ as morphisms.

Let $\mathfrak{B}(X)$ be a basis for the topology $\mathfrak{O}(X)$ of X. It is obvious that the basis $\mathfrak{B}(X)$ gives rise to a category defined in the same way that we consider the topology $\mathfrak{O}(X)$ as being the category $\mathbf{Open}_X$. By a slight abuse of notations, we will also denote such a category as $\mathfrak{B}(X)$. In the same manner as above, we define a presheaf $\mathscr{F}$ of sets on the topology basis $\mathfrak{B}$ as a *contravariant functor* on the category $\mathfrak{B}$ with values in the category of sets $\mathbf{Set}$.

Namely, for every basis open $U \in \mathfrak{B}(X)$, the presheaf $\mathscr{F}$ attaches a set $\mathscr{F}(U)$, and so we are given a map

$$U \mapsto \mathscr{F}(U) \tag{4}$$

defined on the basis $\mathfrak{B}(X)$ for a topology on X. Also, for every pair of opens $U, V \in \mathfrak{B}(X)$ such that $U \subseteq V$, the presheaf $\mathscr{F}$ attaches a map $\mathrm{res}_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$, and so we are given a map

$$\{U \subseteq V\} \mapsto \{\mathrm{res}_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)\} \tag{5}$$

with the properties of identity preserving and transitivity:

(i) $\mathrm{id}_V \mapsto \mathrm{id}_{\mathscr{F}(V)}$ for all opens $V \in \mathfrak{B}(X)$;

(ii) $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ for all nested basis opens $U \subseteq V \subseteq W$ of $\mathfrak{B}(X)$.

Given a basis $\mathfrak{B}(X)$ for a topology on X, the data of $(\mathscr{F}(V), \mathrm{res}_{V,\,U})_{V,U\in\mathfrak{B}(X)}$ satisfying these properties is called presheaf of sets over the basis $\mathfrak{B}(X)$ for the topology on X. In the case of an admissible text X, the topological basis $\mathfrak{B}(X)$ consists of all fragments of the kind U_x, that may be considered as empirical data.

Let $\mathscr{F}$ be a presheaf of sets over a basis $\mathfrak{B}(X)$ for the topology $\mathfrak{O}(X)$ on X. This presheaf $\mathscr{F}$ is said to be a *sheaf* over the topological basis $\mathfrak{B}(X)$ if the following Claims **Sb** and **Cb** are satisfied:

Claim Sb. *Let U be any open of the basis $\mathfrak{B}(X)$ for the topology on X, and let $s, t \in \mathscr{F}(U)$ be two elements of U. If there exists an open covering $U = \bigcup_{j\in J} U_j$ by basis open sets $U_j \in \mathfrak{B}(X)$ such that for each U_j of this covering, we have $\mathrm{res}_{U,\,U_j}(s) = \mathrm{res}_{U,\,U_j}(t)$. Then $s = t$.*

Claim Cb. *Let U be any open of the basis $\mathfrak{B}(X)$ for the topology on X, and let $U = \bigcup_{j\in J} U_j$ be a covering of U by basis open sets $U_j \in \mathfrak{B}(X)$. Suppose we are given a family $(s_j)_{j\in J}$ of elements $s_j \in \mathscr{F}(U_j)$ such that $\mathrm{res}_{U_i,\,U_i\cap U_j}(s_i) = \mathrm{res}_{U_j,\,U_i\cap U_j}(s_j)$. Then there exists an element $s \in \mathscr{F}(U)$ such that $\mathrm{res}_{U,\,U_j}(s) = s_j$ for each open U_j.*

It is obvious that the Claims **Sb** and **Cb** are similar to the Claims **S** and **C** in the definition of a sheaf over a topological space.

Let $\mathscr{F}$ be a presheaf of sets over the basis $\mathfrak{B}(X)$ for a topological space X. For any open $U \in \mathfrak{O}(X)$, the sets $(\mathscr{F}(V))_{\mathfrak{B}\ni V\subseteq U}$ together with maps $\mathrm{res}_{W,\,V}$ (where $W \in \mathfrak{B}(X)$, $V \in \mathfrak{B}(X)$ such that $V \subseteq W \subseteq U$) form a projective system. We can associate with $\mathscr{F}$ a presheaf of sets $\mathscr{F}'$ over X in the ordinary sense by assigning to any open $U \in \mathfrak{O}(X)$, the projective limit

$$\mathscr{F}'(U) = \varprojlim_{\mathfrak{B}\ni V\subseteq U} \mathscr{F}(V), \tag{6}$$

where V are running the set (ordered by $\subseteq$) of all opens $V \in \mathfrak{B}(X)$ such that $V \subseteq U$.

In the *EGA* of A. Grothendieck and J. A. Dieudonné [13, p. 75], there is a general proposition that, for a presheaf with values in the category of sets, is interpreted as the following result.

Proposition 5. *For the presheaf $\mathscr{F}'$ defined over topology $\mathfrak{O}(X)$ by* (6) *to be a sheaf, that is, to verify the Claims* **S** *and* **C***, it is necessary and sufficient that the presheaf $\mathscr{F}$ defined over the basis $\mathfrak{B}(X)$ for $\mathfrak{O}(X)$ verifies the Claims* **Sb** *and* **Cb**.

Let $\mathscr{F}, \mathscr{G}$ be two presheaves of fragmentary meanings defined over the topological basis $\mathfrak{B}(X)$. We define a morphism $\theta\colon \mathscr{F} \to \mathscr{G}$ as a family $(\theta(V))_{V\in\mathfrak{B}}$ of maps $\theta(V)\colon \mathscr{F}(V) \to \mathscr{G}(V)$ satisfying the conditions of compatibility with the corresponding restriction morphisms. With the notation of Proposition 5, we deduce a morphism $\theta'\colon \mathscr{F}' \to \mathscr{G}'$ of presheaves of fragmentary meanings defined on all opens $U \in \mathfrak{O}(X)$ by taking $\theta'(U)$ to be the projective limit of $\theta(V)$ for $V \in \mathfrak{B}(X)$ and $V \subseteq U \in \mathfrak{O}(X)$.

Let $\mathscr{F}$ be a sheaf of fragmentary meanings over $\mathfrak{O}(X)$, and let $\mathscr{F}_1$ be a sheaf over $\mathfrak{B}(X)$ defined by the restriction of $\mathscr{F}$ to $\mathfrak{B}(X)$. Then, the sheaf ${\mathscr{F}_1}'$ over $\mathbf{Open}_X$ that we obtained from $\mathscr{F}_1$ according to the Proposition 5 is canonically isomorphic to $\mathscr{F}$, because of the claims **S** and **C**, and by the uniqueness of the projective limit. Usually, we identify $\mathscr{F}$ and ${\mathscr{F}_1}'$.

For two sheaves $\mathscr{F}, \mathscr{G}$ defined over $\mathfrak{O}(X)$ and a morphism $\theta\colon \mathscr{F} \to \mathscr{G}$, one can show that the data of $\theta(V)\colon \mathscr{F}(V) \to \mathscr{G}(V)$ given only for $V \in \mathfrak{B}(X)$ determines completely the morphism θ. For more details, see *EGA* of A. Grothendieck and J. A. Dieudonné [13, p. 76].

Theoretically speaking, this means that we have a good reason to move the considerations from the level of empirical data, where a phonocentric topology is revealed by the minimal basis $(U_x)_{x\in X}$, to the general level, more abstract but more simple, where a phonocentric topology is defined by the set $\mathfrak{O}(X)$ of all opens according to the classical Hausdorff axioms (t_1), (t_2), and (t_3) of a topological space. In mathematics, the axiomatic view on a topology is particularly useful in all sorts of reasonings where topological structures are concerned. Once we have defined a topological space in terms of its basis, we may continue the reasoning in terms of all open sets of this topology.

5.4. Compositionality of locally defined modes of reading

Note that the class of objects in the category $\mathbf{Schl}(X)$ is not limited to a modest list of sheaves corresponding to *literal*, *allegoric*, *moral*, *psychoanalytical* and other senses mentioned above. In the process of text interpretation, the reader's semantic intentionality changes from time to time, with the result that there is some compositionality (or gluing) of these locally defined sheaves of fragmentary meanings, which we consider in details in [31]. There is a standard way to name the result of such a gluing as, for example, this is the case of *Freudo-Marxist* sense.

As the reader's intentionality to interpret an arbitrary text in a certain sense

$\mathscr{F}$, this particular sense $\mathscr{F}$ yet precedes a reading; for example, one may intend to read a story in a moral sense. But for a given text X, the *intentional object* 'sense $\mathscr{F}$' is represented by the sheaf of sets $(\mathscr{F}(V), \mathrm{res}_{V,U})_{V,U\in\mathfrak{O}(X)}$ of fragmentary meanings.

To analyze the compositionality of senses (or modes of reading) in our sheaf-theoretic formalism, we recall firstly the notion of *induced sheaf*. Let X be a topological space, let U be an open set of X, and let $i\colon U \hookrightarrow X$ be the canonical injection of the open U in X. Then, for any sheaf $\mathscr{F}$ of sets over X, one can define a sheaf of sets over U, which is called *sheaf induced by $\mathscr{F}$ on U*, and which is denoted as $\mathscr{F}|_U$, by setting:

$$(\mathscr{F}|_U)(V) = \mathscr{F}(i(V)) \quad \text{for any open } V \subseteq U;$$
$$(\mathrm{res}|_U)_{W,V} = \mathrm{res}_{i(W),\,i(V)} \quad \text{for all opens } V, W \subseteq U \text{ such that } V \subseteq W.$$

For any morphism $\theta\colon \mathscr{F} \to \mathscr{G}$ of sheaves of sets over X, we note by $\theta|_U$ the morphism $\mathscr{F}|_U \to \mathscr{G}|_U$ consisting of maps $\theta(i(V))$ for opens $V \subseteq U$.

We have a reason to assume that the reading of the whole text X in a sense $\mathscr{F}$ is represented by an open covering $(U_\lambda)_{\lambda\in L}$ of the text X, where each fragment U_λ is read in a sense $\mathscr{F}_\lambda$ that is defined as $\mathscr{F}_\lambda = \mathscr{F}|_{U_\lambda}$.

The obvious concordance of these senses $\mathscr{F}_\lambda$ means that for all pairs of open fragments $U_\lambda, U_\mu \subseteq X$, we have an isomorphism

$$\theta_{\lambda\mu}\colon \mathscr{F}_\mu|_{(U_\lambda\cap U_\mu)} \xrightarrow{\sim} \mathscr{F}_\lambda|_{(U_\lambda\cap U_\mu)}. \tag{7}$$

In other words, in the interpretation of the common part $U_\lambda \cap U_\mu$, we can change the sense $\mathscr{F}_\lambda$ to the sense $\mathscr{F}_\mu$ and vice versa.

It is useful to denote $U_{\lambda\mu} = U_\lambda \cap U_\mu$ and $U_{\lambda\mu\nu} = U_\lambda \cap U_\mu \cap U_\nu$. Then, in this notation, the family of isomorphisms

$$\theta_{\lambda\mu}\colon \mathscr{F}_\mu|_{U_{\lambda\mu}} \xrightarrow{\sim} \mathscr{F}_\lambda|_{U_{\lambda\mu}} \tag{8}$$

satisfies the condition:

$$(\text{for all } U_\lambda, U_\mu, U_\nu) \quad \theta_{\lambda\mu} \circ \theta_{\mu\nu} = \theta_{\lambda\nu} \text{ on } U_{\lambda\mu\nu}. \tag{9}$$

In the theory of sheaves, there is a theorem stating that a family of isomorphisms satisfying the condition (9) allows us to rebuild the sheaf $\mathscr{F}$ uniquely. The following proposition is a linguistic version of this general mathematical result:

Proposition 6. *Let $(U_\lambda)_{\lambda\in L}$ be an open covering of the text X, where each fragment U_λ is read in a sense $\mathscr{F}_\lambda$. Let for each pair of fragments U_λ, U_μ of $(U_\lambda)_{\lambda\in L}$ be given an isomorphism $\theta_{\lambda\mu}\colon \mathscr{F}_\mu|_{U_{\lambda\mu}} \xrightarrow{\sim} \mathscr{F}_\lambda|_{U_{\lambda\mu}}$ of sheaves over $U_{\lambda\mu}$. Assume these isomorphisms are satisfying the condition that for all U_λ, U_μ, U_ν of the covering:*

$$\theta_{\lambda\mu} \circ \theta_{\mu\nu} = \theta_{\lambda\nu} \text{ on } U_{\lambda\mu\nu}. \tag{10}$$

Then, there exists a sheaf $\mathscr{F}$ over X, and for each U_λ of the covering $(U_\lambda)_{\lambda\in L}$ there exists an isomorphism $\theta_\lambda\colon \mathscr{F}|_{U_\lambda} \xrightarrow{\sim} \mathscr{F}_\lambda$ such that $\theta_\mu = \theta_{\mu\lambda} \circ \theta_\lambda$ for U_λ, U_μ of the covering $(U_\lambda)_{\lambda\in L}$. Moreover, $(\mathscr{F}, (\theta_\lambda)_{\lambda\in L})$ is unique up to unique isomorphism.

For the proof, see *EGA* of A. Grothendieck and J. A. Dieudonné [13, p. 77].

The family of isomorphisms $(\theta_{\lambda\mu})$ satisfying the *gluing condition* (10) is called a 1-*cocycle*. One says that the sheaf $\mathscr{F}$ is obtained by *gluing of* sheaves $(\mathscr{F}_\lambda)_{\lambda\in L}$ by means of $\theta_{\lambda\mu}$, and usually one identifies $\mathscr{F}_\lambda$ and $\mathscr{F}|_{U_\lambda}$ by means of θ_λ.

For a finite family of sheaves $(\mathscr{F}_\lambda)_{\lambda\in L}$ and their isomorphisms $\theta_{\lambda\mu}$ satisfying the condition of gluing (10), the sheaf $\mathscr{F}$ is called to be their composition obtained by the gluing of sheaves $(\mathscr{F}_\lambda)_{\lambda\in L}$ by means of the $\theta_{\lambda\mu}$; this describes how we define the compositionality of locally defined modes of reading (senses) understood as sheaves of fragmentary meanings.

The gluing of sheaves is a compositionality method that enables us to obtain a large number of globally defined sheaves from a small number of locally defined ones. In fact, the sense $\mathscr{F}$ as a global mode of reading (or an integral intention during the interpretation of the whole text) is composed of all local modes of reading taken during interpretations of parts.

Example. According to the biblical hermeneutics, the readings of the Scripture in the *literal*, *allegorical*, *moral*, and *anagogical* senses are consistent over each fragment of the type U_x. Suppose that we have read the whole text of the Scripture by fragments, where each fragment was read in one of these four senses (literal, allegorical, moral, anagogical). These partial readings satisfy the gluing condition (10) above. There exists therefore a sense $\mathscr{F}$ of reading of the whole text of the Scripture such that for each of its sentence, there are a neighbourhood and one of these four senses (literal, allegorical, moral, anagogical) that is consistent with the reading of this neighbourhood in the sense $\mathscr{F}$. The sense $\mathscr{F}$ is a composition of these four senses (literal, allegorical, moral, anagogical), but globally it differs from each of these four senses being applied to the whole text. Hence, for the text E of the Scripture, the class of objects of the category of Schleiermacher $\mathbf{Schl}(E)$ contains not only these four senses (literal, allegorical, moral, and anagogical) but much more their compound senses, where each compound sense $\mathscr{F}$ is defined by gluing a family of these four senses according to a particular covering of text by fragments, such that each fragment is read in only one sense of these four.

We summarize the results of our analysis presented in Sect. 5 as the following:

Slogan (Sheaves of Fragmentary Meanings as Semantics). The mathematical study of a natural language texts interpretation in terms of the category of sheaves of fragmentary meanings and their morphisms is a sheaf-theoretic formal semantics.

6. Bundles of contextual meanings

So far, we have considered only the meanings of open sets in the phonocentric topology at any semantic level. In this section, we describe how we have to define the meanings of points in the phonocentric topology at a given semantic level. It should be noticed that in general, not every singleton x is open in T_0-topology, and if this is the case, the meaning of such a point x has not yet been defined.

In 1884, Frege wrote in the *Die Grundlagen der Arithmetik* [9, p. X]: "nach der Bedeutung der Wörter muss im Satzzusammenhange, nicht in ihrer Vereinzelung gefragt werden;" This declaration is traditionally named as Frege's principle of contextuality. Frege stated it eight years before he pointed out his theoretic distinction between *Sinn* and *Bedeutung*; that is why the word 'Bedeutung' here is usually translated in English as 'meaning': "Never ask for the meaning of a word in isolation, but only in the context of a sentence". As we have yet seen in the Sect. 3.4, the context of a whole sentence is the greatest possible at the semantic level of sentence. We may also ask for the meaning of a word x in the context of a clause to which it belongs, or in the context of some lesser part of this clause as, e.g., of the smallest part U_x. This restatement makes Frege's definition more precise. If we try to recast such a contextuality principle to the level of text, then we would have to say: **"Never ask for the meaning of a sentence in isolation, but only in the context of some meaningful fragment of a text"**. Such a fragment may be chosen in many ways to induce the same contextual meaning of the sentence.

To formalize this definition, let us consider the phonocentric topology at the level of text. Let a sentence x belongs to meaningful fragments (opens) U and V. Then fragmentary meanings $s \in \mathscr{F}(U)$, $t \in \mathscr{F}(V)$ are said to *induce the same contextual meaning of a sentence* $x \in U \cap V$ if there exists some open neighbourhood W of x, such that $W \subseteq U \cap V$ and $\mathrm{res}_{U,W}(s) = \mathrm{res}_{V,W}(t) \in \mathscr{F}(W)$. The identity of fragmentary meanings is understood here accordingly to the criterion claimed by **S**.

This relation '*fragmentary meanings s, t induce the same contextual meaning of the sentence* x' is clearly an equivalence relation. The equivalence class so defined by a fragmentary meaning s is called a *germ* at x of this s, and is denoted by $\mathrm{germ}_x(s)$. The equivalence class of fragmentary meanings agreeing in some open neighbourhood of a sentence x is natural to define as a *contextual meaning* of x. Let $\mathscr{F}_x$ be the set of all contextual meanings of x. Following S. Mac Lane and I. Moerdijk [20, pp. 83,84], this $\mathscr{F}_x$ is nothing else but the inductive limit $\mathscr{F}_x = \varinjlim(\mathscr{F}(V), \mathrm{res}_{V,U})_{V,U \in \mathfrak{O}(x)}$, where $\mathfrak{O}(x)$ is the set of all open neighbourhoods of x.

In the bundle-theoretic terms, we summarize the aforesaid as the following:

Frege's Generalized Contextuality Principle. *Let $\mathscr{F}$ be an adopted sense of reading of a fragment U of an admissible text X. For a sentence $x \in U \subseteq X$, its contextual meaning is defined as a $\mathrm{germ}_x(s)$ at x of some fragmentary meaning $s \in \mathscr{F}(U)$. The set $\mathscr{F}_x$ of all contextual meanings of a sentence $x \in X$ is defined as the inductive limit $\mathscr{F}_x = \varinjlim(\mathscr{F}(V), \mathrm{res}_{V,U})_{V,U \in \mathfrak{O}(x)}$, where $\mathfrak{O}(x)$ is the set of all open neighbourhoods of x, that is the set of all meaningful fragments containing x.*

Remark. Note that for an open singleton $\{x\}$, we may canonically identify $\mathscr{F}_x = \mathscr{F}(\{x\})$.

For the coproduct $F = \coprod_{x \in X} \mathscr{F}_x$, we define now a *projection* map $p \colon F \to X$ by setting $p(\mathrm{germ}_x s) = x$. Every fragmentary meaning $s \in \mathscr{F}(U)$ determines a genuine function $\dot{s} \colon x \mapsto \mathrm{germ}_x s$ to be well-defined on U.

We define the topology on F by taking as a basis for this topology all the image sets $\dot{s}(U) \subseteq F$. For an open $U \subseteq X$, a continuous function $t \colon U \to F$ such that $t(x) \in p^{-1}(x)$ for all $x \in U$ is called a *cross-section*. The topology defined on F makes p and every cross-section of the kind of $\dot{s}$ to be continuous.

For a given topological space X, we have so defined a topological spaces F and a continuous surjection $p \colon F \to X$. In topology, this data (F, p) is called a *bundle over the base space* X. A *morphism* of bundles from $p \colon F \to X$ to

$q\colon G \to X$ is a continuous map $h\colon F \to G$ such that the diagram

$$
\begin{tikzcd}
F \arrow[rr, "h"] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\
& X &
\end{tikzcd}
$$

commutes, that is, $q \circ h = p$.

Thus, we have defined a category of bundles over X. A bundle (F, p) over X is called *étale* if $p\colon F \to X$ is a local homeomorphism. It is immediately seen that a bundle of contextual meanings $(\bigsqcup_{x \in X} \mathscr{F}_x, p)$ constructed as above from a given sheaf $\mathscr{F}$ of fragmentary meanings is étale. Thus, for an admissible text X, we have defined the category $\mathbf{Context}(X)$ of étale bundles (of contextual meanings) over X as a framework for the generalized contextuality principle at the level of text.

The similar definition may be formulated at each semantic level. The definition formulated at the level of sentence returns Frege's classic contextuality principle. Once a semantic level is given, the definition of a contextual meaning for a point x of the corresponding topological space X is stated as $\mathrm{germ}_x s$, where s is some fragmentary meaning defined on some neighbourhood U of x.

7. Frege duality

For a given admissible text X, we have defined two categories formalizing the interpretation process, that is, the Schleiermacher category $\mathbf{Schl}(X)$ of *sheaves* of fragmentary meanings and the category $\mathbf{Context}(X)$ of *étale bundles* of contextual meanings. Our intention now is to relate them to each other.

We will firstly define a so-called *germ-functor*

$$\Lambda\colon \mathbf{Schl}(X) \to \mathbf{Context}(X).$$

For each sheaf $\mathscr{F}$, it assigns an étale bundle $\Lambda(\mathscr{F}) = (\bigsqcup_{x \in X} \mathscr{F}_x, p)$, where the projection p is defined as above. For a morphism of sheaves $\phi\colon \mathscr{F} \to \mathscr{F}'$, the induced map of fibers $\phi_x\colon \mathscr{F}_x \to \mathscr{F}'_x$ gives rise to a continuous map $\Lambda(\phi)\colon \bigsqcup_{x \in X} \mathscr{F}_x \to \bigsqcup_{x \in X} \mathscr{F}'_x$ such that $p' \circ \Lambda(\phi) = p$; hence $\Lambda(\phi)$ defines a morphism of bundles. Given another morphism of sheaves ψ, one sees easily that $\Lambda(\psi \circ \phi) = \Lambda(\psi) \circ \Lambda(\phi)$ and $\Lambda(\mathrm{id}_{\mathscr{F}}) = \mathrm{id}_F$. Thus, we have constructed a desired germ-functor $\Lambda\colon \mathbf{Schl}(X) \to \mathbf{Context}(X)$.

We will now define a so-called *section-functor*

$$\Gamma\colon \mathbf{Context}(X) \to \mathbf{Schl}(X).$$

We denote a bundle (F, p) over X simply by F. For a bundle F, we denote the set of all its cross-sections over U by $\Gamma(U, F)$. If $U \subseteq V$ are open sets, one has a restriction map $\mathrm{res}_{V,U}\colon \Gamma(V, F) \to \Gamma(U, F)$ that operates as $s \mapsto s|_U$, where $s|_U(x) = s(x)$ for all $x \in U$. It is clear that $\mathrm{res}_{U,U} = \mathrm{id}_{\Gamma(U,F)}$ for any open U, and that the transitivity $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ holds for all nested opens $U \subseteq V \subseteq W$. So we have constructed obviously a sheaf $(\Gamma(V, F), \mathrm{res}_{V,U})$.

Then for any given morphism of bundles $h\colon E \longrightarrow F$, we have a map $\Gamma(h)(U)\colon \Gamma(U, E) \to \Gamma(U, F)$ defined as $\Gamma(h)(U)\colon s \mapsto h \circ s$, which is obviously a morphism of sheaves. Thus, we have constructed a desired section-functor $\Gamma\colon \mathbf{Context}(X) \to \mathbf{Schl}(X)$.

The fundamental theorem of topology states that the section-functor Γ and the germ-functor Λ establish a *dual adjunction* between the category of presheaves and the category of bundles (over the same topological space); this dual adjunction restricts to a *dual equivalence* of categories (or *duality*) between corresponding full subcategories of sheaves and of étale bundles (see, e.g., [19, p. 179] or [20, p. 89]). Transferred to linguistics in our [28], it yields the following result:

Theorem (Frege Duality). *The generalized compositionality and contextuality principles are formulated in terms of categories those are in natural duality*

$$\mathbf{Schl}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Context}(X)$$

established by the section-functor Γ *and the germ-functor* Λ*, the pair of adjoint functors.*

Each fragmentary meaning $s \in \mathscr{F}(U)$ determines a function $\dot{s}\colon x \mapsto \mathrm{germ}_x s$ to be well-defined on U; for each $x \in U$, its value $\dot{s}(x)$ is taken in the stalk $\mathscr{F}_x$. This gives rise to a functional representation

$$\eta(U)\colon s \mapsto \dot{s} \tag{11}$$

defined for all fragmentary meanings $s \in \mathscr{F}(U)$. This representation of a fragmentary meaning s as a genuine function $\dot{s}$ provides an insight into the nature of fragmentary meanings. Each fragmentary meaning $s \in \mathscr{F}(U)$, which has been described in Sect. 5 as an abstract entity, may now be thought of as a genuine function $\dot{s}$ defined on the fragment U of a given text. At the argument (sentence) $x \in U$, this function $\dot{s}$ (representing s) takes its value $\dot{s}(x)$ to be the contextual meaning $\mathrm{germ}_x s$ of this sentence x

$$x \mapsto \dot{s}(x) = \mathrm{germ}_x s \tag{12}$$

Remark. Due to the functional representation (11), the Frege duality is of a great theoretical importance because it allows us to consider any fragmentary meaning s as a genuine function $\dot{s}\colon x_i \mapsto \mathrm{germ}_{x_i} s$ that assigns to each sentence $x_i \in U$ its contextual meaning $\mathrm{germ}_{x_i} s$, and which is continuous on U. It allows us to develop a kind of *dynamic theory of meaning* [28,31,34] describing how, during the reading of the text $X = (x_1, x_2, x_3, \ldots, x_n)$, the understanding proceeds through the discrete time $i = 1, 2, 3, \ldots, n$ as a sequence of grasped contextual meanings $(\dot{s}(x_1), \dot{s}(x_2), \dot{s}(x_3), \ldots, \dot{s}(x_n))$. That gives rise to a genuine function $\dot{s}$ on X representing some $s \in \mathscr{F}(X)$; this s is one of possible meanings of the whole text X interpreted in the sense $\mathscr{F}$.

Moreover, this duality gives a solution to an old problem concerning delicate relations between Frege's compositionality and contextuality principles, in revealing that the acceptance of one of them implies the acceptance of the other (see, e.g., [31]).

8. Sheaf-theoretic dynamic semantics

We sketch now a formal model of a natural language text understanding, which is a kind of dynamic semantics we proposed in [29,31,34]. Our approach describes the dynamics of interpretation process that results in the understanding of a certain meaning of the whole text in its integrity. With the notations used above, for a given text $X = (x_1, \ldots, x_n)$ interpreted in a sense $\mathscr{F}$, we have to describe how a reader finally grasps some global section $s \in \mathscr{F}(X)$ of a sheaf $\mathscr{F}$ of fragmentary meanings.

We consider first a particular case of reading from the very beginning of an admissible text $X = (x_1, x_2, x_3, \ldots, x_n)$ whose size is short enough to allow a reading at one sitting. The general case will be reduced to this particular case by means of the generalized Frege's compositionality principle.

The first sentence x_1 in the order $\leq$ of writing must obviously be understood in the context that consists of its own data. This means that a first sentence

x_1 constitutes an open one-point set $\{x_1\}$. Thus $U_{x_1} = \{x_1\}$, and hence the sentence x_1 should be a minimal element in the specialization order; therefore $\mathscr{F}_{x_1} = \mathscr{F}(\{x_1\})$.

This means that the grasping of a contextual meaning of x_1 is equivalent to the grasping of a fragmentary meaning of the fragment $\{x_1\}$ reduced to this sentence x_1. It is obviously equivalent to the grasping of a global meaning of this sentence x_1 at the semantic level of a sentence considered as a sequence of words. We understand first the theme (topic) of this sentence x_1, and then we understand the rheme (comment) as what is being said in the sense $\mathscr{F}$ concerning this theme. Thus, we have done a descent from the level of text to the level of sentence. In our reasoning, it is the *basis of induction*.

Let us now do the *induction step*. Let us suppose that we have read and understood the text X in the sense $\mathscr{F}$ from the beginning x_1 up to the sentence x_k, $1 < k < n$. That is, we suppose that we have already endowed $X = (x_1, \ldots, x_k)$ with a phonocentric topology and we have built a suite $(\dot{s}_{x_1}, \ldots, \dot{s}_{x_k})$ of contextual meanings of sentences of the open set $U = (x_1, \ldots, x_k)$ of a given text $X = (x_1, \ldots, x_k, \ldots, x_n)$. The suite $(\dot{s}_{x_1}, \ldots, \dot{s}_{x_k})$ of contextual meanings is a continuous function that represents some fragmentary meaning $s \in \mathscr{F}(U)$.

We consider the interpretation process at its $(k+1)$-th step as the choice of an appropriate context $U_{x_{k+1}}$ for x_{k+1} that endows the initial segment $(x_1, \ldots x_{k+1})$ with a particular phonocentric topology among many possible, and allows us to extend the function s defined on the open $(x_1, \ldots, x_k)$ to a function defined on the open $(x_1, \ldots, x_{k+1})$.

The phrase x_{k+1} is read in the context of the fragment $(x_1, \ldots, x_{k+1})$ of the text X. This neighbourhood is the most large context among possible ones we dispose to understand the contextual meaning of x_{k+1}. To grasp the same contextual meaning of x_{k+1}, it suffices to understand only its minimal neighbourhood $U_{x_{k+1}}$. It may be two cases:

Case 1°. It may happen that the understanding of the sentence x_{k+1} is independent of the understanding of $U = (x_1, \ldots, x_k)$, for it constitutes alone its own context $\{x_{k+1}\} = U_{x_{k+1}}$ because there is here a turning point in the narrative, what may be confirmed by various morphologic markers such as the beginning of a new chapter, etc. The contextual meaning $\dot{s}_{x_{k+1}}$ is defined at a point x_{k+1}, and as such it is a continuous function because $\{x_{k+1}\}$ constitutes an open set.

The process of understanding of x_{k+1} is therefore conducted in the same way as that one of the first sentence x_1 whose case we have considered above as the basis of induction.

Note that the interval $U = (x_1, \dots, x_k)$ is open. We can therefore extend the suite $(\dot{s}_{x_1}, \dots, \dot{s}_{x_k})$ we supposed to be a continuous function on $U = (x_1, \dots, x_k)$ to the suite $(\dot{s}_{x_1}, \dots, \dot{s}_{x_{k+1}})$ that is a continuous function on $(x_1, \dots, x_{k+1})$.

Case 2°. The understanding of x_{k+1} is reached with the support of the understanding of the preceding sentences of the interval $U = (x_1, \dots, x_k)$. Not all the sentences in $U = (x_1, \dots, x_k)$ are required to determine the understanding of x_{k+1}, but only some subsequence of U. Let V be a subsequence of U, such that V contains only sentences those are required for the understanding of x_{k+1}. We define a phonocentric topology on $(x_1, \dots, x_{k+1})$ by defining $U_{x_{k+1}} = V \cup \{x_{k+1}\}$.

Now we transform the subsequence V into one sentence in such a way that each sentence of V, except the first in the order $\leq$ of writing, begins with "and then" that assembles it to the preceding sentence in order to get a compound sentence. This single lengthy sentence x is made up of all sentences of V in order to get the thematic context that allows the sentence x_{k+1} to express its communicative content. Finally, we join x_{k+1} to x by means of "and then" inserted at the beginning of the sentence x_{k+1}, that transforms x_{k+1} into another sentence x'_{k+1}.

In the text $(x_1, \dots, x_k, x'_{k+1})$ so defined, the sentence x'_{k+1} constitutes an open one-point set $\{x'_{k+1}\}$ that is understandable in the context of its own data. A contextual meaning of x'_{k+1} is grasped when we understand the rheme of x_{k+1} as being what is said in the sense $\mathscr{F}$ concerning the theme of x_{k+1} in the context defined by the sentences of V. But obviously the contextual meaning of x'_{k+1} is the same as the contextual meaning of x_{k+1}. So we have extended the sequence of contextual meanings $(\dot{s}_{x_1}, \dots, \dot{s}_{x_k})$ to the sequence $(\dot{s}_{x_1}, \dots, \dot{s}_{x_{k+1}})$.

Thus, we have done a descent from the level of text to the level of sentence. This trick is inspired by Russell's work *How I write* [38], where he discuss advises he received at the beginning of his career of a writer.

We consider now a general case of reading of an admissible text X whose size does not allow us to finish reading at one sitting. In this case, we consider the reading process of a text X as its covering by some family of meaningful fragments $(U_j)_{j\in J}$ already read, that is $X = \bigcup_{j\in J} U_j$ is an open covering.

Let us suppose given a family $(s_j)_{j\in J}$, where $s_j \in \mathscr{F}(U_j)$ such that all genuine functions $\dot{s_j}\colon x \mapsto \mathrm{germ}_x s_j$ of the corresponding family $(\dot{s_j})_{j\in J}$ are pairwise compatible, that is $\dot{s_i}\,\big|_{U_i\cap U_j}(x) = \dot{s_j}\,\big|_{U_i\cap U_j}(x)$ for all $x \in U_i \cap U_j$.

Let us define the function t on $X = \bigcup_{j\in J} U_j$ as $t(x) = \dot{s_j}(x)$ if $x \in U_j$ for some j. The Frege duality theorem states that $t = \dot{s}$ where $s \in \mathscr{F}(X)$ is a composition of the family $(s_j)_{j\in J}$, whose existence is ensured by the generalized Frege's compositionality principle.

The formalization of the interpretation process as an extension of a function introduces a *dynamic* view of semantics, and its theory deserves the term *inductive* because the domain of a considered function is naturally endowed with two order structures, that is, the linear order of writing $\leq$ and the specialization order $\preceq$ of context-dependence. We have outlined so a sheaf-theoretic framework for the dynamic semantics of a natural language, where the understanding of a text X in some sense $\mathscr{F}$ is described as a process of step-by-step grasping for each sentence x_i of only one contextual meaning $\dot{s}(x_i)$ from the fiber $\mathscr{F}_{x_i}$ lying over x_i in the étale bundle $\mathbf{Context}(X)$ of contextual meanings.

9. Algebraic semantics versus sheaf-theoretic semantics

According to T. M. V. Janssen, the compositionality principle is a basis for *Montague grammar, Generalized phrase structure grammar, Categorial grammar* and *Lexicalized tree adjoining grammar*. These theories propose the different notions of meaning, but follow the compositionality principle in its *standard interpretation*:

> A technical description of the standard interpretation is that syntax and semantics are algebras, and meaning assignment is a homomorphism from syntax to semantics. (T. M. V. Janssen [15, p. 116])

Let us consider this conception of standard interpretation as an algebraic homomorphism $f\colon A \to B$, where the algebra A is representing Syntax, and the algebra B is representing Semantics.

Whatever the algebras A and B would be, the homomorphism f is a function in a set-theoretic paradigm. Given the function f, we define the relation $\mathfrak{q}$ on A so that $\langle x, y\rangle \in \mathfrak{q}$, if and only if $f(x) = f(y)$. Clearly, this $\mathfrak{q}$ is an equivalence relation on A. Any given element $a \in A$ lies in precisely one equivalence class; if $f(a) = b \in B$, then the equivalence class of a is $f^{-1}(b)$. The set of equivalence classes is denoted by $A/\mathfrak{q}$ and called the *quotient set* of A by $\mathfrak{q}$. Let the equivalence classes of a be denoted by $a^{\mathfrak{q}}$. If with each $x \in A$ we associate $x^{\mathfrak{q}}$, we obtain a function $\varepsilon \colon A \to A/\mathfrak{q}$, called the identification associated with $\mathfrak{q}$. Clearly the function ε is surjective, by definition. Following the Theorem 3.1 of [4, p. 15], there is a decomposition of f:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\varepsilon}\big\downarrow & & \big\uparrow{\scriptstyle\mu} \\ A/\mathfrak{q} & \xrightarrow{f'} & f(A), \end{array}$$

where $\varepsilon \colon A \to A/\mathfrak{q}$ is a surjection, $f' \colon A/\mathfrak{q} \to f(A)$ is a bijection, and $\mu \colon f(A) \to B$ is an injection.

In the category of algebras, an injective homomorphism is called a monomorphism; a surjective homomorphism is called an epimorphism; every bijective homomorphism should be an isomorphism (usually defined as an invertible homomorphism). The above decomposition theorem remains valid in the category of algebras; moreover, $A/\mathfrak{q}$ and $f(A)$ may be endowed with the structures of algebras in such a way that ε, f', μ become homomorphisms.

Linguistically speaking, the Syntax and the Semantics should not be one and the same theory. Thus, the meaning assignment homomorphism $f \colon A \to B$ should not be an isomorphism. Nor should this homomorphism f be a monomorphism; otherwise the Syntax A would be isomorph with a proper part of the Semantics B. Hence, f should be an epimorphism with a non-trivial kernel that is defined to be the congruence relation $\mathfrak{q}$ described above. Two different elements of an algebra A representing Syntax are congruent if and only if they are mapped to the same element of an algebra B representing Semantics. Thus, the different syntactical objects will have one and the same meaning as their value under such a homomorphism $f \colon A \to B$. Thus, an algebraic approach is pertinent in the study of synonymy, but the problems of polysemy do resist to algebraic semantic theories. Moreover, an algebraic semantic, of whatever kind, is always static because the meaning $f(x) \in B$ of a syntactic element $x \in A$ under the homomorphism f is calculated in the algebra B just after the calculation of meanings of all syntactic components of x was done.

However, when studying the process of interpretation of a natural language text, we are confronted with a quite another situation. Any admissible text is really a great universe of meanings to be disclosed or reconstructed in the process of reading and interpretation. But these multiple meanings are offered to a reader as got identified in a single text. Thus, in the process of interpretation of a natural language text, the reader is confronted with a surjection Semantics $\to$ Syntax. Note that we have turned the arrow round, and this is a **paradigmatic turn**.

From a sheaf-theoretic point of view, a discourse interpretation activity proceeds as the following: The text X under interpretation is a given sequence of its sentences $x_1, x_2, x_3, \ldots, x_n$; this is a finite combinatorial object from the universe of Syntax. Over these sentences, there is another sequence of stalks of their contextual meanings $\mathscr{F}_{x_1}, \mathscr{F}_{x_2}, \mathscr{F}_{x_3}, \ldots, \mathscr{F}_{x_n}$; this is a potentially infinite and, in some degree, a virtual object from the universe of Semantics. The total disjoint union of all these stalks, that is, the coproduct $F = \bigsqcup_{x \in X} \mathscr{F}_x$ is projected by a local homeomorphism p on the text X. Thus, we have the surjective projection $p\colon F \to X$ from Semantics to Syntax. The challenge of text interpretation is to create a global cross-section s of the projection p; this s is constructed as a sequence of grasped step-by-step contextual sentences' meanings $(\dot{s}(x_1), \dot{s}(x_2), \dot{s}(x_3), \ldots, \dot{s}(x_n))$; it gives rise to a genuine function $\dot{s}$ on X representing some global cross-section $s \in \mathscr{F}(X)$; this s is one of all possible meanings of the whole text X interpreted in the sense $\mathscr{F}$.

The proposed sheaf-theoretic semantics answers to crucial questions about *what* the fragmentary meanings are and *how* they are formally composed. That is, we consider the reading process of a fragment U in a sense $\mathscr{F}$ as its covering by some family of subfragments $(U_j)_{j \in J}$, each read in a unique session. Any family $(s_j)_{j \in J}$ of pairwise compatible fragmentary meanings $s_j \in \mathscr{F}(U_j)$ under a functional representation (11) gives rise to a family $(\dot{s}_j)_{j \in J}$ of genuine functions (where each $\dot{s}_j$ is defined on U_j by (12)), those are pairwise compatible in the sense that $\dot{s}_i \big|_{U_i \cap U_j} (x) = \dot{s}_j \big|_{U_i \cap U_j} (x)$ for all $x \in U_i \cap U_j$. Let a cross-section s be defined on $U = \bigcup_{j \in J} U_j$ as $s(x) = \dot{s}_j(x)$ if $x \in U_j$ for some j. Then this cross-section s over U is clearly a composition of the family $(\dot{s}_j)_{j \in J}$ as it is claimed by the generalized Frege's compositionality principle.

The sheaf-theoretic conception of compositionality serves as the basis for the dynamic semantics we discussed in the Sect. 8. This approach has an advantage because $\mathbf{1}^\circ$ it extends the area of semantics from the level of sentence or phrase to the level of text or discourse, and it gives a uniform treatment of discourse interpretation at each semantic level (word, sentence, paragraph, text); $\mathbf{2}^\circ$ it takes into theoretical consideration the polysemy of words, sentences and texts.

10. Sheaf-theoretic formal hermeneutics

Our approach provides a mathematical model of a text interpretation process while rejecting attempts to codify interpretative practice as a kind of calculus. In a series of previous papers [28, 29, 31–33], we named this text interpretation theory as *formal hermeneutics*. It presents a formal framework for syntax and semantics of texts written in some unspecified natural language, say for us English, French, German, Russian considered as a means of communication. The object of study in this formal hermeneutics are couples $(X, \mathscr{F})$ made up of an admissible text X and a sheaf $\mathscr{F}$ of its fragmentary meanings; we call any such a couple *textual space*. But this representation is possible only in the realm of a language following the famous slogan of Wittgenstein, "to understand a text is to understand a language". Rigorously, this claim may be formulated in the frame of category theory. Likewise, the present sheaf-theoretic formal semantics describes a natural language in the *category of textual spaces* $\mathbf{Logos}$. The objects of this category are couples $(X, \mathscr{F})$, where X is a topological space naturally attached to an admissible text and $\mathscr{F}$ is a sheaf of fragmentary meanings defined on X; the morphisms are couples $(f, \theta)\colon (X, \mathscr{F}) \to (Y, \mathscr{G})$ made up of a continuous map $f\colon X \to Y$ and a f-morphism of sheaves θ that respects the concerned sheaves; such an f-morphism is formally defined as $\theta\colon \mathscr{G} \to f_*\mathscr{F}$, where f_* is a well-known *direct image* functor (see, e.g., [31]).

Given any admissible text E considered to be fixed forever as, for instance, the Scripture, it yields a full subcategory $\mathbf{Schl}(E)$ in the category $\mathbf{Logos}$ of all textual spaces. Named after Schleiermacher, the category $\mathbf{Schl}(E)$ describes the exegesis of this particular text.

The *topological syntax* and the *dynamic sheaf-theoretic semantics* based on Frege duality, as well as different categories and functors related to discourse and text interpretation process are the principal objects of study in the *sheaf-theoretic formal hermeneutics* as we understand it.

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