Topologies and Sheaves Appeared as Syntax and Semantics of Natural Language

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The aim of this talk is to present mathematical structures of topologies and sheaves appearing in natural language.

We use language to communicate with each others and to represent aspects of the world. Accordingly, there are two main theoretical preconceptions about a human language in linguistic research: as a means of communication and as a means of representation. If the latter involves some mental phenomena conceptualized in terms of cognition and mind, the former deals with more observable phenomena, that is, with messages in oral and written forms.

We outline a sheaf-theoretic framework to study the process of interpretation of text written in some unspecified natural language, say in English, considered as a means of communication. Our analysis concerns the only texts written with good grace and intended for human understanding; we call them *admissible*. All sequences of words written in order to imitate some human writings are cast aside as irrelevant to linguistic communication.

We argue that an admissible text is naturally endowed with a structure of a finite T_0-topological space, that is, with a structure of partial order, and that the meanings of an admissible text are congregated in a particular category of sheaves. We may summarize the main trusts of our view on linguistics in the form of slogan:

Syntax is the study of Topology, Semantics is the study of Sheaves

appeared quite natural in natural language texts intended for communication. All the grammarians and semanticists speak about topologies and sheaves without knowing it, just like monsieur Jourdain who spoke prose without knowing it.

We consider the class of basic communicative units of a language as made up of texts, which is wider than the class of all stand-alone sentences studied in traditional grammars.

A text $\boldsymbol{X}$ is a finite sequence $(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \ldots \boldsymbol{x_n})$ of its constituent sentences, and so it is formally identified with a graph of function. When reading a particular distinguished part of text, we delete mentally the other sentences, but follow the induced order of remaining ones. Important is the induced order of their reading and not the concrete index numbers of their occupied places. Thus, any part of text is a subsequence whose graph is a subset of the whole sequence graph. Likewise for a sentence considered as a finite sequence of its words.

Basic Semantic Concepts

We distinguish the notions *sense*, *meaning* and *reference*. The term *fragmentary meaning* of some fragment of a given text is accepted as the communicative content grasped in some particular situation of reading guided by the reader's presuppositions, preferences and prejudices, which we denominate by the term *sense* (or *mode of reading*). In our acceptance, the sense is a kind of semantic intentionality in the interpretative process, and in some degree, it is a secular remake of the term *sense* in the medieval exegesis (St. Thomas). At the level of text, it may be, for example, *literal*, *allegoric*, *moral*, *psychoanalytical*, etc.

In the present work, we assume a total *referential* competence of an idealized reader who knows the lexicon of a language and follows the rules of common usage. That is why we are less interested in the problem of understanding of denotative expressions, and the ontological status of objects thus defined.

Topologies on Text and Meaningfulness

When reading a text, the understanding is not postponed until the final sentence. So the text should have the meaningful parts and the meanings of these parts determine the meaning of the whole as it is postulated by the principle of *hermeneutic circle*. We argue (Prosorov 2003 - 2010) that in agreement with our linguistic intuition:

(i) *an arbitrary union of meaningful parts of an admissible text is meaningful;*

(ii) *a non-empty intersection of two meaningful parts of an admissible text is meaningful.*

The first property expresses the *principle of hermeneutic circle*, which requires to understand the whole (the union) through the understanding of its parts. The second property expresses the contextual mechanism of understanding. To understand a meaningful part $\boldsymbol{U}$ of text is to understand contextually all its sentences $\boldsymbol{x} \in \boldsymbol{U}$, where the context of a particular sentence $\boldsymbol{x}$ is some meaningful part lying in $\boldsymbol{U}$.

For the least meaningful part U_x containing x, we have that $x \in U \cap V$ implies $x \in U_x \subseteq U \cap V$; hence, $U \cap V$ is meaningful as a union $\cup_{x \in U \cap V} U_x$ of meaningful parts.

Since an admissible text is supposed to be meaningful as a whole by the very definition, it remains only to define formally the meaning of its empty part (for example, as a singleton) in order to endow it with some *topology* in a mathematical sense, where the set of open sets $\mathfrak{O}(X)$ is defined to be the set of all meaningful parts.

Any explicitly stated *concept of meaning* or *criterion of meaningfulness* satisfying conditions (i) and (ii) allows us to define some type of *semantic topology* on texts. Then we may interpret several tasks of discourse analysis in topological terms (Prosorov 2002, 2006, 2008).

Phonocentric Topology

In what follows, we consider only admissible texts endowed with a particular type of semantic topology corresponding to the criterion of meaningfulness conveying an idealized reader's linguistic competence meant as ability to grasp a communicative content. The topology so defined is called *phonocentric*.

The natural process of reading supposes that any sentence x of a text X should be understood on the basis of the text's part already read, because the interpretation cannot be postponed, for "it is compulsive and uncontrollable" according to F. Rastier (1995).

Thus for every pair of distinct sentences x, y of a text X, there is an open U, that contains one of them (to be read first in the natural order $\leq$ of sentences reading) but not the other. Whence a phonocentric topology should satisfy the *separation axiom* T_0 of Kolmogoroff.

Phonocentric Topology and Partial Order

An admissible text $\boldsymbol{X}$ gives rise to a finite space; hence an arbitrary intersection of its open sets is open and so it is an *Alexandroff space*. For a sentence $\boldsymbol{x} \in \boldsymbol{X}$, we define $\boldsymbol{U_x}$ to be the intersection of all the meaningful parts that contain $\boldsymbol{x}$, that is the smallest open neighborhood of $\boldsymbol{x}$. The *specialization relation* $\boldsymbol{x} \preceq \boldsymbol{y}$ (read as '$\boldsymbol{x}$ is more special than $\boldsymbol{y}$') on a topological space $\boldsymbol{X}$ is defined by setting:

$$\boxed{\boldsymbol{x} \preceq \boldsymbol{y} \textit{ if and only if } \boldsymbol{x} \in \boldsymbol{U_y} \textit{ or, equally, } \boldsymbol{U_x} \subseteq \boldsymbol{U_y}.}$$

It is clear that $\boldsymbol{x} \in \boldsymbol{U_y}$ if and only if $\boldsymbol{y} \in \mathrm{cl}(\{\boldsymbol{x}\})$, where $\mathrm{cl}(\{\boldsymbol{x}\})$ denote the closure of a one-point set $\{\boldsymbol{x}\}$.

Key properties of these notions are summarized in the following proposition, which is a linguistic version of general results concerning interplay of topological and ordered structures on a finite set (May 2003).

Proposition. *For an admissible text* $\boldsymbol{X}$*, the set of all opens of the kind of least open neighborhood* $\boldsymbol{U_x}$ *of* $\boldsymbol{x}$*, is a basis of the phonocentric topology. Since the phonocentric topology on* $\boldsymbol{X}$ *satisfies the separation axiom* $\boldsymbol{T_0}$*, it defines a partial order* $\preceq$ *on* $\boldsymbol{X}$ *by means of specialization. The initial phonocentric topology can be recovered from this partial order* $\preceq$ *in a unique way as the topology with the basis constituted of all sets of the kind* $\boldsymbol{U_x = \{z \colon z \preceq x\}}$*.*

The relationships between topological and ordered structures are very manifold. For our investigation, it is essential that the category of finite topological $\boldsymbol{T_0}$-spaces and continuous maps is isomorphic to the category of finite partially ordered sets (*posets*) and monotone maps.

Graphical Representation of a Finite Poset

There is a simple intuitive tool for graphical representation of a finite poset, called Hasse diagram. For a poset $(X, \preceq)$, the *cover relation* $x \prec y$ (read as 'x is covered by y') is defined by setting:

$$\boxed{x \prec y \text{ if and only if } x \preceq y \text{ and there is no } z \text{ such that } x \preceq z \preceq y.}$$

For a given poset $(X, \preceq)$, its Hasse diagram is defined as the graph whose vertices are the elements of X and whose edges are those pairs $\langle x, y \rangle$ for which $x \prec y$. In the picture, the vertices of Hasse diagram are labeled by the elements of X and the edge $\langle x, y \rangle$ is drawn by an arrow going from x to y (or sometimes by an indirected line connecting x and y, but in this case the vertex y is displayed lower than the vertex x); moreover, the vertices are displayed in such a way that each line meets only two vertices.

Phonocentric Topology at the Level of Text

The usage of some kind of Hasse diagram named *Leitfaden* is widely spread in the mathematical textbooks to facilitate the understanding of logical dependence of its chapters or paragraphs. Mostly, the partially ordered set is constituted of all chapters of the book. So, in the *Local Fields* by J.-P. Serre, there is the following Hasse diagram:

$$
\begin{tikzcd}
 & 1 \arrow[d] & & & \\
 & 2 \arrow[d] & & & \\
 & 3 \arrow[d] & & 7 \arrow[dl] \arrow[dr] & \\
 & 4 \arrow[d] & 10 \arrow[dd] & & 8 \arrow[d] \\
 & 5 \arrow[dl] \arrow[dr] & & & 9 \arrow[d] \\
6 & & 12 \arrow[dr] & & 11 \arrow[dl] \\
 & & & 13 \arrow[dl] \arrow[dr] & \\
 & & 14 & & 15
\end{tikzcd}
$$

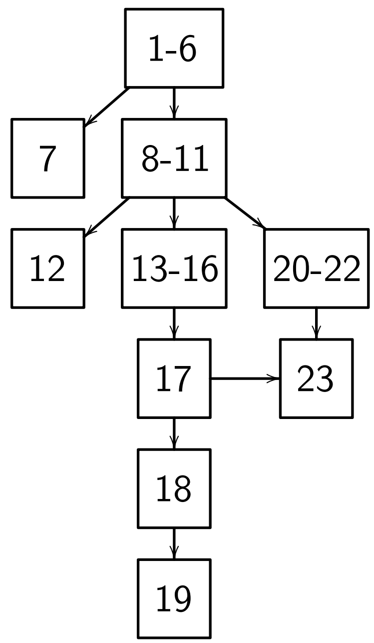
In *A Course in Mathematical Logic* by Yu. I. Manin, there is another diagram of chapters' interdependence:

$$
\begin{tikzcd}
 & 1 \arrow[d] & & & 5 \arrow[d] & & \\
 & 2 \arrow[dl] \arrow[d] \arrow[dr] & & & 6 \arrow[dll] \arrow[drr] & & \\
4 & 3 & 7 & & & & 8
\end{tikzcd}
$$

Yet another diagram, whose vertices are labeled with chapter number and title, is presented in *Symmetric bilinear forms* by J. Milnor and D. Husemoller:

$$
\begin{tikzcd}
 & \boxed{\text{I. Arbitrary commutative rings}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\boxed{\text{II. The ring of integers}} & \boxed{\text{III. Fields}} \arrow[d] & \boxed{\text{V. Miscellaneous examples}} \\
 & \boxed{\text{IV. Dedekind domains}} &
\end{tikzcd}
$$

These Leitfäden surely presuppose the linear reading of paragraphs within each chapter. However, its vertices may be “split” in order to draw the Leitfäden whose vertices are all the paragraphs like it’s done in *Differential forms in algebraic topology* by R. Bott and L. W. Tu:



This way, one may go further and do the next step. For every sentence x of a given admissible text $\boldsymbol{X}$, one can find basic open set of the kind of its least open neighborhood $\boldsymbol{U_x}$ in order to define the phonocentric topology at the semantic level of text (where points are sentences), and then to draw the Hasse diagram of the corresponding poset.

Phonocentric Topology at the Level of Sentence

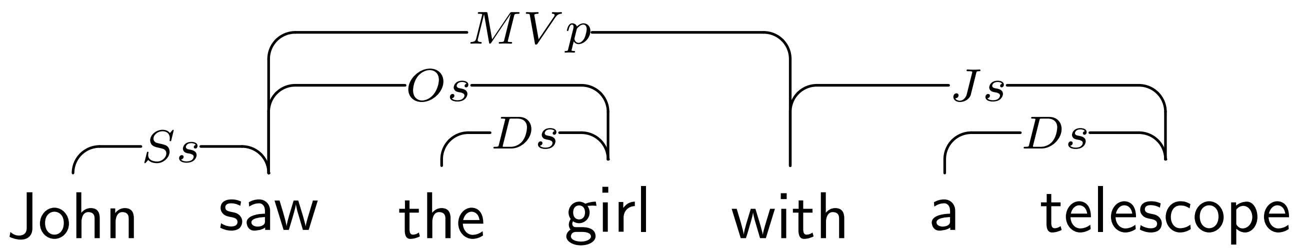
Likewise, we may go further by doing the next step. In order to define a phonocentric topology at the semantic level of sentence (where points are words), we must distinguish there the meaningful fragments those are similar to meaningful fragments at the level of text. Let x, y be any two words such that $x \preceq y$ in the specialization order at the level of sentence that is similar to the specialization order at the level of text. This relation $x \preceq y$ means that the word x should necessary be an element of the least part U_y required to understand the meaning of the word y in the interpreted sentence. So we have $x \leq y$ in the order of writing and there should be some syntactic dependence between them. It means that a grammar in which the notion of dependence between pairs of words plays an essential role will be closer to our topological framework than a grammar of Chomsky's type.

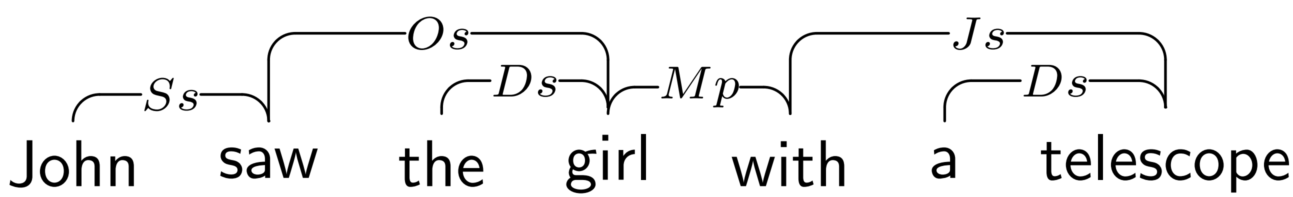
There are many formal grammars focused on links between words. We think that the theoretical approach of a *special link grammar* of D. Sleator and D. Temperley is more relevant to define a phonocentric topology at the level of sentence, because in whose formalism "[t]he grammar is distributed among the words" (1991, p. 3), and "the links are not allowed to form cycles" (1991, p. 13) comparing with *dependency grammars* which draw syntactic structure of sentence as a planar tree with one distinguished root word.

Given a sentence, the link grammar assigns to it a syntactic structure (linkage diagram) which consists of a set of labeled links connecting pairs of words. We use this diagram to define phonocentric topology on a sentence. To explain how to do it, let us consider a sentence

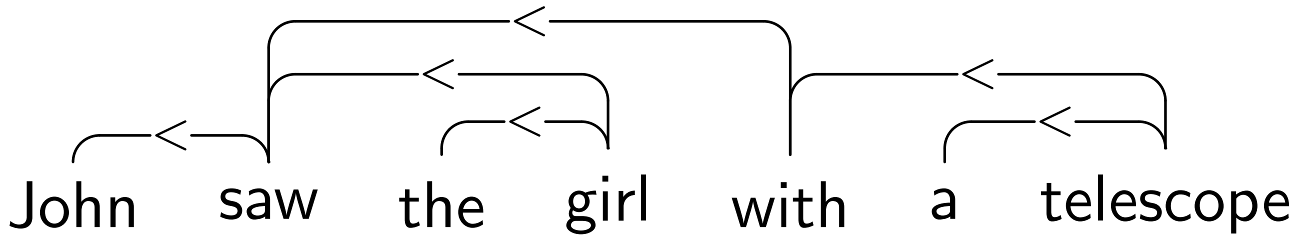
(1) John saw the girl with a telescope.

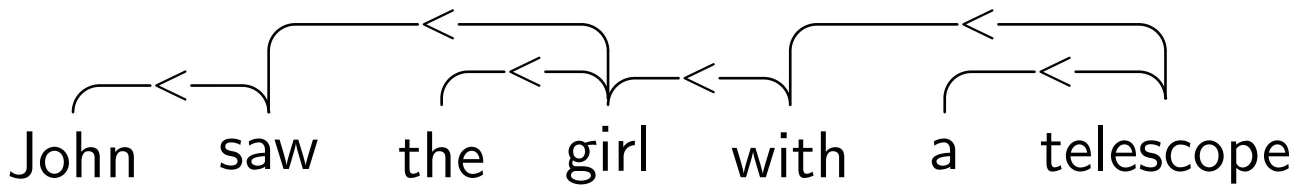
The analysis of this sentence by the *Link Parser 4.0* of D. Temperley, D. Sleator and J. Lafferty (2008) gives the following two linkage diagrams:





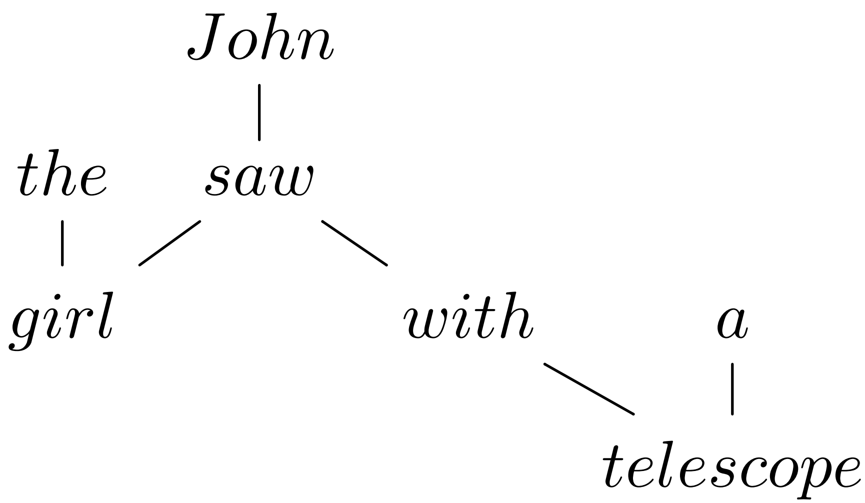
These two diagrams rewritten with arrows that indicate the direction of context dependence in which the connectors match (instead of connector names) have the following appearance:

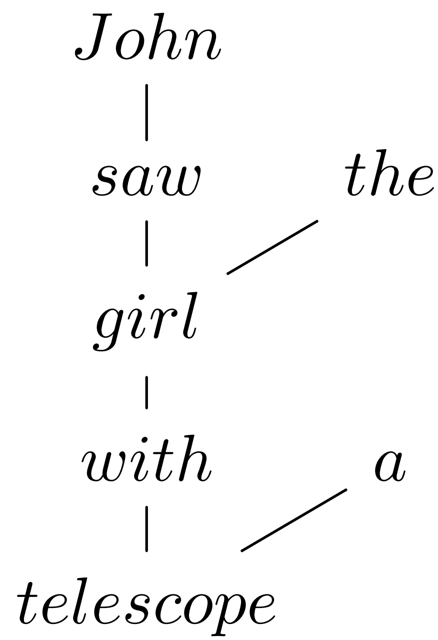




It is clear that the transitive closure $x \preceq y$ of this relation $<$ between pairs of words defines two partial order structures on the sentence (1).

In recovering the phonocentric topology from this partial order $\preceq$ as the topology with the basis constituted of all $\boldsymbol{U_x = \{z \colon z \preceq x\}}$, we can endow the sentence (1) with a phonocentric topology in two different ways. The Hasse diagrams of corresponding posets are:





To understand the sentence (1), the reader should do an *ambiguity resolution* when arriving to the word $\boldsymbol{x} =$“with” by choosing between basis sets:

$$\boldsymbol{U_x = \{\langle 1, \mathrm{John}\rangle, \langle 2, \mathrm{saw}\rangle, \langle 5, \mathrm{with}\rangle\}},$$

$$\boldsymbol{U_x = \{\langle 1, \mathrm{John}\rangle, \langle 2, \mathrm{saw}\rangle, \langle 3, \mathrm{the}\rangle, \langle 4, \mathrm{girl}\rangle, \langle 5, \mathrm{with}\rangle\}}.$$

In general case, the step by step choice of an appropriate context $\boldsymbol{U_x}$ for each word $\boldsymbol{x}$ results in endowing the interpreted sentence with a particular phonocentric topology among many possible.

Once the phonocentric topology and the specialization order are determined at a certain semantic level, the systematic interpretation of linguistic concepts in terms of topology and order and their geometric studies is a kind of $formal\ syntax$ at this semantic level, for the word $\sigma\acute{\upsilon}\nu\tau\alpha\xi\iota\zeta$ is derived from $\sigma\acute{\upsilon}\nu$ (together) and $\tau\acute{\alpha}\xi\iota\zeta$ (order).

Sheaves of Fragmentary Meanings

Let $\boldsymbol{X}$ be an admissible text endowed with a phonocentric topology, and let $\boldsymbol{\mathcal{F}}$ be an adopted sense of reading. In a Platonic manner, for each non-empty open (that is meaningful) part $\boldsymbol{U} \subseteq \boldsymbol{X}$, we collect in the set $\boldsymbol{\mathcal{F}}(\boldsymbol{U})$ all fragmentary meanings of this part $\boldsymbol{U}$ read in the sense $\boldsymbol{\mathcal{F}}$; also we define $\boldsymbol{\mathcal{F}}(\varnothing)$ to be a singleton $\boldsymbol{pt}$. Thus we are given a map

$$\boxed{\boldsymbol{U} \mapsto \boldsymbol{\mathcal{F}}(\boldsymbol{U})} \tag{1}$$

defined on the set $\mathfrak{O}(\boldsymbol{X})$ of all open sets in a phonocentric topology on $\boldsymbol{X}$.

Following the precept of hermeneutic circle **“to understand a part in accordance with the understanding of the whole”**, for each inclusion $U \subseteq V$ of non-empty opens, $\mathcal{F}$ assigns a restriction map $\mathbf{res}_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)$. Thus we are given a map

$$\boxed{\{U \subseteq V\} \mapsto \{\mathbf{res}_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)\}} \tag{2}$$

with the properties:

(i) identity preserving: $\mathbf{id}_V \mapsto \mathbf{id}_{\mathcal{F}(V)}$, for any open V;

(ii) transitivity: $\mathbf{res}_{V,U} \circ \mathbf{res}_{W,V} = \mathbf{res}_{W,U}$, for all nested opens $U \subseteq V \subseteq W$, which means that two consecutive restrictions may be done by one step.

As for the empty part $\varnothing$ of X, the restriction maps $\mathbf{res}_{\varnothing,\varnothing}$ and $\mathbf{res}_{V,\varnothing}$ with the same properties are obviously defined.

Any topological space $(X, \mathfrak{O}(X))$ gives rise to a category $\mathbf{Open}(X)$ with open sets $U \in \mathfrak{O}(X)$ as objects and their inclusions $U \subseteq V$ as morphisms.

From the mathematical point of view, the assignments (1) and (2) give rise to a presheaf $\mathcal{F}$ defined as a contravariant functor $\mathcal{F}\colon \mathbf{Open}(X) \to \mathbf{Sets}$, acting as $U \mapsto \mathcal{F}(U)$ on open sets, and acting as $U \subseteq V \mapsto \mathbf{res}_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)$ on their inclusions.

In sheaf theory, the elements of $\mathcal{F}(V)$ are called *sections* (*over* V); sections over the whole X are said to be *global*.

We consider the reading process of a fragment U as its covering by some family of subfragments $(U_j)_{j \in J}$ already read, that is $U = \bigcup_{j \in J} U_j$.

Following Quine (1977), there is no entity without identity. Any reasonable *identity criterion* should define two fragmentary meanings as equal globally if and only if they are equal locally. It motivates the following:

Claim S (Separability). *Let X be an admissible text, and let U be a fragment of X. Suppose that $s, t \in \mathcal{F}(U)$ are two fragmentary meanings of U and there is an open covering $U = \bigcup_{j \in J} U_j$ such that $\mathbf{res}_{U, U_j}(s) = \mathbf{res}_{U, U_j}(t)$ for all fragments U_j. Then $s = t$.*

In other words, a kind of local-global principle holds for the identity of fragmentary meanings so defined.

According to the precept of hermeneutic circle **"to understand the whole by means of understandings of its parts"**, a presheaf $\mathcal{F}$ of fragmentary meanings should satisfy the following:

Claim C (Compositionality). *Let* $\boldsymbol{X}$ *be an admissible text, and let* $\boldsymbol{U}$ *be a fragment of* $\boldsymbol{X}$*. Suppose that* $\boldsymbol{U} = \bigcup_{j \in J} \boldsymbol{U_j}$ *is an open covering of* $\boldsymbol{U}$*; suppose we are given a family* $(\boldsymbol{s_j})_{j \in J}$ *of fragmentary meanings,* $\boldsymbol{s_j} \in \mathcal{F}(\boldsymbol{U_j})$ *for all fragments* $\boldsymbol{U_j}$*, such that* $\mathbf{res}_{U_i, U_i \cap U_j}(\boldsymbol{s_i}) = \mathbf{res}_{U_j, U_i \cap U_j}(\boldsymbol{s_j})$*. Then there exists some meaning* $\boldsymbol{s}$ *of the whole fragment* $\boldsymbol{U}$ *such that* $\mathbf{res}_{U, U_j}(\boldsymbol{s}) = \boldsymbol{s_j}$ *for all* $\boldsymbol{U_j}$*.*

Thus a family of locally compatible fragmentary meanings may be composed in a global one.

Thus any presheaf of fragmentary meanings defined as above should satisfy the claims (S) and (C), and so it is a *sheaf* by the very definition. It motivates the following:

Frege's Generalized Compositionality Principle. *A presheaf of fragmentary meanings naturally attached to any sense* (*mode of reading*) *of an admissible text is really a sheaf; its sections over a fragment of the text are its fragmentary meanings; its global sections are the meanings of the text as a whole.*

The claim (S) implies the meaning $\boldsymbol{s}$, whose existence is claimed by (C), to be unique as such.

Category of Schleiermacher

We suppose that any part of text which is meaningful in one sense of reading should remain meaningful after the passage to any other sense of reading. We suppose also that the transfer from the understanding in one sense (e.g., literal) to the understanding in another sense (e.g., moral) commutes with the restriction maps. Formally, this idea is well expressed by the notion of *morphism* of the corresponding sheaves $\phi\colon \mathcal{F} \mapsto \mathcal{F}'$ defined as a family of maps $\phi(V)\colon \mathcal{F}(V) \to \mathcal{F}'(V)$, such that the following diagrams commute for all opens $U \subseteq V$ of X:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{F}'(V) \\ {\scriptstyle \mathrm{res}_{V,U}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{res}'_{V,U}} \\ \mathcal{F}(U) & \xrightarrow[\phi(U)]{} & \mathcal{F}'(U) \end{array} .$$

So, for an admissible text X, the data of sheaves of fragmentary meanings and its morphisms constitutes a category $\mathbf{Schl}(X)$ in a strict mathematical sense, we call *category of Schleiermacher*.

The category of Schleiermacher supplies a formal framework for the part-whole structure of natural language text understanding established by Schleiermacher, and called later by Dilthey as the "hermeneutic circle".

Note that the class of objects in the category of Schleiermacher $\mathbf{Schl}(\boldsymbol{X})$ is not limited to a modest list of sheaves corresponding to $\mathit{literal}$, $\mathit{allegoric}$, moral, $\mathit{psychoanalytical}$ senses mentioned above. In the process of text interpretation, the reader's semantic intentionality changes from time to time, with the result that there is some compositionality (or gluing) of sheaves which are defined only locally. There is a standard way to name the result of such a gluing as, for example, in the case of $\mathit{Freudo\text{-}Marxist}$ sense. This compositionality of senses is discussed in details in (Prosorov 2008).

Contextuality in Sheaf-Theoretic Framework

So far, we have considered only the meanings of open sets in the phonocentric topology. It may happen that a particular point (sentence) $x \in X$ constitutes an open one-point set $\{x\}$, and so the set $\mathcal{F}(\{x\})$ of its fragmentary meanings have yet been defined. But in general, not every singleton is open in T_0-topology. Now we describe how to define the meanings of each point in the phonocentric topology.

Two fragmentary meanings $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$ are said to *induce the same contextual meanings of a sentence* $x \in U \cap V$ if there exists some open neighborhood W of x, such that $W \subseteq U \cap V$ and $\mathbf{res}_{U,W}(s) = \mathbf{res}_{V,W}(t) \in \mathcal{F}(W)$. This relation is clearly an equivalence relation. Any equivalence class of fragmentary meanings agreeing in some open neighborhood of a sentence x is natural to define as a *contextual meaning* of x. The equivalence class defined by a fragmentary meaning s is called a *germ* at x of this s and is denoted by $\mathbf{germ}_x(s)$.

Paraphrasing Frege, we say: **“Never ask for the meaning of a sentence in isolation, but only in the context of some fragment of a text”**. In sheaf-theoretic terms, it gives the following:

Frege’s Generalized Contextuality Principle. *Let* $\mathcal{F}$ *be an adopted sense of reading of a fragment* U *of an admissible text* X*. For a sentence* $x \in U \subseteq X$*, its contextual meaning is defined as* $\mathbf{germ}_x(s)$*, that is as a germ at* x *of some fragmentary meaning* $s \in \mathcal{F}(U)$*. The set* $\mathcal{F}_x$ *of all contextual meanings of a sentence* $x \in X$ *is defined as the inductive limit* $\mathcal{F}_x = \varinjlim(\mathcal{F}(U), \mathbf{res}_{V,U})_{U,V \in \mathfrak{O}(x)}$*, where* $\mathfrak{O}(x)$ *is a set of all open neighborhoods of* x*.*

Note that for an open singleton $\{x\}$, we may canonically identify its contextual meanings with the fragmentary ones, that is $\mathcal{F}_x = \mathcal{F}(\{x\})$.

Bundles of Contextual Meanings

For the coproduct $\boldsymbol{F} = \bigsqcup_{\boldsymbol{x} \in \boldsymbol{X}} \boldsymbol{\mathcal{F}_x}$, we define now a *projection* map $\boldsymbol{p} \colon \boldsymbol{F} \to \boldsymbol{X}$ by setting $\boldsymbol{p}(\mathbf{germ}_{\boldsymbol{x}} \boldsymbol{s}) = \boldsymbol{x}$.

Every fragmentary meaning $\boldsymbol{s} \in \boldsymbol{\mathcal{F}}(\boldsymbol{U})$ determines a genuine function $\dot{\boldsymbol{s}} \colon \boldsymbol{x} \mapsto \mathbf{germ}_{\boldsymbol{x}} \boldsymbol{s}$ to be well-defined on $\boldsymbol{U}$. It gives rise to a *functional representation* $\boldsymbol{s} \mapsto \dot{\boldsymbol{s}}$ of fragmentary meanings which clarifies the nature of abstract entity $\boldsymbol{s} \in \boldsymbol{\mathcal{F}}(\boldsymbol{U})$ as being represented by a genuine function $\dot{\boldsymbol{s}}$.

We define the topology on $\boldsymbol{F}$ by taking as a basis of open sets all the image sets $\dot{\boldsymbol{s}}(\boldsymbol{U}) \subseteq \boldsymbol{F}$, $\boldsymbol{U} \in \boldsymbol{\mathfrak{O}}(\boldsymbol{X})$. Given a fragment $\boldsymbol{U} \subseteq \boldsymbol{X}$, a continuous function $\boldsymbol{t} \colon \boldsymbol{U} \to \boldsymbol{F}$ such that $\boldsymbol{t}(\boldsymbol{x}) \in \boldsymbol{p}^{-1}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{U}$ is called a *cross-section*. The topology defined on $\boldsymbol{F}$ makes $\boldsymbol{p}$ and every cross-section of the kind of $\dot{\boldsymbol{s}}$ to be continuous. So we have defined topological spaces $\boldsymbol{F}$, $\boldsymbol{X}$ and a continuous map $\boldsymbol{p} \colon \boldsymbol{F} \to \boldsymbol{X}$.

In topology, this data (F, p) is called a *bundle over the base space* X. A *morphism* of bundles from $p\colon F \to X$ to $q\colon G \to X$ is a continuous map $h\colon F \to G$ such that $q \circ h = p$, that is, the following diagram commutes:

$$\begin{tikzcd} F \arrow[rr, "h"] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\ & X & \end{tikzcd} .$$

We have so defined a category of bundles over X. A bundle (F, p) over X is called *étale* if $p\colon F \to X$ is a local homeomorphism. It is immediately seen that a bundle of contextual meanings $(\bigsqcup_{x \in X} \mathcal{F}_x, p)$ constructed as above from a given sheaf $\mathcal{F}$ of fragmentary meanings is étale. Thus, for an admissible text X, we have defined the category $\mathbf{Context}(X)$ of étale bundles (of contextual meanings) over X as a framework for the Frege's generalized contextuality principle at the level of text.

Frege Duality

The fundamental theorem of topology states that there is a *duality* between the category of sheaves and the category of étale bundles (Lambek & Scott 1986; Mac Lane & Moerdijk 1992). Transferred to linguistics, this important result yields at the level of text the following

Theorem (Frege Duality). *The generalized compositionality and contextuality principles are formulated in terms of categories that are in natural duality*

$$\mathbf{Schl}(\boldsymbol{X}) \underset{\boldsymbol{\Gamma}}{\overset{\boldsymbol{\Lambda}}{\rightleftarrows}} \mathbf{Context}(\boldsymbol{X})$$

established by the section-functor $\boldsymbol{\Gamma}$ *and the germ-functor* $\boldsymbol{\Lambda}$*, which are the pair of adjoint functors.*

Due to the functional representation $s \mapsto \dot{s}$, Frege duality is of a great theoretical importance because it allows us to understand a fragmentary meaning s as a genuine function $\dot{s}\colon x_i \mapsto \mathbf{germ}_{x_i} s$ which assigns to each sentence $x_i \in U$ its contextual meaning $\mathbf{germ}_{x_i} s$, and which is continuous on U. It allows us to develop a kind of *inductive* or *dynamic theory of meaning* (Prosorov 2005b, 2008) describing how in reading of the text $X = (x_1, x_2, x_3 \ldots x_n)$ the understanding process runs in a discrete time $i = 1, 2, 3 \ldots n$ as a sequence of grasped contextual meanings $(\dot{s}(x_1), \dot{s}(x_2), \dot{s}(x_3) \ldots \dot{s}(x_n))$ that gives a genuine function $\dot{s}$ on X representing some $s \in \mathcal{F}(X)$ which is one of meanings of the whole text X interpreted in the sense $\mathcal{F}$.

Moreover, it gives a solution to an old problem concerning delicate relations between Frege's compositionality and contextuality principles, in revealing that the acceptance of one of them implies the acceptance of the other (Prosorov 2003, 2008).

Sheaf-Theoretic Dynamic Semantics

For a given text $X = (x_1, \dots x_k, \dots x_n)$ interpreted in a sense $\mathcal{F}$, we have to describe how a reader finally grasps some global section $s \in \mathcal{F}(X)$ of a sheaf $\mathcal{F}$ of fragmentary meanings.

We consider first a particular case of reading from the very beginning of an admissible text $X = (x_1, x_2, x_3 \dots x_n)$ whose size allows us to finish reading at one sitting. The general case will be reduced to this particular case by means of the generalized Frege's compositionality principle.

The first sentence x_1 in the order $\leq$ of writing must obviously be understood in the context which consists of its own data. This means that a first sentence x_1 constitutes an open one-point set $\{x_1\}$. Thus $U_{x_1} = \{x_1\}$ and therefore sentence x_1 should be a minimal element in the specialization order, and therefore $\mathcal{F}_{x_1} = \mathcal{F}(\{x_1\})$.

This means that having grasped a contextual meaning of $\boldsymbol{x_1}$ is equivalent to having grasped a fragmentary meaning of the fragment $\{\boldsymbol{x_1}\}$ reduced to one sentence $\boldsymbol{x_1}$. It is obviously equivalent to having grasped a global meaning of this sentence $\boldsymbol{x_1}$ at the semantic level of a sentence considered as a sequence of words. We understand first the theme of this sentence $\boldsymbol{x_1}$, and then we understand the rheme as being what is said in the sense $\boldsymbol{\mathcal{F}}$ concerning to this theme. Thus we have done a descent from the level of text to the level of sentence. This part of our reasoning is the *basis of induction*.

Let us now do the *induction step*. Let us suppose that we have read and understood the text $\boldsymbol{X}$ in the sense $\boldsymbol{\mathcal{F}}$ from the beginning $\boldsymbol{x_1}$ up to the sentence $\boldsymbol{x_k}$, $\boldsymbol{1 < k < n}$. That is, we suppose that we have already endow $\boldsymbol{X = (x_1, \ldots x_k)}$ with a phonocentric topology and we have built a suite $\boldsymbol{(s_{x_1}, \ldots s_{x_k})}$ of contextual meanings of sentences of the open set $\boldsymbol{U = (x_1, \ldots x_k)}$ of a given text $\boldsymbol{X = (x_1, \ldots x_k, \ldots x_n)}$. The suite $\boldsymbol{(s_{x_1}, \ldots s_{x_k})}$ of contextual meanings is a continuous function which represents some fragmentary meaning $\boldsymbol{s \in \mathcal{F}(U)}$.

We consider the interpretation process at its $\boldsymbol{(k+1)}$-th step as the choice of an appropriate context $\boldsymbol{U_{x_{k+1}}}$ for $\boldsymbol{x_{k+1}}$ that endows the interval $\boldsymbol{(x_1, \ldots x_{k+1})}$ with a particular phonocentric topology among many possible, and as an extension of the function $\boldsymbol{s}$ defined on the open $\boldsymbol{(x_1, \ldots x_k)}$ to a function defined on the open $\boldsymbol{(x_1, \ldots x_{k+1})}$.

The phrase x_{k+1} is read in the context of the fragment $(x_1, \ldots x_{k+1})$ of the text X. This neighborhood is the most large context as possible we dispose to understand the contextual meaning of x_{k+1}.

To grasp the same contextual meaning of x_{k+1}, it suffices to understand only its minimal neighborhood $U_{x_{k+1}}$. It may be two cases:

Case 1°: It may happen that the understanding of the sentence x_{k+1} is independent of the understanding of $U = (x_1, \ldots x_k)$, for it constitutes alone its own context $\{x_{k+1}\} = U_{x_{k+1}}$ because there is here a turning point in the narrative, what may be confirmed by various morphologic markers such as the beginning of a new chapter, etc. The contextual meaning $s_{x_{k+1}}$ is defined at a point x_{k+1}, and as such it is a continuous function because $\{x_{k+1}\}$ constitutes an open set.

The process of understanding of x_{k+1} is therefore conducted in the same way as that one of the first sentence x_1 whose case we have considered above as the basis of induction.

Note that the interval $U = (x_1, \dots x_k)$ is open. We can therefore extend the suite $(s_{x_1}, \dots s_{x_k})$ which is a continuous function on $U = (x_1, \dots x_k)$, to the suite $(s_{x_1}, \dots s_{x_{k+1}})$ which is continuous on $(x_1, \dots x_{k+1})$.

Case 2°: The understanding of x_{k+1} is reached with the support of the understanding of the preceding sentences of the interval $U = (x_1, \dots x_k)$. Not all the sentences in $U = (x_1, \dots x_k)$ are required to determine the understanding of x_{k+1}, but only some subsequence of U. Let it be the minimal such a subsequence $V \subseteq U$. We define a phonocentric topology on $(x_1, \dots x_{k+1})$ by defining $U_{x_{k+1}} = V \cup \{x_{k+1}\}$.

Now one can imagine that we do a transformation of the text so that each sentence of V, except the first in the order $\leq$ of writing, begins with "and then" which assembles it to the preceding sentence as its extension in a compound sentence.

So, this transformation reduces all sentences of $\boldsymbol{V}$ into a single lengthy sentence $\boldsymbol{x}$ that forms a thematic context concerning which was written the sentence $\boldsymbol{x_{k+1}}$ to express its communicative content. Finally, we join $\boldsymbol{x}$ by “and then” with the beginning of the sentence $\boldsymbol{x_{k+1}}$ that transforms it to another sentence $\boldsymbol{x'_{k+1}}$.

In the text $(\boldsymbol{x_1, \dots x_k, x'_{k+1}})$ so defined, the sentence $\boldsymbol{x'_{k+1}}$ constitutes an open one-point set $\{\boldsymbol{x'_{k+1}}\}$ which is understandable in the context of its own data. A contextual meaning of $\boldsymbol{x'_{k+1}}$ is grasped when we understand the rheme of $\boldsymbol{x_{k+1}}$ as being what is said in the sense $\mathcal{F}$ concerning the theme of $\boldsymbol{x_{k+1}}$ in the context defined by the sentences of V. But obviously the contextual meaning of $\boldsymbol{x'_{k+1}}$ is the same as the contextual meaning of $\boldsymbol{x_{k+1}}$. So we have extended the sequence of contextual meanings $(\boldsymbol{s_{x_1}, \dots s_{x_k}})$ to the sequence $(\boldsymbol{s_{x_1}, \dots s_{x_{k+1}}})$.

Thus we have done a descent from the level of text to the level of sentence. This trick is inspired by Russell's work "How I write" (1983), where he discuss advises he received at the beginning of his carrier of a writer.

In the general case, we consider the reading process of a fragment U as its covering by some family of meaningful fragments $(U_j)_{j\in J}$ already read, that is $U = \bigcup_{j\in J} U_j$ is an open covering.

Let X be an admissible text, and let $X = \bigcup_{j\in J} U_j$ be an open covering of X by read fragments $(U_j)_{j\in J}$. Let us suppose given a family $(s_j)_{j\in J}$, where $s_j \in \mathcal{F}(U_j)$ such that all genuine functions $\dot{s}_j\colon x \mapsto \mathbf{germ}_x s_j$ of the corresponding family $(\dot{s}_j)_{j\in J}$ are pairwise compatible, that is $\dot{s}_i\,\big|_{U_i\cap U_j}(x) = \dot{s}_j\,\big|_{U_i\cap U_j}(x)$ for all $x \in U_i \cap U_j$.

Let us define the function t on $X = \bigcup_{j\in J} U_j$ as $t(x) = \dot{s}_j(x)$ if $x \in U_j$ for some j. The Frege duality theorem states that $t = \dot{s}$ where $s \in \mathcal{F}(X)$ is a composition of the family $(s_j)_{j\in J}$, whose existence is guaranteed by the generalized Frege's compositionality principle.

The formalization of the interpretation process as an extension of a function introduces a *dynamic* view of semantics, and its theory deserves the term *inductive* because the domain of a considered function is naturally endowed with two order structures: the linear order of writing $\leq$ and the specialization order $\preceq$ of context-dependence.

We have outlined so a sheaf-theoretic framework for dynamic semantics of natural language, where the understanding of a text $\boldsymbol{X}$ in the sense $\boldsymbol{\mathcal{F}}$ is described as a process of step by step grasping for each sentence $\boldsymbol{x_i}$ of only one contextual meaning $\dot{\boldsymbol{s}}(\boldsymbol{x_i})$ from the fiber $\boldsymbol{\mathcal{F}_{x_i}}$ lying over $\boldsymbol{x_i}$ in the étale bundle $\mathbf{Context}(\boldsymbol{X})$ of contextual meanings.

Sheaf-Theoretic Semantics

Thus the true object of study in the natural language semantics should be a couple $(\boldsymbol{X}, \boldsymbol{\mathcal{F}})$ made up of an admissible text $\boldsymbol{X}$ and a sheaf $\boldsymbol{\mathcal{F}}$ of its fragmentary meanings; any such a couple is called *textual space*. But this representation is possible only in the realm of a language following the famous slogan of Wittgenstein "to understand a text is to understand a language". Rigorously, this claim may be formulated in the frame of category theory. Likewise the present sheaf-theoretic formal semantics describes a natural language in the *category of textual spaces* **Logos**. The objects of this category are couples $(\boldsymbol{X}, \boldsymbol{\mathcal{F}})$, where $\boldsymbol{X}$ is a topological space naturally attached to an admissible text and $\boldsymbol{\mathcal{F}}$ is a sheaf of fragmentary meanings defined on $\boldsymbol{X}$; the morphisms are couples $(\boldsymbol{f}, \boldsymbol{\theta})\colon (\boldsymbol{X}, \boldsymbol{\mathcal{F}}) \to (\boldsymbol{Y}, \boldsymbol{\mathcal{G}})$ made up of a continuous map $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ and a $\boldsymbol{f}$-morphism of sheaves $\boldsymbol{\theta}$ which respects the concerned sheaves, i.e., $\boldsymbol{\theta}\colon \boldsymbol{\mathcal{G}} \to \boldsymbol{f}_*\boldsymbol{\mathcal{F}}$, where $\boldsymbol{f}_*$ is a well-known *direct image* functor.

Given any admissible text $\boldsymbol{X}$ considered as fixed forever, it yields very naturally a full subcategory $\mathbf{Schl}(\boldsymbol{X})$ in the category $\mathbf{Logos}$ of all textual spaces. This category of Schleiermacher $\mathbf{Schl}(\boldsymbol{X})$ describes the exegesis of some particular text as, for example, Sacred Scripture.

The dynamic theory of meaning based on Frege duality, and also the different categories and functors related to discourse and text interpretation processes are the principal objects of study in the *sheaf-theoretic formal semantics* as we understand it.

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