FORMAL HERMENEUTICS AND FREGE DUALITY

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ABSTRACT

In this article we continue to develop the formal hermeneutics intended to be a kind of discourse interpretation theory. Our approach will provide the common categorical framework for generalized Frege's compositionality and contextuality principles. Thus for any given admissible text X, we introduce the Schleiermacher category $\mathbf{Schl}(X)$ of sheaves of fragmentary meanings in terms of which the general compositionality principle is formulated. We also introduce another category $\mathbf{Context}(X)$ of étale bundles of contextual meanings in terms of which the general contextuality principle is formulated. We have considered these categories in our previous works [1], [2], [3]. This categorical point of view leads to the important Frege Duality obtained by the same procedure as many of well-known important classic dualities and defined as an equivalence of categories

$$\mathbf{Schl}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Context}(X)$$

established by the well-known section-functor Γ and germ-functor Λ. Moreover, this equivalence gives rise to some kind of functional representation for any fragmentary meaning which allows to establish some kind of inductive theory of meaning describing the creative process of text understanding. This inductive theory of meaning based on Frege Duality, and also the different categories and functors related to discourse interpretation are the principal objects of study in the formal hermeneutics as we understand it.

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Key words and phrases: formal hermeneutics, hermeneutic circle, admissible text, fragmentary meaning, contextual meaning, Frege's principle of compositionality of meaning, Frege's principle of contextuality, Frege Duality, category, functor, phonocentric topology, logocentric topology, sheaf, bundle, étale bundle, textual space, formal discourse scheme.

Contents

Introduction

In recent years, the discourse interpretation has become a field of intensive investigations in logic, linguistics and the philosophy of language. Despite the great progress in this area, the central problem about the key theoretical structures the discourse interpretation theory should be based upon remains still unsettled. The object of this work is to give an outline of some theory of discourse interpretation named as *formal hermeneutics* and intended to reveal the existence of mathematical structures that underlie the process of discourse or text understanding. So the term formal hermeneutics does not mean hermeneutics of any formal system but concerns with the application of formal mathematical methods to analysis of natural language understanding. This article develops some ideas from our previous works [1], [2], [3]. Our approach allows to generalize the classic Frege's compositionality and contextuality principles. So revised, these principles are reconciled in some dual equivalence called Frege Duality between the categories that provide a frame for their explicit formulation.

The classic approaches to semantics of natural language, both the philological ones and their mathematical formalizations, are based on the implicit premise that any language is nothing more than the set of all its correct sentences (and yet only of all its propositions, i. e. the sentences having truth-value). These approaches are very restrictive and yet inadequate to everyday human practice of language communication. When a person wants to express his thoughts to somebody, he needs to utter some discourse or to write some text, and to understand this data is quite another thing than to understand the set of all sentences it was made up. This is why the semantics of natural language should be defined as a discipline studying the discourse and text understanding. Since antiquity, there exists a concept of discourse interpretation that goes back to Greek mythology there Hermes interprets the cryptic messages of the gods to mortals. Derived from the Greek verb *hermeneuein* which means "to make clear and understandable", the term *hermeneutics* was first used in the 17th century to mean scriptural exegesis. The Protestant Reformation had a need in the interpretation of Scripture based on the selfsufficiency of the holy text. With the plurality of possible interpretation, it results in a need to establish the principles of correct interpretation. As the theory of textual interpretation, hermeneutics began with biblical exegesis and was closely allied to philology. The scope of hermeneutics was widely extended in the works of Protestant theologian Friedrich Schleiermacher who created a general theory of interpretation applicable not only to religious texts but also to a great variety of secular ones. Schleiermacher formulated what is known as the *hermeneutic circle*: the part is understood in terms of the whole and the whole in terms of the parts. This part-whole structure in the understanding, he claimed, is principal in matter of interpretation of any written expression of human phenomena. The theoretical principle of *hermeneutic circle* is a precursor to these of *compositionality* and *contextuality* formulated later in 19th century. *Grosso modo*, the hermeneutics as a discourse interpretation theory is based on the hermeneutic circle principle in according to which the meaning of the whole text is sought in terms of the meanings of its constitutive parts. This is a sort of compositionality that is meant implicitly to hold at the level of text. In any way, the usual semantics at the level of sentence is based on the implicit use of compositionality principle in according to which the meaning of the whole sentence is a function of the meanings of its constitutive parts. So the hermeneutics may be defined as semantics at the level of text which covers a usual semantics at the level of sentence. It is a reason why we have called *formal hermeneutics* our sheaf-theoretical approach to discourse interpretation theory which provides a mathematical account of the text understanding process while rejecting the attempt to codify interpretative practice as a kind of calculus. We consider the meaning as being composed during the interpretative process, contrary e.g. to propositional calculus or to Gentzen's natural inference theory there one yields a theorem D as a final sequent in some *proof* described as a finite series $C_1, \dots C_n$ of sequents such that $C_n = D$. The understanding of a text is not postponed to a final sentence, but it is present at all semantic levels during the reading process.

1. Compositionality

The logically minded linguists relate nowadays the compositionality principle at the level of sentence with the name of Gottlob Frege. This principle in its *standard interpretation* is a theoretical basis for a Montague grammar, Generalized phrase structure grammar, Categorial grammar and Lexicalized tree adjoining grammar; these theories propose the different notions of meaning but a meaning is assigned to words in isolation as T. M. V. Janssen claims in [4, p. 116]: "A technical description of the standard interpretation is that syntax and semantics are algebras, and meaning assignment is a homomorphism from syntax to semantics". To apply such a homomorphism (as a function) at some element of its domain, one needs to neglect the plurality of meanings of a word. But every dictionary confirms the contrary.

From a general point of view, any semantic theory would explain how the local understandings (the meanings of the constitutive parts) produce the global understanding (the meaning of the whole). In other words, how the local data gives the global one! The modern mathematics knows such an engine under the name of *sheaf*! But the *general hermeneutics* of Schleiermacher disposes a key theoretical notion closely related to that of sheaf and named as hermeneutic circle that prescribes: 1° to understand the whole text by means of understanding of its parts and 2° to understand any part of text in accordance with the understanding of the whole text!

First of all, we need to define rigorously what is a text in our formalism. Clearly any text is not just a set of its sentences as the sentence is not a set of its words. Important is the order they ought to be read. In addition, the same words may occur in several places of one sentence and the same sentences may occur in several places of one text. So from a mathematical point of view, we ought to consider a given sentence as a sequence of its words and a given text as a sequence of its sentences. Likewise any part of a considered text is simply a subsequence of a given sequence. Any mathematical structure on a given text, such as topology, sheaves etc., is to be defined on the *graph* of the corresponding sequence viewed as a function on some interval of natural numbers. Henceforth, we shall simply identify a given text with the graph of its corresponding sequence.

We want to stress, from the very beginning, that we distinguish the notions of *sense, meaning* and that of *reference*. These notions serve for the purposes of formal semantic analysis of text understanding. This triad of concepts formalizes a certain distinction that seems to appear in various forms all over the history of traditional logic and semantics. There is a difference between them and a classic Frege's *Sinn/Bedeutung* distinction intended to solve the problems which differs from those we try to solve. Our general aim is the construction of the concepts suitable as instruments for rigorous formal analysis of discourse interpretation process there the celebrated Frege's compositionality and contextuality principles are involved. Following Janssen [*Op. cit.*], the history of Frege's principles is rather difficult to trace and study because he had never tried to formalize them in his writings. We have a firm conviction that these both Frege's principles are closely related to that of hermeneutic circle which is of fundamental importance in the biblical exegesis. Our aim is not to propose some competitive concepts vs. Frege's *Sinn/Bedeutung* distinction but to find some adequate distinction for precise formulation and generalization of the classic Frege's compositionality and contextuality principles.

Given a text X in natural language, we consider the process of reading of its fragment U to be successful if the reader has understood one of its *meaning s*. But it depends of so many factors such as personality of reader, situation of reading, many kinds of presuppositions summed up in the reader's attitude, etc., which we call *sense* or *mode of reading*; every reading is only an interpretation there are involved the historicity of the reader and the historicity of the text, whence the multiplicity of meanings for any meaningful fragment of text.

Our approach to the notion *sense* (or *sens* in French) may be considered as the secular remake of exegetical approach to this notion in the medieval theology. The Fathers of the Church have distinguished the four senses of Sacred Scripture: "littera gesta docet, quid credas allegoria, moralis quid agas, quo tendas anagogia". In other words, our approach defines the term *sense* as a kind of semantic orientation in the interpretative process which relates to the totality of message to understand, as some *mode of reading*. At the level of text, it may be literal, allegoric, moral, naïve, psychoanalytical, etc. At the level of sentence, it may be literal or metaphoric. At the level of syntagm or word, it may be literal or figurative.

In our approach to the term *meaning* (or *signification* in French), the reader grasps it in result of the interpretative process guided by some *mode of reading* or *sense* adopted in accordance with his attitude and based on the linguistic competence, which is rooted in the social practice of communication with others using the medium of language. Note that following this terminology, we can read two different texts in one and the same sense (moral for example) to realize in result that we have grasped their different meanings. Likewise for historical, psychoanalytical, etc., senses. It seems that these acceptances of key terms *sense, meaning* are in accordance with its everyday usage as common words (likewise for the French terms *sens* and *signification*).

As for term *sense*, it should be mentioned that in French the word "sens" literally equals to "direction" and as figurative it may be *littéral*, *strict*, *large*, *naïf*, *bon*, *platonicien*, *leibnitzéen*, *frégéen*, *kripkéen*, etc. In English, a figurative sense may also be *literal*, *narrow*, *wide*, *naïve*, *common*, *platonistic*, *Leibnizian*, *Fregean*, *Kripkean*, etc. In this usage, the term *sense* deals with the totality of discourse, text, expression or word and involves our subjective premises that what is to be understood constitutes a meaningful whole. In this acceptance, the term *sense* or *mode of reading* concerns the reader's interest in the subject matter of text; it's a kind of questioning that allows a reader to enter into the dialogue with the author.

As for term *meaning*, it has been used in various different ways in different theories. To avoid the possible misunderstanding from the very beginning, we would like to precise our acceptance of these key terms *sense, meaning* and give a detailed account of their distinction. Now let us consider the Frege's opinions. In the famous work of 1892 *Über Sinn und Bedeutung* he has introduced the distinction between *Sinn* and *Bedeutung* essential for his further research. These terms are translated in English (and in French) quite differently following the adopted point of view. So A. Church translates *Sinn* with "sense" but B. Russell uses "meaning".[1] To translate another Frege's term *Bedeutung*, Church and Russell use "denotation";[2] it is likewise for *Bedeutung* in the article of E. N. Zalta in *Stanford Encyclopedia of Philosophy on the Web*[3] but he translates *Sinn* with "sense". Whereas in the article[4] analyzing Frege's and Russell's views, K. Bach translates Frege's *Sinn* with *sense* and *Bedeutung* with *reference*. Likewise in [4], T. M. V. Janssen translates *Sinn* with *sense* but he uses "meaning" for *Bedeutung* following Long and White.

There is a similar discordance for the French translations. So *Sinn* is translated in French sometimes with "sens" as it done by C. Imbert[5] and by J.-F. Malherbe[6], or sometimes with *signification* as translated by F. Rastier in [5, ch. I, s. 1]. For *Bedeutung*, J.-F. Malherbe and some French authors use "référent" with the object denoted, and "référence" or "dénotation" with the relation of denoting.[7]

Perhaps it explains that the position of Frege on what is *Sinn* is not univocal. There are many theses about *Sinn* that Frege asserted in his writings. To understand his opinions about *Sinn*, we shall quote some of his definitions in French and/or in English translations. Following one definition:[8]

> Il est assez naturel d'associer à un signe (nom, groupe de mots, caractères), outre ce qu'il désigne et qu'on pourrait appeler son référent (*Bedeutung*), ce que je voudrais appeler le sens (*Sinn*) du signe où est contenu le mode de donation de l'objet.

So he maintains here that, e. g., for some group of words their *Sinn* is the mode of presentation of reference. Following another definition:[9]

> Avec le signe (qui est la graphie du nom), on exprime le sens (*Sinn*) du nom propre et on en désigne le référent (*Bedeutung*).

This acceptance of *Sinn* corresponds perhaps more closely than that of *Bedeutung* to ordinary acceptance of common English word "meaning" (and to ordinary acceptance of common French word "signification"); it comprises, not the object named by a name, but rather what we *do understand* by this name. If we have grasped it we are able to say about any object whether or not it is the reference of the name. Again in the work *Über Begriff und Gegenstand* of 1892, Frege writes: "Il ne reste qu'à inviter par quelque signe le lecteur ou l'auditeur à mettre sous le mot ce que l'on veut lui faire entendre".[10] In this acceptance, the meaning of an expression is the content that the reader or listener grasps when he understands it; and this can be done regardless of the ontological status of its referent. So Frege writes that "the thought remains the same whether 'Odysseus' has reference or not".[11]

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[1] R. Carnap, *Meaning and Necessity*, Chicago, The University of Chicago Press, 1948, p.118.

[2] R. Carnap, *Op. cit.*, p. 118.

[3] Zalta, Edward N., "Gottlob Frege", *The Stanford Encyclopedia of Philosophy (Spring 2002 Edition)*, Edward N. Zalta (ed.), URL=http://plato.stanford.edu/archives/spr2002/entries/frege/

[4] K. Bach, *Comparing Frege and Russell*, URL=http://online.sfsu.edu/ kbach/FregeRus.html

[5] Frege, G., *Ecrits logiquees et philosophiques*, Paris, Seuil, 1968.

[6] J.-F. Malherbe, *Epistémologies Anglo-Saxonnes*, Namur, Presse universitaires de Namur, 1981, p.15.

[7] J.-F. Malherbe, *Op. cit.*, p. 15.

[8] Frege, G., *Über Sinn und Bedeutung*, French translation by C. Imbert in Frege, G., *Ecrits logiquees et philosophiques*, Paris, Seuil, 1968, p. 103; quoted from J.-F. Malherbe, *Op. cit.*, p. 17.

[9] *Ibid.*, p. 107; quoted from J.-F. Malherbe *Op. cit.*, p. 19.

[10] Frege, G., *Über Begriff und Gegenstand*, French translation by C. Imbert in Frege, G., *Ecrits logiquees et philosophiques*, Paris, Seuil, 1968, p. 128; quoted from J.-F. Malherbe, *Op. cit.*, p. 19.

[11] Frege, G., *Über Sinn und Bedeutung*, Reprinted in P. Geach and M. Black, eds., *Translations from the Philosophical Writings of Gottlob Frege*, Oxford, Blackwell, 1960, p. 63; quoted from K. Bach, *Op. cit.*

It is very interesting to observe that Frege uses *Sinn* as an ordinary German word in these two different acceptances. So for *Sinn* as mode of presentation we read on the page 105^{e} in Austin's translation[12] of *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl* of 1884:

> In the spatial sense (*Sinn*) they are, in any case, neither inside nor outside either the subject or any object. But, of course, they are outside the subject in the sense (*Sinn*) that they are not subjective.

But we find its another use in the §101 of the same work and yet its both uses in the same sentence:[13]

> That would have to result from the sense (*Sinn*) of $a + bi$, which we are here taking to have been made available. [...] Well, perhaps it is indeed possible to assign a whole variety of different meanings (*Bedeutung*) to $a + bi$, and to sum and product, all of them such that those laws continue to hold good; but it is not immaterial whether we can or cannot find *some* such a sense (*Sinn*) for those expressions.

Remark. Perhaps the presence of different acceptances of *Sinn* in Frege's writings explain his hesitations about the compositionality principle described in the interesting article of Janssen [4] on the history of Frege's contextuality and compositionality principles. If we use the term *Sinn* to formalize a notion of *mode of presentation of reference*, it seems to be very doubtful that, for example, two such *modes of presentation* for the subexpressions were composable in any other *mode of presentation* of reference for the whole expression. On the contrary, it seems to be very natural that if we pose under the subexpressions what we have understood after have read it, we can understand from this data what ought to mean the whole expression. So one needs to precise his terminological convention on *sense* and *meaning* if he wants to discuss rigorously compositionality or contextuality principles.

We accept the *sense* (or *sens* in French) to be rather the mode of reading of any textual fragment, whereas we accept the *meaning* (or *signification* in French) of any textual fragment to be rather something which is grasped when we understand it. Using this terminology, we can speak in English: "I've understood what does this text mean in the allegorical sense" or equally: "In the allegorical sense, I've grasp its meaning"; or in French: "J'ai compris ce que signifie ce texte au sens allégorique", or equally: "Au sens allégorique, j'en ai saisi une signification". Likewise for the moral, historical, psychoanalytical and others senses. In according to this terminological acceptance, we can apply different mode of reading (or senses) to one and the same text. There is here the basic idea of the *sense/meaning* distinction essential for our terminological convention; so our acceptance of *sense* as a *mode of reading* is similar to that posed in the exegetic concept of four senses of Sacred Scripture, whereas the terminological acceptance of *meaning* of expression as the content, which we grasp when we understand it, corresponds well to the common usage as an ordinary English word.

We would like to stress here the difference between this acceptance and that of Fregean acceptance of *Sinn* as the "mode of presentation" of reference which is often illustrated by the famous example of "morning star" and "evening star". We consider it as an example of two different texts or expressions; each of them may be interpreted in many different senses or *modes of reading* and, following a chosen mode of reading, we can grasp the different meanings of it. For a moment, it leaves open the possibility of comparing of such the meanings for the different expressions. This intertextuality problem is treated at length in our work [3, ch. 8] where we

[12] G. Frege, *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*, Breslau, Verlag von W.Koebner, 1884, Repr. by Basil Blackwell, Oxford, 1953, English translation by J. L. Austin.

[13] *Ibid.* p. 111^{e}.

define a notion of morphism in the category of textual spaces **Logos** involving a well-known notion of the *direct image* functor.

In accordance with our acceptance, we consider a meaning of a given expression that is grasped in some particular situation of communication (real or fictive, direct or mediated) to be rather what the reader or listener understands as a response to an implicit question. One finds such a meaning by asking himself: "What does it mean in this or that sense?" or ("Qu'est-ce que cela signifie dans tel ou tel sens ?"). Words, expressions, texts mean what the members of a given linguistic community at a given time understand them to mean. The later Wittgenstein expressed this point of view in his famous slogan: "The meaning is use". So it is an odd misconception to think that a word (an expression, a text) has only one true meaning. This is why we consider understanding of the fragment U of some text X under some mode of reading $\mathcal{F}$ as the choice of a meaning s from a set $\mathcal{F}(U)$. Being composed during the interpretative process, some meaning s of the fragment U is rooted in the *use* and is motivated by the *mode of reading* $\mathcal{F}$. This fragmentary meaning s should not be understood as some mental state of the reader because the mental states of two readers could not be identified nor compared in some reasonable way; not either it should be identified with some truth-condition in accordance with a vericonditional model because the understanding of e. g. fairy tales or science fictions is achieved regardless of any assumption about verifiability.

We do not relate the meaning with any kind of truth-value or truth-conditions. According to Frege's remark to the work of Jourdain of 1912 on the history of logic, reproduced as footnote n° 6 at p. 11 in the Heijenoort's edition of *Begriffsschrift*:[14]

> We must be able to express a thought without affirming that it is true. If we want to characterize a thought as false, we must first express it without affirming it, then negate it, and affirm as true the thought thus obtained. We cannot correctly express a hypothetical connection between thoughts at all if we cannot express thoughts without affirming them, for in the hypothetical connection neither the thought appearing as antecedent nor that appearing as consequent is affirmed.

According to another his formulation from the famous work *Über Sinn und Bedeutung*:[15]

> A judgement is not mere grasping of a [t]hought, but the recognition of its truth.

So, following Frege, we can express a thought and we can grasp a thought without affirming or recognition of its truth. Likewise for any (admissible) text, we can understand its sentences and its fragments regardless of truth-values or truth-conditions.

In discussing three fundamental relations *reference*, *inference*, and *difference* in [5, chap. I, sec. 4], F. Rastier emphasizes the difficulty concerning reference at the level of text:

> Pour les textes dits "non-fictionnels", on pose dès le palier de la proposition le problème de la vérité ; mais on ne prétend cependant pas qu'un texte ait une valeur de vérité, à moins qu'il ne soit idéalement composé que de propositions vraies. Pour les textes "fictionnels", le problème du réalisme (au sens non philosophique du terme, tel qu'il est employé dans la critique littéraire) doit être abordé en fonction de leur mode mimétique et des impressions référentielles qu'il induit. L'opposition entre fiction et "non fiction", tenue pour acquise et utilisée sur le mode de l'évidence [...], nous semble cependant devoir être évitée tant qu'elle oblitère le problème des modes mimétiques. Et comme nous ne partageons pas ses attendus implicites, nous préférons écarter le problème de la vérité : philologiquement, un texte n'est ni vrai ni faux, mais authentique ou non. Le problème de la vérité dépend d'autres disciplines (histoire, théologie, etc.), qui le traitent chacune à leur manière.

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[14]G. Frege, *Begriffsschrift*, English translation in J. van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Massachusetts, Harvard University Press, 1967.

[15]English translation quoted from G. Sundholm, *A Century of Inference: 1837-1936* in P. Gärdenfors et al. (ed.), *In the Scope of Logic, Methodology and Philosophy of Science*, Dordrecht, Kluwer, 2002, p. 578.

Note that our approach differs from that proposed by the possible world semantics; it seems to be a rather difficult problem to distinguish the way the world is from some another way it might have been, whereas our approach is based on the criteria of equality formulated explicitly for two kinds of meaning *fragmentary* and *contextual* we shall consider in the present work. In our acceptance, the term *meaning* should not be understood in the Tarski/Montague style as the relation between word and world. The interpretation of text has a purpose to understand its meaning. Cognition of reality (world) has a purpose to know the truth about it. Understanding of meaning and knowledge of truth relate both with the objectivity but in a different way. We observe that a meaning s of some fragment U of a given text X (its *fragmentary meaning*) is understood by the reader as something objective as the result of interpretation of this passage U; its "objectivity" carries no claim of correspondence to reality but is grounded in the conviction that this meaning s may be discussed with anybody in some kind of dialogue (actual or imaginary) where such a meaning s may be finally shared by the participants or may be compared with any other meaning t of the same fragment U. We shall consider later the criterion for such a comparison procedure formulated as some equality condition (S). This kind of objectivity is based not only on the shared language but principally on the shared experience as a common life-world and it deals so with the reality. Following Gadamer, this being-with-each-other is a general building principle both in life and in language. The understanding results from being together in a common world. This understanding as a presumed agreement on "what this fragment U wants to say" becomes its meaning s; the process of coming to such a meaning s may be thought of as an exercise of human capacity of naming and understanding; it is a fundamental characteristic of the human linguistic behaviour.

This is a first step in our definition; to move towards the second step, we remark that this meaning varies with each concrete situation of reading. Etymologically, interpretation is the "presence between" (inter-pretatio). To interpret a passage from a given text, is a seeking and finding "between its lines" the meaning that the author where supposed to express. The challenge, then, is to apply all reader's linguistic competence, in realizing that there can never be a final, closed interpretation. Perhaps the process of interpretation happens to give an understanding that differs a little in meaning every time and for every reader. Supposing a model for linguistically competent reader, we collect all these (fragmentary) meanings of the fragment U in the set $\mathcal{F}(U)$ in some kind of platonistic manner, and this is the second step in our definition. Thus for any mode of reading $\mathcal{F}$, we are given a map $U \mapsto \mathcal{F}(U)$ defined on all meaningful parts U of the text X. The abstraction of this kind is usual however for the most of mathematical reasonings concerning sets, groups, topological spaces, etc., and for its applications to other sciences, engineering and everyday life. This platonistic way of reasoning was successfully applied in many problems to justify the power of mathematical approach despite the fact that it may lead to some set-theoretical paradoxes. Carnap analyzes the problems concerning the role of abstract entities in semantics in the classic work *Empirism, Semantics, and Ontology*. In this difficult question, we would like to follow his appeal to "... be cautious in making assertions and critical in examining them, but tolerant in permitting linguistic forms".[16] In the case of interpretative process, the set-theoretical idealization of its result is equal in some degree to the abstraction of reader's linguistic competence which ought to be supposed in any semantic theory.

Formulated not only for the whole text X but more generally for any meaningful fragment V of it, the precept of hermeneutic circle "to understand some part of text as the restriction (to this part) of the understanding of the whole text" gives the definition of a family of maps

[16] R. Carnap, *Empirism, Semantics, and Ontology* in P. Benacerraf, H. Putnam (eds.), *Philosophy of Mathematics*, Englewood Cliffs, Prentice-Hall, 1964, p. 248.

$\mathrm{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$, where U is a fragment of V, and $\mathcal{F}(V)$ is the set of all meanings of V. So the *meaning* s of a non-void fragment V of the text X defines the meaning $\mathrm{res}_U^V(s)$ for any non-void subfragment $U \subset V$, with the obvious property of identity preserving 1° $\mathrm{res}_V^V = \mathrm{id}_V$ and that of transitivity 2° $\mathrm{res}_U^V \circ \mathrm{res}_V^W = \mathrm{res}_U^W$, for any non-void nested subfragments $U \subset V \subset W$. At first sight, this precept of hermeneutic circle seems to be "topology free". But the reading of text as well as the utterance of discourse is always a process that develops in time, and so it inherits in some way its topological structure.

So we need only a topology on a given text to gain there a structure of presheaf. It's not a question to graft some topology onto the given text but to observe that any text has an underlying topological structure which arises quite naturally in accordance with one of two paradigms of reading related with two appearance of sign: as something phonetic or as something graphic.

One of these topologies is defined by the natural order structure the text bears on. From a linguistic point of view, this order structure is known as a notion of *linearity* or that of *words order*. In fact, it is a structure of genuine order and so it defines a topology. A text can be treated as a written speech and so their common distinctive feature is a temporality, implicit for the former and explicit for the latter. The natural temporality of phonetic phenomena is a reason to call this topology *natural* or *phonocentric*. The open sets in this topology are the fragments of text related to the process of reading and are naturally turned out to be meaningful.

Another kind of topology that we name *logocentric* is a specific example of general notion of *Grothendieck topology* [9]. In fact, besides the paradigm of reading which treats a written text as an uttered discourse, there exists another one which intervenes in its structure by using the advantage of text as being totally given.

In accordance with the choice of one of these two topologies, we need to use the appropriate concept of sheaf. In the present work, we consider only the phonocentric topology on text; the logocentric topology and the corresponding generalization of Frege's compositionality principle are discussed in [3].

In the following, we shall consider only texts written with good grace and intended for human understanding. We call *admissible* any such text. All sequences of words written in order to confuse the reader or to imitate some human writings by using a computer or any random procedure, will be cast aside as having no deals with understanding of written expressions of human phenomena. Hence they could not constitute the object of study in any hermeneutics, yet formal one. For some reasons that will be explained later, we exclude from admissible any collection of articles united typographically in one edition. Equally, any library does not constitute an admissible text!

For any given admissible text, we may consider its phonocentric topological structure at the level of text, at the level of sentence and even at the level of word. This division on levels is essential for the process of text understanding. At each level, we distinguish its *primitive elements* called equally *loci* (or *locus* in a singular form) which are the points of corresponding topological space considered to be the whole at this level. At each level, we define a topology by specifying its open sets (called *fragments*) to be some meaningful parts of the whole. The basis of phonocentric topology at a given level is defined by describing the topological basis at each point. The passage from one level to another immediately superior consists in gluing of the whole space into a point of the higher level space.

So at the level of sentence, we consider a given sentence as a sequence of its morphemes and the phonocentric topology is defined on the graph of this sequence by specifying at each morpheme some set of syntagms to be its basic open neighbourhoods (see [3, p. 35]).

When we speak of a phonocentric topology at the level of text, we consider a given text X as a sequence of its sentences; the phonocentric topology is defined on the graph of this sequence in such a way that the basis of topology at a locus $x \in X$ is defined as the class of intervals of the following type: $I_{e_i}(x) = \{\, l : e_i \leqslant l \leqslant x \,\}$, where e_i is the first sentence in the paragraph containing x or the first one in any paragraph which precedes that containing x. It's clear that arbitrary final intersection of basis sets is some basis set or empty. So the open sets are all the arbitrary unions of basis sets. It is easy to see that the open sets in the phonocentric topology (at the level of text) are nothing more than the meaningful fragments of text considered in the majority of philological investigations. In accordance to our definition in [3], we shall often use the term *fragment* as equivalent to this of an *open subset* in the topological space related to text. The detailed definitions of phonocentric topology at different semantic levels are given in our precedent works (see e.g., [3]).

It may happen that some fragment of a given text needs many resumption of reading process, because of its length being of some hundred pages. So we need to consider the reading process for any fragment U as its covering by some family of subfragments $(U_j)_{j\in J}$ already read. Such a covering of U is said to be open if $U = \bigcup_{j\in J} U_j$ and each U_j is open in X. The question naturally arises to compare any two meanings s, t of a given fragment U. Otherwise, it were impossible to consider the fragmentary meaning to be well-defined object of our intuition or of our thought. The definition of equality quite adequate to our intuition is claimed by the following:

Condition S (Separability). *Let X be an admissible text, and let U be a fragment of X. Suppose that s, t are two fragmentary meanings of U and there is an open covering $U = \bigcup_{j\in J} U_j$ such that $\mathrm{res}^U_{U_j}(s) = \mathrm{res}^U_{U_j}(t)$ for all fragments U_j. Then $s = t$.*

In other words, the meanings s, t are considered to be identical meanings of the whole fragment (i. e. globally) if and only if they are identical locally. It should be noticed that Frege has never considered the notion of equality between the meanings of a given expression.[17] Following Quine, there is no entity without identity; so we need some norm of identity between meanings if we want to consider the set-theoretical operations and quantifications with them. The condition (S) defines the criterion of equality between the fragmentary meanings, which corresponds well to our intuition. The condition (S) as above is posed for the case of phonocentric topology at the level of text. The analogous conditions may be formulated at the level of sentence and at the level of syntagm. These criteria of identity together with a special functional representation of fragmentary meanings give a background for some recursive procedure allowing to compare any two meanings of the same textual fragment.

This having been done, we return now to examine the properties 1° and 2° of the maps $\mathrm{res}^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$ defined above only on the non-void open subsets of X. All we need is to define $\mathcal{F}(\varnothing)$, for the map $U \mapsto \mathcal{F}(U)$ to be defined on all the open sets in the phonocentric topology on X. Let $\mathcal{F}(\varnothing) = \mathrm{pt}$ to be a singleton, i. e. the one-element set (e.g., the meaning of the title of X if there is). We need also that the family of maps res^V_U to be defined on all the open sets in the phonocentric topology on X. It's clear that the maps $\mathrm{res}^V_\varnothing$, $\mathrm{res}^\varnothing_\varnothing$ are uniquely defined in an obvious manner. For any open set U in the phonocentric topology on X, we have defined now a non-void set $\mathcal{F}(U)$ of its (fragmentary) meanings, and for any pair of open sets $U \subset V$, we have defined a restriction map $\mathrm{res}^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$, so that the conditions 1° and 2° as above are verified for all nested opens $U \subset V \subset W$ of X. Thus, the data of $(\mathcal{F}(V), \mathrm{res}^V_U)$ satisfying to the conditions of identity preserving 1° and that of transitivity 2° as above, is supposed to be determined for all open subsets U, V of a given admissible text X. In mathematics, this data defines a *presheaf (of sets)* on X. For a given presheaf $\mathcal{F}$, the elements of $\mathcal{F}(U)$ are called

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[17] D. Marconi, *La philosophie du langage au XXème siècle*, French translation by M. Valensi, 1997, ch. 14, URL=http://www.lyber-eclat.net/lyber/marconi/langage.html

sections (over) U; the elements of $\mathcal{F}(X)$ are called *global sections*.

What really we have done at this point, may be recapitulated by saying that for an admissible text X, the condition (S) claims that any *mode of reading* $\mathcal{F}$, as above, defines some *separated presheaf* [7, p. 14] of fragmentary meanings over X.

We come now to the precept of hermeneutic circle 'to understand the whole text by means of understandings of its parts'. For any admissible text, it is nothing more than the compositionality principle for the fragmentary meanings at the semantic level of text. Strictly speaking, it may be formulated by saying that the fragmentary meanings satisfy together the following:

Condition C (Compositionality). *Let X be an admissible text, and let U be a fragment of X. Suppose that $U = \bigcup_{j \in J} U_j$ is an open covering of U; suppose we are given a family $(s_j)_{j \in J}$ of fragmentary meanings, $s_j \in \mathcal{F}(U_j)$ for all fragments U_j, such that $\mathrm{res}^{U_i}_{U_i \cap U_j}(s_i) = \mathrm{res}^{U_j}_{U_i \cap U_j}(s_j)$. Then there exists some meaning s of the whole fragment U such that $\mathrm{res}^{U}_{U_j}(s) = s_j$ for all fragments U_j.*

In other words, the condition (C) claims that locally compatible fragmentary meanings are composable in some global meaning. This claim (C) may be considered as a generalization to the level of text for the classic Frege's principle of compositionality of meaning stated at the level of sentence. In mathematics, there exist many examples of separated presheaves that do not satisfy the condition (C), but they are not so interesting to study. Similarly, there are many printed editions such as newspapers, magazines, journals, bulletins, notes, etc., to give the examples of presheaves of fragmentary meanings which do not satisfy obviously the condition (C) of compositionality. So we have excluded they from a class of admissible texts because their understanding reduces to that of admissible texts they are made up.

In mathematics, a presheaf satisfying to the conditions (S) and (C) is called *sheaf*. In our formal hermeneutics, the condition (C) together with the condition (S) claim that any presheaf of fragmentary meanings attached naturally to an admissible text is really a sheaf! Note that the presence of (S) guarantees the meaning s, whose existence is claimed by (C), to be unique as such. The sheaves arise whenever some consistent local data glues in a global one.

It is not so hard to see that these two conditions needed for presheaf to be a sheaf are analogous to those two conditions needed for a binary relation to be functional. So the true formulation of Frege's compositionality principle does not demand functionality but its sheaf-theoretical generalization, i. e. that any textual presheaf ought *de facto* to be a sheaf. Thus for the phonocentric paradigm of reading, we have revised the formulation of the classic Frege's compositionality principle by augmenting it with an identity criterion for any two fragmentary meanings. So we give the following sheaf-theoretical generalization of the Frege's compositionality principle:

Definition (Frege's Compositionality Principle). *A presheaf of fragmentary meanings naturally attached with a mode of reading of an admissible text is really a sheaf; its sections over any textual fragment are its fragmentary meanings; its global sections are the meanings of the whole text.*

From a mathematical point of view, for a given text X, any mode $\mathcal{F}$ of reading defines *de facto* a sheaf of sets (of fragmentary meanings) over X. It's really a (contravariant) functor in the strict mathematical sense because any topological space X defines a very simple category

$\mathcal{O}(X)$; the objects of $\mathcal{O}(X)$ are the open subsets of X, the morphisms are the canonical inclusions $U \subset V$; all axioms of category are obviously satisfied. Called *sense* (or *mode of reading*), the functor $\mathcal{F}$ associates with any fragment U (object in $\mathcal{O}(X)$) the set of its fragmentary meanings $\mathcal{F}(U)$ and with any inclusion $U \subset V$ (morphism in $\mathcal{O}(X)$) the map $\mathrm{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$. We will make free use of the classic category theory; for basic terminology, our principal references are [8], [9], [10]. In the notation of the French school, we denote $\widehat{\mathcal{O}(X)}$ a category of presheaves of sets over the topological space X, whereas in the articles [2], [3], we use exponential notation $\mathbf{Ens}^{\mathbf{Ouv}_X^{\mathrm{op}}}$ for this category.

At the level of sentence, the same reasoning gives a classic Frege's compositionality principle but with morphemes as primitive elements and with syntagms instead of words as meaningful fragments.

Let us consider now any two modes of readings $\mathcal{F}$, $\mathcal{G}$ of a given text X. The reader should become at home with these functors although we call them as 'modes of readings' instead of 'senses' not only to stress the character of historicity of each actual process of reading but rather to avoid a possible confusion which may be caused by another technical acceptation of the term 'sense'. So one can think, for example, about the historical sense $\mathcal{F}$ and the moral sense $\mathcal{G}$ of some biographical text. Let $U \subset V$ be any two fragments of the text X. It seems to be very natural to consider that any meaning s of fragment V understood in the historical sense $\mathcal{F}$ gives a certain well-defined meaning $\phi(V)(s)$ of the same fragment V understood in the moral sense $\mathcal{G}$. Hence, for each $V \subset X$, we are given a map $\phi(V) : \mathcal{F}(V) \to \mathcal{G}(V)$. To transfer from the meaning s of V in the historical sense to its meaning in the moral sense and then to restrict the latter to a subfragment $U \subset V$ is the same operation as to make first the restriction from V to U of the meaning s in the historical sense, and to make then a change of the historical sense to the moral one. This can be expressed in a simple way by saying that the following diagram

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\mathrm{res}_U^V"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}'^V_U"] \\
\mathcal{F}(U) \arrow[r, "\phi(U)"] & \mathcal{G}(U)
\end{tikzcd}$$

commutes for any fragments $U \subset V$ of X. We meet this situation of somebody's interpretation transfer from one mode of reading to another or from understanding in one sense to understanding in some another sense many times a day.

This notion of morphism is very near to that of *transformation incorporelle* of G. Deleuze and F. Guattari which they illustrate by several examples, one of which is the following:[18]

> Dans un détournement d'avion, la menace du pirate qui brandit un revolver est évidemment une action ; de même l'exécution des otages si elle a lieu. Mais la transformation des passagers en otages, et du corps-avion en corps-prison, est une transformation incorporelle instantanée, un *mass-media act* au sens où les Anglais parlent de *speech-act*.

To adapt this example, we need only to transform it into some written story about a hijacking.

Hence, the family of maps $(\phi(U))_{U \in \mathcal{O}(X)}$ defines a change of mode of reading of a given text X, or simply a morphism $\phi : \mathcal{F} \mapsto \mathcal{G}$. It is obvious that a family of identical maps $\mathrm{id}_{\mathcal{F}(V)} : \mathcal{F}(V) \to \mathcal{F}(V)$ given for each open $V \subset X$ defines the identical morphism of the sheaf $\mathcal{F}$ which will be denoted as $\mathrm{id}_{\mathcal{F}}$. The composition of morphisms is defined in an obvious manner:

[18] G. Deleuze, F. Guattari, *Capitalisme et schizophrénie 2, Mille plateaux*, Coll. « Critique », Paris, Minuit, 1980, pp. 102, 103.

for any two morphism $\mathcal{F} \xrightarrow{\psi} \mathcal{G} \xrightarrow{\phi} \mathcal{H}$, we define $(\phi \circ \psi)(U) = \phi(U) \circ \psi(U)$. It is clear that this composition is associative every time it can be defined. Thus, the data of all sheaves $\mathcal{F}$ on the same text X considered together with all its morphisms constitutes some *category* in the mathematical sense of this term. In the honour of Friedrich Schleiermacher, we name this category of the particular sheaves defined on text X as category of Schleiermacher and denote it as $\mathbf{Schl}(X)$. This category $\mathbf{Schl}(X)$ describes the situation then the reader is interested in the exegesis of some particular text X as, for example, Sacred Scripture.

We have a firm conviction that the key theoretical structure on which any discourse interpretation theory should depend is the mathematical notion of sheaf! It constitutes nowadays one of the most powerful tools in many domains of modern mathematics and is expected to be a pertinent tool in semantics.

2. Contextuality

So far, we have defined only a notion of fragmentary meanings, i. e. we have defined the meaning of any open part of a given text in the phonocentric topology. It is clear that a class of objects of the category $\mathcal{O}(X)$ (i. e. all the open parts of X) contains the kind of fragments usually used in the philological considerations. So the question arises whether the other parts of text (i. e. non-open ones) would be meaningful. First of all, we are interested to define the notion of meaning for a primitive element (point or locus) at each level of semantic consideration, as for a sentence at the level of text and for a morpheme (word) at the level of sentence.

The classic precept of *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl* of 1884: "nach der Bedeutung der Wörter muss im Satzzusammenhange, nicht in ihrer Vereinzelung gefragt werden"[19] is called usually as Frege's *contextuality principle*. Frege had written it eight years before his famous work *Über Sinn und Bedeutung* of 1892, where Frege had introduced in semantics a terminology distinction between *Sinn* and *Bedeutung*, which was essential for his further research. In connection with this distinction of Frege's work of 1892, Church and Russell translate the term *Bedeutung* as *denotation*. To indicate this *Sinn/Bedeutung* distinction, one translates in English *Bedeutung* as *denotation* or sometimes as *reference* if it stands for the named object and as *denoting* if it stands for the relation of naming. It's clear that a word in isolation (e.g., when in a dictionary) does not refer to a particular object (with precision up to several exceptions). But the Frege's *contextuality principle* claims more, as expressed in the accurate English translation of Janssen in [4, p. 115]: "Never ask for the meaning of a word in isolation, but only in the context of a sentence". Likewise for the English translation of Austin: "never ask for the meaning of a word in isolation, but only in the context of a proposition".[20] In the standard French translation of C. Imbert, this principle is expressed so: "On doit rechercher ce que les mots veulent dire non pas isolément, mais pris dans leur contexte". We note that "Satzzusammenhange" means literally 'in relation to phrase'. This celebrated formulation of contextuality principle is quoted from Introduction to *Die Grundlagen der Arithmetik*. In the foregoing considerations of §60, Frege gives more explanations upon. So he writes:[21]

> Only in a proposition have the words really a meaning (*Bedeutung*). It may be that mental pictures float before us all the while, but these need not correspond to the logical elements in the judgement. It is enough

[19] G. Frege, *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*, Breslau, Verlag von W.Koebner, 1884, p. X. Repr. by Basil Blackwell, Oxford, 1953, English translation by J. L. Austin.

[20] *Ibid.,* p. X$^{\mathrm{e}}$.

[21] *Ibid.* p. 71$^{\mathrm{e}}$.

if the proposition taken as a whole has a sense (*Sinn*); it is this that confers on its parts also their content.

It seems however that the *Sinn/Bedeutung* distinction from *Über Sinn und Bedeutung* of 1892 is not relevant here. This passage justifies rather the translation of *Bedeutung* with *meaning* and exemplifies also one of the Fregean acceptances of *Sinn* which we translate with *meaning*: for some meaningful expression, "it is this that confers on its parts also their content".

So we consider this Fregean formulation as an implicit definition which we have to precise if we want to recover the contextuality as a rigorous notion. On the other hands, we will generalize it so as to define the notion of *contextual meaning* that completes our theory by allowing to consider (at each semantic level) not only the meanings of subexpressions but also the meanings of its primitive (elementary) parts. We consider the compositionality and contextuality principles together to guide the interpretative process of text understanding at each its semantic level.

To begin with the first question, we start by reformulating the classic Frege's contextuality principle at the level of text as the claim that a given sentence has a meaning in relation to the whole text. But at the level of text this maximal contextualization seems to be excessive because our everyday practice of text or discourse understanding reveals that our understanding progress usually with the reading or listening, and the meanings of its sentences are caught during this process. In other words, the understanding of any sentence is not postponed until the reading of the final word of the whole text. To understand a given sentence x, we need a context constituted by some fragment containing it, i. e. by some open neighbourhood of x. Let U, V are two neighbourhoods of x and let $\mathcal{F}$ is some mode of reading. Two fragmentary meanings $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ seem to be giving the same contextual meaning to the sentence x if s and t agree on some smaller neighbourhood of x. It seems to be conform with the common reader's intuition about what would it means then two given fragmentary meanings s and t define the same contextual meaning for x. So these two fragmentary meanings $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are called to induce the same contextual meaning at x when there is some open neighbourhood W of x, such that $W \subset U \cap V$ and $\mathrm{res}^U_W(s) = \mathrm{res}^V_W(t) \in \mathcal{F}(W)$. This property should be demanded by any reasonable definition for the notion of contextual meaning. This relation 'induce the same contextual meaning at x' is obviously an equivalence relation, and the equivalence class of fragmentary meanings agreeing in some neighbourhood of x is called the *contextual meaning* of x. We denote by $\mathcal{F}_x$ the set of all contextual meanings of x, i. e. the set of all equivalence classes. This definition for the set $\mathcal{F}_x$ of all contextual meanings of the sentence x is a variant of what is well-known as a construction for *inductive limit* explained in any standard source on the category theory. According to standard terminology, the elements of $\mathcal{F}_x$ are referred as *germs* at x. Given a sentence x, the canonical image in $\mathcal{F}_x$ of a fragmentary meaning $s \in \mathcal{F}(U)$, i. e. the equivalence class of this fragmentary meaning s, is called a *germ* of s at x and denoted as $\mathrm{germ}_x s$. In other words, all the contextual meanings of a sentence x are united in the set $\mathcal{F}_x$ and for every neighbourhood U of x, we are given a map $\mathrm{germ}^U_x : \mathcal{F}(U) \to \mathcal{F}_x$, where $\mathrm{germ}^U_x : s \mapsto \mathrm{germ}_x s$.

Intuitively, the notion of contextual meaning is almost obvious. The problem is that the same sentence may occur in many quite different texts. Following Frege, in seeking the meaning of a word, we must consider it in the context of some sentence; likewise, in seeking the meaning of a sentence, we must consider it in the context of a text. Suppose we want to assign a contextual meaning to some sentence x of a text X. Given a (fragmentary) meaning s of a fragment U containing x, we dispose a piece of data which creates some context to determine a *contextual meaning* of x. Basically, what we really want is to define a mapping ρ which transforms each fragmentary meaning s of a neighbourhood U of x to some contextual meaning $\rho(s)$ of x. It's clear that another fragmentary meaning t of another neighbourhood V of x may supply yet something concerning the sought contextual meaning of x. So this map ρ would be one's own for each neighbourhood U of x. Hence we need to define a family of maps $\rho_x^U : \mathcal{F}(U) \to T$, where T ought to be a set that unites all the presumed contextual meanings of a sentence x considered relative to a given text X.

Let U, V be two neighbourhoods of x, such that $V \subset U$. Recall that $\mathcal{F}(U)$ and $\mathcal{F}(V)$ are related by the map of restriction $\mathrm{res}_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ which determines how each meaning s of the fragment U gives a meaning $\mathrm{res}_V^U(s)$ of its subfragment V. It seems to be very natural that two fragmentary meanings s and $\mathrm{res}_V^U(s)$ define the same contextual meaning for x. In other words, for any s, we have an obvious compatibility $\rho_x^V(\mathrm{res}_V^U(s)) = \rho_x^U(s) = f$ for some $f \in T$, or simply $\rho_x^V \circ \mathrm{res}_V^U = \rho_x^U$. This compatibility condition needs to be satisfied by any candidate T for the set of all contextual meanings of x. So for all nested neighbourhoods $V \subset U$ of the sentence x, the following diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[d, "\mathrm{res}_V^U"'] \arrow[dr, "\rho_x^U"] & \\ & T \\ \mathcal{F}(V) \arrow[ur, "\rho_x^V"'] & \end{tikzcd}$$

commutes.

In the standard terminology of [7, def. 3.4, p. 4], the set T with a family $(\rho_x^U)_{U \in \mathbb{V}(x)}$ making commutative the above-mentioned diagram is called the *target* of *direct (or inductive) system of sets* $(\mathcal{F}(U), \mathrm{res}_V^U)_{U,V \in \mathbb{V}(x)}$, where $\mathbb{V}(x)$ denotes the ordered set of all open neighbourhoods of x.

This condition is clearly satisfied by a set $\mathcal{F}_x$ constructed above. It is easy to prove that this target $\mathcal{F}_x$ has the property to be *universal* in terms of the same definition of [7, def. 3.4, p. 4]. Really, if two fragmentary meanings $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ induce the same contextual meaning at x (i. e. they are equivalent) when there exists some open neighbourhood W of x, such that $W \subset U \cap V$ and $\mathrm{res}_W^U(s) = \mathrm{res}_W^V(t) \in \mathcal{F}(W)$. This property implies that $\rho_x^U(s) = \rho_x^W \circ \mathrm{res}_W^U(s) = \rho_x^W \circ \mathrm{res}_W^V(t) = \rho_x^V(t)$. So equivalent fragmentary meanings have the same image in T and hence there exists some map $\tau : \mathcal{F}_x \to T$ such that for all neighbourhood U of x, one has $\tau \circ \mathrm{germ}_x^U = \rho_x^U$. Hence $\mathcal{F}_x$ is an universal target. In accordance to this well-known definition, any universal target of inductive system of sets is called its *direct limit*. In several sources on category theory, this *direct limit* is also called *inductive limit* or *colimit*, so we'll do. The standard theorem of category theory claims that any universal objet is unique up to isomorphism. This justifies the functional notation $\varinjlim(\ )$ for inductive limit so defined. Hence the set $\mathcal{F}_x$ constructed above (as the set of equivalence classes together with the obvious family of equalizing maps) is isomorphic to the inductive limit for the inductive system of sets $(\mathcal{F}(U), \mathrm{res}_V^U)_{U,V \in \mathbb{V}(x)}$, where $\mathbb{V}(x)$ denotes the ordered set of open neighbourhoods of x; so we write $\mathcal{F}_x = \varinjlim(\mathcal{F}(U), \mathrm{res}_V^U)_{U,V \in \mathbb{V}(x)}$ and, as usual, we call $\mathcal{F}_x$ the *stalk* of the sheaf $\mathcal{F}$ at x and the elements of $\mathcal{F}_x$ are referred as *germs* at x. In other words, all the contextual meanings of a sentence x are united in the stalk $\mathcal{F}_x$.

It is clear that we get the same inductive limit if instead of $\mathbb{V}(x)$ we consider the inductive system $\mathbb{B}(x)$ including only all the basic neighbourhoods of x.

Recall that in accordance with the inductive limit characterizing theorem [7, th. 3.8, p. 5], for a given inductive system of sets, any target is isomorphic to inductive limit if and only if the conjunction of certain two conditions holds. For the inductive system $(\mathcal{F}(U), \mathrm{res}_V^U)_{U,V \in \mathbb{V}(x)}$ defined by some admissible text X, these two characteristic conditions formulated for a target $\mathcal{F}_x$ are expressed by claiming that the following two conditions (Ct) and (E) hold:

Condition Ct (Contextuality). *Let $\mathcal{F}$ be a mode of reading (sense) for a given text X, then for any contextual meaning f of a sentence x, there exist a neighbourhood U of x and a fragmentary meaning $s \in \mathcal{F}(U)$ such that $f = \mathrm{germ}_x s$.*

For any locus x of a given text X, this condition (Ct) claims that the set $\mathcal{F}_x$ unites only the contextual meanings proper for the text X and there are no other superfluous meanings where. We have satisfied the requirements of classic Frege's contextuality principle. Hence, the claim (Ct) may be considered as a generalization in a narrow sense at the level of text for the classic contextuality principle formulated by Frege himself at the level of sentence.

Condition E (Equality). *Let U, V be two open neighbourhoods of a sentence x and let $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$ be two fragmentary meanings for a given mode of reading (sense) $\mathcal{F}$. Then the equality $germ_x s = germ_x t$ between induced contextual meanings of the sentence x holds if and only if there exists an open neighbourhood W of x such that $W \subset U$, $W \subset V$ and $res^U_W(s) = res^V_W(t)$.*

In other words, the condition (E) claims that if two fragmentary meanings s, t give rise to one and the same contextual meaning of a sentence x, they should do it along the way. So the condition (E) may be treated as a criterion of equality between contextual meanings that corresponds well to our intuition. It should be noticed that Frege had never considered the notion of equality between the meanings of a given word in the context of a given sentence.[22]

The conditions (Ct) and (E) as above are formulated for the case of phonocentric topology at the level of text. Their conjunction (Ct)&(E) will answer the question how the set $\mathcal{F}_x$ of all contextual meanings for a sentence $x \in X$ should be formally defined if we want to follow the classic Frege's precept called nowadays as *contextuality principle*. Thus, for the phonocentric paradigm of reading, we have the following generalization in a wide sense of the classic contextuality principle:

Definition (Frege's Contextuality Principle). *Any sentence x within a fragment U of the text X has a contextual meaning defined as the germ at x of some fragmentary meaning $s \in \mathcal{F}(U)$, where $\mathcal{F}$ is the mode of reading (sense) adopted; the set $\mathcal{F}_x$ of all contextual meanings of a sentence $x \in X$ is defined as the stalk of $\mathcal{F}$ at x, i. e. as the inductive limit $\mathcal{F}_x = \varinjlim(\mathcal{F}(U), res^U_V)_{U,V \in \mathbb{V}(x)}$.*

The reader should interpret the above definition of contextual meaning in this way: If we have grasped some fragmentary meaning s of the fragment $U \subset X$, then for any sentence $x \in U$ we are given a canonical way of finding corresponding contextual meaning $\mathrm{germ}_x s$ of it.

The contextuality principle posed above is an explicit definition of contextual meaning for a given locus at the semantic level of text. The similar definition may be formulated at each semantic level. This one formulated at the level of sentence gives the classic contextuality principle of Frege. As soon as the semantic level is fixed, the corresponding definition of contextual meaning for a locus x is given as $\mathrm{germ}_x s$, where s is some fragmentary meaning defined on some neighbourhood U containing x.

Remark. More generally, one can consider all the open neighbourhoods of some arbitrary part $A \subset X$ to define all its contextual meanings as $\mathcal{F}_A = \varinjlim_{U \supset A}(\mathcal{F}(U))$; however for any open part A, this definition provides its contextual meanings as the fragmentary ones already given, i. e. in this case $\varinjlim_{U \supset A}(\mathcal{F}(U)) = \mathcal{F}(A)$; in particular, for the first sentence x of any paragraph being open in the phonocentric topology, we have $\mathcal{F}_x = \mathcal{F}(\{x\})$.

[22] D. Marconi, *Op. cit.*, ch. 14.

We would like to stress the difference between two kinds of meaning we consider in the interpretative process at some semantic level:

– the notion of *fragmentary meaning* supplies a relevant frames to characterize the successful understanding of some textual fragment as a whole (at a given semantic level) within the current interpretative process.

– the notion of *contextual meaning* supplies a relevant frames to characterize the successful contextual understanding of any primitive textual element (at a given semantic level) within the current interpretative process.

The difference between these notions is of the same nature as that between being 'some property of a subset' and being 'some property of an element' in the naïve set theory. Any admissible text is a kind of a structured whole where the parts are given in quite another manner than, for example, in the case of the shadow parts of a landscape or in the case of the submerged part of a ship. On the contrary, the same ship, viewed as a technical construction, has some part-whole structure, there, e.g., the engine is one of its parts, also structured. So we consider the part-whole textual structure at different semantic levels. The text understanding consists in incessant passage from one semantic level to another in a kind of inductive interpretative process. At each semantic level, this process may be modelled as a sheaf of fragmentary meanings on the one hand, and it may be modelled as a bundle of contextual meanings on the other hand. So the understanding of text is achieved in some kind of inductive process where we may distinguish its inductive step and its inductive basis at each semantic level.

Suppose that we are given a fixed fragmentary meaning $s \in \mathcal{F}(U)$. This fragmentary meaning s determines a function $\dot{s} : x \mapsto \mathrm{germ}_x s$ to be well-defined on U. The domain of this function is the fragment U, and for a given $x \in U$, its value is taken in $\mathcal{F}_x$. We define now the total coproduct (disjoint union) $F = \coprod_{x \in X} \mathcal{F}_x$ and consider it as a codomain for the function $\dot{s}$. We speak about the coproduct to avoid formally a possibility for any two sets $\mathcal{F}_x$ and $\mathcal{F}_y$ to have some elements in common. Hence, every fragmentary meaning $s \in \mathcal{F}(U)$ gives rise to some partial function $\dot{s} : U \to F$ defined on the open part $U \subset X$. This gives a kind of functional representation $\eta(U) : s \mapsto \dot{s}$ defined for all fragmentary meanings $s \in \mathcal{F}(U)$. This representation of a fragmentary meaning s as an actual function $\dot{s}$ is of a great theoretical importance to explain the nature of fragmentary meanings! Each fragmentary meaning s may be thought of as a partial function $\dot{s}$ defined on some fragment U of a given text; the value $\dot{s}(x)$ taken at a given locus $x \in U$ by this function is $\mathrm{germ}_x s$, i. e. the contextual meaning of x defined by s.

For $F = \coprod_{x \in X} \mathcal{F}_x$ defined above, we consider a map called projection $p : F \to X$ which sends each $\mathrm{germ}_x s$ to the point x where it is taken. We call *cross-section* any function $t : U \to F$, which has the property: $t(x) \in p^{-1}(x)$ for all $x \in U$. Any function of the kind $\dot{s}$ defined on some open U (i.e. determined by some fragmentary meaning s) is obviously a cross-section. For any cross-section $t : U \to F$, the projection p has the obvious property $p(t(x)) = x$ for all $x \in U$, namely, $p \circ t = \mathrm{id}_U$.

We define a topology on F by taking as a basis of open sets all the image sets $\dot{s}(U) \subset F$; thus an open set in F is a union of images of the cross-sections of the type $\dot{s}$. This topology makes obviously both p and every function $\dot{s}$ continuous.

This situation may be resumed by saying that we are given two topological spaces F, X and a continuous map p called *projection*. In topology, this data (F, p) is called *bundle over the base* X in English [9] or *espace découpé* in French [6]; such a bundle may be thought of as an X-indexed family of *fibers* $p^{-1}(x)$ varying continuously with x. Considered together, these bundles are the objects of the *slice* category $\mathbf{Top}/X$ (see [9, pp. 12, 79]), where $\mathbf{Top}$ is a category of all topological spaces and continuous maps; while a morphism between the bundles $p : F \to X$ and

$q : G \to X$ is a continuous map $h : F \to G$ preserving the fibers, i. e. such that the diagram

$$\begin{tikzcd} F \arrow[rr, "h"] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

is commutative.

Thus, for any text X, we may consider all bundles of contextual meanings over X, each bundle represents some mode of reading; a morphism h between two bundles over the same text (F, p) and (G, q) represents a certain transfer of senses, which is coherent locally, i. e. $h : \mathcal{F}_x \to \mathcal{G}_x$. We define so a category of bundles of contextual meanings which we denote by $\mathbf{Context}(X)$. Evidently that $\mathbf{Context}(X)$ is a full subcategory in $\mathbf{Top}/X$.

3. Frege Duality

We have defined above a functor of germs $\Lambda : \mathbf{Schl}(X) \to \mathbf{Top}/X$. For any sheaf $\mathcal{F}$, a bundle $\Lambda(\mathcal{F})$ is defined as $(\coprod_{x\in X} \mathcal{F}_x, p)$; for any morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, the induced map of stalks $\mathcal{F}_x \to \mathcal{G}_x$ gives rise to some morphism of bundles $\Lambda(\phi) : \coprod_{x\in X} \mathcal{F}_x \to \coprod_{x\in X} \mathcal{G}_x$.

The bundle $p : F \to X$ so constructed is a local homeomorphism, in the sense that each point of F has an open neighbourhood which is mapped by p homeomorphically onto an open subset of X. Namely, each point $\mathrm{germ}_x s$ has the open neighbourhood $\dot{s}(U)$, and p restricted to $\dot{s}(U)$ has $\dot{s} : U \to \dot{s}(U)$ as a two-sided inverse, hence p is a homeomorphism to U.

In topology, a bundle (F, p) is called *étale* if p is a local homeomorphism (see e.g., [6] or [9]). It is easy to prove that a bundle constructed above from any textual sheaf is always étale. So we have defined a category $\mathbf{Context}(X)$ of all étale bundles of contextual meanings for a given admissible text X. The same construction may be equally applied to any presheaf of sets to define a so-called *germ-functor* in a more general situation $\Lambda : \widehat{\mathcal{O}(X)} \to \mathbf{Top}/X$.

We will define now a so-called *section-functor* Γ. We begin with a category of bundles $\mathbf{Top}/X$. For simplicity, we denote a bundle (F, p) over X by F. For a bundle F, we denote $\Gamma(U, F)$ the set of all its cross-sections over U. If $U \subset V$ are open, one has a restriction operation $\mathrm{res}_U^V : \Gamma(V, F) \to \Gamma(U, F)$. It's clear that $\mathrm{res}_U^U = \mathrm{id}_U$ for any open U, and that the transitivity $\mathrm{res}_U^V \circ \mathrm{res}_V^W = \mathrm{res}_U^W$ holds for all nested opens $U \subset V \subset W$. So we have constructed a presheaf $(\Gamma(V, F), \mathrm{res}_U^V)$ or simply $\Gamma(F)$. For any given morphism of bundles $h : E \to F$, we have at once a map $\Gamma(U, E) \to \Gamma(U, F)$ defined in the obvious way as $s \mapsto h \circ s$. It's clear that the diagram

$$\begin{tikzcd} \Gamma(V, E) \arrow[r] \arrow[d, "{\mathrm{res}'}_U^V"'] & \Gamma(V, F) \arrow[d, "\mathrm{res}_U^V"] \\ \Gamma(U, E) \arrow[r] & \Gamma(U, F) \end{tikzcd}$$

is commutative for all opens $U \subset V$, whence a morphism of presheaves $\Gamma(E) \xrightarrow{\Gamma(h)} \Gamma(F)$. Thus, we have constructed a desired section-functor $\Gamma : \mathbf{Top}/X \to \widehat{\mathcal{O}(X)}$.

To sum up, for a given admissible text X, we have defined two categories formalizing the interpretative process:

– the Schleiermacher category $\mathbf{Schl}(X)$ of *sheaves* of fragmentary meanings;

– the category $\mathbf{Context}(X)$ of *étale bundles* of contextual meanings.

Our goal now is to study their relations.

One can find in many sources [7, pp. 18-27], [9, th. 2, p. 89], [10, th. 10.3 p. 179] a general formulation of the following important well-known result about a dual adjunction between presheaves and bundles which we give in some linguistic version:

Theorem (Dual Adjunction). *For any admissible text X equipped with a phonocentric topology, there is a pair of adjoint functors*

$$\widehat{\mathcal{O}(X)} \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Top}/X,$$

where Γ assigns to each bundle $p : F \to X$ the sheaf of all cross-sections of F, while its left adjoint Λ assigns to each presheaf $\mathcal{F}$ on X the étale bundle of germs of $\mathcal{F}$. There are natural transformations

$$\eta(\mathcal{F}) : \mathcal{F} \to \Gamma\Lambda(\mathcal{F}), \qquad \epsilon(F) : \Lambda\Gamma(F) \to F$$

for a presheaf $\mathcal{F}$ and a bundle F. If $\mathcal{F}$ is a sheaf, $\eta(\mathcal{F})$ is an isomorphism; while if F is étale, $\epsilon(F)$ is an isomorphism. The functors Γ and Λ restrict to an equivalence of categories

$$\mathbf{Schl}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Context}(X),$$

where $\mathbf{Schl}(X)$ is the Schleiermacher category of sheaves of fragmentary meanings, $\mathbf{Context}(X)$ is the category of étale bundles of contextual meanings.

Following S. MacLane: “adjoints occur almost everywhere in many branches of mathematics” [8, p. 103]. Completions, free constructions, Galois connections, polarities and important classic dualities, such as Stone, Gelfand-Naimark, and Pontrjagin-van Kampen Duality, all these examples illustrate the general concept of adjunction and confirm the Slogan V enunciated in the work [10, p. 18] of J. Lambek and P. J. Scott: “Many equivalence and duality theorems in mathematics arise as an equivalence of fixed subcategories induced by a pair of adjoint functors.”

Likewise in linguistics, for any admissible text X, the category $\mathbf{Schl}(X)$ of sheaves of fragmentary meanings and the category $\mathbf{Context}(X)$ of étale bundles of contextual meanings yield an important example of dual adjunction which explains interplay between Frege’s compositionality and contextuality principles. So we have the following:

Proposition-Definition (Frege Duality). *The generalized compositionality and contextuality principles are formulated in terms of categories that are in natural dual adjunction*

$$\mathbf{Schl}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Context}(X)$$

established by the section-functor Γ and the germ-functor Λ.

Naturally, this adjunction may be thought of as a long-awaited reconciliation of compositionality with contextuality. The proof of the Dual Adjunction Theorem in the general case may be found in many standard sources on category theory. However one can formulate this theorem as concerning only the duality between the category of sheaves and the category of étale bundles without mentioning of the category of all presheaves $\widehat{\mathcal{O}(X)}$ and the category of all bundles $\mathbf{Top}/X$. Consider, in particular, what does it mean the existence of adjunction $\eta(\mathcal{F}) : \mathcal{F} \to \Gamma\Lambda(\mathcal{F})$. In that case, the Frege Duality states that every sheaf of fragmentary meanings is a sheaf of cross-sections of some étale bundle of contextual meanings.

Let X be an admissible text. For a given sheaf $\mathcal{F}$ of fragmentary meanings of X, consider the sheaf $\Gamma\Lambda(\mathcal{F})$ of cross-sections of the bundle $\Lambda(\mathcal{F})$. Recall that for each open $U \subset X$ there is a function

$$\eta(U) : \mathcal{F}(U) \to (\Gamma\Lambda(\mathcal{F}))(U), \qquad \eta(U)(s) = \dot{s}. \tag{*}$$

The map of restricting s to an open subset of U matches the η's, so η is a natural transformation of functors $\eta : \mathcal{F} \to \Gamma\Lambda(\mathcal{F})$, where we write η instead of $\eta(\mathcal{F})$.

The Frege Duality states that η is isomorphism, i. e. that a given sheaf $\mathcal{F}$ is a sheaf of cross-sections of étale bundle $\Lambda(\mathcal{F})$. To show that η is isomorphism, we need, for arbitrary open $U \subset X$, to prove that $\eta(U) : \mathcal{F}(U) \to (\Gamma\Lambda(\mathcal{F}))(U)$ is bijection.

1°. First we show that $\eta(U)$ is an injection; that is

$$\text{for all} \quad s, t \in \mathcal{F}(U) : \quad \dot{s} = \dot{t} \quad \text{implies} \quad s = t. \tag{**}$$

For $\dot{s} = \dot{t}$ means that $\mathrm{germ}_x s = \mathrm{germ}_x t$ for each $x \in U$. So for each x there exists an open set $V_x \subset U$ with $\mathrm{res}^U_{V_x}(s) = \mathrm{res}^U_{V_x}(t)$. These open sets V_x cover U, so that the given fragmentary meanings s and t have the same image in each $\mathcal{F}(V_x)$. By the condition (S) of separability satisfied by the sheaf $\mathcal{F}$, we have $s = t$.

2°. To complete the proof, we show that $\eta(U)$ is a surjection. Let $h : U \to F = \coprod_{x\in X} \mathcal{F}_x$ be any cross-section of the bundle of germs $\Lambda(\mathcal{F}) = (F, p)$ over some open set U. Then for each point (sentence) $x \in U$ there is an open set U_x and a fragmentary meaning $s_x \in \mathcal{F}(U_x)$ such that

$$h(x) = \mathrm{germ}_x(s_x), \quad x \in U_x, \quad s_x \in \mathcal{F}(U_x).$$

By the definition of topology in F, $\dot{s}_x(U_x)$ is an open subset of F which contains $h(x)$. But h is continuous and so there exists an open set $V_x \subset U$ with $x \in V_x \subset U_x$ and $h(V_x) \subset \dot{s}_x(U_x)$; that is, with $h = \dot{s}_x$ on V_x. Thus, we have a covering of the open set U by open sets V_x and an element $t_x = \mathrm{res}^{U_x}_{V_x}(s_x)$ in each $\mathcal{F}(V_x)$. On each pairwise intersection $V_x \cap V_y$, the functions $\dot{s}_x$ and $\dot{s}_y$ agree with h and hence with each other. This means that $\mathrm{germ}_z t_x = \mathrm{germ}_z s_x = \mathrm{germ}_z s_y = \mathrm{germ}_z t_y$ for z in $V_x \cap V_y$, so that $\mathrm{res}^{V_x}_{V_x\cap V_y}(t_x) = \mathrm{res}^{V_y}_{V_x\cap V_y}(t_y)$ by (**) above. The fragmentary meanings $t_x \in \mathcal{F}(V_x)$ thus have the same image under restrictions to pairwise intersections $V_x \cap V_y$ of a covering $U = \bigcup_{x\in U} V_x$. Therefore, by the compositionality condition (C) satisfied by the sheaf $\mathcal{F}$, there exists a fragmentary meaning $t \in \mathcal{F}(U)$ with $\mathrm{res}^U_{V_x}(t) = t_x$. Then at each x, $h(x) = \mathrm{germ}_x(s_x) = \mathrm{germ}_x(t_x) = \mathrm{germ}_x(t)$, so $h = \dot{t}$; the arbitrary cross-section is thus in the image of η. So η is a surjection and this proves that η is an isomorphism.

Thus Frege Duality states that η is isomorphism, i. e. a given sheaf $\mathcal{F}$ is a sheaf of cross-sections of étale bundle $\Lambda(\mathcal{F})$. So we have a functional representation (*) of fragmentary meanings $\eta(U) : s \mapsto \dot{s}$. This result is of a great theoretical importance to clarify the nature of a fragmentary meaning and to give a response to a principal question: what are the fragmentary meanings. This functional representation η of fragmentary meanings is natural in the sense that it is established for each admissible text. We have formulated it at the semantic level of text but the similar representation clearly holds at each semantic level. Such a functional representation answers a question concerning the relationship between the fragmentary and the contextual meanings at each semantic level. Recall that at each semantic level we distinguish the whole, viewed as a sequence of primitive or atomic elements, and the meaningful subexpressions (or fragments) of the whole considered as the subsequences of a given sequence. Any fragmentary meaning s of a fragment U may be represented as a function $\dot{s} : x \mapsto \mathrm{germ}_x s = \dot{s}(x)$ defined on the fragment U following the generalized contextuality principle of Frege so that its value at each primitive constituent x of U is the contextual meaning of x induced by the fragmentary meaning s. In the classic case of a phrase taken in isolation, Frege's contextuality principle defines the contextual meanings of all its constituents (words) by means of the global meaning of the given phrase as a whole. Reciprocally, the meaning of the whole phrase is determined by the sequence of contextual meanings of all its constituents (words).

The statement of the Dual Adjunction Theorem allows to give another equivalent formulation to the generalized compositionality principle. For any admissible text, this theorem claims that a presheaf of fragmentary meanings is a sheaf if and only if the corresponding contextual bundle is étale. So for the phonocentric paradigm of reading, we may give the following bundle-theoretical generalization of the Frege's compositionality principle:

Definition (Frege's Compositionality Principle). *A bundle of contextual meanings naturally attached to any mode of reading of an admissible text is étale.*

For any contextual bundle, this claim is equal to the conjunction of the conditions (E) and (Ct). So we have that (E)&(Ct) is equal to (S)&(C). Recall that the condition (Ct) is a generalization in the narrow sense of the classic Frege's contextuality principle and the condition (C) is a generalization in the narrow sense of the classic Frege's compositionality principle. Put separately, they seem to be in rather difficult relations, but augmented with the corresponding notions of equality (E) and (S), they become equivalent. We prefer to name this resulting Frege's principle as the compositional one because the understanding of the whole text is achieved via composing its local understandings.

4. Outline of text understanding as some inductive process

We shall try now to answer how an admissible text X is interpreted in adopted mode of reading $\mathcal{F}$. Supposing arbitrary length of the text X, the reading process consists in its covering by some family $(U_j)_{j\in J}$ of fragments each having been read during a single action. So one starts the $(i+1)$th resumption of the reading process by keeping in mind some fragmentary meaning $s \in \mathcal{F}(U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_i})$, where U_{j_1} is a fragment read firstly, and so on, and finally U_{j_i} is a fragment read lastly. This fragmentary meaning s were composed as an intermediate result of interpretation process according to sheaf-theoretical formulation of compositionality principle, and one starts to read the $(i+1)$th fragment $U_{j_{i+1}}$ in the context of having grasped s. So we need to describe the process of understanding of the fragment $U_{j_{i+1}}$. Usually one reads a given text in the normal order it bears on, i. e. by beginning from the first sentence, then passing to the second, etc. It may occur to begin a reading from the passage already read. If this is the case, one arrives quickly to a coherent understandings for $(U_{j_1} \cup \cdots \cup U_{j_i}) \cap U_{j_{i+1}}$, and continues the usual reading process. Nevertheless, we shall consider the general case of arbitrary fragment $U_{j_{i+1}}$ because it may happen to read a text by fits and starts, as for example in a library or in a book shop when deciding to get it. In this case, we consider some kind of standard covering of the fragment $U_{j_{i+1}}$ by chapters. So we can reduce the interpretative process for $U_{j_{i+1}}$ to the special case of being a subfragment of some chapter. It is well known that any topological space is a disjoint union of its irreducible subspaces. Recall that X is irreducible if and only if any two non-empty opens of X have a non-empty intersection. So the problem has been reduced to that of grasping some fragmentary meaning of an irreducible fragment. Recall [3] that in a phonocentric topology any irreducible fragment is an interval of the form $[x_1, x_m]$ (denoted sometimes as $I_{x_1}(x_m)$); in other words, it is a sequence of sentences $x_1, x_2, \ldots x_m$, where all x_i

belong to the same paragraph and x_1 is the first sentence of it. The problem is now to explain how the reader does grasp some fragmentary meaning $s \in \mathcal{F}(I_{x_1}(x_m))$. As we have explained in [3, pp. 23-25], an irreducible fragment needs to be read from the beginning to the end. So its reading starts from the sentence x_1. Being a primitive element at the level of text, the first sentence x_1 constitutes an open $\{x_1\}$ which has only one nonempty subset, i. e. itself. So its fragmentary meanings at the level of text are exactly the global meanings of x_1 considered as a space at the level of sentence. So the grasping of some fragmentary meaning of the first sentence x_1 at the level of text may be considered as the process that passes at the semantic level of sentence. This is an inductive basis and equally a step in down recursion from semantic level of text to lower semantic level of sentence where we may continue similarly our considerations. On the other hand, for $\{x_1\}$ being open, the set of its contextual meanings $\mathcal{F}_{x_1}$ coincides with the set of its fragmentary meanings $\mathcal{F}(\{x_1\})$ following our *Remark* on the page 16. So the reading of the first sentence x_1 gives some contextual meaning $s(x_1)$ of it at the level of text. One passes then to the reading of the second sentence x_2 if there is. During this process, the meaning $s(x_1)$ constitutes some context for the understanding of x_2. In result, the reader has caught some contextual meaning $s(x_2)$ of the sentence x_2. And so on, suppose that we have constructed a sequence of contextual meanings $s(x_1), \dots s(x_i)$ and we are going to make an inductive step. In the process of reading of the sentence x_{i+1}, we have in mind the context done by the sentences $x_1, \dots x_i$ and a part of x_{i+1} actually been read, and finally by the interval $I_{x_1}(x_{i+1})$. In result, we grasp some contextual meaning $s(x_{i+1})$ of the sentence of x_{i+1}. So we construct a partial function s on the interval $I_{x_1}(x_{i+1})$ as a sequence $s(x_1), \dots s(x_{i+1})$. This partial function s is really some cross-section over the fragment $I_{x_1}(x_{i+1})$, that is $s \in (\Gamma\Lambda(\mathcal{F}))(I_{x_1}(x_{i+1}))$ and may be thought of as a function on (discrete) time. By Frege Duality, there exists a unique fragmentary meaning $t \in \mathcal{F}(I_{x_1}(x_{i+1}))$ such that $\eta(t) = s$. According to our notations, we have $s = \dot{t}$. So the process of reading gives rise to some fragmentary meaning t to be an element of $\mathcal{F}(I_{x_1}(x_{i+1}))$. This fragmentary meaning t is caught as some kind of *gestalt* organizing the temporal sequence of values $s(x_1), \dots s(x_{i+1})$ into a whole as functional representation $\dot{t}$ of some fragmentary meaning t. And so on, the process continues up to grasping some fragmentary meaning of the whole interval $I_{x_1}(x_m)$. It may happens afterwards, that this fragmentary meaning t as a whole will serve to restore a function $\dot{t} = s$ or some its value $s(x_j)$ in accordance with the functional representation of fragmentary meanings described above. So we have described the process of understanding of an irreducible part of text. As we have treated in [3, ch. 5], the understanding of some sequence of irreducible parts may be thought of as a passage to inductive limit following some inductive system of closed immersions. So the understanding of text is achieved in inductive process where the understanding of a whole text is not postponed to the reading of the last word but is composed during the process of reading because the meaning of the whole text is obtained by means of *extension by title* using *direct image* functor as described in [3]. Following R. Bartes: "the meaning is not at the end of a story but traverses it".[23]

Note that in this inductive process at each level, the whole is understood by means of corresponding compositionality principle applied to get an appropriate fragmentary meanings (as claimed by the condition (C)), whereas any fragmentary meaning is understood by means of functional representation as a cross-section taking its value at x in the corresponding stalk $\mathcal{F}_x$ of contextuality meanings; the latter are determined by means of contextuality principle claimed by the condition (Ct). This interplay between contextuality and compositionality may be thought of as a rigorous version of hermeneutic circle expressed formally by Frege Duality.

[23] R. Barthes, *Introduction à l'analyse structurale des récits*, in *Communications, 1966*, Paris, Seuil, 1981, p. 12; it is our English translation of this quotation.

5. The realm of language

So in the phonocentric paradigm, the true object of study in the natural language semantics should be a pair $(X, \mathcal{F})$, i. e. a text with a sheaf of its fragmentary meanings; any such a couple is called *textual space*. But this representation is possible only in the realm of a language. Following the famous slogan of Wittgenstein, we can say that "to understand a text is to understand a language". Rigorously, this claim may be formulated in the frame of category theory. Likewise our formal hermeneutics describes semantics of a natural language in the category of textual spaces **Logos**. The objects of this category are couples $(X, \mathcal{F})$, where X is a topological space attached naturally to a text and $\mathcal{F}$ is a sheaf of fragmentary meanings defined on X; the morphisms are couples $(f, \theta) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ made of a continuous map $f : X \to Y$ and an f-morphism which respects the given sheaves, i. e. $\theta : \mathcal{G} \to f_*\mathcal{F}$, where f_* is a well-known *direct image* functor. All these notions are discussed at length in our work [3, ch. 8].

Given any admissible text X considered as fixed forever, it yields very naturally a full subcategory $\mathbf{Schl}(X)$ in the category **Logos** of all textual spaces. This category of Schleiermacher $\mathbf{Schl}(X)$ describes the situation then the reader is interested in the exegesis of some particular text as, for example, Sacred Scripture.

Any particular literary genre of texts or discourses defines some finite set of model spaces in the category of textual spaces. Any particular text (or discourse) of a given genre is considered as the 'global variety' (called *formal discourse scheme*) obtained by pasting these model spaces in a certain way. Thus any literary genre defines a corresponding full subcategory (of formal discourse schemes of this genre) in the category **Logos** of all textual spaces. We define an arbitrary formal discourse scheme of a particular genre to be textual space which locally is isomorphic to one of the model textual space of this genre. This definition follows that one usually given to variety of some type in geometry and formalizes in some way the celebrated semantic studies of V. Propp. For the details, we refer the reader to our work [3, ch. 8].

Conclusion

To sum up, in the phonocentric paradigm of reading, we have proposed two equivalent generalizations of a well-known *Frege's principle of compositionality of meaning* in accordance with two model categories $\mathbf{Schl}(X)$ and $\mathbf{Context}(X)$ corresponding naturally to any admissible text X; each of these two formulations of compositionality principle has an advantage over the classic one because it: 1° extends its area from the level of individual sentence to that of a whole text or discourse and 2° takes into account the multiplicity of senses and meanings. The compositionality principle provides so a basis for the whole formal theory of meaning called formal hermeneutics. The categorical point of view leads to the important Frege Duality which is obtained by the same procedure as many of well-known important classic dualities such as Stone, Gelfand-Naimark, and Pontrjagin-van Kampen Duality, completions, free constructions, Galois connections, polarities and many others, and which is defined as an equivalence of categories $\mathbf{Schl}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Context}(X)$ established by the well-known section-functor Γ and germ-functor Λ. Moreover, this equivalence gives rise to some functional representation for any fragmentary meaning which allows to establish some kind of inductive theory of meaning describing the creative process of text understanding. This inductive theory of meaning based on Frege Duality, and also the different categories and functors related to discourse and text interpretation process are the principal objects of study in the formal hermeneutics as we understand it.

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