In: Emar Maier, Corien Bary & Janneke Huitink, eds. (2005) "Proceedings of SuB9"
www.ru.nl/ncs/sub9

# FORMAL HERMENEUTICS BASED ON FREGE DUALITY

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**Abstract**

We outline a sheaf-theoretical framework for the discourse interpretation theory called *formal hermeneutics* in our previous works (1997, 2001, 2002, 2003, 2004). This approach will provide a common categorical paradigm to generalize the classic Frege's compositionality and contextuality principles at each semantic level (text, sentence, word). So revised, these principles are reconciled in a natural dual equivalence between two key categories, called Frege duality, that gives rise to some important *functional representations* of fragmentary meanings. As its application, we develop an *inductive meaning theory* formalizing the creative process of text understanding. Based on Frege duality, our formal hermeneutics intends to reveal the mathematical structures that underlie the natural language understanding process.

## 1 Introduction

In recent years, the discourse interpretation has become a field of intensive investigations in logic, linguistics and the philosophy of language. Despite the great progress in this area, the central problem about the key theoretical structures the discourse interpretation theory should be based upon remains still unsettled. The object of this work is to give an outline of some theory of discourse interpretation named as *formal hermeneutics* and intended to reveal the existence of sheaf-theoretical structures that underlie the process of discourse or text understanding. So the term formal hermeneutics does not mean hermeneutics of any formal system but concerns with the application of formal mathematical methods to analysis of natural language understanding.

We consider some unspecified Indo-European language as a means of communication. This article is mainly concerned with a written type of linguistic communication and so its basic units are texts. All the texts are supposed to be written with good grace and intended for a human understanding; we call them *admissible*.

The classic approaches to semantics of natural language are based on the implicit premise that any language is nothing more than the set of all its correct sentences (and yet only of all its propositions, i. e. the sentences having truth-value). These approaches are very restrictive and yet inadequate to everyday human practice of language communication. When a person wants to express his thoughts to somebody, he needs to utter some discourse or to write some text, and to understand this data is quite another thing than to understand the set of all sentences it was made up. This is why the semantics of natural language should be defined as a discipline studying the discourse and text understanding.

Since antiquity, there exists a concept of discourse interpretation that goes back to Greek mythology. Derived from the Greek verb *hermeneuein*, the term *hermeneutics* was firstly used in the 17th century to mean scriptural exegesis. The Protestant Reformation had a need in the interpretation of Scripture based on the self-sufficiency of the holy text. With the plurality of possible interpretation, it results in a requirement to establish the principles of correct interpretation. As the theory of textual interpretation, hermeneutics began as biblical exegesis and was closely related to philology. The domain of hermeneutics was widely extended in the works of Protestant theologian Friedrich Schleiermacher who created a *general hermeneutics* as a theory of interpretation applicable not only to religious texts but also to a great variety of secular ones. Schleiermacher stressed the importance of the *hermeneutic circle* principle according to which the part is understood in terms of the whole and the whole is understood in terms of its constitutive parts. Following Schleiermacher, this part-whole structure is principal in matter of texts interpretation. It is a kind of compositionality that is meant implicitly to hold at the level of text. So, the theoretical principle of *hermeneutic circle* is a precursor to these of *compositionality* and *contextuality* formulated later in 19th century.

In any way, the usual semantics at the level of sentence is based on the implicit use of compositionality principle according to which the meaning of the whole sentence is a function of the meanings of its constitutive parts. So the hermeneutics may be defined as semantics at the level of text which covers a usual semantics at the level of sentence. It is a reason to call *formal hermeneutics* the sheaf-theoretical discourse interpretation theory which provides a mathematical account of the text understanding process while rejecting the attempt to codify interpretative practice as a kind of calculus.

## 2 Basic Concepts

This chapter will describe the basic concept and the basic constructions which are relevant to the mathematical structures that underlie the natural language understanding process.

First of all, we need to define rigorously what is a text in our formalism. Clearly any text is not just a set of its sentences as the sentence is not a set of its words. Important is the order they ought to be read. In addition, the same words may occur in several places of one sentence and the same sentences may occur in several places of one and the same text. So from a mathematical point of view, we ought to consider a given sentence as a sequence of its words and a given text as a sequence of its sentences. Likewise any part of a considered text is simply a subsequence of a given sequence. Any mathematical structure at a given semantic level (text, sentence, word) is to be defined on the *functional graph* of the corresponding sequence. Henceforth, we shall simply identify a given text with the graph of its corresponding sequence.

### 2.1 Sense, Meaning and Reference

We distinguish the semantic notions of *sense*, *meaning* and *reference* considered to be the basic ones and instead of analysis of these notions in terms of more basic concepts, we seek for key mathematical structures that underlie the process of text understanding.

This triad of concepts formalize a certain distinction that seems to appear in various forms all over the history of language studies. To avoid the possible misunderstanding from the very beginning, we would like to precise our acceptance of these key terms and to point out that our distinction *sense*/*meaning* differs from a classic Frege's *Sinn*/*Bedeutung* distinction, whereas we accept *reference* to be an English translation of Frege's *Bedeutung*. Our aim is not to propose some competitive alternative to Frege's *Sinn*/*Bedeutung* distinction but to find some adequate semantic concepts pertinent as instruments for rigorous formal analysis of text interpretation process. However, one can find our distinction *sense*/*meaning* in the different usage of the word *Sinn* in early Frege's writings before he had formalized *Sinn*/*Bedeutung* distinction in his classic work of 1892.

We consider the meaning as being composed in the interpretative process, where the understanding of a text is not postponed until the end of a text but is present at all semantic levels during the reading process. So the text should have the meaningful parts and the meanings of these parts should determine the meaning of the whole text as it's claimed by the principle of *hermeneutic circle*. We accept the term *fragmentary meaning* of some fragment of a given text to be the content grasped when the reader has understood this fragment in some particular situation of reading, which depends of personality of reader, situation of reading, presuppositions and prejudices summed up in the reader's attitude, that we call by the term *sense* (or *mode of reading*); this *sense* is a kind of semantic orientation in the interpretative process that relates to the totality of text or its fragment, sentence or its syntagma, and involves the reader's subjective premises that what is to be understood constitutes a meaningful whole. At the level of text, it may be historical, moral, allegorical, psychoanalytical, etc. At the level of sentence, it may be literal or metaphoric. At the level of word, it may be literal or figurative. So our acceptance of the term *sense* as a mode of reading is near to that posed in the exegetic concept of the four senses of Sacred Scripture.

### 2.2 Semantic Topology

For the understanding is not postponed until the final word of the final sentence of a given text, the meaning is not at the end of a story but traverses it. So the text should have the meaningful parts and the meanings of these parts determine the meaning of the whole as it is postulated by the principle of *hermeneutic circle*. Any semantic theory tries to explain how these local understandings of the constitutive meaningful parts produce the global understanding of the whole text. The philological investigations are abound in examples of the meaningful fragments cited from the studied texts. It is clear that not all the subsets of a given text are meaningful. Contrary, any meaningful fragment became understood in the process of reading. But the reading of text as well as the utterance of discourse is always a process that develops in time, and so it inherits in some way its order structure. From a linguistic point of view, this order structure is known as a notion of *linearity* or that of *words order*. In fact, it is a natural linear order $\leq$ of sentences the text bears on. It is well-known that any order structure carries several standard topological structures as for example classical *interval topology* generalizing *Euclidean topology* on the real line or other topologies like *upper topologies*, the *Scott topology* or the *Alexandroff topology* (Erné 1991). But it's not a question to graft some topology onto a given text but to observe that any admissible text has an underlying topological structure which arises quite naturally. It seems to be in agreement with our linguistic intuition that

(i) *an arbitrary union of meaningful parts of an admissible text is meaningful;*

(ii) *non-empty intersection of two meaningful parts of an admissible text is meaningful.*

For an admissible text is supposed to be meaningful by definition, it remains only to define the meaning of the empty part of a given text in order to provide it with a topology, where the open sets are all the meaningful parts. Given an admissible text $X$, let the meaning of its empty part $\varnothing$ to be a one-element set pt (e. g. the meaning of the title of $X$ if there is). It shows that this kind of *semantic topology* may be defined so on an arbitrary admissible text (Prosorov 2004, sec. 1.2). A text can be treated as a written speech and so their distinctive feature is a temporality, implicit for the former and explicit for the latter. The natural temporality of phonetic phenomena is a reason to call this semantic topology *phonocentric*.

In (Prosorov 2002, chap. 3), we have defined a *phonocentric* topology at the level of text by specifying in a constructive manner the basis of topology at each sentence $x \in X$ to be the class of intervals $\{l : e \leq l \leq x\}$, where $e$ is the first sentence of the paragraph that contains $x$ or the first one in any paragraph which precedes that containing $x$. In this approach, the opens of a topological base are defined by means of explicit semantic markers the text is endowed with. This definition allows to take into consideration the anaphora and theme/rheme semantic relations. As any constructive definition, it has some advantage of being concrete, but not all semantic relations can be formally recovered by means of explicit text division into paragraph, section, chapter, etc. However this definition covers the majority of examples of meaningful fragments cited in the philological investigation. An uttered discourse has many other expressive means such as stress patterns, intonation patterns, rhythm and pause, which disappear in a written text. Moreover, that constructive definition disregards the influence of the author's vocabulary choice produced on the reader's understanding process. So that definition may be considered as a first approximation to a more fine topological structure the arbitrary admissible text should bears on. Here we will follow our approach of (Prosorov 2004) to define a phonocentric topology in a general axiomatic setting.

Recall that a topological space $X$ is an $A$-space (or *Alexandroff space*) if the set $\mathcal{O}(X)$ of all its open sets is closed under arbitrary intersections. For admissible text being finite, it defines a finite space and thus it is an $A$-space. As we have mentioned above, not all the subsets of an admissible text are meaningful, and hence the semantic topology is not *discrete*. On the other hand, there are certainly the proper meaningful parts in an admissible text, hence the semantic topology is not *coarse*. Moreover, a natural style of text writing should respect good order and arrangement, as each part ought to fall into its right place; the natural process of reading (from right to left and from top to bottom) supposes that understanding of any sentence $x$ of the text $X$ should be achieved on the base of its part already read, because the interpretation cannot be postponed, although it may be made more precise and corrected in further reading and rereading. This is a fundamental feature of a competent reader's linguistic behavior. Following F. Rastier

> Alors que le régime herméneutique des langages formels est celui du suspens, car leur interprétation peut se déployer après le calcul, les textes ne connaissent jamais le suspens de l'interprétation. Elle est compulsive et incoercible. Par exemple, les mots inconnus, les noms propres, voire les non-mots sont interprétés, validement ou non, peu importe. (Rastier 1995, pp. 165-166)

Thus for every pair of distinct sentences $x, y$ of an admissible text $X$, there exists an open (i.e. meaningful) part of $X$ that contains one of them (to be read first) and does not contain the other. Hence the admissible text endowed with the phonocentric topology should satisfy the *separation axiom* $T_0$ of Kolmogoroff and so it is a $T_0$-*space*.

Let $X$ be an admissible text. For a sentence $x \in X$, we define $U_x$ to be the intersection of all the meaningful parts that contain $x$. In other words, for a given sentence $x$, the part $U_x$ is a minimal open neighborhood of $x$.

It is clear that $x \in U_y$ if and only if $y \in \mathrm{cl}(\{x\})$, where $\mathrm{cl}(\{x\})$ denotes the closure of the one-element set $\{x\}$. This relation "$x$ is contained in all open sets that contain $y$" is usually called a *specialization*, and some authors denote it as $y \preceq x$ or $y \leq x$ (e.g. Erné 1991, p. 59) contrary to others who denote it as $x \preceq y$ or $x \leq y$ (e.g. May 2003, p. 2). As for the notation choice, we follow rather May to define a relation $\preceq$ on the text $X$ by setting $x \preceq y$ if and only if $x \in U_y$ or, equivalently, $U_x \subset U_y$. Note that in this notation, for all $x, y \in X$, $x \preceq y$ implies that $x \leq y$, where $\leq$ define the usual order of sentences reading.

The following properties of phonocentric topology and its close relation with partial order structure are the simple translation to linguistic situation of the well-known results for the finite topological spaces.

**Lemma 1.** *The set of all open sets of the kind $U_x$ is a basis of phonocentric topology for $X$. Moreover, it is the unique minimal basis of phonocentric topology for $X$.*

*Proof.* Clearly, for each $x \in U \in \mathcal{O}(X)$, indeed $x \in U_x \subset U$. If $\mathcal{B}$ is another basis, then there is a $B \in \mathcal{B}$ such that $x \in B \subset U_x$. Hence $B = U_x$, so that $U_x \in \mathcal{B}$ for all $x \in X$.

**Lemma 2.** *The relation $\preceq$ is a partial order on $X$.*

*Proof.* The relation $\preceq$ is clearly reflexive and transitive. It is also antisymmetric, because $x \preceq y$ and $y \preceq x$ means $U_x = U_y$; for $X$ being $T_0$-space, it implies that $x = y$.

**Proposition.** *For an admissible text $X$, the phonocentric topology on it defines a partial order structure $\preceq$ on it; the topology can be reconstructed from this partial order in a unique way.*

This is a linguistic variant of a well-known general theorem concerning the relationships between topological and order-theoretical structures on a finite set (Erné 1991, May 2003). Given a partial order $\preceq$ on a finite set $X$, one defines where a $T_0$-topology by means of the basis constituted of all sets $\{l : l \preceq x\}$. The given order structure is reconstructed from this topology by means of specialization.

All these considerations might be repeated with a slight modifications in order to define a *phonocentric* topology at each semantic level of a given admissible text. At each level (text, sentence, word), we distinguish its *primitive elements* which are the points of corresponding topological space considered to be the *whole* at this level. The passage from one semantic level to another immediately superior consists in gluing of the whole space into a point of the higher level space. In the following, we consider mainly a phonocentric topology at the level of text. The open sets of a phonocentric topology on $X$ will be further referred to as *fragments* of $X$.

As soon as we have defined a phonocentric topology, we may seek to interpret some linguistic notion in the topological terms and then to study it by the topological means.

Take for example a well-known property of a literary work to be the communicative unity of meaning. So for any two novels $X$ and $Y$ yet of the same kind, say historical, detective or biographical, their concatenation $Z$ under one and the same cover does not constitute a new one. What does it mean, topologically speaking? We see that for any $x \in X$ there exists an open neighborhood $U$ of $x$ that does not meet $Y$, and for any $y \in Y$ there exists an open neighborhood $V$ of $y$ that does not meet $X$. Thus $Z = X \bigsqcup Y$, that is $Z$ a disjoint union of two non-empty open subsets $X$ and $Y$. Recall that a space $X$ is said to be *connected* if it is not the disjoint union of two non-empty open subsets. It is clear that each minimal basic open set $U_x$ is connected.

## 3 Compositionality

Understanding of some fragment carries no claim of correspondence to reality but is grounded in the conviction that its meaning may be discussed with anybody in some dialogue where it may be finally shared or may be compared with any other one of the same fragment. Following Gadamer, the understanding is based not only on the shared language but principally on the shared experience as a common life-world. This understanding as a presumed agreement on 'what this fragment wants to say' (that is on its communicative content) becomes for the reader its *meaning*. So the reading process involves the historicity of reader and the historicity of text, whence the multiplicity of meanings for any meaningful fragment.

Let $X$ be an admissible text, and let $\mathcal{F}$ be an adopted *sense* or *mode of reading*. From a mathematical point of view, we consider that $\mathcal{F}$ assigns to each open (called further *fragment*) $U$ the set $\mathcal{F}(U)$ of its fragmentary meanings and, following the precept of hermeneutic circle "to understand any part of text in accordance with the understanding of the whole text" taken in a wide sense, that $\mathcal{F}$ assigns to each inclusion $U \subset V$ a map $\mathrm{res}^V_U\colon \mathcal{F}(V) \to \mathcal{F}(U)$, such that $\mathrm{res}^V_V = \mathrm{id}_{\mathcal{F}(V)}$ and $\mathrm{res}^V_U \circ \mathrm{res}^W_V = \mathrm{res}^W_U$ for all nested fragments $U \subset V \subset W$. Mathematically, the data $(\mathcal{F}(U), \mathrm{res}^U_V)$ defines a *presheaf* of fragmentary meanings over $X$ endowed with the phonocentric topology.

It may happen that some fragment of a given text needs many resumption of reading process. So we have to consider the reading process for any fragment $U$ as its covering by some family of subfragments $(U_j)_{j\in J}$ already read. Such a covering of $U$ is said to be *open* if $U = \bigcup_{j\in J} U_j$ and each $U_j$ is open in $X$.

Following Quine, there is no entity without identity; so we need some notion of identity between fragmentary meanings accepted technically as the content grasped during the reading process. Otherwise, it were impossible to consider the fragmentary meanings to be well-defined objects susceptible to set theoretic operations and quantifications with them. The *explicit criterion of equality* between fragmentary meanings that seems to be quite adequate to our linguistic intuition is defined by the following:

**Claim S (Separability).** *Let $X$ be an admissible text, and let $U$ be a fragment of $X$. Suppose that $s, t \in \mathcal{F}(U)$ are two fragmentary meanings of $U$ and there is an open covering $U = \bigcup_{j\in J} U_j$ such that $res^U_{U_j}(s) = res^U_{U_j}(t)$ for all fragments $U_j$. Then $s = t$.*

In other words, two fragmentary meanings of the same fragment are considered to be identical globally if and only if they are identical locally. This definition determines an effective procedure to decide whether two given fragmentary meaning $s$, $t$ of one and the same $U \subset X$ are equal. Following a standard sheaf-theoretical terminology (Tennison 1975, p. 14), a presheaf satisfying the claim (S) is called *separated*. Thus any sense (mode of reading) $\mathcal{F}$ defines some *separated presheaf* of fragmentary meanings over an admissible text $X$.

The precept of hermeneutic circle "to understand the whole text by means of understandings of its parts" is a kind of compositionality principle at the level of text that generalizes the classic Frege's one; so the fragmentary meanings should satisfy the following

**Claim C (Compositionality).** *Let $X$ be an admissible text, and let $U$ be a fragment of $X$. Suppose that $U = \bigcup_{j \in J} U_j$ is an open covering of $U$; suppose we are given a family $(s_j)_{j \in J}$ of fragmentary meanings, $s_j \in \mathcal{F}(U_j)$ for all fragments $U_j$, such that $res^{U_i}_{U_i \cap U_j}(s_i) = res^{U_j}_{U_i \cap U_j}(s_j)$. Then there exists some meaning $s$ of the whole fragment $U$ such that $res^{U}_{U_j}(s) = s_j$ for all fragments $U_j$.*

In other words, the locally compatible fragmentary meanings of an **admissible** text are composable in some global one. Note that we are specifically excluding arbitrary word sequences in our discourse interpretation theory for their lack of this property. This agrees with our intuitive idea of a text written with good grace and intended for a human understanding, which we call *admissible*.

This claim (C) may be considered as a generalization in the narrow sense to the level of text for the classic Frege's principle of compositionality of meaning stated at the level of sentence.

In mathematics, a separated presheaf satisfying the claim (C) of compositionality is called a *sheaf*. Note that for any sheaf, the presence of (S) guarantees the meaning $s$, whose existence is claimed by (C), to be unique as such. So we have motivated the following sheaf theoretic

**Definition (Generalized Frege's Compositionality Principle).** *A separated presheaf of fragmentary meanings naturally attached to any sense (mode of reading) of an admissible text is really a sheaf; its sections over any fragment of the text are the fragmentary meanings; its global sections are the meanings of the whole text.*

We have not yet defined morphisms for these sheaves. To illustrate this notion by means of example, consider e. g. the historical sense $\mathcal{F}$ and the moral sense $\mathcal{G}$ of some biographical text $X$. Let $U \subset V$ be any two fragments of the text $X$. It seems to be very natural that any meaning $s$ of fragment $V$ understood in the historical sense $\mathcal{F}$ gives a certain well-defined meaning $\phi(V)(s)$ of the same fragment $V$ understood in the moral sense $\mathcal{G}$. Hence, for each $V \subset X$, we are given a map $\phi(V)\colon \mathcal{F}(V) \to \mathcal{G}(V)$. To transfer from the meaning $s$ of $V$ in the historical sense to its meaning in the moral sense and then to restrict the latter to a subfragment $U \subset V$, following the precept of hermeneutic circle, is the same operation as to make first the restriction from $V$ to $U$ of the meaning $s$ in the historical sense, and then to transfer from the understanding in the historical sense to the understanding in the moral one. This property of a family $(\phi(V))_{V \in \mathcal{O}(X)}$ may be expressed simply by claiming that the following diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \\ {\scriptstyle \mathrm{res}_U^V}\downarrow & & \downarrow{\scriptstyle \mathrm{res'}_U^V} \\ \mathcal{F}(U) & \xrightarrow[\phi(U)]{} & \mathcal{G}(U) \end{array}$$

commutes for all fragments $U \subset V$ of $X$ (that is $\phi(U) \circ \mathrm{res}_U^V = \mathrm{res'}_U^V \circ \phi(V)$ for all fragments $U \subset V$). This kind of transfer from the understanding in one sense $\mathcal{F}$ to the understanding in some another sense $\mathcal{F}'$ is a usual matter of linguistic communication. Hence, such a family of maps $(\phi(V))_{V \in \mathcal{O}(X)}$ defines a natural transformation of senses $\phi\colon \mathcal{F} \mapsto \mathcal{G}$ considered as functors and hence defines their morphism as sheaves.

Thus, given an admissible text $X$, the data of all sheaves $\mathcal{F}$ of fragmentary meanings together with all its morphisms constitutes some *category* in a strict mathematical sense of the term. Called *category of Schleiermacher* and denoted by $\mathbf{Schl}(X)$, this category supplies a mathematical framework for the part-whole structure in the text understanding formulated by Schleiermacher as the theoretical principle of *hermeneutic circle*. The category of Schleiermacher $\mathbf{Schl}(X)$ describes the exegesis of a given particular text $X$ as, for example, Sacred Scripture.

At the level of sentence, the same considerations generalize the classic Frege's compositionality principle but with words as primitive elements and syntagmas as meaningful fragments (see Prosorov 2002, chap. 4, p. 35).

## 4 Contextuality

So far, we have defined only a notion of fragmentary meaning. To consider (at each semantic level) not only the meanings of fragments but also the meanings of its primitive elements (points of a corresponding topological space), we define a notion of *contextual meaning*. Let $U$, $V$ be two neighborhoods of $x$ and let $\mathcal{F}$ be an adopted sense. Two fragmentary meanings $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are said to induce the same contextual meaning at $x$ if there exists some smaller open neighborhood $W$ of $x$, such that $W \subset U \cap V$ and $\mathrm{res}_W^U(s) = \mathrm{res}_W^V(t) \in \mathcal{F}(W)$. This relation "induce the same contextual meaning at $x$" is an equivalence relation, and any equivalence class of fragmentary meanings agreeing in some neighborhood of $x$ is called a *contextual meaning* of $x$. The set of all equivalence classes is called a *stalk* of $\mathcal{F}$ at $x$ and denoted by $\mathcal{F}_x$. The equivalence class of a fragmentary meaning $s \in \mathcal{F}(U)$ in $\mathcal{F}_x$ is called the *germ of* $s$ *at* $x$ and denoted by $\mathrm{germ}_x s$. Recalling classic Frege's *contextuality principle*, we give the following

**Definition (Generalized Frege's Contextuality Principle).** *A sentence $x$ within a fragment $U$ of an admissible text $X$ has a contextual meaning defined as the germ at $x$ of some fragmentary meaning $s \in \mathcal{F}(U)$, where the sheaf $\mathcal{F}$ is the adopted sense (mode of reading); the set $\mathcal{F}_x$ of all contextual meanings of a sentence $x \in X$ is defined as the stalk of $\mathcal{F}$ at $x$, i. e. as the inductive limit $\mathcal{F}_x = \varinjlim(\mathcal{F}(U), res_V^U)_{U,V \in \mathbb{V}(x)}$.*

In other words, if we have grasped some fragmentary meaning of a given fragment then, for any its sentence, we obtain a corresponding contextual meaning in a canonical way.

According to a well-known *inductive limit characterizing theorem* (see e.g. Tennison 1975, Th. 3.8, p. 5), this contextuality principle, stated at the level of text, is equivalent to the conjunction (E)&(Ct) of the following two claims (E) and (Ct) (see Prosorov 2003, chap. 2).

**Claim E (Equality).** *Let $U$, $V$ be two open neighborhoods of a sentence $x$ and let $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$ be two fragmentary meanings for a given sense (mode of reading) $\mathcal{F}$. Then the equality germ$_x s =$ germ$_x t$ in $\mathcal{F}_x$ between induced contextual meanings of the sentence $x$ holds if and only if there exists an open neighborhood $W$ of $x$ such that $W \subset U$, $W \subset V$ and res$^U_W(s) =$ res$^V_W(t)$.*

The claim (E) is an *explicit criterion of equality* between contextual meanings of a given sentence in the context of a given text.

**Claim Ct (Contextuality).** *Let $\mathcal{F}$ be a sense (mode of reading) adopted for a given text $X$, then for any contextual meaning $f \in \mathcal{F}_x$ of a sentence $x$, there exist a neighborhood $U$ of $x$ and a fragmentary meaning $s \in \mathcal{F}(U)$ such that $f =$ germ$_x s$.*

Stated at the level of text, the claim (Ct) is a generalization in the narrow sense of the classic Frege's contextuality principle; it may be paraphrased as "ask for the meaning of a sentence only in the context of some fragment of a given text".

Our next aim is to describe a bundle-theoretical frame for the generalized Frege's contextuality principle.

For the coproduct $F = \bigsqcup_{x \in X} \mathcal{F}_x$, a map $p \colon F \to X$ defined as $p(\text{germ}_x s) = x$ will be referred to as *projection*.

Every fragmentary meaning $s \in \mathcal{F}(U)$ determines a function $\dot{s} \colon x \mapsto \text{germ}_x s$ to be well-defined on $U$; for each $x \in U$, its value $\dot{s}(x)$ is taken in $\mathcal{F}_x$. This gives rise to the *functional representation* $\eta(U) \colon s \mapsto \dot{s}$ for all fragmentary meanings $s \in \mathcal{F}(U)$.

We define the topology on $F$ by taking as a basis of open sets all the image sets $\dot{s}(U) \subset F$. Given a fragment $U \subset X$, a continuous function $t \colon U \to F$ such that $t(x) \in p^{-1}(x)$ for all $x \in U$ is called a *cross-section*. For any cross-section $t \colon U \to F$, the projection $p$ has the obvious property $p(t(x)) = x$ for all $x \in U$. The topology defined on $F$ makes $p$ and every cross-section of the kind $\dot{s}$ continuous. So we have defined two topological spaces $F$, $X$ and a continuous map $p \colon F \to X$. In topology, this data $(F, p)$ is called a *bundle over the base space* $X$. A *morphism* of bundles from $p \colon F \to X$ to $q \colon G \to X$ is a continuous map $h \colon F \to G$ such that the following diagram

$$\begin{tikzcd} F \arrow[rr, "h"] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

commutes. We have so defined a category of bundles over $X$. A bundle $(F, p)$ over $X$ is called *étale* if $p \colon F \to X$ is a local homeomorphism. The étale bundles constitute a full subcategory in the category of bundles over $X$. It is immediately seen that a bundle of contextual meanings $(\bigsqcup_{x \in X} \mathcal{F}_x, p)$ constructed as above from a given sheaf $\mathcal{F}$ of fragmentary meanings is étale.

Thus, the explicit notion of contextual meaning allows, for any admissible text $X$, to define the category $\mathbf{Context}(X)$ of étale bundles of contextual meanings over $X$ as a framework for the generalized contextuality principle at the level of text.

## 5 Frege Duality

Our aims now is to relate the two key categories underlying the text understanding process.

We define firstly a *germ-functor* $\Lambda\colon \mathbf{Schl}(X) \to \mathbf{Context}(X)$. For each sheaf $\mathcal{F}$, it assigns a bundle $\Lambda(\mathcal{F}) = (F, p)$, where $F = \bigsqcup_{x \in X} \mathcal{F}_x$ and $p$ is projection defined as above; for each morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, the induced map of stalks $\mathcal{F}_x \to \mathcal{G}_x$ gives rise to the morphism of bundles $\Lambda(\phi)\colon \bigsqcup_{x \in X} \mathcal{F}_x \to \bigsqcup_{x \in X} \mathcal{G}_x$.

Secondly, we define a *section-functor* $\Gamma\colon \mathbf{Context}(X) \to \mathbf{Schl}(X)$. We denote a bundle $(F, p)$ over $X$ simply by $F$. For a bundle $F$, we denote the set of all its cross-sections over $U$ by $\Gamma(U, F)$. If $U \subset V$ are open, one has a restriction operation $\mathrm{res}^V_U\colon \Gamma(V, F) \to \Gamma(U, F)$. It's clear that $\mathrm{res}^U_U = \mathrm{id}_{\Gamma(U,F)}$ for any open $U$, and that the transitivity $\mathrm{res}^V_U \circ \mathrm{res}^W_V = \mathrm{res}^W_U$ holds for all nested opens $U \subset V \subset W$. So we have constructed obviously a sheaf $(\Gamma(V, F), \mathrm{res}^V_U)$. For any given morphism of bundles $h\colon E \to F$, we have at once a map $\phi(U)\colon \Gamma(U, E) \to \Gamma(U, F)$ defined for $U \subset X$ as $\phi(U)\colon s \mapsto h \circ s$, which is obviously a morphism of sheaves.

The fundamental theorem of topology states that the section-functor $\Gamma$ and the germ-functor $\Lambda$ establish a *dual adjunction* between the category of presheaves and the category of bundles (over the same topological space) which restricts to a *dual equivalence* of categories (or *duality*) between corresponding full subcategories of sheaves and of étale bundles (see e.g. Lambek and Scott 1986, Mac Lane and Moerdijk 1992). In the linguistic situation, this result yields at the level of text the following

**Theorem (Frege Duality).** *The generalized compositionality and contextuality principles are formulated in terms of categories being in natural duality*

$$\mathbf{Schl}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Context}(X)$$

*established by the section-functor* $\Gamma$ *and the germ-functor* $\Lambda$ *, which are the pair of adjoint functors.*

As many of well-known classic dualities arising from dual adjunctions, Frege duality may be proven as an equivalence between full subcategories of sheaves and étale bundles arising from a dual adjunction between the category of presheaves and the category of bundles. However, a formal translating of such a proof into our linguistic situation compels us to give a semantic interpretation for the latter too vast categories. In a work (Prosorov 2004, p.36), the interested reader will find a sketched proof of Frege duality which restrains within the framework of the categories of sheaves and étale bundles. One of the key point of this proof is the functional representation $\eta(U)\colon s \mapsto \dot{s}$ for all fragmentary meanings $s \in \mathcal{F}(U)$ defined above in chapter 4. This representation $\eta(U)$ is really a bijective correspondence between all the fragmentary meanings $s$ of $U$ and all the genuine functions on $U$ (see Prosorov 2004, p. 36) of the type $x \mapsto \mathrm{germ}_x(s)$, which matches with the restriction operations $s \mapsto \mathrm{res}_U^V(s)$ and $\dot{s} \mapsto \dot{s}|_U$.

This will answer the fundamental question about the nature of fragmentary meanings: namely, each fragmentary meaning $s$ of $U$ is represented as a sequence $(\dot{s}(x))_{x \in U}$ of the contextual meanings of its sentences $x \in U$ grasped during a particular reading process, and *vice versa*.

### 5.1 Inductive Theory of Meaning

The similar Frege duality may be formulated also at the semantic level of sentence and even of word, that gives rise to some *functional representation* of fragmentary meanings at each semantic level; it allows to develop an *inductive theory of meaning* (see Prosorov 2003, chap. 4) describing how runs the process of text understanding. We outline here our exposition of (Prosorov 2003, Prosorov 2004) with a slight modifications.

Consider an admissible text $X$ of arbitrary length interpreted in some adopted mode of reading $\mathcal{F}$. The reading process consists in the open covering of $X$ by some family $(U_j)_{j \in J}$ of fragments, each having been read during a single action. So one starts the $(i+1)$th resumption of the reading process by keeping in mind some fragmentary meaning $s \in \mathcal{F}(U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_i})$, where $U_{j_1}$ is a fragment read firstly, and so on, and finally $U_{j_i}$ is a fragment read lastly. This fragmentary meaning $s$ were composed as an intermediate result of interpretation process according to sheaf-theoretical formulation of compositionality principle, and one starts to read the $(i+1)$th fragment $U_{j_{i+1}}$ in the context of having grasped $s$. So we need to describe the process of understanding of the fragment $U_{j_{i+1}}$. Recall that following our terminological convention, the open $U_{j_{i+1}}$ is a union of the minimal basic opens of the kind $U_x$. So the problem is reduced to explain how the reader grasps some fragmentary meaning of a minimal basic open of the kind $U_x$.

Usually one reads a given text in the normal order $\leq$ it bears on. It may occur to begin a reading from the passage already read. If this is the case, one arrives quickly to a coherent understandings for the part already read and continues the usual reading process. So we may suppose that $U_x \subset U_{j_{i+1}} \cap I(x)$, where $I(x) = \{l : l \leq x\}$.

Suppose that we have explained how the reader has grasped some fragmentary meaning $s'$ of $U_x \cap I(x')$, where $x'$ is the sentence immediately preceding $x$ in the natural sentence order $\leq$. Following the functional representation of fragmentary meanings, this $s'$ is represented by some sequence of contextual meanings of the sentences containing in $U_x \cap I(x')$. So we need to explain how the reader grasps some contextual meaning of $x$ with the purpose to extend the sequence of grasped contextual meanings on the whole $U_x$. But during the process of reading of the sentence $x$ at the level of sentence, where the corresponding Frege compositionality principle holds, the reader grasps some its global meaning at the level of sentence, which is apparently one of its literal meaning. This literal meaning of the sentence $x$ together with the fragmentary meaning $s'$ of $U_x \cap I(x')$ allows to grasp some fragmentary meaning of the whole $U_x$, and whence the contextual meaning of $x$. So the reader has extended the sequence of grasped contextual meanings to the whole $U_x$.

This was the inductive step. As for the basis of induction, note that all the minimal elements of $U_x$ (in the sense of the order $\preceq$) are open singletons; for any such open singleton

$\{y\}$, the set of all its contextual meanings at the level of text is in the bijective correspondence to the set of all its fragmentary meanings at the level of text, which is evidently the set of all global meanings of the sentence $y$ at the level of sentence. This is a recursive step to the inferior semantic level, where the corresponding Frege compositionality principle holds to explain how the reader grasps one of its literary meaning. For $\{y\}$ being open, that is meaningful at the level of text, its literal meaning grasped at the level of sentence is apparently its fragmentary (and equally contextual) meaning at the level of text. Note that the first sentence of a novel is always supposed to be understood in his own context, that is supposed to be open in the phonocentric topology.

### 5.2 Compositionality versus Contextuality

Recall that for any presheaf the property of being sheaf is equivalent to the conjunction (S)&(C); similarly, the conjunction (E)&(Ct) implies the property of being étale for the corresponding bundle. We have noticed above that the claim (Ct) is a generalization in the narrow sense of the classic Frege's contextuality principle and the claim (C) is a generalization in the narrow sense of the classic Frege's compositionality principle. Separately, they seem to be in rather difficult relations, but augmented with the corresponding notions of equality (E) and (S), they give rise to equivalent categories being in adjunction. It's exactly in this sense that we consider compositionality and contextuality as adjoint principles.

At the level of sentence, the same considerations generalize the classic Frege's compositionality and contextuality principles but with words as primitive elements and syntagmas as meaningful fragments.

## 6 The Realm of Language

Thus the true object of study in the natural language semantics should be a pair $(X, \mathcal{F})$, i. e. a text with a sheaf of its fragmentary meanings; any such a couple is called *textual space*. But this representation is possible only in the realm of a language following the famous slogan of Wittgenstein "to understand a text is to understand a language". Rigorously, this claim may be formulated in the frame of category theory. Likewise our formal hermeneutics describes semantics of a natural language in the category of textual spaces $\mathbf{Logos}$. The objects of this category are couples $(X, \mathcal{F})$, where $X$ is a topological space attached naturally to an admissible text and $\mathcal{F}$ is a sheaf of fragmentary meanings defined on $X$; the morphisms are couples $(f, \theta)\colon (X, \mathcal{F}) \to (Y, \mathcal{G})$ made of a continuous map $f\colon X \to Y$ and an $f$-morphism $\theta$ which respects the given sheaves, i. e. $\theta\colon \mathcal{G} \to f_*\mathcal{F}$, where $f_*$ is a well-known *direct image* functor (see e.g. Mac Lane and Moerdijk 1992).

For an admissible text $X$ considered as fixed forever, it yields naturally a full subcategory $\mathbf{Schl}(X)$ in the category $\mathbf{Logos}$. This category of Schleiermacher $\mathbf{Schl}(X)$ describes the exegesis of this particular text $X$ as, for example, Sacred Scripture.

Any particular literary genre of texts (discourses) defines some finite set of model spaces in the category of textual spaces. Any particular text (discourse) of a given genre is considered as the 'global variety' (called *formal discourse scheme*) obtained by pasting these model spaces in a certain way. We define an arbitrary formal discourse scheme of a particular genre to be textual space which locally is isomorphic to one of the model textual space of this genre (see Prosorov 2002, chap. 8). Thus any literary genre defines a corresponding full subcategory (of formal discourse schemes of this genre) in the category $\mathbf{Logos}$ of all textual spaces. This definition follows that one usually given to variety of some type in geometry and formalizes in some way the celebrated semantic studies of V. Propp.

The aforesaid inductive theory of meaning based on Frege duality and the different categories and functors related to discourse and text interpretation process are the principal objects of study in the formal hermeneutics as we understand it.

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