Functional a posteriori error estimates for PDE's

This note gives a short overview of the results obtained in the framework of the functional approach to a posteriori error estimation. Its main goal is to obtain computable guaranteed error bounds that do not involve mesh dependent constants or/and exact solutions of other differential problems.

General concept.
Functional a posteriori error estimates are derived on purely functional grounds without attracting any specific properties of approximations or method used (such as, e.g., Galerkin orthogonality, higher regularity, superconvergence effects). They have one common form

$$\underline{\mathfrak{M}}\ (\mathcal{D}, v) \leq \|u - v\|_V \leq \overline{\mathfrak{M}}\ (\mathcal{D}, v) \qquad \forall v \in V, \tag{1}$$

where u is the exact solution of a problem, v is an approximation from the admissible (energy) space V, $\mathcal{D}$ denotes the set of known data of a problem and other functions and parameters which are in our disposal. The functionals $\overline{\mathfrak{M}}$ (error majorant) and $\underline{\mathfrak{M}}$ (error minorant) must be **directly computable** and such that

$$\overline{\mathfrak{M}}\ (\mathcal{D}, u) = \underline{\mathfrak{M}}\ (\mathcal{D}, u) = 0, \tag{2}$$

$$\overline{\mathfrak{M}}\ (\mathcal{D}, v_k) \to 0, \quad \underline{\mathfrak{M}}\ (\mathcal{D}, v_k) \to 0 \quad \text{as } k \to +\infty. \tag{3}$$

Two–sided functional a posteriori estimates establishes **guaranteed error bounds for conforming approximations of all the types**. They contain **no local constants** depending on a mesh/type of approximations used. If (1) is obtained for a class of boundary–value problems, then approximations computed for a problem in the class are **fully controllable**.

Variational method.
First functional a posteriori estimates were derived for **convex variational problems** with the help of the duality methods in the calculus of variations (see [Re97a, Re97b, ReXa, ReENUMATH97, Re99a]). A consequent exposition of this a posteriori error control theory can be found in [Re00] and in the book [NeRe04].

Transformation of integral identities. In [Re03a], it was introduced another method based on **transformations of integral identities**. It was shown that for linear elliptic problems derived by the variational and non–variational methods approach lead to identical a posteriori estimates. New method was applied to linear parabolic problems in [Re02b, GaRe], to the Maxwell's equation in [Re07a, Re07b], to the reaction–diffusion problem in [ReSa], and to mixed formulations of elliptic problems [Re07b].

Functional a posteriori estimates for mixed approximations.
If a solution is approximated with the help of a **mixed scheme**, then in (1)–(3) u is replaced by a pair (u, p) and v by (v, q), where p is the exact solution of the dual formulation and q is its approximation (see [ReSm, ReSaSm05, Re06]).

Functional a posteriori estimates for incompressible viscous fluids. Estimates for the **Stokes problem** were derived in [Re02a], later in [Re04b] these estimates were obtained by the non–variational method. For different versions of the generalized Stokes problem the estimates were derived in [**?**, ReSt07]. Estimates for flow problems in the rotating coordinate system were derived in [GoMaNeRe07] and for generalized Newtonian fluids in [BiFuRe, FuRe06, Re04b].

Functional a posteriori estimates for modeling errors.
In [Re01], functional a posteriori estimates were applied to the estimation of the error of the 2D plane stress model with respect to a 3D model of linear elasticity. **Dimension reduction errors** for the diffusion elliptic problem has been studied in [ReSaSm04]. The errors caused by **data indeterminacy** were investigated in [Re03b].

Functional a posteriori estimates for variational inequalities. First functional a posteriori estimate for elliptic type obstacle problem was obtained in [BuRe] with the help of the variational method. In [Re00a] this result was generalized to the case of two–sided obstacles and the estimates were also obtained for *elesto–plastic torsion* problem and for a model problem with *friction type boundary conditions*. In [ReVa], the estimates were derived for the general class of nonlinear boundary conditions described in terms of a convex boundary potential.

Functional a posteriori estimates for locally supported quantities.
Functional methods are also applicable to the analysis of **local norms** or other nonnegative quantities (see [Re04a, Re06]).

Estimates for nonconforming approximations. In [ReSaSm03], it was obtained a form of the estimate applicable to such approximations that violate the main boundary conditions. In [LaReTo]), functional a posteriori estimates vere applied to Discontinuous Galerkin method.

Functional a posteriori estimates for Maxwell's equation. Estimates for elliptic Maxwell's model were derived in [Re06].

Nonlinear elliptic problems. General a posteriori error estimation theory is exposed in [Re97c, Re00, NeRe04]. Error analysis of elasto–plastic problems is exposed in [Re96, ReXa]. A posteriori estimates for a class of nonconvex variational problems were derived in [Re99b]. For problems with power growth (or for p–Laplacian type equations) functional a posteriori estimates were derived in [BiRe, Re97a] for power models of viscous fluids in [BiFuRe].

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