The Navier–Stokes Equations and Backward Uniqueness

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We consider the open problem of regularity for $L_{3,\infty}$-solutions to the Navier–Stokes equations. We show that the problem can be reduced to a backward uniqueness problem for the heat operator with lower order terms.

1. Introduction

In this paper, we deal with the classical Cauchy problem for the Navier–Stokes equations:

$$\begin{aligned} &\partial_t v(x,t) + \operatorname{div} v(x,t) \otimes v(x,t) - \Delta\, v(x,t) = -\nabla p(x,t), \\ &\operatorname{div} v(x,t) = 0 \end{aligned} \tag{1.1}$$

for $x \in \mathbb{R}^3$, $t > -T_0$ and

$$v(x,-T_0) = a(x), \quad x \in \mathbb{R}^3. \tag{1.2}$$

The problem (1.1), (1.2) has at least one weak solution v in the so-called Leray–Hopf class (cf. [**1, 2**]).

It is known (cf. [**3**]–[**10**]) that, under the additional condition

$$v \in L_{s,l}(-T_0,T;\mathbb{R}^3), \quad \frac{3}{s} + \frac{2}{l} \leqslant 1, \quad s \geqslant 3, \quad l \geqslant 2,$$

the weak Leray–Hopf solution is unique on the interval $]-T_0,T[$. Moreover, this solution is smooth if $s > 3$. It is an open problem whether weak solutions remain smooth if $s = 3$ and $l = +\infty$ (cf. [**11**]–[**13**] for results related to this problem).

In this paper, we connect the above problem to a backward uniqueness problem for the heat equation. This problem is of independent interest from the point of view of control theory.

We outline the main idea. Assume that $(0,0) \in \mathbb{R}^3 \times]-T_0, +\infty[$ is a singular point of a solution v satisfying

$$\operatorname*{ess\,sup}_{-T_0<t<+\infty} \int\limits_{\mathbb{R}^3} |v(x,t)|^3 dx < +\infty. \tag{1.3}$$

For $\lambda > 0$ we consider the functions

$$v_\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t) \tag{1.4}$$

defined in $\mathbb{R}^3 \times]-T_0/\lambda, +\infty[$. The crucial point is that both (1.1) and (1.3) (with T_0 replaced by T_0/λ) are invariant under scaling (1.4). By the compactness properties of a weak solution, it is possible to pass to the limit as $\lambda \to 0+0$ along a suitable subsequence $\lambda_j \to 0+0$.

The result of this procedure is a solution $u = \lim v_{\lambda_j}$ to the Navier–Stokes equations which is nontrivial (unless (0,0) is a regular point of v), is defined on $\mathbb{R}^3 \times \mathbb{R}$, and vanishes for $t > 0$. Moreover, u is regular in space-time domains of the form

$$\{\mathbb{R}^3 \setminus B(0,R)\} \times]-T_1, +\infty[,$$

where $R = R(T_1)$. We consider the equation for the vorticity $\omega = \nabla \wedge u$

$$\partial_t \omega + \omega_{,k} u_k - u_{,k} \omega_k - \Delta \omega = 0. \tag{1.5}$$

We regard (1.5) as a linear heat equation for ω with lower-order terms

$$\partial_t \omega - \Delta \omega = A_k \omega_{,k} + B\omega, \tag{1.6}$$

where $A = (A_k)$ and $B = (B_{ij})$ are given functions.

Conjecture. [1)] *Assume that A and B have reasonable regularity properties and suitable decay at ∞. Assume that ω is a bounded solution to* (1.6) *in $\{\mathbb{R}^3 \setminus B(0,R)\} \times]-T_1, +\infty[$ which vanishes for $t > 0$. Then $\omega \equiv 0$ in $\{\mathbb{R}^3 \setminus B(0,R)\} \times]-T_1, +\infty[$.*

The main point here is that we do not make any assumptions about ω on $\partial B(0,R)$. In fact, we can consider $\omega\big|_{\partial B(0,R)}$ as a "control," and try to drive ω to zero by prescribing $\omega\big|_{\partial B(0,R)}$. Our conjecture says that exact controllability is never possible in this case. Even the case $A = 0$ and $B = 0$ seems to be interesting, and we have not found it in the literature. One of the results of

[1)] *Added in proof*: The conjecture has been confirmed by Luis Escauriaza and the authors under the assumption that A and B are bounded. In particular, Conjecture H in Sec. 2 is true.

this paper is a proof of the conjecture for $A = 0$ and $B = 0$. We believe that the general case might be approachable by existing methods in the theory of unique continuation. By our results here, such a proof would give a solution to the regularity problem for the Navier–Stokes equations under the condition (1.3).

2. The Notation and Main Results

We denote by $\mathbb{M}^3$ the space of all real 3×3 matrices. Adopting summation over repeated Latin indices, running from 1 to 3, we use the following notation:

$$u \cdot v = u_i v_i, \quad |u| = \sqrt{u \cdot u}, \quad u = (u_i) \in \mathbb{R}^3, \quad v = (v_i) \in \mathbb{R}^3,$$
$$A : B = \operatorname{tr} A^* B = A_{ij} B_{ij}, \quad |A| = \sqrt{A : A},$$
$$A^* = (A_{ji}), \quad \operatorname{tr} A = A_{ii}, \quad A = (A_{ij}) \in \mathbb{M}^3, \quad B = (B_{ij}) \in \mathbb{M}^3,$$
$$u \otimes v = (u_i v_j) \in \mathbb{M}^3, \quad Au = (A_{ij} u_j) \in \mathbb{R}^3, \quad u, v \in \mathbb{R}^3, \quad A \in \mathbb{M}^3.$$

Let ω be a domain in some finite-dimensional space. We denote by $L_m(\omega; \mathbb{R}^l)$ and $W^1_m(\omega; \mathbb{R}^l)$ the Lebesgue and Sobolev spaces respectively of functions from ω into $\mathbb{R}^l$. The norm of the space $L_m(\omega; \mathbb{R}^l)$ is denoted by $\| \cdot \|_{m,\omega}$. If $m = 2$, then we use the abbreviation $\| \cdot \|_\omega \equiv \| \cdot \|_{2,\omega}$.

Let T and T_1 be two parameters such that $T_1 < T$, and let Ω be a domain in $\mathbb{R}^3$. We denote by $Q_{T_1,T} \equiv \Omega \times]T_1, T[$ the space-time cylinder. Space-time points are denoted by $z = (x, t)$, $z_0 = (x_0, t_0)$, etc. Let $L_{m,n}(Q_{T_1,T}; \mathbb{R}^l)$ be the space of measurable $\mathbb{R}^l$-valued functions with the norm

$$\|f\|_{m,n,Q_{T_1,T}} = \begin{cases} \Big(\int\limits_{T_1}^{T} \|f(\cdot,t)\|^n_{m,\Omega}\, dt \Big)^{1/n}, & n \in [1, +\infty[, \\ \operatorname*{ess\,sup}\limits_{t \in]T_1,T[} \|f(\cdot,t)\|_{m,\Omega}, & n = +\infty. \end{cases}$$

In the special case $\Omega = \mathbb{R}^3$ and $T_1 = -T_0$ and $T = +\infty$, we abbreviate

$$L_m(\Omega; \mathbb{R}^3) = L_m, \quad W^1_2(\Omega; \mathbb{R}^3) = H^1, \quad L_{m,n}(Q_{T_1,T}; \mathbb{R}^3) = L_{m,n},$$
$$L_m(T_1, T; W^1_2(\Omega; \mathbb{R}^3)) = L_m(H^1).$$

For integrable in Q_T scalar-valued, vector-valued, and tensor-valued functions we use the following differential operators

$$\partial_t v = \frac{\partial v}{\partial t}, \quad v_{,i} = \frac{\partial v}{\partial x_i}, \quad \nabla p = (p_{,i}), \quad \nabla\, u = (u_{i,j}),$$
$$\operatorname{div}\, v = v_{i,i}, \quad \operatorname{div}\, \tau = (\tau_{ij,j}), \quad \Delta\, u = \operatorname{div}\, \nabla\, u,$$

which are understood in the sense of distributions. Here x_i, $i = 1, 2, 3$, are the Cartesian coordinates of a point $x \in \mathbb{R}^3$ and $t \in]0, T[$ is the time variable.

We recall to the reader the definition of the weak Leray–Hopf solution to the following Cauchy problem (cf. [**2, 1**]):

$$\begin{aligned}&\partial_t v(x,t)+\operatorname{div} v(x,t)\otimes v(x,t)-\Delta\, v(x,t)=-\nabla p(x,t),\\&\operatorname{div} v(x,t)=0\end{aligned}\tag{2.1}$$

for $x\in\mathbb{R}^3$ and $t>-T_0$ and

$$v(x,-T_0)=a(x),\quad x\in\mathbb{R}^3.\tag{2.2}$$

Here, T_0 is a given positive parameter and a is a given divergence-free function from $W^1_2(\mathbb{R}^3;\mathbb{R}^3)$.

Definition 2.1. A divergence-free function

$$v\in L_{2,\infty}\cap L_2(H^1)$$

is called the *weak Leray–Hopf solution* to the Cauchy problem (2.1) and (2.2) if the following conditions holds:

$$\begin{aligned}&\text{for each } w\in L_2 \text{ the function } t\mapsto\int\limits_{\mathbb{R}^3} v(x,t)\cdot w(x)\,dx\quad\text{is}\\&\text{continuous at any point } t\in[-T_0,+\infty[;\end{aligned}\tag{2.3}$$

$$\begin{aligned}&\int\limits_{Q_{-T_0,+\infty}}\{-v\cdot w_t-v\otimes v:\nabla w+\nabla v:\nabla w\}\,dz=0\\&\text{for any divergence-free function } w\in C^\infty_0(Q_{-T_0,+\infty};\mathbb{R}^3);\end{aligned}\tag{2.4}$$

for any $t\in[-T_0,+\infty\,[$ the following energy inequality is valid:

$$\int\limits_{\mathbb{R}^3}|v(x,t)|^2dx+2\int\limits_{-T_0}^{t}\int\limits_{\mathbb{R}^3}|\nabla v|^2\,dxdt'\leqslant\int\limits_{\mathbb{R}^3}|a|^2dx;\tag{2.5}$$

$$\|v(\cdot,t)-a(\cdot)\|_{L_2}\quad\text{as } t\to+0.\tag{2.6}$$

One can show (cf., for example, [**14**]) that if, for a given weak solution, we introduce (normalized) pressure

$$p(x,t)\equiv\frac{1}{4\pi}\int\limits_{\mathbb{R}^3}\frac{1}{|x-y|}\operatorname{div}\operatorname{div}\Big(v(y,t)\otimes v(y,t)\Big)\,dy,\tag{2.7}$$

then the pair v,p satisfies the Navier–Stokes equations in the sense of distributions.

We assume that

$$v\in L_{3,\infty}.\tag{2.8}$$

This allows us to improve properties (2.3) and (2.5). Namely, instead of (2.3), we have

$$\text{for each } w \in L_{3/2} \text{ the function } t \mapsto \int\limits_{\mathbb{R}^3} v(x,t)\cdot w(x)\,dx \quad \text{is continuous at any point } t \in [-T_0,+\infty[. \tag{2.9}$$

The inequality (2.5) takes the form

$$\int\limits_{\mathbb{R}^3} |v(x,t)|^2 dx + 2\int\limits_{-T_0}^{t}\int\limits_{\mathbb{R}^3} |\nabla v|^2\,dxdt' = \int\limits_{\mathbb{R}^3} |a|^2 dx \tag{2.10}$$

for any $t \in [-T_0,+\infty[$. To see that, we note that

$$v \in L_4 \tag{2.11}$$

and

$$\mathrm{div}\,(v\otimes v) \in L_{\frac{4}{3}} \cap L_{\frac{6}{5},2}, \tag{2.12}$$

and, by the coercive $L_{s,l}$-estimates for solutions to the Cauchy problem for the linearized Navier–Stokes equations (cf. [**15, 16, 4, 5**] and [**17**] in the case $s = l$),

$$|\partial_t v|,\ |\nabla^2 v|,\ |\nabla p| \in L_{\frac{4}{3}}(Q_{-T_0+\delta,T}) \cap L_{\frac{6}{5},2}(Q_{-T_0+\delta,T}) \tag{2.13}$$

for any positive numbers δ and T such that $-T_0+\delta < T$. Then (2.10) easily follows from (2.6), (2.11), and (2.13). We also note that (2.3) and (2.10) imply

$$v \in C([-T_0,T];L_2). \tag{2.14}$$

Another important consequence of the assumption (2.8) is the time-continuity of v from the right with values in L_3, i.e.,

$$\|v(\cdot,t)-v(\cdot,t_0)\|_{L_3} \to 0 \quad \text{as } t\to t_0 \text{ and } t > t_0. \tag{2.15}$$

In turn, according to (2.7), (2.8), and (2.15), we see that

$$p \in L_{\frac{3}{2},\infty} \tag{2.16}$$

and

$$\|p(\cdot,t)-p(\cdot,t_0)\|_{L_{\frac{3}{2}}} \to 0 \quad \text{as } t\to t_0 \text{ and } t > t_0. \tag{2.17}$$

Given positive numbers T_1 and R_1, we let

$$\widetilde{Q}(T_1,R_1) = \{R^3 \setminus B(0,R_1)\}\times]-T_1,+\infty[.$$

Definition 2.2. Assume that $A = (A_i)$ and $B = (B_{ij})$ are measurable and bounded functions on $\widetilde{Q}(T_1,R_1)$. We say that the pair (A,B) belongs to the class $\mathcal{C}(T_1,R_1)$ if the following condition holds. Whenever a function $\omega : \widetilde{Q}(T_1,R_1) \to \mathbb{R}^3$ satisfies the conditions

(i) ω and $\nabla\omega$ are bounded and continuous,

$$\lim_{|x|\to+\infty} |\omega(x,t)| = 0 \quad \text{uniformly in } t,$$

(ii) $\partial_t\omega - \Delta\omega = A_k\omega_{,k} + B\omega$ in $\widetilde{Q}(T_1, R_1)$ in the sense of distributions,

(iii) if $\omega(x,t) = 0$, $x \in \mathbb{R}^3 \setminus B(0,R_1)$, and $t > 0$,

we have $\omega(x,t) \equiv 0$ in $\widetilde{Q}(T_1, R_1)$.

Conjecture H. *Assume that for $k = 0, 1, \dots$ the functions $\nabla^k A$ and $\nabla^k B$ are Hölder continuous and bounded in $\widetilde{Q}(T_1, R_1)$,*

$$\lim_{R\to+\infty} \sup_{x\in\mathbb{R}^3\setminus B(0,R)} \sup_{-T_1<t<+\infty} |\nabla^k A(x,t)| + |\nabla^k B(x,t)| = 0, \tag{2.18}$$

and

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^3\setminus B(0,R_1)} |\nabla^k A(x,t)| + |\nabla^k B(x,t)| = 0. \tag{2.19}$$

Then $(A, B) \in \mathcal{C}(T_1, R_1)$.

Conjecture G (restricted conjecture H)**.** *Assume that for a solenoidal vector-valued function w the functions $A = -w$ and $B = \nabla w$ satisfy the assumptions of Conjecture* H. *Then $(A, B) \in \mathcal{C}(T_1, R_1)$.*

The main result of this paper is contained in the following assertion.

Theorem 2.3. *Suppose that Conjecture* G *is true. Then any Leray–Hopf solution to the Cauchy problem* (2.1), (2.2) *belonging to $L_{3,\infty}$ is smooth.*

We believe that Conjecture H is true although we have a proof of it only for the case $A = 0$ and $B = 0$ (cf. Sec. 4 for details).

3. Blow-Up

Assume that the statement of Theorem 2.3 is false. Without loss of generality, we can assume that a singular point appears at the time $t = 0$ and is located at the origin.

Under our assumptions on v and p, the pair (v, p) is a *suitable weak solution* (for the definition we refer the reader to [**18**]), i.e., it satisfies the local energy inequality. In our case, it satisfies even the local energy identity. Since $z = 0$ is a singular point of our weak solution, the theory of partial regularity for suitable weak solutions to the Navier–Stokes equations says (cf. [**14**]) that there exists a sequence of positive numbers R_k such that $R_k \to 0$ as $k \to +\infty$ and

$$A(R_k) \equiv \sup_{-R_k^2\leqslant t\leqslant 0} \frac{1}{R_k} \int\limits_{B(0,R_k)} |v(x,t)|^2 dx > \varepsilon_\star \tag{3.1}$$

for all $k \in \mathbb{N}$. Here, $\varepsilon_\star$ is an absolute positive constant and $B(x,R)$ stands for the three-dimensional ball of radius R with the center at the point x.

We extend functions v and p to the whole space $\mathbb{R}^{3+1}$ in the following way:

$$\widetilde{v}(x,t) = \begin{cases} v(x,t), & t \geqslant -T_0, \\ 0, & t < -T_0, \end{cases} \quad x \in \mathbb{R}^3,$$

$$\widetilde{p}(x,t) = \begin{cases} p(x,t), & t \geqslant -T_0, \\ 0, & t < -T_0, \end{cases} \quad x \in \mathbb{R}^3.$$

Let

$$v^{R_k}(x,t) = R_k \widetilde{v}(R_k x, R_k^2 t), \quad p^{R_k}(x,t) = R_k \widetilde{p}(R_k x, R_k^2 t)$$

for $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. It is obvious that for any $t \in \mathbb{R}$

$$\int\limits_{\mathbb{R}^3} |v^{R_k}(x,t)|^3 dx = \int\limits_{\mathbb{R}^3} |\widetilde{v}(x,t)|^3 dx \tag{3.2}$$

and

$$\int\limits_{\mathbb{R}^3} |p^{R_k}(x,t)|^{3/2} dx = \int\limits_{\mathbb{R}^3} |\widetilde{p}(x,t)|^{3/2} dx. \tag{3.3}$$

Hence, without loss of generality, we can assume that

$$v^{R_k} \overset{\star}{\rightharpoonup} u \quad \text{in } L_\infty(\mathbb{R}; L_3) \quad \text{as } k \to +\infty, \tag{3.4}$$

where $\operatorname{div} u = 0$ in $\mathbb{R}^3 \times \mathbb{R}$ and

$$p^{R_k} \overset{\star}{\rightharpoonup} q \quad \text{in} \quad L_\infty(\mathbb{R}; L_{3/2}) \quad \text{as } k \to +\infty. \tag{3.5}$$

To get more information about the boundedness of various norms for functions v^{R_k} and p^{R_k}, we fix a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^{3+1})$ and introduce a function φ^{R_k} as follows:

$$\varphi(y,\tau) = R_k \varphi^{R_k}(R_k y, R_k^2 \tau), \quad y \in \mathbb{R}^3, \quad \tau \in \mathbb{R}.$$

Let R_k be so small that

$$\operatorname{spt} \varphi \subset \left\{ (y,\tau) \parallel \tau > -\frac{T_0}{R_k} \right\} \Longrightarrow \operatorname{spt} \varphi^{R_k} \subset \{(x,t) \parallel t > -T_0\}.$$

Then

$$2 \int\limits_{-T_0}^{+\infty} \int\limits_{\mathbb{R}^3} \varphi^{R_k} |\nabla v|^2 \, dz = \int\limits_{-T_0}^{+\infty} \int\limits_{\mathbb{R}^3} \{ |v|^2 (\Delta \varphi^{R_k} + \partial_t \varphi^{R_k}) + v \cdot \nabla \varphi^{R_k} (|v|^2 + 2p) \} dz. \tag{3.6}$$

Making the change of variables, we obtain the identity

$$2\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^3}\varphi|\nabla\, v^{R_k}|^2\,dz=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^3}\{|v^{R_k}|^2(\Delta\varphi+\partial_t\varphi)$$
$$+\,v^{R_k}\cdot\nabla\varphi(|v^{R_k}|^2+2p^{R_k})\}dz. \qquad (3.7)$$

¿From (3.2), (3.3), and (3.7) it follows that for any domain $Q \Subset \mathbb{R}^{3+1}$

$$\int\limits_Q |\nabla v^{R_k}|^2\,dz \leqslant c_1(Q) < +\infty. \qquad (3.8)$$

We emphasize that the constant in (3.8) is independent of R_k. By standard arguments, including multiplicative inequalities and $L_{s,l}$-coercive estimates for solutions to the nonstationary Stokes equations, we obtain the bound

$$\int\limits_Q (|v^{R_k}|^4+|\partial_t v^{R_k}|^{\frac43}+|\nabla^2 v^{R_k}|^{\frac43}+|\nabla p^{R_k}|^{\frac43})\,dz \leqslant c_2(Q), \qquad (3.9)$$

which, together with (3.4) and (3.5), implies

$$v^{R_k}\longrightarrow u \quad \text{in} \quad L_3(Q;\mathbb{R}^3) \qquad (3.10)$$

for $Q \Subset \mathbb{R}^{3+1}$. By (3.4), (3.5), and (3.9), we find

$$v^{R_k}\longrightarrow u \quad \text{in} \quad C([a,b];L_2(\Omega;\mathbb{R}^3)) \qquad (3.11)$$

for any $-\infty<a<b<+\infty$ and $\Omega\Subset\mathbb{R}^3$.

Combining all information about limit functions u and q (cf. (3.2)–(3.11)), we conclude that

$$\int\limits_Q (|u|^4+|\nabla u|^2+|\partial_t u|^{\frac43}+|\nabla^2 u|^{\frac43}+|\nabla q|^{\frac43})\,dz \leqslant c_3(Q) \qquad (3.12)$$

for any $Q \Subset \mathbb{R}^{3+1}$ and

$$u\in C([a,b];L_2(\Omega;\mathbb{R}^3)) \qquad (3.13)$$

for any $-\infty<a<b<+\infty$ and $\Omega\Subset\mathbb{R}^3$;

functions u and q satisfy the Navier–Stokes equations a.e. in $\mathbb{R}^3$; (3.14)

$$2\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^3}\varphi|\nabla\, u|^2\,dz=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^3}\{|u|^2(\Delta\varphi+\partial_t\varphi)+u\cdot\nabla\varphi(|u|^2+2q)\}dz \qquad (3.15)$$

for all functions $\varphi\in C_0^\infty(\mathbb{R}^{3+1})$.

By (2.11)–(2.14), the pair (u,q) is a suitable weak solution to the Navier–Stokes equation in $\mathbb{R}^{3+1}$.

Our next observation on limit functions comes from (2.15) and (2.17) for $t_0 = 0$. For any positive numbers R and t (2.15) implies

$$\int\limits_{B(0,R)} |v^{R_k}(y,t)|^3 dy = \int\limits_{B(0,R_kR)} |v(x,R_k^2 t)|^3 dx \longrightarrow 0$$

as $R_k \to 0$. This means that

$$u(\cdot,t) = 0, \quad t > 0. \tag{3.16}$$

In the same way, we deduce from (2.17) that

$$q(\cdot,t) = 0, \quad t > 0. \tag{3.17}$$

Finally, according to (3.1),

$$\sup_{-R_k^2 \leqslant t \leqslant 0} \frac{1}{R_k} \int\limits_{B(0,R_k)} |v(x,t)|^2 dx = \sup_{-1 \leqslant t \leqslant 0} \int\limits_{B(0,1)} |v^{R_k}(x,t)|^2 dx > \varepsilon_\star$$

for all $k \in \mathbb{N}$ and, by (3.11), we obtain

$$\sup_{-1 \leqslant t \leqslant 0} \int\limits_{B(0,1)} |u(x,t)|^2 dx > \varepsilon_\star. \tag{3.18}$$

Proof of Theorem 2.3. First, we show that there exits positive numbers R_1 and T_1 such that for any $k = 0, 1, \ldots$ the function $\nabla^k u$ is Hölder continuous and bounded on the set

$$\widetilde{Q}(2T_1, R_1/2) = \mathbb{R}^3 \setminus B(0, R_1/2) \times]-2T_1, +\infty[.$$

To this end, we fix an arbitrary number $T_1 > 2$ and note that

$$\int\limits_{-4T_1}^{+\infty} \int\limits_{\mathbb{R}^3} (|u|^3 + |q|^{3/2}) dz = \int\limits_{-4T_1}^{0} \int\limits_{\mathbb{R}^3} (|u|^3 + |q|^{3/2}) dz < +\infty.$$

Therefore,

$$\int\limits_{-4T_1}^{+\infty} \int\limits_{\mathbb{R}^3 \setminus B(0,R)} (|u|^3 + |q|^{\frac{3}{2}}) dz \to 0 \quad \text{as } R \to +\infty.$$

This means that for given $\varepsilon > 0$ there exists a number $R_1(\varepsilon, T_1) > 4$ such that

$$\int\limits_{-4T_1}^{+\infty} \int\limits_{\mathbb{R}^3 \setminus B(0,R_1/24)} (|u|^3 + |q|^{\frac{3}{2}}) dz < \varepsilon. \tag{3.19}$$

Now, assume that $z_0 = (x_0, t_0) \in \widetilde{Q}(2T_1, R_1/2)$. Then

$$Q(z_0, 1) \equiv B(x_0, 1) \times]t_0 - 1, t_0[\subset \{\mathbb{R}^3 \setminus B(0, R_1/4)\} \times]-4T_1, +\infty[.$$

By (3.19), one can claim that for any ε

$$\int\limits_{t_0-1}^{t_0}\int\limits_{B(x_0,1)}(|u|^3+|q|^{\frac{3}{2}})dz<\varepsilon \tag{3.20}$$

for $z_0\in\widetilde{Q}(2T_1,R_1/2)$, where $T_1>2$ and $R_1(\varepsilon,T_1)>4$. It follows from (3.20), the Caffarelli–Kohn–Nirenberg theorem, and the regularity theory for solutions to the Stokes equations (we refer to [**18**] and [**11**, Proposition 2.1] for details) that for each $k=0,1,\dots$ there exists a number $c^{(k)}$ independent of z_0 such that

$$\sup_{z\in Q(z_0,1/2)}|\nabla^k u(z)|\leqslant c^{(k)}<+\infty.$$

The Hölder continuity of $\nabla^k u$ on $\widetilde{Q}(2T_1,R_1/2)$ is also a consequence of the regularity theory for the Stokes equations and bootstrap arguments.

It remains to show that the functions $A=-u$ and $B=\nabla u$ satisfy the conditions (2.18) and (2.19). To see this, we introduce the sequence of functions

$$v_e^m(x,t)=u(x+me,t),\quad p_e^m(x,t)=q(x+me,t)$$

for $x\in B(0,2)$, $e\in B(0,1)$, and $-2T_1<t<+\infty$. Obviously, for each fixed $m\in\mathbb{N}$ and for each fixed $e\in B(0,1)$ the pair (v_e^m,p_e^m) is a suitable weak solution to the Navier–Stokes equations in $B(0,2)\times]-2T_1,+\infty[$. Moreover,

$$\lim_{m\to+\infty}\int\limits_{t_0-1}^{t_0}\int\limits_{B(0,1)}\big(|v_e^m|^3+|p_e^m|^{\frac{3}{2}}\big)dxdt=0.$$

By the above arguments, one can claim that

$$\|\nabla^k v_e^m\|_{C^\alpha(\overline{Q}(z_0,1/2))}\leqslant\widetilde{c}_k$$

for some $\alpha\in]0,1[$. Here, $z_0=(0,t_0)$, $C^\alpha(\overline{Q})$ is the space of functions that are continuous on the compact set $\overline{Q}$ with respect to the usual parabolic distance. It is important to note that the constants $\widetilde{c}_k$, $k=0,1,\dots$, are independent of m, e, and t_0. Thus, we see that for $k=0,1,\dots$

$$\sup_{e\in B(0,1)}\ \sup_{B(0,1/2)}\ \sup_{-T_1\leqslant t<+\infty}|\nabla^k v_e^m(x,t)|\to 0$$

as $m\to+\infty$. This implies (2.18). Now, (2.19) follows from (2.18) and the Hölder continuity of $\nabla^k u$ on $\widetilde{Q}(2T_1,R_1/2)$.

Now, let ω be the vorticity of u, i.e.,

$$\omega=\nabla\wedge u.$$

The function ω satisfies the equation

$$\partial_t\omega+u_k\omega_{,k}-\omega_k u_{,k}-\Delta u=0,\quad \text{in } \mathbb{R}^3\times]-T_1,\infty[.$$

By the conjecture G,

$$\omega(z) = 0 \quad \text{if } z \in \widetilde{Q}(T_1, R_1). \tag{3.21}$$

On the other hand, there exists an open subset $\mathcal{O} \subset]-T_1, 0[$ such that $|\mathcal{O}| = T_1$ and, for each $t \in \mathcal{O}$, the function u is analytic in spatial variables. But then ω is also an analytic function in the same variables. Therefore, (3.21) implies that

$$\omega(\cdot, t) = 0, \quad t \in \mathcal{O}. \tag{3.22}$$

This means that for each $t \in \mathcal{O}$ the function $u(\cdot, t)$ is harmonic and has the finite $L_3(\mathbb{R}^3; \mathbb{R}^3)$-norm. It turn, this fact leads to the identity

$$u(\cdot, t) = 0, \quad t \in \mathcal{O}.$$

Hence $u = 0$ a.e. in $\mathbb{R}^3 \times]-T_1, \infty[$. This contradicts (3.18). Theorem 2.3 is proved.

4. A Backward Uniqueness Theorem for the Heat Equation

In this section, we introduce additional notation:

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,\|\, x = (x_1, x_2, \dots, x_n) = (x_i),\ x_n > 0\},$$
$$Q_T = \mathbb{R}^n_+ \times]0, T[,$$

where T is a positive fixed number.

Theorem 4.1. *Let $u : Q_T \to \mathbb{R}$ be a bounded smooth function satisfying the heat equation $\partial_t u = \Delta u$ in Q_T. Assume that there exists a nonempty open set $\Omega \subset \mathbb{R}^n_+$ such that*

$$\lim_{t \to T-0} \int\limits_{\Omega} |u(x,t)| dx = 0.$$

Then $u \equiv 0$ in Q_T.

Proof. Using the regularity theory for the heat equation and the fact that smooth solutions to the heat equation are analytic in spatial variables, we see that one can extend u by zero to the set $Q = \mathbb{R}^n_+ \times]0, +\infty[$, and the extension, also denoted by u, is smooth, satisfies the heat equation in Q, and vanishes for $t \geqslant T$. Also, replacing $u(x,t)$ by $u(x_1, x_2, \dots, x_{n-1}, x_n + y_n, t + s)$ for small $y_n > 0$ and $s > 0$, we can assume that all the derivatives of u are well defined, bounded, and continuous in the closure $\overline{Q}$ of Q. Making these simplifying assumptions, we now prove the theorem in several steps.

Step 1. Reduction to the case $n = 1$. The obvious idea here is to use the Fourier transformation along $x' = (x_1, x_2, \dots, x_{n-1})$. For each $t > 0$ and $x_n \geqslant 0$ we define a distribution $\widetilde{u}(\cdot, x_n, t)$ on $\mathbb{R}^{n-1}$ by the formula

$$\langle \widetilde{u}(\cdot, x_n, t), \varphi(\cdot)\rangle = \int\limits_{\mathbb{R}^{n-1}} dx' u(x', x_n, t) \int\limits_{\mathbb{R}^{n-1}} d\xi' e^{t|\xi|^2 - ix'\cdot\xi'} \varphi(\xi').$$

Here, $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ and $\xi' = (\xi_1, \xi_2, \dots, \xi_{n-1})$. Under suitable assumptions, $\widetilde{u}(\cdot, x_n, t)$ is a function and we have

$$\widetilde{u}(\xi', x_n, t) = e^{t|\xi'|^2} \int\limits_{\mathbb{R}^{n-1}} u(x', x_n, t) e^{-ix'\cdot\xi'} dx'.$$

A simple calculation shows that for each fixed $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ the function

$$\widetilde{u}_\varphi(x_n, t) = \langle \widetilde{u}(\cdot, x_n, t), \varphi(\cdot)\rangle$$

is bounded in $\mathbb{R}_+ \times]0, +\infty[$, satisfies the heat equation in $\mathbb{R}_+ \times]0, +\infty[$, and vanishes for $t > T$. We now see that this is enough to prove the case $n = 1$.

In what follows, we use the notation $Q = \mathbb{R}_+ \times]0, +\infty[$ and denote by (x, t) points of Q.

Step 2. Reduction to the case $|u(x,t)| \leqslant Ce^{-\alpha x}$. This can be achieved by the following change of variables:

$$u(x, t) = v(x + 2\alpha t, t) e^{\alpha x + \alpha^2 t}, \quad \alpha > 0.$$

The function v is defined in a domain different from Q_T, but we can obviously achieve by a suitable shift that the domain of v contains a domain of the form Q in which the theorem is violated if v does not vanish identically. Moreover, v has the required decay as $x \to +\infty$.

Step 3. Proof in the case $n = 1$ and $|u(x,t)| \leqslant Ce^{-\alpha x}$. We extend u to all $\mathbb{R} \times \mathbb{R}$ by requiring that the extension be an even function of x vanishing for $t \in] - \infty, 0[\cup]T, +\infty[$. The extended function has a discontinuity in t at $t = 0$, but it is smooth in t (for a fixed x) for $t \in]0, +\infty[$.

Let $a(x) = u(x, 0)$, and let

$$g(t) = \lim_{x \to 0+0} 2\frac{\partial u}{\partial x}(x, t).$$

Clearly,

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = -\delta(x) g(t) + \delta(t) a(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{4.1}$$

where δ denotes the Dirac distribution.

Denoting by $\widehat{g}$ and $\widehat{a}$ the Fourier transformations of g and a respectively, we note that (4.1) implies

$$\widehat{g}(i\xi^2) = \widehat{a}(\xi), \quad \xi \in \mathbb{R}. \tag{4.2}$$

We now look at the function $\widehat{g}$ in more detail. We have

$$g(\tau) = \int_{\mathbb{R}} g(t)e^{-i\tau t}\,dt = \int_0^T g(t)e^{-i\tau t}\,dt. \tag{4.3}$$

Hence $\widehat{g}$ is defined for each $\tau \in \mathbb{C}$ and is holomorphic in $\mathbb{C}$. Moreover, standard calculations together with (4.2) and the decay property of a imply that

$$\widehat{g}(\tau) = \mathcal{O}\Big(\frac{1}{|\tau|}\Big) \quad \text{as } \tau \to \infty \text{ in } K, \tag{4.4}$$

where $K = \{\tau \in \mathbb{C} | \operatorname{Re}\tau = 0 \text{ or } \operatorname{Im}\tau = 0\}$, i.e., K is the union of the real and imaginary axes. In fact, (4.4) along the real axis and along the negative part of the imaginary axis follows from (4.3) and integration by parts, whereas (4.4) along the positive part of imaginary axis follows from (4.2) and the fact that

$$a(\xi) = \mathcal{O}\Big(\frac{1}{|\xi|^2}\Big) \quad \text{as } \xi \to \infty \text{ and } \xi \in \mathbb{R}.$$

STEP 4. The last step in the proof is a simple lemma about holomorphic functions.

Lemma 4.2. *Let $K \subset \mathbb{C}$ be the union of the real and imaginary axes. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying the following two conditions*:

$$|f(z)| \leqslant Ae^{a|z|}, \quad z \in \mathbb{C},$$

for some positive constants a and A,

$$f(\tau) = \mathcal{O}\Big(\frac{1}{|\tau|}\Big) \quad \textit{as } \tau \to \infty \quad \textit{in } K.$$

Then $f \equiv 0$.

The proof of the lemma is left to the reader as an exercise. We note that Lemma 4.2 follows from the Phragmén–Lindelöf theorem for an angle (we refer to [**19**, Theorem 7.5] for details). Theorem 4.1 is proved. □

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