# On Smoothness of $L_{3,\infty}$-Solutions to the Navier-Stokes Equations up to Boundary

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**Abstract** We show that $L_{3,\infty}$-solutions to the three-dimensional Navier-Stokes equations near a flat part of the boundary are smooth.

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**Key Words**: boundary regularity, the Navier-Stokes equations, suitable weak solutions, backward uniqueness.

## 1 Introduction

In the present paper, we are going to prove smoothness of the so-called $L_{3,\infty}$-solutions to the Navier-Stokes equations up to a flat part of the boundary. In particular, Theorem 1.1 proved below implies the result announced in [3]. It was stated there that $L_{3,\infty}$-solutions to the initial boundary value problems for the Navier-Stokes equations in a half space are smooth if the initial data are smooth. As in the case of the Cauchy problem, we deduce this statement from the theorem on local regularity of $L_{3,\infty}$-solutions near a flat part of the boundary.

The main idea how to treat boundary regularity of $L_{3,\infty}$-solutions is similar to the case of interior regularity: reduction to a backward uniqueness problem for the heat operator, see [15], [2], and [4]. The second part of such analysis has been already done in [3], where the backward uniqueness result for the heat operator in a half space was established.

However, serious difficulties occur if we scale and blow up the Navier-Stokes equations at singular boundary points. In particular, since $L_{3,\infty}$-norm is invariant with respect to the natural scaling, the global $L_{3,\infty}$-norm of the blow-up velocity is bounded. In the interior case, we were able to prove global boundedness of $L_{\frac{3}{2},\infty}$-norm of the the blow-up pressure. We do not know whether the same is true near the boundary. If it would be so, the proof of boundary regularity could be essentially simplified. Unfortunately, we cannot even show that there is a reasonable global norm of the blow-up pressure which is finite. This makes our proof a bit tricky. Key points are Lemma 4.1 and suitable decomposition of the pressure.

The main result of the paper is as follows.

**Theorem 1.1** *Let a pair of functions $v$ and $p$ has the following differentiability properties:*

$$v \in L_{2,\infty}(Q^+) \cap W^{1,0}_2(Q^+) \cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(Q^+), \qquad p \in W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+). \tag{1.1}$$

*Here, $Q^+ = \{z = (x,t) \parallel |x| < 1,\ x_3 > 0,\ -1 < t < 0\}$.*

*Suppose that $v$ and $p$ satisfy the Navier-Stokes equations a.e. in $Q^+$, i.e.:*

$$\left.\begin{array}{c}\partial_t v + \operatorname{div} v \otimes v - \Delta v = -\nabla p \\ \operatorname{div} v = 0\end{array}\right\} \quad in \quad Q^+ \tag{1.2}$$

*and the boundary condition*

$$v(x,t) = 0, \qquad x_3 = 0 \ and \ -1 \leq t \leq 0. \tag{1.3}$$

*Assume, in addition, that*

$$v \in L_{3,\infty}(Q^+). \tag{1.4}$$

*Then $v$ is Hölder continuous in the closure of the set*

$$Q^+(1/2) = \{z = (x,t) \parallel |x| < 1/2,\ x_3 > 0,\ -(1/2)^2 < t < 0\}.$$

Explanations why conditions (1.1) are natural can be found in paper [13], see Theorem 2.2 there. We just briefly note that any weak Leray-Hopf solution to initial boundary value problems in a half space together with the associated pressure satisfies (1.1). So, the real additional assumption of Theorem 1.1 is condition (1.4).

# 2 Some preliminary estimates for the pressure

We denote by $\mathbb{M}^3$ the space of all real $3 \times 3$ matrices. Adopting summation over repeated Latin indices, running from 1 to 3, we shall use the following notation

$$u \cdot v = u_i v_i, \quad |u| = \sqrt{u \cdot u}, \quad u = (u_i) \in \mathbb{R}^3,\ v = (v_i) \in \mathbb{R}^3;$$

$$A : B = \mathrm{tr} A^* B = A_{ij} B_{ij}, \quad |A| = \sqrt{A : A},$$

$$A^* = (A_{ji}), \quad \mathrm{tr} A = A_{ii}, \quad A = (A_{ij}) \in \mathbb{M}^3,\ B = (B_{ij}) \in \mathbb{M}^3;$$

$$u \otimes v = (u_i v_j) \in \mathbb{M}^3, \quad Au = (A_{ij} u_j) \in \mathbb{R}^3, \quad u, v \in \mathbb{R}^3,\ A \in \mathbb{M}^3.$$

Let $\omega$ be a domain in some finite-dimensional space. We denote by $L_m(\omega)$ and $W^l_m(\omega)$ the known Lebesgue and Sobolev spaces. The norm of the space $L_m(\omega)$ is denoted by $\|\cdot\|_{m,\omega}$. If $m = 2$, then we use the abbreviation $\|\cdot\|_\omega \equiv \|\cdot\|_{2,\omega}$.

Let $T$ be a positive parameter, $\Omega$ be a domain in $\mathbb{R}^3$. We denote by $Q_T \equiv \Omega \times ]0, T[$ the space-time cylinder. Space-time points are denoted by $z = (x, t)$, $z_0 = (x_0, t_0)$, and etc.

For summable in $Q_T$ scalar-valued, vector-valued, and tensor-valued functions, we shall use the following differential operators

$$\partial_t v = \frac{\partial v}{\partial t}, \quad v_{,i} = \frac{\partial v}{\partial x_i}, \quad \nabla p = (p_{,i}), \quad \nabla\, u = (u_{i,j}),$$

$$\mathrm{div}\, v = v_{i,i}, \quad \mathrm{div}\, \tau = (\tau_{ij,j}), \quad \Delta\, u = \mathrm{div}\, \nabla\, u,$$

which are understood in the sense of distributions. Here, $x_i$, $i = 1, 2, 3$, are the Cartesian coordinates of a point $x \in \mathbb{R}^3$, and $t \in ]0, T[$ is a moment of time.

Let $L_{m,n}(Q_T)$ be the space of measurable $\mathbb{R}^l$-valued functions with the following norm

$$\|f\|_{m,n,Q_T} = \begin{cases} (\int\limits_0^T \|f(\cdot, t)\|^n_{m,\Omega}\, dt)^{\frac{1}{n}}, & n \in [1, +\infty[ \\ \mathrm{ess\,sup}_{t \in [0,T]} \|f(\cdot, t)\|_{m,\Omega}, & n = +\infty. \end{cases}$$

Now, we can define the following Sobolev spaces with the mixed norm:

$$W^{1,0}_{m,n} = \{v \in L_{m,n}(Q_T) \parallel \|v\|_{m,n,Q_T} + \|\nabla\, v\|_{m,n,Q_T} < +\infty\},$$

$$W^{2,1}_{m,n} = \{v \in L_{m,n}(Q_T) \parallel \|v\|_{m,n,Q_T} + \|\nabla\, v\|_{m,n,Q_T} + \|\nabla^2 v\|_{m,n,Q_T} + \|\partial_t v\|_{m,n,Q_T} < +\infty\}.$$

Setting $x' = (x_1, x_2) \in \mathbb{R}^2$, we introduce the additional notation:

$$B(x_0, R) \equiv \{x \in \mathbb{R}^3 \parallel |x - x_0| < R\},$$

$$B^+(x_0,R)\equiv\{x\in B(x_0,R)\parallel x=(x',x_3),\quad x_3>x_{03}\},$$
$$B(\theta)\equiv B(0,\theta),\quad B\equiv B(1),\quad B^+(\theta)\equiv B^+(0,\theta),\quad B^+\equiv B^+(1),$$
$$\Gamma(x_0,R)\equiv\{x\in B(x_0,R)\parallel x_3=x_{30}\},\quad \Gamma(\theta)\equiv\Gamma(0,\theta),\quad \Gamma\equiv\Gamma(1),$$
$$Q(z_0,R)\equiv B(x_0,R)\times]t_0-R^2,t_0[,\quad z_0=(x_0,t_0),$$
$$Q^+(z_0,R)\equiv B^+(x_0,R)\times]t_0-R^2,t_0[,$$
$$Q(\theta)\equiv Q(0,\theta),\quad Q\equiv Q(1),\quad Q^+(\theta)\equiv Q^+(0,\theta),\quad Q^+\equiv Q^+(1).$$

Various mean values of integrable functions are denoted as follows

$$[p]_\Omega(t)\equiv \fint_\Omega p(x,t)\,dx\equiv\frac{1}{|\Omega|}\int\limits_\Omega p(x,t)\,dx,$$

$$(v)_\omega\equiv\fint_\omega v\,dz\equiv\frac{1}{|\omega|}\int\limits_\omega v\,dz.$$

We denotes by $c$ all universal positive constants.

In this section, we shall prove a couple of propositions about the pressure in the Navier-Stokes equations provided that conditions (1.4) holds. To this end, we are going to use two results established in [12] and [13]. For the reader convenience, they are formulated below. Moreover, since the first lemma is slightly different from Lemma 3.1 in [12], we shall prove it.

**Lemma 2.1** *Let $v\in L_3(Q(z_0,R))$ and $p\in L_{\frac32}(Q(z_0,R))$ satisfy the Navier-Stokes equations in the sense of distributions. Then, for $0<r\leq\rho\leq R$, we have*

$$D(z_0,r;p)\leq c\Big[\Big(\frac{r}{\rho}\Big)^{\frac52}D(z_0,\rho;p)+\Big(\frac{\rho}{r}\Big)^2C(z_0,\rho;v)\Big],\qquad(2.1)$$

*where*

$$C(z_0,r;v)\equiv\frac{1}{r^2}\int\limits_{Q(z_0,r)}|v|^3\,dz,\qquad D(z_0,r;p)\equiv\frac{1}{r^2}\int\limits_{Q(z_0,r)}|p-[p]_{B(x_0,r)}|^{\frac32}\,dz.$$

**Lemma 2.2** *Let a pair of functions $v$ and $p$ satisfy the following conditions. They have the differentiability properties*

$$v\in L_{2,\infty}(Q^+(z_0,R))\cap W^{1,0}_2(Q^+(z_0,R))\cap W^{2,1}_{\frac98,\frac32}(Q^+(z_0,R)),$$
$$p\in W^{1,0}_{\frac98,\frac32}(Q^+(z_0,R)).\qquad(2.2)$$

*The pair $v$ and $p$ satisfies the Navier-Stokes equations a.e. in $Q^+(z_0, R)$ and the boundary condition*

$$v(x,t) = 0, \qquad x_3 = x_{30} \text{ and } t_0 - R^2 \leq t \leq t_0. \tag{2.3}$$

*For a.a. $t \in ]t_0 - R^2, t_0[$ and for all nonnegative cut-off functions $\varphi \in C_0^\infty(\mathbb{R}^4)$, vanishing in a neighborhood of the parabolic boundary*

$$\partial' Q(z_0, R) = B(x_0, R) \times \{t = t_0 - R^2\} \cup \partial B(x_0, R) \times [t_0 - R^2, t_0]$$

*of the cylinder $Q(z_0, R)$, $v$ and $p$ satisfy the local energy inequality*

$$\int\limits_{B(x_0,R)} \varphi(x,t)|v(x,t)|^2\,dx + 2 \int\limits_{B(x_0,R)\times]t_0-R^2,t[} \varphi|\nabla v|^2\,dxdt'$$

$$\leq \int\limits_{B(x_0,R)\times]t_0-R^2,t[} \Big[|v|^2(\partial_t\varphi + \nabla\varphi) + v\cdot\nabla\varphi(|v|^2 + 2p)\Big]\,dxdt'. \tag{2.4}$$

*Then, for any $0 < r \leq \rho \leq R$, we have*

$$D_1^+(z_0, r; p) \leq c\Big\{\Big(\frac{r}{\rho}\Big)^2\Big[D_1^+(z_0,\rho;p) + (E^+)^{\frac{3}{4}}(z_0,\rho;v)\Big]$$

$$+\Big(\frac{\rho}{r}\Big)^{\frac{3}{2}}(A^+)^{\frac{1}{2}}(z_0,\rho;v)E^+(z_0,\rho;v)\Big\}, \tag{2.5}$$

*where*

$$A^+(z_0,\rho;v) \equiv \operatorname{ess}\sup_{t_0-R^2<t<t_0} \frac{1}{\rho}\int\limits_{B^+(x_0,\rho)} |v(x,t)|^2\,dx,$$

$$E^+(z_0,\rho;v) \equiv \frac{1}{\rho}\int\limits_{Q^+(z_0,\rho)} |\nabla v|^2\,dz,$$

$$D_1^+(z_0,\rho;p) \equiv \frac{1}{\rho^{\frac{3}{2}}}\int\limits_{t_0-\rho^2}^{t_0}\Big(\int\limits_{B^+(x_0,\rho)} |\nabla p|^{\frac{9}{8}}\,dx\Big)^{\frac{4}{3}}\,dt.$$

**Remark 2.3** *Lemma 2.2 was proved in [13], see Lemma 7.2 there.*

**Remark 2.4** *According to the definition introduced in [13], see Definition 2.1 there, the pair $v$ and $p$, satisfying condition (2.2)–(2.4), is called a suitable weak solution to the Navier-Stokes equations in $Q^+(z_0, R)$ near $\Gamma(x_0, R) \times [t_0 - R^2, t_0]$.*

Proof of Lemma 2.1 We just outline our proof because it is essentially the same as the proof of Lemma 3.1 in [12].

For a.a. $t \in ]t_0 - \rho^2, t_0[$, the pressure $p$ meets the equation

$$\Delta p(\cdot,t) = -\mathrm{div}\,\mathrm{div}\, v(\cdot,t) \otimes v(\cdot,t) \qquad \text{in} \quad B(x_0,\rho)$$

in the sense of distributions. We decompose it so that

$$p = p_1 + p_2,$$

where $p_1$ is defined as follows:

$$\int\limits_{B(x_0,\rho)} p_1(x,t)\Delta\varphi(x)\,dx = \int\limits_{B(x_0,\rho)} v(x,t) \otimes v(x,t) : \nabla^2(x)\,dx$$

for any $\varphi \in W^2_3(\mathbb{R}^3)$ such that $\varphi = 0$ on $\partial B(x_0,\rho)$. Regarding $p_2$, we have

$$\Delta p(\cdot,t) = 0 \qquad \text{in} \quad B(x_0,\rho) \tag{2.6}$$

for a.a. $t \in ]t_0 - \rho^2, t_0[$. By the known regularity theory results,

$$\int\limits_{B(x_0,\rho)} |p_1(x,t)|^{\frac{3}{2}}\,dx \leq c \int\limits_{B(x_0,\rho)} |v(x,t)|^3\,dx. \tag{2.7}$$

Let $0 < r \leq \rho/2$. We have

$$\begin{aligned} D(z_0,r;p) &\leq c\Big[D(z_0,r;p_1) + D(z_0,r;p_2)\Big] \\ &\leq c\Big[\frac{1}{r^2}\int\limits_{Q(z_0,\rho)} |p_1|^{\frac{3}{2}}\,dz + D(z_0,r;p_2)\Big] \\ &\leq c\Big[\Big(\frac{\rho}{r}\Big)^2 C(z_0,\rho;v) + D(z_0,r;p_2)\Big]. \end{aligned} \tag{2.8}$$

Since $p_2$ is a harmonic function, we see that

$$\begin{aligned} \sup_{x\in B(x_0,r)} |p_2(x,t) - [p_2]_{B(x_0,r)}(t)| &\leq cr \sup_{x\in B(x_0,\rho/2)} |\nabla p_2(x,t)| \\ &\leq cr\frac{1}{\rho^4}\int\limits_{B(x_0,\rho)} |p_2(x,t) - [p_2]_{B(x_0,\rho)}(t)|dx \\ &\leq c\Big(\frac{r}{\rho}\Big)\frac{1}{\rho^2}\Big(\int\limits_{B(x_0,\rho)} |p_2(x,t) - [p_2]_{B(x_0,\rho)}(t)|^{\frac{3}{2}}\,dx\Big)^{\frac{2}{3}}. \end{aligned}$$

Therefore,

$$D(z_0,r;p_2)\leq c\frac{1}{r^2}\Big(\frac{r}{\rho}\Big)^3\Big(\frac{r}{\rho}\Big)^{\frac{3}{2}}\int\limits_{t_0-r^2}^{t_0}dt\int\limits_{B(x_0,\rho)}|p_2-[p_2]_{B(x_0,\rho)}|^{\frac{3}{2}}\,dx$$

$$\leq c\Big(\frac{r}{\rho}\Big)^{3+\frac{3}{2}-2}D(z_0,\rho;p_2)$$

$$\leq c\Big(\frac{r}{\rho}\Big)^{\frac{5}{2}}\Big[D(z_0,\rho;p)+D(z_0,\rho;p_1)\Big]$$

$$\leq c\Big(\frac{r}{\rho}\Big)^{\frac{5}{2}}\Big[D(z_0,\rho;p)+C(z_0,\rho;v)\Big]. \qquad (2.9)$$

Combining (2.8) and (2.9), we easily arrived at (2.1). Lemma 2.1 is proved.

Now, our goal is to prove two auxiliary propositions.

**Proposition 2.5** *Assume that all conditions of Lemma 2.1 are fulfilled. And let, in addition,*

$$\|v\|_{3,\infty,Q(z_0,R)}\leq L<+\infty. \qquad (2.10)$$

*Then, for any $\gamma\in]0,1[$, there exists a constant $c_1$ depending on $\gamma$ and $L$ only such that, for $0<r\leq R$, we have*

$$D(z_0,r;p)\leq c_1\Big[\Big(\frac{r}{R}\Big)^{\frac{5}{2}\gamma}D(z_0,R;p)+1\Big]. \qquad (2.11)$$

Proof It can be derived from (2.1) that:

$$D(z_0,\tau^{k+1}R;p)\leq c\Big[\tau^{\frac{5}{2}}D(z_0,\tau^kR;p)+\frac{1}{\tau^2}L^3\Big] \qquad (2.12)$$

for any $0<\tau<1$. We may choose $\tau\in]0,1[$ so that

$$c\tau^{\frac{5}{2}(\gamma-1)}\leq 1.$$

Then we find (2.11) from (2.12) just by iterations. Proposition 2.5 is proved.

**Proposition 2.6** *Assume that all conditions of Lemma 2.2 are fulfilled. And let, in addition,*

$$\|v\|_{3,\infty,Q^+(z_0,R)}\leq L<+\infty. \qquad (2.13)$$

*Then, for any $\gamma\in]0,1[$, there exists a constant $c_2$ depending on $\gamma$ and $L$ only such that, for $0<r\leq R$, we have*

$$D^+_1(z_0,r;p)\leq c_2\Big[\Big(\frac{r}{R}\Big)^{2\gamma}D^+_1(z_0,R;p)+1\Big]. \qquad (2.14)$$

Proof Let $\rho\leq R/2$. Then local energy inequality (2.4) gives us the following estimate

$$E^+(z_0,\rho;v)\leq c\Big[(C^+)^{\frac23}(z_0,2\rho;v)+(D^+)^{\frac23}(z_0,2\rho;p)(C^+)^{\frac13}(z_0,2\rho;v)$$
$$+C^+(z_0,2\rho;v)\Big],$$

where

$$D^+(z_0,r;p)\equiv\frac{1}{r^2}\int\limits_{Q^+(z_0,r)}|p-[p]_{B^+(x_0,r)}|^{\frac32}\,dz.$$

Now, using condition (2.13) and the embedding theorem, we show

$$E^+(z_0,\rho;v)\leq c\Big[L^2+(D_1^+)^{\frac23}(z_0,2\rho;p)L+L^3\Big].$$

And thus, from Lemma 2.2, see (2.5), we find

$$D_1^+(z_0,r;p)\leq c_3(L)\Big[\Big(\frac{r}{\rho}\Big)^2\Big(D_1^+(z_0,2\rho;p)+1\Big)$$
$$+\Big(\frac{\rho}{r}\Big)^3\Big((D_1^+)^{\frac23}(z_0,2\rho;p)+1\Big)\Big]$$
$$\leq c_3'(L)\Big[\Big(\frac{r}{\rho}\Big)^2D_1^+(z_0,2\rho;p)+\Big(\frac{\rho}{r}\Big)^{13}\Big]$$

for any $0<r\leq\rho\leq R/2$. But the latter immediately implies

$$D_1^+(z_0,r;p)\leq c_4(L)\Big[\Big(\frac{r}{\rho}\Big)^2D_1^+(z_0,\rho;p)+\Big(\frac{\rho}{r}\Big)^{13}\Big]$$

for all $0<r\leq\rho\leq R$. Using the same arguments as in the proof of the previous proposition, we establish (2.14). Proposition 2.6 is proved.

# 3 $\varepsilon$-regularity results for suitable weak solutions

First, let us show that the pair $v$ and $p$ from Theorem 1.1 forms the so-called suitable weak solution to the Navier-Stokes equations in $Q^+$ near $\Gamma\times[-1,0]$. The corresponding definition was introduced in [13]. It is a natural modification of the known definitions of suitable weak solutions, discussed in

[10], [1], and [8], for the interior case. In our case, this means that the pair $v$ and $p$ must subject to the following conditions:

$$v\in L_{2,\infty}(Q^+)\cap W^{1,0}_2(Q^+)\cap W^{2,1}_{\frac98,\frac32}(Q^+),\qquad p\in W^{1,0}_{\frac98,\frac32}(Q^+);\tag{3.1}$$

$$v \text{ and } p \text{ satisfy the Navier-Stokes equations a.e. in } Q^+;\tag{3.2}$$

$$v=0\qquad\text{on}\quad \Gamma\times[-1,0];\tag{3.3}$$

for a.a. $t\in]-1,0[$ and for all nonnegative functions $\varphi\in C^\infty_0(\mathbb{R}^4)$, vanishing a neighborhood of the parabolic boundary $\partial'Q$ of $Q$, $v$ and $p$ satisfy the inequality

$$\int\limits_{B^+}\varphi(x,t)|v(x,t)|^2\,dx+2\int\limits_{B^+\times]-1,t[}\varphi|\nabla v|^2\,dxdt'$$
$$\leq\int\limits_{B^+\times]-1,t[}\Big[|v|^2(\partial_t\varphi+\Delta\varphi)+v\cdot\nabla\varphi(|v|^2+2p)\Big]\,dxdt'.\tag{3.4}$$

(3.1)–(3.3) hold by the assumptions of Theorem 1.1. We should just verify that they satisfy local energy inequality (3.4). To this end, it is sufficient to show that

$$v\in W^{2,1}_{\frac43}(Q^+(\tau)),\qquad p\in W^{1,0}_{\frac43}(Q^+(\tau))\tag{3.5}$$

for any $\tau\in]0,1[$. If (3.5) is proved, then (3.4) holds as identity.

Fix a domain $\widetilde{B}$ with smooth boundary such that

$$B^+((1+\tau)/2)\subset\widetilde{B}\subset B^+$$

and consider the following initial boundary value problem

$$\left.\begin{array}{c}\partial_t v^1-\Delta v^1=\widetilde{f}-\nabla p^1\\ \operatorname{div} v^1=0\end{array}\right\}\quad\text{in}\quad \widetilde{Q}=\widetilde{B}\times]-1,0[\tag{3.6}$$

$$v^1|_{\partial'\widetilde{Q}}=0,\tag{3.7}$$

where $\widetilde{f}=-\operatorname{div} v\otimes v=-v_iv_{,i}$. It is easy to check that

$$\widetilde{f}\in L_{\frac43}(Q^+)\cap L_{\frac98,\frac32}(Q^+).\tag{3.8}$$

By the coercive estimates for solutions to the Stokes system, see [5] and [9],

$$v^1 \in W^{2,1}_{\frac{4}{3}}(\widetilde{Q}) \cap W^{2,1}_{\frac{9}{8},\frac{3}{2}}(\widetilde{Q}), \qquad p^1 \in W^{1,0}_{\frac{4}{3}}(\widetilde{Q}) \cap W^{1,0}_{\frac{9}{8},\frac{3}{2}}(\widetilde{Q}). \tag{3.9}$$

On the other hand, functions $v^2 = v - v^1$ and $p^2 = p - p^1$ satisfy the following equations:

$$\left.\begin{array}{c} \partial_t v^2 - \Delta\, v^2 = -\nabla\, p^2 \\ \operatorname{div} v^2 = 0 \end{array}\right\} \quad \text{in} \quad Q^+((1+\tau)/2)$$

$$v^2 = 0 \qquad \text{on} \quad \Gamma((1+\tau)/2) \times ] - ((1+\tau)/2)^2, 0[.$$

As it was shown in [11], see Proposition 2 there,

$$v^2 \in W^{2,1}_{s,\frac{3}{2}}(Q^+(\tau)), \qquad p^2 \in W^{1,0}_{s,\frac{3}{2}}(Q^+(\tau)) \tag{3.10}$$

for any $s > 9/8$. (3.5) follows from (3.9), (3.10), and the obvious inequality $3/2 > 4/3$.

Since $v$ and $p$ are a suitable weak solution, we may apply various conditions of the so-called $\varepsilon$-regularity. First, we would like to note that pairs $v$, $p$ and $v$, $p - [p]_{B^+}$ are suitable weak solutions to the Navier-Stokes equations in $Q^+$ near $\Gamma \times [-1, 0]$ simultaneously. Therefore, the main result of [14], see Theorem 1.2 in [14], may be formulated in the following way.

**Lemma 3.1** *There exist universal positive constants $\varepsilon_1$ and $c_0^1$ with the following property. Let a pair $v$ and $p$ be an arbitrary suitable weak solution to the Navier-Stokes equations in $Q^+$ near $\Gamma \times [-1, 0]$ and satisfy the additional condition*

$$C^+(0, 1; v) + D^+(0, 1; p) < \varepsilon_1. \tag{3.11}$$

*Then, the function $v$ is Hölder continuous in the closure of the set $Q^+(1/2)$ and*

$$\sup_{z \in Q^+(1/2)} |v| \leq c_0^1.$$

By the embedding theorem, we can reformulate Lemma 3.1 in the following way.

**Lemma 3.2** *There exist universal positive constants $\varepsilon_2$ and $c_0^2$ with the following property. Let a pair $v$ and $p$ be an arbitrary suitable weak solution to the Navier-Stokes equations in $Q^+$ near $\Gamma \times [-1, 0]$ and satisfy the additional condition*

$$C^+(0, 1; v) + D_1^+(0, 1; p) < \varepsilon_2. \tag{3.12}$$

*Then, the function $v$ is Hölder continuous in in the closure of the set $Q^+(1/2)$ and*

$$\sup_{z\in Q^+(1/2)}|v|\leq c_0^2.$$

Finally, we would like to use another condition of $\varepsilon$-regularity in terms of the velocity $v$ only.

**Lemma 3.3** *There exists an universal positive constant $\varepsilon_3$ with the following property. Let a pair $v$ and $p$ be an arbitrary suitable weak solution to the Navier-Stokes equations in $Q^+$ near $\Gamma\times[-1,0]$. Assume that, for some $R_0\in]0,1[$, $v$ satisfies the additional condition*

$$\sup_{0<R\leq R_0}\frac{1}{R^2}\int\limits_{Q^+(R)}|v|^3\,dz<\varepsilon_3. \tag{3.13}$$

*Then, there exists $r_0\in]0,R_0[$ such that the function $v$ is Hölder continuous in the closure of the set $Q^+(r_0)$.*

Proof Our arguments are similar to those used in the proof of Proposition 2.6. By Lemma 2.2, we have

$$D_1^+(r)\leq c\Big\{\Big(\frac{r}{\rho}\Big)^2\Big[D_1^+(\rho)+(E^+)^{\frac34}(\rho)\Big]+\Big(\frac{\rho}{r}\Big)^3(A^+)^{\frac12}(\rho)E^+(\rho)\Big\} \tag{3.14}$$

for any $0<r\leq\rho\leq R_0/2$. Here,

$$A^+(\rho)\equiv A^+(0,\rho;v),\qquad C^+(\rho)\equiv C^+(0,\rho;v)$$
$$D_1^+(\rho)\equiv D_1^+(0,\rho;p),\qquad E^+(\rho)\equiv E^+(0,\rho;v).$$

In addition, the local energy inequality gives us:

$$\begin{aligned}A^+(\rho)+E^+(\rho)&\leq c\Big[(C^+)^{\frac23}(2\rho)+(C^+)^{\frac13}(2\rho)(D^+)^{\frac23}(2\rho)+C^+(2\rho)\Big]\\&\leq c\Big[\varepsilon_3^{\frac23}+\varepsilon_3^{\frac13}(D_1^+)^{\frac23}(2\rho)+\varepsilon_3\Big].\end{aligned} \tag{3.15}$$

Without loss of generality, we may assume that $\varepsilon_3\leq1$. Combining (3.14) and (3.15), we find

$$D_1^+(r)\leq c\Big\{\Big(\frac{r}{\rho}\Big)^2\Big[D_1^+(\rho)+\Big(\varepsilon_3^{\frac23}+\varepsilon_3^{\frac13}(D_1^+)^{\frac23}(2\rho)+\varepsilon_3\Big)^{\frac34}\Big]$$

$$
\begin{aligned}
&+\Big(\frac{\rho}{r}\Big)^3\Big(\varepsilon_3^{\frac{2}{3}}+\varepsilon_3^{\frac{1}{3}}(D_1^+)^{\frac{2}{3}}(2\rho)+\varepsilon_3\Big)^{\frac{3}{2}}\Big\}\\
&\leq c\Big\{\Big(\frac{r}{\rho}\Big)^2\Big[D_1^+(2\rho)+\varepsilon_3^{\frac{1}{2}}\Big]+\varepsilon_3^{\frac{1}{2}}\Big(\frac{\rho}{r}\Big)^3 D_1^+(2\rho)+\varepsilon_3^{\frac{1}{2}}\Big(\frac{\rho}{r}\Big)^3\Big\}\\
&\leq c\Big\{\Big[\Big(\frac{r}{\rho}\Big)^2+\varepsilon_3^{\frac{1}{2}}\Big(\frac{\rho}{r}\Big)^3\Big]D_1^+(2\rho)+\varepsilon_3^{\frac{1}{2}}\Big(\frac{\rho}{r}\Big)^3\Big\}
\end{aligned}
$$

for any $0<r\leq\rho\leq R_0/2$. After some simple calculations, the latter can be rewritten as follows.

$$
D_1^+(\tau R)\leq c\Big\{\Big[\tau^2+\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}\Big]D_1^+(R)+\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}\Big\}
$$

for any $0<R\leq R_0$ and for any $0<\tau<1$. Next, we fix $\tau$ so that

$$
c\tau<1/2
$$

and assume that

$$
c\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}<\frac{\tau}{2}\Leftrightarrow(\varepsilon_3<\Big(\frac{\tau^4}{2c}\Big)^2).
$$

Then we have

$$
D_1^+(\tau R)\leq\tau D_1^+(R)+c\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}
$$

for any $0<R\leq R_0$. Making iterations, we find

$$
D_1^+(\tau^k R_0)\leq\tau^k D_1^+(R_0)+c\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}\frac{1}{1-\tau}
$$

for any natural $k$. Therefore,

$$
\begin{aligned}
C^+(\tau^k R_0)+D_1^+(\tau^k R_0)&\leq\varepsilon_3+\tau^k D_1^+(R_0)+c\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}\frac{1}{1-\tau}\\
&\leq\tau^k D_1^+(R_0)+2c\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}\frac{1}{1-\tau}
\end{aligned}
$$

Choose $\varepsilon_3$ so small that

$$
2c\frac{\varepsilon_3^{\frac{1}{2}}}{\tau^3}\frac{1}{1-\tau}<\frac{\varepsilon_2}{3}
$$

and then fix $k$ so that

$$\tau^k D_1^+(R_0) < \frac{\varepsilon_2}{3}.$$

Hence, a $\varepsilon$-regularity condition holds. In particular, we have

$$C^+(\tau^k R_0) + D_1^+(\tau^k R_0) < \varepsilon_2.$$

By scaling and Lemma 3.2, we can take $r_0 = \frac{1}{2}\tau^k R_0$. Lemma 3.3 is proved.

# 4 Proof of Theorem 1.1

We let

$$L \equiv \|v\|_{3,\infty,Q^+} < +\infty. \tag{4.1}$$

Using known arguments, (3.5), and (4.1), we can assert that

$$\sup_{-(3/4)^2\leq t\leq 0} \|v(\cdot,t)\|_{3,B^+(3/4)} \leq L. \tag{4.2}$$

Assume now that the statement of Theorem 1.1 is false. Let $z_0 \in \overline{Q}^+(1/2)$ be a singular point. As it was shown in [4], Theorem 1.4, $z_0$ must belong to $\overline{\Gamma}(1/2)$. Without loss of generality (just by translation and by scaling), we may assume that $z_0 = 0$. It follows from Lemma 3.3 that a sequence $R_k \downarrow 0$ exists such that

$$\frac{1}{R_k}\int\limits_{Q^+(R_k)} |v|^3\, dz \geq \varepsilon_3 \tag{4.3}$$

for any natural $k$.

Extending functions $v$ and $p$ outside $Q^+$ to zero, we introduce scaled functions

$$u^k(y,s) = R_k v(R_k y, R_k^2 s), \qquad q^k(y,s) = R_k^2 p(R_k y, R_k^2 s)$$

for $y \in \mathbb{R}^3_+$ and for $s \in \mathbb{R}_- = \{s < 0\}$.

Our first observation is that

$$u^k \overset{\star}{\rightharpoonup} u \qquad \text{in} \qquad L_{3,\infty}(\mathbb{R}^3_+ \times \mathbb{R}_-) \tag{4.4}$$

(at least for a subsequence).

Fix $a > 0$ and let $k$ be so that

$$R_k a < \frac{1}{8}. \tag{4.5}$$

By Proposition 1 in [11], we have two estimates:

$$\begin{aligned}
&\|\nabla^2 u^k\|_{\frac{9}{8},\frac{3}{2},Q^+(a)} + \|\nabla q^k\|_{\frac{9}{8},\frac{3}{2},Q^+(a)} \\
&\leq c_1(a)\Big[\|u^k_i u^k_{,i}\|_{\frac{9}{8},\frac{3}{2},Q^+(2a)} + \|u^k\|_{W^{1,0}_{\frac{9}{8},\frac{3}{2}}(Q^+(2a))} \\
&\qquad + \|q^k - [q^k]_{B^+(2a)}\|_{\frac{9}{8},\frac{3}{2},Q^+(2a)}\Big] \\
&\leq c_1'(a)\Big[\|u^k\|_{3,\infty,Q^+(2a)}\|\nabla u^k\|_{2,Q^+(2a)} + \|\nabla u^k\|_{2,Q^+(2a)} \\
&\qquad + \|u^k\|_{3,Q^+(2a)} + \|q^k - [q^k]_{B^+(2a)}\|_{\frac{9}{8},\frac{3}{2},Q^+(2a)}\Big] \\
&\leq c_1''(a,L)\Big[\|\nabla u^k\|^2_{2,Q^+(2a)} + 1 + \|q^k - [q^k]_{B^+(2a)}\|^{\frac{3}{2}}_{\frac{9}{8},\frac{3}{2},Q^+(2a)}\Big] \\
&\leq c_1'''(a,L)\Big[E^+(0,2a;u^k) + 1 + D_1^+(0,2a;q^k)\Big]
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\|\nabla^2 u^k\|_{\frac{4}{3},Q^+(a)} + \|\nabla q^k\|_{\frac{4}{3},Q^+(a)} \\
&\leq c_2(a)\Big[\|u^k_i u^k_{,i}\|_{\frac{4}{3},Q^+(2a)} + \|u^k\|_{W^{1,0}_{\frac{4}{3}}(Q^+(2a))} \\
&\qquad + \|q^k - [q^k]_{B^+(2a)}\|_{\frac{4}{3},Q^+(2a)}\Big] \\
&\leq c_2'(a,L)\Big[\|\nabla u^k\|^{\frac{3}{2}}_{2,Q^+(2a)} + \|\nabla u^k\|_{2,Q^+(2a)} \\
&\qquad + 1 + \|q^k - [q^k]_{B^+(2a)}\|_{\frac{3}{2},Q^+(2a)}\Big] \\
&\leq c_2''(a,L)\Big[E^+(0,2a;u^k) + 1 + D_1^+(0,2a;q^k)\Big].
\end{aligned} \tag{4.7}$$

On the other hand, by the inverse scaling and by the local energy inequality, we find (see the proof of Proposition 2.6)

$$\begin{aligned}
E^+(0,2a;u^k) + D_1^+(0,2a;q^k) &= E^+(0,2aR_k;v) + D_1^+(0,2aR_k;p) \\
&\leq c_3(L)\Big[1 + D_1^+(0,4aR_k;p)\Big].
\end{aligned} \tag{4.8}$$

To establish uniform boundedness with respect to $k$, let us make use of Proposition 2.6. As a result, we have

$$D^+_1(0,4aR_k;p)\leq c_4(L)\Big[(4aR_k)D^+_1(0,1;p)+1\Big]$$
$$\leq c_4(L)\Big[D^+_1(0,1;p)+1\Big]. \qquad (4.9)$$

Now, by (4.6)–(4.9) and by the diagonal Cantor process, we can select subsequences (still denoted by $u^k$ and $q^k$) with the following properties:

$$u^k \rightharpoonup u \quad \text{in} \quad W^{2,1}_{\frac43}(Q^+(a))\cap W^{2,1}_{\frac98,\frac32}(Q^+(a)), \qquad (4.10)$$

$$\nabla q^k \rightharpoonup \nabla q \quad \text{in} \quad L_{\frac43}(Q^+(a))\cap L_{\frac98,\frac32}(Q^+(a)), \qquad (4.11)$$

$$\nabla u^k \rightharpoonup \nabla u \quad \text{in} \quad L_2(Q^+(a)) \qquad (4.12)$$

for any $a>0$. Moreover, by (4.4), (4.10), (4.11), and by the known multiplicative inequality, we can state that:

$$\left.\begin{array}{lll} u^k \rightharpoonup u & \text{in} & L_{\frac{10}{3}}(Q^+(a)) \\ u^k \to u & \text{in} & L_3(Q^+(a)) \end{array}\right\} \qquad (4.13)$$

for any $a>0$.

According to (4.10)–(4.13), the pair $u$ and $q$ forms a suitable weak solution to the Navier-Stokes equations in $Q^+(a)$ near $\Gamma(a)\times[-a^2,0]$ for any $a>0$. This solution possesses the additional property

$$\|u\|_{3,\infty,\mathbb{R}^3_+\times\mathbb{R}_-}\leq L. \qquad (4.14)$$

Moreover, using (4.10) and interpolation, we can show that

$$u^k\to u \quad \text{in} \quad C([-a^2,0];L_2(B^+(a))), \qquad (4.15)$$

see details in the proof of (3.23) in [4]. Letting $d=x_{03}/2$ for an arbitrary point $x_0\in\mathbb{R}^3_+$ and using (4.2) and (4.15), we find

$$\Big(\int\limits_{B(x_0,d)}|u(x,0)|^2\,dx\Big)^{\frac12}\leq\Big(\int\limits_{B(x_0,d)}|u(x,0)-u^k(x,0)|^2\,dx\Big)^{\frac12}$$
$$+\Big(\int\limits_{B(x_0,d)}|u^k(x,0)|^2\,dx\Big)^{\frac12}\leq\Big(\int\limits_{B(x_0,d)}|u(x,0)-u^k(x,0)|^2\,dx\Big)^{\frac12}$$

$$+cd^{\frac{1}{2}}\Big(\int\limits_{B(x_0,d)}|u^k(x,0)|^3\,dx\Big)^{\frac{2}{3}}=$$

$$\Big(\int\limits_{B(x_0,d)}|u(x,0)-u^k(x,0)|^2\,dx\Big)^{\frac{1}{2}}+cd^{\frac{1}{2}}\Big(\int\limits_{B(x_0R_k,dR_k)}|v(x,0)|^3\,dx\Big)^{\frac{2}{3}}\to 0$$

as $k\to+\infty$. So,

$$u(\cdot,0)=0\qquad\text{in}\quad\mathbb{R}^3_+.\tag{4.16}$$

However, $u$ is not trivial solution. This directly follows from (4.3) and (4.13):

$$\frac{1}{R_k^2}\int\limits_{Q^+(R_k)}|v|^3\,dz=\int\limits_{Q^+}|u^k|^3\,dz\to\int\limits_{Q^+}|u|^3\,dz\geq\varepsilon_3.\tag{4.17}$$

Now, our goal is to show that in fact

$$u=0\qquad\text{in}\quad\mathbb{R}^3_+\times]-1,0[.\tag{4.18}$$

This would contradict with (4.17) and complete the proof of Theorem 1.1.

First, we would like to point out that by interior regularity results of [4], see Theorem 1.4 there, $u$ may have singular points on the plane $x_3=0$ only. To apply backward uniqueness arguments (as it was done in [4]), we need to know if $u$ and $\nabla\,u$ are bounded on certain sets. We shall show that it is so on sets of the form $(\mathbb{R}^3_++hi_3)\times]-T,0[$, where $h>0$ and $T>0$ are arbitrarily fixed and $i_3=(0,0,1)$. To this end, let us prove the following statement.

**Lemma 4.1** *There exists a positive constant $c_5$, depending on $L$ and $D^+_1(0,1;p)$ only, with the following property. Fix $h>0$ and $T>0$ arbitrarily, then*

$$D(e_0,2h;q)\leq c_5\tag{4.19}$$

*for any $e_0=(y_0,s_0)\in(\mathbb{R}^3_++3hi_3)\times]-T,0[$.*

Proof From (4.11), we know that

$$\limsup_{k\to+\infty}D(e_0,2h;q^k)\geq D(e_0,2h;q).\tag{4.20}$$

So, it is sufficient to prove the following bound

$$D(e_0,2h;q^k)\leq c_5(L,D^+_1(0,1;p))\tag{4.21}$$

provided that

$$x_0^k = y_0 R_k \in B^+(1/4), \qquad t_0^k = s_0 R_k^2 > -(1/4)^2. \tag{4.22}$$

Obviously, (4.20) and (4.21) imply (4.19).

We have

$$D(e_0, 2h; q^k) = D(z_0^k, 2hR_k; p), \qquad z_0^k = (x_0^k, t_0^k). \tag{4.23}$$

Further, if $d_k = x_{03}^k = \mathrm{dist}(x_0^k, \Gamma) > 2hR_k$, then $Q(z_0^k, d_k) \subset Q^+(1/2)$ and, therefore, we may use Proposition 2.5. As a result, we find

$$\begin{aligned} D(z_0^k, 2hR_k; p) &\leq c_6(L)\Big[\Big(\frac{2hR_k}{d_k}\Big)^{\frac{5}{4}} D(z_0^k, d_k; p) + 1\Big] \\ &\leq c_6(L)\Big[D(z_0^k, d_k; p) + 1\Big]. \end{aligned} \tag{4.24}$$

On the other hand,

$$Q(z_0^k, d_k) \subset Q^+(\overline{z}_0^k, 2d_k), \qquad \overline{z}_0^k = (\overline{x}_0^k, t_0^k)$$

where $\overline{x}_0^k = (x_{01}^k, x_{02}^k, 0)$, and, moreover,

$$D(z_0^k, d_k; p) \leq c D^+(\overline{z}_0^k, 2d_k; p).$$

Therefore, we have (see (4.24))

$$\begin{aligned} D(z_0^k, 2hR_k; p) &\leq c_7(L)\Big[D^+(\overline{z}_0^k, 2d_k; p) + 1\Big] \\ &\leq c_7'(L)\Big[D_1^+(\overline{z}_0^k, 2d_k; p) + 1\Big]. \end{aligned} \tag{4.25}$$

Now, taking into account

$$Q^+(\overline{z}_0^k, 2d_k) \subset Q^+(\overline{z}_0^k, 1/2) \subset Q^+,$$

we apply Proposition 2.6 which says that

$$\begin{aligned} D_1^+(\overline{z}_0^k, 2d_k; p) &\leq c_8(L)\Big[\Big(\frac{2d_k}{1/2}\Big) D_1^+(\overline{z}_0^k, 1/2; p) + 1\Big] \\ \leq c_8(L)\Big[D_1^+(\overline{z}_0^k, 1/2; p) + 1\Big] &\leq c_8'(L)\Big[D_1^+(0, 1; p) + 1\Big]. \end{aligned} \tag{4.26}$$

Obviously, (4.21) follows from (4.23), (4.25), and (4.26). Lemma 4.1 is proved.

Now, we proceed the proof of Theorem 1.1. Fix $h \in ]0,1[$ arbitrarily and let $T = 100$. Take an arbitrary point $z_0 = (x_0, t_0)$ so that

$$z_0 \in (\mathbb{R}^3_+ + 3hi_3)\times]-100, 0[.$$

In the ball $B(x_0, 2h)$, we decompose pressure

$$q = q_1 + q_2$$

in the following way:

$$\int\limits_{B(x_0,2h)} q_1(x,t)\Delta\varphi(x)\,dx = -\int\limits_{B(x_0,2h)} u(x,t)\otimes u(x,t) : \nabla^2\varphi(x)\,dx$$

for any $\varphi \in C^2(\overline{B}(x_0, 2h))$ such that $\varphi|_{\partial B(x_0,2h)} = 0$, and

$$\Delta q_2(\cdot, t) = 0 \qquad \text{in} \quad B(x_0, 2h).$$

For $q_1$ and $q_2$, the following estimates are valid:

$$\int\limits_{B(x_0,2h)} |q_1(x,t)|^{\frac{3}{2}}\,dx \le c \int\limits_{B(x_0,2h)} |u(x,t)|^3\,dx \tag{4.27}$$

and

$$\sup_{x\in B(x_0,h)} |\nabla\, q_2(x,t)| \le c\frac{1}{h}\Big(\frac{1}{h^3}\int\limits_{B(x_0,2h)} |q_2(x,t) - [q_2]_{B(x_0,2h)}(t)|^{\frac{3}{2}}\,dx\Big)^{\frac{2}{3}}. \tag{4.28}$$

For any $0 < \rho < 1$, using (4.27) and (4.28), we can derive the estimate:

$$D(z_0, h\rho; q) \le c\Big[D(z_0, h\rho; q_1) + D(z_0, h\rho; q_2)\Big]$$

$$\le c\Big[\frac{1}{(h\rho)^2}\int\limits_{Q(z_0,2h)} |q_1|^{\frac{3}{2}}\,dz + (h\rho)^{\frac{5}{2}}\int\limits_{t_0-(h\rho)^2}^{t_0} \sup_{x\in B(x_0,h)} |\nabla\, q_2(x,t)|^{\frac{3}{2}}\,dt\Big]$$

$$\le c\Big[\frac{1}{(h\rho)^2}\int\limits_{Q(z_0,2h)} |u|^3\,dz + \rho^{\frac{5}{2}} D(z_0, 2h; q_2)\Big]$$

$$\leq c\Big[\frac{1}{(h\rho)^2}\int\limits_{Q(z_0,2h)}|u|^3\,dz+\rho^{\frac{5}{2}}D(z_0,2h;q)+\rho^{\frac{5}{2}}D(z_0,2h;q_1)\Big]$$

$$\leq c\Big[\frac{1}{(h\rho)^2}\int\limits_{Q(z_0,2h)}|u|^3\,dz+\rho^{\frac{5}{2}}D(z_0,2h;q)\Big].$$

To evaluate the last term on the right hand side of the latter inequality, we need Lemma 4.1. It gives us:

$$C(z_0,h\rho;v)+D(z_0,h\rho;q)\leq c\Big[\frac{1}{(h\rho)^2}\int\limits_{Q(z_0,2h)}|u|^3\,dz+\rho^{\frac{5}{2}}c_5\Big]$$

for any $z_0\in(\mathbb{R}^3_++3hi_3)\times]-100,0[$.

Now, take an arbitrary number $\varepsilon>0$ and fix $\rho(L,\varepsilon,D^+_1(0,1;p))\in]0,1[$ in the following way:

$$c\rho^{\frac{5}{2}}c_5<\varepsilon/3.$$

Then, we find $R_1>100$ so that

$$\frac{1}{(h\rho)^2}\int\limits_{-200}^{0}dt\int\limits_{\mathbb{R}^3_+\setminus B(R_1/4)}|u|^3\,dx<\varepsilon/3.$$

Since $Q(z_0,2h)\subset(\mathbb{R}^3_+\setminus B(R_1/4))\times]-200,0[$ for $|x_0|>R_1/2$, the latter implies

$$\frac{1}{(h\rho)^2}\int\limits_{Q(z_0,2h)}|u|^3\,dz<\varepsilon/3$$

for all $z_0\in(\mathbb{R}^3_++3hi_3)\times]-100,0[$ such that $|x_0|>R_1/2$. It is known (see, for instance, [4], Lemma 2.2) that, for any $k=0,1,...$, the function $\nabla^k u$ is bounded on the set $(\mathbb{R}^3_++6hi_3)\setminus B(R_1)\times[-50,0]$. Smoothness (and boundedness) of $\nabla^k u$ on the set $(\mathbb{R}^3_++6hi_3)\cap\overline{B}(R_1)\times[-50,0]$ is already known.

So, if we introduce the vorticity $\omega=\nabla\wedge u$, then $\omega$ satisfies the relations:

$$|\partial_t\omega-\Delta\omega|\leq M(|\omega|+|\nabla\omega|),$$
$$|\omega|\leq M$$

on the set $(\mathbb{R}^3_+ + 6hi_3) \times [-50, 0]$ for some $M > 0$, and

$$\omega(\cdot, 0) = 0 \qquad \text{in} \quad \mathbb{R}^3_+.$$

In [3], it was shown that these three conditions imply

$$\omega = 0 \qquad \text{in} \quad (\mathbb{R}^3_+ + 6hi_3) \times [-50, 0].$$

Since $h$ was taken arbitrarily, the latter means that

$$\omega = 0 \qquad \text{in} \quad \mathbb{R}^3_+ \times [-50, 0].$$

Hence, for a.a. $t \in [-50, 0]$, $u$ is a harmonic function, which satisfies the boundary condition $u(x, t) = 0$ if $x_3 = 0$. But, for a.a. $t \in [-50, 0]$, $L_3$-norm of $u$ over $\mathbb{R}^3_+$ is finite. This leads to the conclusion that, for the same $t$, $u(\cdot, t) = 0$ in $\mathbb{R}^3_+$ . Theorem 1.1 is proved.

# 5 Application to the initial boundary value problem in a half space

Fix an arbitrary $T > 0$ and consider the following initial boundary value problem:

$$\left.\begin{array}{c} \partial_t v + \operatorname{div} v \otimes v - \Delta v = -\nabla p \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in} \quad Q_T = \mathbb{R}^3_+ \times ]0, T[; \tag{5.1}$$

$$v(x, t) = 0, \qquad x_3 = 0 \quad \text{and} \quad 0 \leq t \leq T; \tag{5.2}$$

$$v(x, 0) = a \qquad x \in \mathbb{R}^3_+, \tag{5.3}$$

where a solenoidal vector-valued field $a$ belongs to $L_2(\mathbb{R}^3_+)$. For any $T > 0$, problem (5.1)–(5.3) has at least one the so-called weak Leray-Hopf solution $v$ having the following properties (see, for instance, [6] and [7]):

$$v \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T);$$

$$t \mapsto \int\limits_{\mathbb{R}^3_+} v(x, t) \cdot u(x)\, dx \text{ is continuous on } [0, T] \text{ for any } u \in L_2(\mathbb{R}^3_+);$$

$$\int_{Q_T} (v\cdot\partial_t w + v\otimes v : \nabla w - \nabla v : \nabla w)\,dz = 0$$

for any divergence free test function $w \in C_0^\infty(Q_T)$;

$$\|v(\cdot,t) - a(\cdot)\|_{L_2(\mathbb{R}^3_+)} \to 0$$

as $t \downarrow 0$;

$$\int_{\mathbb{R}^3_+} |v(x,t)|^2\,dx + 2\int_{\mathbb{R}^3_+\times]0,t[} |\nabla v|^2\,dxdt' \leq \int_{\mathbb{R}^3_+} |a(x)|^2\,dx$$

for any $t \in [0,T]$.

**Theorem 5.1** *Assume that*

$$v \in L_{3,\infty}(Q_T).$$

*Then, $v \in L_5(Q_T)$ and, moreover, $v$ is smooth and unique.*

Proof To this end, as it was shown in [4], it sufficient to prove the estimate

$$\sup_{z\in\mathbb{R}^3_+\times[\delta,T]} |v(z)| \leq M(\delta) < \infty, \qquad \forall\delta > 0.$$

With the help of linear theory, the associated pressure $p$ can be introduced so that:

$$p \in L_{\frac{3}{2}}(B^+(R)\times]\delta,T[)$$

for any $R > 0$;

$$\nabla^2 v,\ \partial_t v,\ \nabla p \in L_{\frac{9}{8},\frac{3}{2}}(\mathbb{R}^3_+\times]\delta,T[);$$

$$\partial_t v + \operatorname{div} v\otimes v - \Delta v = -\nabla p$$

a.a. in $Q_T$.

From Theorem 1.1 and from Theorem 1.4 in [4], it follows that $v$ has no singular point. We must prove the global boundedness only. Obviously, for $R \to +\infty$,

$$\int_0^T dt \int_{\mathbb{R}^3_+\setminus B^+(R)} |v|^3\,dx \to 0$$

and

$$\int_{\delta}^{T} dt\Big(\int_{\mathbb{R}^3_+\setminus B^+(R)} |\nabla p|^{\frac{9}{8}}\,dx\Big)^{\frac{4}{3}} \to 0.$$

Next, using these facts as well as various conditions of $\varepsilon$-regularity, see Lemmas 3.1 and 3.2 and Lemma 2.2 in [4], we conclude that

$$\sup_{z\in(\mathbb{R}^3_+\setminus B^+(R))\times[\delta,T]} |v(z)| \leq M_1(\delta,R) < \infty.$$

Another estimate

$$\sup_{z\in\overline{B}^+(R)\times[\delta,T]} |v(z)| \leq M_2(\delta,R) < \infty$$

is already known since our solution $v$ is locally smooth. Theorem 5.1 is proved.

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