

$L_{3,\infty}$ -Solutions to the Navier-Stokes Equations and Backward Uniqueness

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Dedicated to Olga Alexandrovna Ladyzhenskaya

Abstract We show that $L_{3,\infty}$ -solutions to the Cauchy problem for the three-dimensional Navier-Stokes equations are smooth.

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1 Introduction

In this expository paper, we review some recent results in the regularity theory for the Navier-Stokes equations. We consider the classical Cauchy problem for these equations:

$$\begin{aligned} \partial_t v(x, t) + \operatorname{div} v(x, t) \otimes v(x, t) - \Delta v(x, t) &= -\nabla p(x, t), \\ \operatorname{div} v(x, t) &= 0 \end{aligned} \tag{1.1}$$

for $x \in \mathbb{R}^3$ and $t \geq 0$, together with the initial condition

$$v(x, 0) = a(x), \quad x \in \mathbb{R}^3. \quad (1.2)$$

We assume for the moment that a is a smooth divergence-free vector field in \mathbb{R}^3 which decays “sufficiently fast” as $x \rightarrow \infty$. (We will return later to the important case when the initial velocity field a belongs to more general classes of functions.) In the classical paper [19], Leray proved the following results:

(i) There exists a $T_\star > 0$ such that the Cauchy problem (1.1), (1.2) has a unique smooth solution with “reasonable properties at ∞ ”.

(ii) Problem (1.1), (1.2) has at least one global weak solution satisfying a natural energy inequality. Moreover, the weak solutions coincide with the smooth solution in $\mathbb{R}^3 \times]0, T_\star[$.

(iii) If $]0, T_\star[$ is the maximal interval of the existence of the smooth solution, then, for each $p > 3$, there exists $\varepsilon_p > 0$ such that

$$\left(\int_{\mathbb{R}^3} |u(x, t)|^p dx \right)^{\frac{1}{p}} \geq \frac{\varepsilon_p}{(T_\star - t)^{\frac{1}{2}(1 - \frac{3}{p})}}$$

as $t \uparrow T_\star$.

(iv) For a given weak solution, there exists a closed set $S \in]0, +\infty[$ of measure zero such that the solution is smooth in $\mathbb{R}^3 \times (]0, \infty[\setminus S)$. (In fact, Leray’s proof gives us S with $\mathcal{H}^{\frac{1}{2}}(S) = 0$, although it is not mentioned explicitly.)

The modern definition of the weak solutions (often called Leray-Hopf weak solutions due to important contributions of E. Hopf in the case of bounded domains) is as follows. We denote by \dot{C}_0^∞ the space of all infinitely differentiable solenoidal vector fields with compact support in \mathbb{R}^3 ; $\overset{\circ}{J}$ and $\overset{\circ}{J}_2^1$ are the closures of the set \dot{C}_0^∞ in the spaces L_2 and W_2^1 , respectively. (We use the standard notation for the Lebesgue and Sobolev spaces.)

In what follows we will use the notation $Q_T = \mathbb{R}^3 \times]0, T[$.

A *Leray-Hopf weak solution* of the Cauchy problem (1.1) and (1.2) in Q_T is a vector field $v : Q_T \rightarrow \mathbb{R}^3$ such that

$$v \in L_\infty(0, T; \overset{\circ}{J}) \cap L_2(0, T; \overset{\circ}{J}_2^1); \quad (1.3)$$

the function $t \rightarrow \int_{\mathbb{R}^3} v(x, t) \cdot w(x) dx$ can be continuously extended to $[0, T]$ for any $w \in L_2$; (1.4)

$$\int_{Q_T} (-v \cdot \partial_t w - v \otimes v : \nabla w + \nabla v : \nabla w) dx dt = 0, \quad \forall w \in \dot{C}_0^\infty(Q_T); \quad (1.5)$$

$$\frac{1}{2} \int_{\mathbb{R}^3} |v(x, t_0)|^2 dx + \int_{\mathbb{R}^3 \times]0, t_0[} |\nabla v|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^3} |a(x)|^2 dx, \quad \forall t_0 \in [0, T]; \quad (1.6)$$

$$\|v(\cdot, t) - a(\cdot)\|_2 \rightarrow 0 \quad \text{as } t \rightarrow 0. \quad (1.7)$$

The definition is meaningful also for $T = +\infty$ if we replace the closed interval $[0, T]$ by $[0, \infty[$ throughout the definition.

Leray's result (ii) above can now be stated as follows. (See [19], [12], [14], and [16].)

Theorem 1.1 *Assume that*

$$a \in \dot{J}. \quad (1.8)$$

Then there exists at least one Leray-Hopf weak solution to the Cauchy problem (1.1) and (1.2) in $\mathbb{R}^3 \times]0, \infty[$.

At the time of this writing, both uniqueness and regularity of Leray-Hopf weak solutions remain open problems.

Important extensions of Leray's results were later obtained by many workers. In particular, the works of Prodi [30], Serrin [42], and Ladyzhenskaya [15] lead to the following generalizations of (ii).

Theorem 1.2 *Suppose that condition (1.8) holds. Let v and v_1 be two weak Leray-Hopf solutions to the Cauchy problem (1.1) and (1.2). Assume that, for some $T > 0$ the velocity field v satisfies the so-called Ladyzhenskaya-Prodi-Serrin condition, i.e.,*

$$v \in L_{s,l}(Q_T) \quad (1.9)$$

with

$$\frac{3}{s} + \frac{2}{l} = 1, \quad s \in]3, +\infty]. \quad (1.10)$$

Then, $v = v_1$ in Q_T and, moreover, v is a smooth function in $\mathbb{R}^3 \times (0, T]$.

The uniqueness was proved by Prodi in [30] and Serrin in [42] and the smoothness was established by Ladyzhenskaya in [15]. Further extension of Theorem 1.2 can be found in paper of Giga [10]. A local version of this theorem was

proved by Serrin [41] for $\frac{3}{s} + \frac{2}{l} < 1$ and Struwe [45] for $\frac{3}{s} + \frac{2}{l} = 1$. We recall that the norm in the mixed Lebesgue space $L_{s,l}(Q_T)$ is given as follows:

$$\|f\|_{s,l,Q_T} = \begin{cases} \left(\int_0^T \|f(\cdot, t)\|_s^l dt \right)^{\frac{1}{l}}, & l \in [1, +\infty[\\ \operatorname{ess\,sup}_{t \in]0, T[} \|f(\cdot, t)\|_s, & l = +\infty. \end{cases}$$

If $s = l$, we abbreviate $\|f\|_{s,Q_T} = \|f\|_{s,s,Q_T}$.

We note that, by standard imbeddings, functions of the Leray-Hopf class satisfy

$$v \in L_{s,l}(Q_T) \tag{1.11}$$

with

$$\frac{3}{s} + \frac{2}{l} = \frac{3}{2}, \quad s \in [2, 6]. \tag{1.12}$$

Hence there is a substantial gap between what we have according to the existence theorem and what we need for uniqueness.

An important step towards understanding regularity properties of the weak Leray-Hopf solutions was a ‘‘localization in x ’’ of Leray’s results (iv). This program was started by Scheffer [31]–[34] and developed further by Caffarelli-Kohn-Nirenberg [2]. Recently, Lin [20] outlined significant simplifications in the proof of these results (see also [17] for more detail proofs).

In this paper, we address the problem of regularity for the weak Leray-Hopf solutions v satisfying the additional condition

$$v \in L_{3,\infty}(Q_T). \tag{1.13}$$

We prove that Leray’s result (iii) has the following analogue for $p = 3$. If $]0, T_\star[$ is the maximal interval of the existence of the smooth solution to problem (1.1), (1.2) and $T_\star < +\infty$, then

$$\limsup_{t \uparrow T_\star} \int_{\mathbb{R}^3} |v(x, t)|^3 dx = +\infty.$$

In other words, the spatial L_3 -norm of v must blow-up if the solution develops a singularity. We can also view this result as an extension of Theorem 1.2 to the case

$$s = 3, \quad l = +\infty.$$

More precisely, we have

Theorem 1.3 *Assume that v is a weak Leray-Hopf solution to the Cauchy problem (1.1) and (1.2) in Q_T and satisfies the additional condition (1.13). Then,*

$$v \in L_5(Q_T) \tag{1.14}$$

and hence it is smooth and unique in Q_T .

The uniqueness of v under condition (1.13) has already been known, see [22, 23, 26, 43]. Some partial results on smoothness of such solutions can be found in [1, 29].

In fact, we prove the following local result.

Theorem 1.4 *Consider two functions v and p defined in the space-time cylinder $Q = B \times]-1, 0[$, where $B(r) \subset \mathbb{R}^3$ stands for the ball of radius r with the center at the origin and $B = B(1)$. Assume that v and p satisfy the Navier-Stokes equations in Q in the sense of distributions and have the following differentiability properties:*

$$v \in L_{2,\infty}(Q) \cap L_2(-1, 0; W_2^1(B)), \quad p \in L_{\frac{3}{2}}(Q). \tag{1.15}$$

Let, in addition,

$$\|v\|_{3,\infty,Q} < +\infty. \tag{1.16}$$

Then the function v is Hölder continuous in the closure of the set

$$Q(1/2) = B(1/2) \times]-(1/2)^2, 0[.$$

The main interest of the above results comes from the fact that they seem out to be of reach of “standard methods”. By those methods, we mean various conditions on (local) “smallness” of various norms of v which are invariant with respect to the natural scaling

$$u(x, t) \rightarrow \lambda u(\lambda x, \lambda^2 t), \quad p(x, t) \rightarrow \lambda^2 p(\lambda x, \lambda^2 t)$$

of the equations.

We note that finiteness of a norm $\|f\|_{s,l}$ with $s, l < \infty$ implies “local smallness” of f in this norm. This is not the case for $L_{3,\infty}$ -norm (which is still invariant under the scaling). This possible “concentration effect” was the main obstacle to proving regularity. To rule out concentration, we use a new method based on the reduction of the regularity problem to a backward uniqueness problem, which is then solved by finding suitable Carleman-type

inequalities. The backward uniqueness results are new and seem to be of independent interest, see Section 5 and Section 6.

Our methods can be probably easily adopted to other parabolic problems with critical non-linearities. In fact, one could speculate that the general idea of the approach might be applicable to an even larger class of interesting equations with critical non-linearities, such as non-linear Schrödinger equations or non-linear wave equations. However, the local regularity issues arising in these cases would be slightly harder than in the parabolic case.

The plan of the paper is as follows. In Section 2, we discuss known results about regularity of so-called suitable weak solutions. In the third section, we reduce the regularity problem to the backward uniqueness for the heat operator with variable lower order terms. This proves Theorem 1.4 and therefore Theorem 1.3. In Section 4, we discuss known facts from the theory of the unique continuation of solutions to parabolic equations through spatial boundaries. In the next section, we prove the backward uniqueness result used in Section 3. The sixth section is devoted to the derivation of two Carleman-type inequalities, which play the crucial role in our proof of the backward uniqueness theorem. Finally, just for completeness, we present the known theorem on the short time solvability of the Cauchy problem with the initial data from $L_3 \cap \dot{J}$ in the class $C([0, T_*]; L_3) \cap L_5(Q_{T_*})$ in the Appendix.

2 Suitable Weak Solutions

In this section, we are going to discuss smoothness of the so-called suitable weak solutions to the Navier-Stokes equations. The definition of suitable weak solutions was introduced in [2], see also [31]-[34], [20], and [17]. Our version is due to [17].

Definition 2.1 *Let ω be a open set in \mathbb{R}^3 . We say that a pair u and q is a suitable weak solution to the Navier-Stokes equations on the set $\omega \times]-T_1, T[$ if it satisfies the conditions:*

$$u \in L_{2,\infty}(\omega \times]-T_1, T[) \cap L_2(-T_1, T; W_2^1(\omega)); \quad (2.1)$$

$$q \in L_{\frac{3}{2}}(\omega \times]-T_1, T[); \quad (2.2)$$

$$u \text{ and } q \text{ satisfy the Navier-Stokes equations} \\ \text{in the sense in distributions;} \quad (2.3)$$

$$\left. \begin{aligned}
& u \text{ and } q \text{ satisfy the local energy inequality} \\
& \int_{\omega} \varphi(x, t) |u(x, t)|^2 dx + 2 \int_{\omega \times]-T_1, t[} \varphi |\nabla u|^2 dx dt' \\
& \leq \int_{\omega \times]-T_1, t[} (|u|^2 (\Delta \varphi + \partial_t \varphi) + u \cdot \nabla \varphi (|u|^2 + 2q)) dx dt'
\end{aligned} \right\} \quad (2.4)$$

for a.a. $t \in]-T_1, T[$ and for all nonnegative functions $\varphi \in C_0^\infty(\mathbb{R}^{3+1})$, vanishing in the neighborhood of the parabolic boundary $\partial'Q \equiv \omega \times \{t = -T_1\} \cup \partial\omega \times [-T_1, T]$.

The main result of the theory of suitable weak solutions, which we are going to use, is as follows.

Lemma 2.2 *There exist absolute positive constants ε_0 and c_{0k} , $k = 1, 2, \dots$, with the following property. Assume that the pair U and P is suitable weak solution to the Navier-Stokes equations in Q and satisfies the condition*

$$\int_Q (|U|^3 + |P|^{\frac{3}{2}}) dz < \varepsilon_0. \quad (2.5)$$

Then, for any natural number k , $\nabla^{k-1}U$ is Hölder continuous in $\overline{Q}(\frac{1}{2})$ and the following bound is valid:

$$\max_{z \in \overline{Q}(\frac{1}{2})} |\nabla^{k-1}U(z)| < c_{0k}. \quad (2.6)$$

To formulate Lemma 2.2, we exploit the notation:

$$z = (x, t), \quad z_0 = (x_0, t_0); \quad B(x_0, R) = \{|x - x_0| < R\};$$

$$Q(z_0, R) = B(x_0, R) \times]t_0 - R^2, t_0[;$$

$$B(r) = B(0, r), \quad Q(r) = Q(0, r), \quad B = B(1), \quad Q = Q(1).$$

Remark 2.3 *For $k = 1$, Lemma 2.2 was proved essentially in [2], see Corollary 1. For alternative approach, we refer the reader to [17], see Lemma 3.1. Cases $k > 1$ were treated in [28], see Proposition 2.1, with the help of the case $k = 1$ and regularity results for linear Stokes type systems.*

In fact, for the case $k = 1$, Lemma 2.2 is a consequence of scaling and the following statement.

Proposition 2.4 *Given numbers $\theta \in]0, 1/2[$ and $M > 3$, there exist two positive constants $\varepsilon_1(\theta, M)$ and $c_1(M)$ such that, for any suitable weak solution v and p to the Navier-Stokes equations in Q , satisfying the additional conditions*

$$|(v)_{,1}| < M, \quad Y_1(v, p) < \varepsilon_1, \quad (2.7)$$

the following estimate is valid:

$$Y_\theta(v, p) \leq c_1 \theta^{\frac{2}{3}} Y_1(v, p). \quad (2.8)$$

Here and in what follows, we use the notation:

$$\begin{aligned} Y(z_0, R; v, p) &= Y^1(z_0, R; v) + Y^2(z_0, R; p), \\ Y^1(z_0, R; v) &= \left(\frac{1}{|Q(R)|} \int_{Q(z_0, R)} |v - (v)_{z_0, R}|^3 dz \right)^{\frac{1}{3}}, \\ Y^2(z_0, R; p) &= R \left(\frac{1}{|Q(R)|} \int_{Q(z_0, R)} |p - [p]_{z_0, R}|^{\frac{3}{2}} dz \right)^{\frac{2}{3}}, \\ (v)_{z_0, R} &= \frac{1}{|Q(R)|} \int_{Q(z_0, R)} v dz, \quad [p]_{x_0, R} = \frac{1}{|B(R)|} \int_{B(x_0, R)} p dx, \\ Y_\theta^1(v) &= Y^1(0, \theta; v), \quad Y_\theta^2(p) = Y^2(0, \theta; p), \\ Y_\theta(v, p) &= Y(0, \theta; v, p), \quad (v)_{, \theta} = (v)_{0, \theta}, \quad [p]_{, \theta} = [p]_{0, \theta}. \end{aligned}$$

PROOF OF PROPOSITION 2.4 Assume that the statement of the proposition is false. This means that a number $\theta \in]0, 1/2[$ and a sequence of suitable weak solutions v^k and p^k (in Q) exist such that:

$$Y_1(v^k, p^k) = \varepsilon_{1k} \rightarrow 0 \quad (2.9)$$

as $k \rightarrow +\infty$,

$$Y_\theta(v^k, p^k) > c_1 \varepsilon_{1k} \theta^{\frac{3}{2}}. \quad (2.10)$$

The constant c_1 will be chosen later in order to get a contradiction. We introduce new functions

$$u^k = (v^k - (v^k)_{,1})/\varepsilon_{1k}, \quad q^k = (p^k - [p^k]_{,1})/\varepsilon_{1k}.$$

They satisfy the following relations

$$Y_1(u^k, q^k) = 1, \quad (2.11)$$

$$Y_\theta(u^k, q^k) > c_1 \theta^{\frac{2}{3}}, \quad (2.12)$$

and the system

$$\left. \begin{aligned} \partial_t u^k + \frac{1}{\varepsilon_{1k}} \operatorname{div} ((v^k)_{,1} + \varepsilon_{1k} u^k) \otimes ((v^k)_{,1} + \varepsilon_{1k} u^k) \\ - \Delta u^k = -\nabla q^k, \quad \operatorname{div} u^k = 0 \end{aligned} \right\} \quad \text{in } Q \quad (2.13)$$

in the sense of distributions.

Without loss of generality, we may assume that:

$$\left\{ \begin{array}{lll} u^k \rightharpoonup u & \text{in} & L_3(Q) \\ q^k \rightharpoonup q & \text{in} & L_{\frac{3}{2}}(Q) \\ (v^k)_{,1} \rightarrow b & \text{in} & \mathbb{R}^3 \end{array} \right. \quad (2.14)$$

and

$$\left. \begin{aligned} \partial_t u + \operatorname{div} u \otimes b - \Delta u = -\nabla q \\ \operatorname{div} u = 0 \end{aligned} \right\} \quad \text{in } Q \quad (2.15)$$

in the sense of distributions. By (2.11) and (2.14), we have

$$|b| < M, \quad Y_1(u, q) \leq 1, \quad [q(\cdot, t)]_{,1} = 0 \quad \text{for all } t \in]-1, 0[. \quad (2.16)$$

From the regularity theory for solutions to the Stokes system, see, for instance, [38], and from (2.15), (2.16), it follows that the function u is Hölder continuous in $Q(3/4)$ and the following estimate is valid:

$$Y_\theta^1(u) \leq \tilde{c}_1(M) \theta^{\frac{2}{3}}. \quad (2.17)$$

On the other hand, choosing a cut-off function φ in an appropriate way in the local energy inequality, we find

$$\|u^k\|_{2,\infty,Q(3/4)} + \|\nabla u^k\|_{2,Q(3/4)} \leq c_3(M). \quad (2.18)$$

Using the known multiplicative inequality, we derive from (2.18) another estimate

$$\|u^k\|_{\frac{10}{3},Q(3/4)} \leq c_4(M). \quad (2.19)$$

It remains to make use of system (2.13) and the inequalities in (2.16). As a result, we have

$$\|\partial_t u^k\|_{L_{\frac{3}{2}}(-\frac{3}{4})^2, 0; (\dot{W}_2^2(B(\frac{3}{4})))'} \leq c_5(M). \quad (2.20)$$

By well-known compactness arguments, we select a subsequence with the property

$$u^k \rightarrow u \quad \text{in} \quad L_3(Q(\frac{3}{4})). \quad (2.21)$$

Now, taking into account (2.21) and (2.17), we pass to the limit in (2.12) and find

$$c_1 \theta^{\frac{2}{3}} \leq \tilde{c}_1 \theta^{\frac{2}{3}} + \limsup_{k \rightarrow \infty} Y_\theta^2(q^k). \quad (2.22)$$

To take the limit of the last term in the right hand side of (2.22), we decompose the pressure q^k so that (see [35, 36, 37])

$$q^k = q_1^k + q_2^k, \quad (2.23)$$

where the function q_1^k is defined as a unique solution to the following boundary value problem: find $q_1^k(\cdot, t) \in L_{\frac{3}{2}}(B)$ such that

$$\int_B q_1^k(x, t) \Delta \psi(x) dx = -\varepsilon_{1k} \int_B u^k(x, t) \otimes u^k(x, t) : \nabla^2 \psi(x) dx$$

for all smooth test functions ψ subjected to the boundary condition $\psi|_{\partial B} = 0$. It is easy to see that

$$\Delta q_2^k(\cdot, t) = 0 \quad \text{in} \quad B \quad (2.24)$$

and, by the coercive estimates for Laplace's operator, we have the bound for q_1^k :

$$\int_B |q_1^k(x, t)|^{\frac{3}{2}} dx \leq c_6 \varepsilon_{1k}^{\frac{3}{2}} \int_B |u^k(x, t)|^3 dx. \quad (2.25)$$

Here, c_6 is an absolute positive constant. Passing to the limit in (2.22), we show with the help of (2.25)

$$c_1 \theta^{\frac{2}{3}} \leq \tilde{c}_1 \theta^{\frac{2}{3}} + \limsup_{k \rightarrow \infty} Y_\theta^2(q_2^k). \quad (2.26)$$

By Poincare's inequality, (2.26) can be reduced to the form

$$c_1\theta^{\frac{2}{3}} \leq \tilde{c}_1\theta^{\frac{2}{3}} + c_7\theta^2 \limsup_{k \rightarrow \infty} \left(\frac{1}{|Q(\theta)|} \int_{Q(\theta)} |\nabla q_2^k|^{\frac{3}{2}} dz \right)^{\frac{2}{3}}. \quad (2.27)$$

Since the function $q_2^k(\cdot, t)$ is harmonic in B , we have the estimate

$$\sup_{x \in B(3/4)} |\nabla q_2^k(x, t)|^{\frac{3}{2}} \leq c_8 \int_B |q_2^k(x, t)|^{\frac{3}{2}} dx$$

and therefore

$$\begin{aligned} \frac{1}{|Q(\theta)|} \int_{Q(\theta)} |\nabla q_2^k|^{\frac{3}{2}} dz &\leq \frac{c_9}{\theta^2} \int_Q |q_2^k|^{\frac{3}{2}} dz \\ &\leq c'_9 \left(\frac{1}{\theta^2} + \frac{1}{\theta^2} \int_Q |q_1^k|^{\frac{3}{2}} dz \right). \end{aligned}$$

The latter inequality together with (2.25) allows us to take the limit in (2.27). As a result, we have

$$c_1\theta^{\frac{2}{3}} \leq \tilde{c}_1\theta^{\frac{2}{3}} + c_7(c'_9)^{\frac{2}{3}}\theta^{\frac{2}{3}}. \quad (2.28)$$

If, from the very beginning, c_1 is chosen so that

$$c_1 = 2(\tilde{c}_1 + c_7(c'_9)^{\frac{2}{3}}),$$

we arrive at the contradiction. Proposition 2.4 is proved. \square

Proposition 2.4 admits the following iterations.

Proposition 2.5 *Given numbers $M > 3$ and $\beta \in [0, 2/3]$, we choose $\theta \in]0, 1/2[$ so that*

$$c_1(M)\theta^{\frac{2-3\beta}{6}} < 1. \quad (2.29)$$

Let $\bar{\varepsilon}_1(\theta, M) = \min\{\varepsilon_1(\theta, M), \theta^5 M/2\}$. If

$$|(v)_{,1}| < M, \quad Y_1(v, p) < \bar{\varepsilon}_1, \quad (2.30)$$

then, for any $k = 1, 2, \dots$,

$$\begin{aligned} \theta^{k-1} |(v)_{,\theta^{k-1}}| &< M, \quad Y_{\theta^{k-1}}(v, p) < \bar{\varepsilon}_1 \leq \varepsilon_1, \\ Y_{\theta^k}(v, p) &\leq \theta^{\frac{2+3\beta}{6}} Y_{\theta^{k-1}}(v, p). \end{aligned} \quad (2.31)$$

PROOF We use induction on k . For $k = 1$, this is nothing but Proposition 2.4.

Assume now that statements (2.31) are valid for $s = 1, 2, \dots, k \geq 2$. Our goal is prove that they are valid for $s = k+1$ as well. Obviously, by induction,

$$Y_{\theta^k}(v, p) < \bar{\varepsilon}_1 \leq \varepsilon_1,$$

and

$$\begin{aligned} |(v^k)_{,1}| &= \theta^k |(v)_{,\theta^k}| \leq \theta^k |(v)_{,\theta^k} - (v)_{,\theta^{k-1}}| + \theta^k |(v)_{,\theta^{k-1}}| \\ &\leq \frac{1}{\theta^5} Y_{\theta^{k-1}}(v, p) + \frac{1}{2} \theta^{k-1} |(v)_{,\theta^{k-1}}| < \frac{1}{\theta^5} \bar{\varepsilon}_1 + M/2 \leq M. \end{aligned}$$

Now, we make natural scaling:

$$v^k(y, s) = \theta^k v(\theta^k y, \theta^{2k} s), \quad p^k(y, s) = \theta^{2k} p(\theta^k y, \theta^{2k} s)$$

for $(y, s) \in Q$. We observe that v^k and p^k form suitable weak solution in Q . Since

$$Y_1(v^k, p^k) = \theta^k Y_{\theta^k}(v, p) < \bar{\varepsilon}_1 \leq \varepsilon_1$$

and

$$|(v^k)_{,1}| = \theta^k |(v)_{,\theta^k}| < M,$$

we conclude

$$Y_\theta(v^k, p^k) \leq c_1 \theta^{\frac{2}{3}} Y_1(v^k, p^k) < \theta^{\frac{2+3\beta}{6}} Y_1(v^k, p^k),$$

which is equivalent to the third relation in (2.31). Proposition 2.5 is proved. \square

A direct consequence of Proposition 2.5 and the scaling

$$v^R(y, s) = Rv(x_0 + Ry, t_0 + R^2 s), \quad p^R(y, s) = R^2 p(x_0 + Ry, t_0 + R^2 s)$$

is the following statement.

Proposition 2.6 . *Let M, β, θ , and $\bar{\varepsilon}_1$ be as in Proposition 2.5. Let a pair v and p be an arbitrary suitable weak solution to the Navier-Stokes equations in the parabolic cylinder $Q(z_0, R)$, satisfying the additional conditions*

$$R|(v)_{z_0, R}| < M, \quad RY(z_0, R; v, p) < \bar{\varepsilon}_1. \quad (2.32)$$

Then, for any $k = 1, 2, \dots$, we have

$$Y(z_0, \theta^k R; v, p) \leq \theta^{\frac{2+3\beta}{6} k} Y(z_0, R; v, p). \quad (2.33)$$

PROOF OF LEMMA 2.2 We start with the case $k = 1$. We let

$$A = \int_Q \left(|U|^3 + |P|^{\frac{3}{2}} \right) dz.$$

Then, let $M = 2002$, $\beta = 1/3$, and θ is chosen according to (2.29) and fix.

First, we observe that

$$Q(z_0, 1/4) \subset Q \quad \text{if} \quad z_0 \in \overline{Q}(3/4)$$

and

$$\frac{1}{4}Y(z_0, 1/4; U, P) \leq c_{10}(A^{\frac{1}{3}} + A^{\frac{2}{3}}), \quad \frac{1}{4}|(U)_{z_0, \frac{1}{4}}| \leq c_{10}A^{\frac{1}{3}}$$

for an absolute positive constant c_{10} . Let us choose ε_0 so that

$$c_{10}(\varepsilon_0^{\frac{1}{3}} + \varepsilon_0^{\frac{2}{3}}) < \bar{\varepsilon}_1, \quad c_{10}\varepsilon_0^{\frac{1}{3}} < 2002.$$

Then, by (2.5), we have

$$\frac{1}{4}Y(z_0, 1/4; U, P) < \bar{\varepsilon}_1, \quad \frac{1}{4}|(U)_{z_0, \frac{1}{4}}| < M,$$

and thus, by Proposition 2.6,

$$Y(z_0, \theta^k/4; U, P) \leq \theta^{\frac{k}{2}}Y(z_0, 1/4; U, P) \leq \theta^{\frac{k}{2}}\bar{\varepsilon}_1$$

for all $z_0 \in \overline{Q}(3/4)$ and for all $k = 1, 2, \dots$. Hölder continuity of v on the set $\overline{Q}(2/3)$ follows from Campanato's condition. Moreover, the quantity

$$\sup_{z \in \overline{Q}(2/3)} |v(z)|$$

is bounded by an absolute constant.

The case $k > 1$ is treated with the help of the regularity theory for the Stokes equations and bootstrap arguments, for details, see [28], Proposition 2.1. Lemma 2.2 is proved. \square

3 Proof of the main results

We start with the proof of Theorem 1.3, assuming that the statement of Theorem 1.4 is valid.

Our approach is based on the reduction of the regularity problem to some problems from the theory of unique continuation and backward uniqueness for the heat operator. We follow the paper [40].

PROOF OF THEOREM 1.3 The first observation is a consequence of (1.13) and can be formulated as follows

$$t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) dx \text{ is continuous in } [0, T] \text{ for all } w \in L_{\frac{3}{2}}. \quad (3.1)$$

This means that $\|v(\cdot, t)\|_3$ is bounded for each $t \in [0, T]$.

Using known Ladyzhenskaya's arguments, involving the coercive estimates (see [11, 24, 44]) and the uniqueness theorem for Stokes problem (see [14]), we can introduce the so-called associated pressure p and, since $|\operatorname{div} v \otimes v| \in L_{\frac{4}{3}}(Q_T)$, we find

$$v \in L_4(Q_T), \quad \partial_t v, \nabla^2 v, \nabla p \in L_{\frac{4}{3}}(Q_{\delta_1, T}) \quad (3.2)$$

for any $\delta_1 > 0$, where $Q_{\delta_1, T} = \mathbb{R}^3 \times]\delta_1, T[$. The pair v and p satisfies the Navier-Stokes equations a.e. in Q_T . Moreover, by the pressure equation

$$\Delta p = -\operatorname{div} \operatorname{div} v \otimes v, \quad (3.3)$$

we have

$$p \in L_{\frac{3}{2}, \infty}(Q_T). \quad (3.4)$$

The pair v and p is clearly a suitable weak solution in any bounded cylinders of Q_T . Moreover, by (3.2), the local energy inequality holds as the identity. So, we can apply Theorem 1.4 and state that:

$$\begin{aligned} \text{for any } z_0 \in \mathbb{R}^3 \times]0, T], \text{ there exists a neighborhood } \mathcal{O}_{z_0} \text{ of } z_0 \\ \text{such that } v \text{ is Hölder continuous in } \mathbb{R}^3 \times]0, T] \cap \mathcal{O}_{z_0}. \end{aligned} \quad (3.5)$$

Indeed, for any $z_0 \in \mathbb{R}^3 \times]0, T]$ and for any $R > 0$ such that $Q(z_0, R) \subset Q_T$, making obvious scaling $\tilde{v}(x, t) = Rv(x_0 + Rx, t_0 + R^2t)$ and $\tilde{p}(x, t) = R^2p(x_0 + Rx, t_0 + R^2t)$, we see that the pair \tilde{v} and \tilde{p} satisfies all conditions of Theorem

1.4. This means that \tilde{v} is Hölder continuous in $\overline{Q}(1/2)$ and therefore v is Hölder continuous in $\overline{Q}(z_0, R/2)$. So, (3.5) is a consequence of Theorem 1.4.

Now, we are going to explain that, in turn, (3.5) implies Theorem 1.3. To this end, we note

$$\lim_{|z_0| \rightarrow +\infty} \int_{Q(z_0, R)} (|v|^3 + |p|^{\frac{3}{2}}) dz = 0, \quad Q(z_0, R) \subset Q_T.$$

Therefore, using scaling arguments, Lemma 2.2 and statement (3.5), we observe that

$$\max_{z \in \mathbb{R}^3 \times [\delta, T]} |v(z)| < C_1(\delta) < +\infty \quad (3.6)$$

for all $\delta > 0$. Setting $w = |v|^{\frac{3}{2}}$, we find from (1.13) and (3.6)

$$w \in L_{2,\infty}(Q_T) \cap L_2(\delta, T; W_2^1(\mathbb{R}^3))$$

and then, by the multiplicative inequality

$$\|w(\cdot, t)\|_{\frac{10}{3}} \leq C_2 \|w(\cdot, t)\|_2^{\frac{2}{5}} \|\nabla w(\cdot, t)\|_2^{\frac{3}{5}}, \quad (3.7)$$

we deduce

$$w \in L_{\frac{10}{3}}(Q_{\delta, T}) \iff v \in L_5(Q_{\delta, T})$$

for any $\delta > 0$. On the other hand, since $a \in L_3 \cap \overset{\circ}{J}$ (this is the necessary condition following from (3.1)), we apply Theorem 7.4 and conclude that

$$v \in L_5(Q_{\delta_0})$$

for some $\delta_0 > 0$. So, we have shown that Theorem 1.3 follows from (3.5). Theorem 1.3 is proved. \square

PROOF OF THEOREM 1.4 First, we note that v and p , satisfying conditions (1.15) and (1.16), form a suitable weak solution to the Navier-Stokes equations in Q . This can be verified with the help of usual mollification and the fact $v \in L_4(Q)$. The latter is just a consequence of the known multiplicative inequality.

Second, we are going to prove two facts:

$$t \rightarrow \int_{B(3/4)} v(x, t) \cdot w(x) dx \text{ is continuous in } [-(3/4)^2, 0] \quad (3.8)$$

for any $w \in L_{\frac{3}{2}}(B(3/4))$

and therefore

$$\sup_{-(3/4)^2 \leq t \leq 0} \|v(\cdot, t)\|_{3, B(3/4)} \leq \|v\|_{3, \infty, Q}. \quad (3.9)$$

We can justify (3.8) as follows. Using the local energy inequality, we can find the bound for $\|\nabla v\|_{2, Q(5/6)}$ via $\|v\|_{3, \infty, Q}$ and $\|p\|_{\frac{3}{2}, Q}$ only. Then, by the known multiplicative inequality, we estimate the norm $\|v\|_{4, Q(5/6)}$. Hence, $|\operatorname{div} v \otimes v| \in L_{\frac{4}{3}}(Q(5/6))$. Now, using a suitable cut-off function, the $L_{s,l}$ -coercive estimates for solutions to the non-stationary Stokes system, known duality arguments, we find the following bound

$$\int_{Q(3/4)} \left(|v|^4 + |\partial_t v|^{\frac{4}{3}} + |\nabla^2 v|^{\frac{4}{3}} + |\nabla p|^{\frac{4}{3}} \right) dz \leq c_{01}, \quad (3.10)$$

with a constant c_{01} depending on the norms $\|v\|_{3, \infty, Q}$ and $\|p\|_{\frac{3}{2}, Q}$ only. In particular, it follows from (3.10) that $v \in C([- (3/4)^2, 0]; L_{\frac{4}{3}}(B(3/4)))$ which, in turn, implies (3.8).

As in the proof of Proposition 2.4, we can present the pressure p in the form

$$p = p_1 + p_2,$$

where the function $p_1(\cdot, t)$ is a unique solution to the following boundary value problem: find $p_1(\cdot, t) \in L_{\frac{3}{2}}(B)$ such that

$$\int_B p_1(x, t) \Delta \psi(x) dx = - \int_B v(x, t) \otimes v(x, t) : \nabla^2 \psi(x) dx$$

for all smooth functions ψ satisfying the boundary condition $\psi|_{\partial B} = 0$. Then, $\Delta p_2(\cdot, t) = 0$ in B . The same arguments, as in Section 2, lead to the estimates

$$\|p_1\|_{\frac{3}{2}, \infty, Q} \leq c_1 \|v\|_{3, \infty, Q}^2 \quad (3.11)$$

and

$$\begin{aligned} \|p_2\|_{\infty, \frac{3}{2}, B(3/4) \times]-1, 0[} &= \left(\int_{-1}^0 \sup_{x \in B(3/4)} |p_2(x, t)|^{\frac{3}{2}} dt \right)^{\frac{2}{3}} \\ &\leq c_1 (\|p\|_{\frac{3}{2}, Q} + \|v\|_{3, \infty, Q}^2), \end{aligned} \quad (3.12)$$

where c_1 is an absolute positive constant.

Assume that the statement of Theorem 1.4 is false. Let $z_0 \in \overline{Q}(1/2)$ be a singular point, see the definition of regular points in the proof of Theorem 1.3. Then, as it was shown in [39], there exists a sequence of positive numbers R_k such that $R_k \rightarrow 0$ as $k \rightarrow +\infty$ and

$$A(R_k) \equiv \sup_{t_0 - R_k^2 \leq t \leq t_0} \frac{1}{R_k} \int_{B(x_0, R_k)} |v(x, t)|^2 dx > \varepsilon_\star \quad (3.13)$$

for all $k \in \mathbb{N}$. Here, ε_\star is an absolute positive constant.

We extend functions v and p to the whole space \mathbb{R}^{3+1} by zero. Extended functions will be denoted by \tilde{v} and \tilde{p} , respectively. Now, we let

$$\begin{aligned} v^{R_k}(x, t) &= R_k \tilde{v}(x_0 + R_k x, t_0 + R_k^2 t), & p^{R_k}(x, t) &= R_k^2 \tilde{p}(x_0 + R_k x, t_0 + R_k^2 t), \\ p_1^{R_k}(x, t) &= R_k^2 \tilde{p}_1(x_0 + R_k x, t_0 + R_k^2 t), & p_2^{R_k}(x, t) &= R_k^2 \tilde{p}_2(x_0 + R_k x, t_0 + R_k^2 t), \end{aligned}$$

where \tilde{p}_1 and \tilde{p}_2 are extensions of p_1 and p_2 , respectively.

Obviously, for any $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^3} |v^{R_k}(x, t)|^3 dx = \int_{\mathbb{R}^3} |\tilde{v}(x, t_0 + R_k^2 t)|^3 dx, \quad (3.14)$$

$$\int_{\mathbb{R}^3} |p_1^{R_k}(x, t)|^{\frac{3}{2}} dx = \int_{\mathbb{R}^3} |\tilde{p}_1(x, t_0 + R_k^2 t)|^{\frac{3}{2}} dx \quad (3.15)$$

and, for any $\Omega \Subset \mathbb{R}^3$,

$$\int_{\mathbb{R}} \sup_{x \in \Omega} |p_2^{R_k}(x, t)|^{\frac{3}{2}} dt = R_k \int_{\mathbb{R}} \sup_{x \in \Omega} |\tilde{p}_2(x_0 + R_k x, s)|^{\frac{3}{2}} ds. \quad (3.16)$$

Hence, without loss of generality, one may assume that

$$v^{R_k} \xrightarrow{\star} u \quad \text{in} \quad L_\infty(\mathbb{R}; L_3) \quad \text{as} \quad k \rightarrow +\infty, \quad (3.17)$$

where $\operatorname{div} u = 0$ in $\mathbb{R}^3 \times \mathbb{R}$ and

$$p_1^{R_k} \xrightarrow{\star} q \quad \text{in} \quad L_\infty(\mathbb{R}; L_{\frac{3}{2}}) \quad \text{as} \quad k \rightarrow +\infty, \quad (3.18)$$

$$p_2^{R_k} \rightarrow 0 \quad \text{in} \quad L_{\frac{3}{2}}(\mathbb{R}; L_\infty(\Omega)) \quad \text{as} \quad k \rightarrow +\infty \quad (3.19)$$

for any $\Omega \Subset \mathbb{R}^3$. For justification of (3.18) and (3.19), we take into account identities (3.15), (3.16) and bounds (3.11), (3.12).

To extract more information about boundedness of various norms of functions v^{R_k} and p^{R_k} , let us fix a cut-off function $\phi \in C_0^\infty(\mathbb{R}^{3+1})$ and introduce the function ϕ^{R_k} in the following way

$$\phi(y, \tau) = R_k \phi^{R_k}(x_0 + R_k y, t_0 + R_k^2 \tau), \quad y \in \mathbb{R}^3 \quad \tau \in \mathbb{R}.$$

We choose R_k so small to ensure

$$\begin{aligned} \text{supp} \phi &\subset \{(y, \tau) \mid t_0 + R_k^2 \tau \in] - (3/4)^2, (3/4)^2[, x_0 + R_k y \in B(3/4)\} \\ &\implies \text{supp} \phi^{R_k} \subset B(3/4) \times] - (3/4)^2, (3/4)^2[. \end{aligned}$$

Then, since the pair v and p is a suitable weak solution, we have

$$2 \int_{-1}^0 \int_B \phi^{R_k} |\nabla v|^2 dz \leq \int_{-1}^0 \int_B \left\{ |v|^2 (\Delta \phi^{R_k} + \partial_t \phi^{R_k}) + v \cdot \nabla \phi^{R_k} (|v|^2 + 2p) \right\} dz$$

and after changing variables we arrived at the inequality

$$2 \int_{\mathbb{R}} \int_{\mathbb{R}^3} \phi |\nabla v^{R_k}|^2 dz \leq \int_{\mathbb{R}} \int_{\mathbb{R}^3} \left\{ |v^{R_k}|^2 (\Delta \phi + \partial_t \phi) + v^{R_k} \cdot \nabla \phi (|v^{R_k}|^2 + 2p^{R_k}) \right\} dz.$$

Now, our goal is to estimate $\|p^{R_k}\|_{\frac{3}{2}, \Omega \times]a, b[}$ for all $\Omega \Subset \mathbb{R}^3$ and for all $-\infty < a < b < +\infty$. We find

$$\begin{aligned} \|p^{R_k}\|_{\frac{3}{2}, \Omega \times]a, b[} &\leq \|p_1^{R_k}\|_{\frac{3}{2}, \Omega \times]a, b[} + \|p_2^{R_k}\|_{\frac{3}{2}, \Omega \times]a, b[} \\ &\leq c_2(a, b, \Omega) \left[\|p_1^{R_k}\|_{\frac{3}{2}, \infty, \mathbb{R}^3 \times \mathbb{R}} + \left(\int_a^b \int_{\Omega} \sup_{y \in \Omega} |p_2^{R_k}(y, t)|^{\frac{3}{2}} dx dt \right)^{\frac{2}{3}} \right] \\ &\leq c'_2(a, b, \Omega) \left[\|p_1^{R_k}\|_{\frac{3}{2}, \infty, \mathbb{R}^3 \times \mathbb{R}} + \left(R_k \int_{\mathbb{R}} \sup_{y \in \Omega} |\tilde{p}_2(x_0 + R_k y, s)|^{\frac{3}{2}} ds \right)^{\frac{2}{3}} \right] \\ &\leq c''_2(a, b, \Omega) (\|p\|_{\frac{3}{2}, Q} + \|v\|_{3, \infty, Q}^2). \end{aligned}$$

So, from the last two inequalities, we deduce the bound

$$\int_Q \left(|p^{R_k}|^{\frac{3}{2}} + |\nabla v^{R_k}|^2 \right) dz \leq c_3(Q) < +\infty \quad (3.20)$$

for any domain $Q \in \mathbb{R}^{3+1}$ with a constant c_3 in (3.20) independent of R_k . Then, we apply known arguments, including multiplicative inequalities, the $L_{s,l}$ -coercive estimates for solutions to the non-stationary Stokes equations, and duality. As a result, we find

$$\int_Q (|v^{R_k}|^4 + |\partial_t v^{R_k}|^{\frac{4}{3}} + |\nabla^2 v^{R_k}|^{\frac{4}{3}} + |\nabla p^{R_k}|^{\frac{4}{3}}) dz \leq c_4(Q). \quad (3.21)$$

The latter together with (3.17) implies

$$v^{R_k} \longrightarrow u \quad \text{in } L_3(Q) \quad (3.22)$$

for $Q \in \mathbb{R}^{3+1}$. Let us show that, in addition,

$$v^{R_k} \longrightarrow u \quad \text{in } C([a, b]; L_2(\Omega)) \quad (3.23)$$

for any $-\infty < a < b < +\infty$ and for any $\Omega \in \mathbb{R}^3$. Indeed, by (3.21),

$$v^{R_k} \longrightarrow u \quad \text{in } C([a, b]; L_{\frac{4}{3}}(\Omega))$$

and then (3.23) can be easily derived from the interpolation inequality

$$\begin{aligned} & \|v^{R_k}(\cdot, t + \Delta t) - v^{R_k}(\cdot, t)\|_{2, \Omega} \\ & \leq \|v^{R_k}(\cdot, t + \Delta t) - v^{R_k}(\cdot, t)\|_{\frac{2}{3}, \Omega}^{\frac{2}{3}} \|v^{R_k}(\cdot, t + \Delta t) - v^{R_k}(\cdot, t)\|_{\frac{4}{3}, \Omega}^{\frac{4}{3}} \end{aligned}$$

and from (3.17).

Now, we combine all information about limit functions u and q , coming from (3.14)–(3.23), and conclude that:

$$\int_Q (|u|^4 + |\nabla u|^2 + |\partial_t u|^{\frac{4}{3}} + |\nabla^2 u|^{\frac{4}{3}} + |\nabla q|^{\frac{4}{3}}) dz \leq c_3(Q) \quad (3.24)$$

for any $Q \in \mathbb{R}^{3+1}$;

$$u \in C([a, b]; L_2(\Omega)) \quad (3.25)$$

for any $-\infty < a < b < +\infty$ and for any $\Omega \in \mathbb{R}^3$;

functions u and q satisfy the Navier-Stokes equations a.e. in \mathbb{R}^{3+1} ; (3.26)

$$2 \int_{\mathbb{R}} \int_{\mathbb{R}^3} \phi |\nabla u|^2 dz = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \left\{ |u|^2 (\Delta \phi + \partial_t \phi) + u \cdot \nabla \phi (|u|^2 + 2q) \right\} dz \quad (3.27)$$

for all functions $\phi \in C_0^\infty(\mathbb{R}^{3+1})$. It is easy to show that, according to (3.24)–(3.27), the pair u and q is a suitable weak solution to the Navier-Stokes equations in $\Omega \times [a, b]$ for any bounded domain $\Omega \Subset \mathbb{R}^3$ and for any $-\infty < a < b < +\infty$. Moreover, according to (3.13),

$$\sup_{t_0 - R_k^2 \leq t \leq t_0} \frac{1}{R_k} \int_{B(x_0, R_k)} |v(x, t)|^2 dx = \sup_{-1 \leq t \leq 0} \int_{B(0,1)} |v^{R_k}(x, t)|^2 dx > \varepsilon_\star$$

for all $k \in \mathbb{N}$ and, by (3.23), we find

$$\sup_{-1 \leq t \leq 0} \int_{B(0,1)} |u(x, t)|^2 dx \geq \varepsilon_\star. \quad (3.28)$$

Let us proceed the proof of Theorem 1.4. We are going to show that there exist some positive numbers R_2 and T_2 such that, for any $k = 0, 1, \dots$, the function $\nabla^k u$ is Hölder continuous and bounded on the set

$$(\mathbb{R}^3 \setminus \overline{B}(R_2/2)) \times]-2T_2, 0].$$

To this end, let us fix an arbitrary number $T_2 > 2$ and note that

$$\int_{-4T_2}^0 \int_{\mathbb{R}^3} (|u|^3 + |q|^{\frac{3}{2}}) dz < +\infty.$$

Therefore,

$$\int_{-4T_2}^0 \int_{\mathbb{R}^3 \setminus \overline{B}(0, R)} (|u|^3 + |q|^{\frac{3}{2}}) dz \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

This means that there exists a number $R_2(\varepsilon_0, T_2) > 4$ such that

$$\int_{-4T_2}^0 \int_{\mathbb{R}^3 \setminus \overline{B}(0, R_2/4)} (|u|^3 + |q|^{\frac{3}{2}}) dz < \varepsilon_0. \quad (3.29)$$

Now, assume that $z_1 = (x_1, t_1) \in (\mathbb{R}^3 \setminus \overline{B}(R_2/2)) \times]-2T_2, 0]$. Then,

$$Q(z_1, 1) \equiv B(x_1, 1) \times]t_1 - 1, t_1[\subset (\mathbb{R}^3 \setminus \overline{B}(0, R_2/4)) \times]-4T_2, 0].$$

So, by (3.29),

$$\int_{t_1-1}^{t_1} \int_{B(x_1, 1)} (|u|^3 + |q|^{\frac{3}{2}}) dz < \varepsilon_0 \quad (3.30)$$

for any $z_1 \in (\mathbb{R}^3 \setminus \overline{B}(R_2/2)) \times]-2T_2, 0]$, where $T_2 > 2$ and $R_2 > 4$. Then, it follows from (3.30) and from Lemma 2.2 that, for any $k = 0, 1, \dots$,

$$\max_{z \in Q(z_1, 1/2)} |\nabla^k u(z)| \leq c_{0k} < +\infty \quad (3.31)$$

and $\nabla^k u$ is Hölder continuous on $(\mathbb{R}^3 \setminus \overline{B}(R_2/2)) \times]-2T_2, 0]$.

Now, let us introduce the vorticity ω of u , i.e., $\omega = \nabla \wedge u$. The function ω meets the equation

$$\partial_t \omega + u_k \omega_{,k} - \omega_k u_{,k} - \Delta \omega = 0 \quad \text{in } (\mathbb{R}^3 \setminus \overline{B}(R_2)) \times]-T_2, 0].$$

Recalling (3.31), we see that, in the set $(\mathbb{R}^3 \setminus \overline{B}(R_2)) \times]-T_2, 0]$, the function ω satisfies the following relations:

$$|\partial_t \omega - \Delta \omega| \leq M(|\omega| + |\nabla \omega|) \quad (3.32)$$

for some constant $M > 0$ and

$$|\omega| \leq c_{00} + c_{01} < +\infty. \quad (3.33)$$

Let us show that

$$\omega(x, 0) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{B}(R_2). \quad (3.34)$$

To this end, we take into account the fact that $u \in C([-T_2, 0]; L_2)$ and find

$$\begin{aligned} & \left(\int_{B(x_*, 1)} |u(x, 0)|^2 dx \right)^{\frac{1}{2}} \leq \\ & \leq \left(\int_{B(x_*, 1)} |v^{R_k}(x, 0) - u(x, 0)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{B(x_*, 1)} |v^{R_k}(x, 0)|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \|v^{R_k} - u\|_{C([-T_2, 0]; L_2)} + |B|^{\frac{1}{6}} \left(\int_{B(x_*, 1)} |v^{R_k}(x, 0)|^3 dx \right)^{\frac{1}{3}} \\
&\leq \|v^{R_k} - u\|_{C([-T_2, 0]; L_2)} + |B|^{\frac{1}{6}} \left(\int_{B(x_0 + R_k x_*, R_k)} |v(y, t_0)|^3 dy \right)^{\frac{1}{3}}.
\end{aligned}$$

Since $\|v(\cdot, t)\|_{3, B(3/4)}$ is bounded for any $t \in [-(3/4)^2, 0]$, see (3.9), we show that, by (3.23),

$$\int_{B(x_*, 1)} |u(x, 0)|^2 dx = 0$$

for all $x_* \in \mathbb{R}^3$. So, (3.34) is proved.

Relations (3.32)–(3.34) allow us to apply the backward uniqueness theorem of Section 5, see Theorem 5.1, and conclude that

$$\omega(z) = 0 \quad z \in (\mathbb{R}^3 \setminus \overline{B(R_2)}) \times]-T_2, 0]. \quad (3.35)$$

If we show that

$$\omega(\cdot, t) = 0 \quad \text{in } R^3 \quad (3.36)$$

for a.a. $t \in]-T_2, 0[$, then we are done. Indeed, by (3.36), the function $u(\cdot, t)$ is harmonic and has the finite L_3 -norm. It turn, this fact leads to the identity $u(\cdot, t) = 0$ for a.a. $t \in]-T_2, 0[$. This contradicts with (3.28).

So, our goal is to show that (3.35) implies (3.36).

To simplify our notation, we let $T = T_2/2$, $R = 2R_2$. We know that functions u and q meet the equations:

$$\begin{aligned}
\partial_t u + \operatorname{div} u \otimes u &= -\nabla q, \\
\operatorname{div} u &= 0, \quad \Delta u = 0, \quad \nabla \wedge u = 0
\end{aligned} \quad (3.37)$$

in the set $(\mathbb{R}^3 \setminus \overline{B(R/2)}) \times]-2T, 0]$. From (3.37), we deduce the following bound

$$\max_{z \in Q_0} \left(|\nabla^k u(z)| + |\nabla^k \partial_t u(z)| + |\nabla^k q(z)| \right) \leq c_{0k}^1 < +\infty \quad (3.38)$$

for all $k = 0, 1, \dots$. Here, $Q_0 = (\mathbb{R}^3 \setminus \overline{B(R)}) \times]-T, 0]$.

Next, we fix a smooth cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3)$ subjected to the conditions: $\varphi(x) = 1$ if $x \in B(2R)$, $\varphi(x) = 0$ if $x \notin B(3R)$. Then, we let $w = \varphi u$, $r = \varphi q$. New functions w and r satisfy the system

$$\begin{aligned}
\partial_t w + \operatorname{div} w \otimes w - \Delta w + \nabla r &= g \\
\operatorname{div} w &= u \cdot \nabla \varphi
\end{aligned} \quad (3.39)$$

in $Q_\star = B(4R) \times]-T, 0[$ and

$$w|_{\partial B(4R) \times]-T, 0]} = 0, \quad (3.40)$$

where

$$g = (\varphi^2 - \varphi) \operatorname{div} u \otimes u + u u \cdot \nabla \varphi^2 + q \nabla \varphi - 2 \nabla u \nabla \varphi - u \Delta \varphi.$$

The function g satisfies the conditions:

$$g(x, t) = 0 \quad \text{if } x \in B(2R) \quad \text{or} \quad x \notin B(3R), \quad (3.41)$$

$$\sup_{z \in Q_0} \left(|\nabla^k g(z)| + |\nabla^k \partial_t g(z)| \right) \leq c_{0k}^2 < +\infty \quad (3.42)$$

for all $k = 0, 1, \dots$. Obviously, (3.42) follows from (3.31), (3.38), and (3.41).

Unfortunately, the function w is not solenoidal. For this reason, we introduce functions \tilde{w} and \tilde{r} as a solution to the Stokes system:

$$-\Delta \tilde{w} + \nabla \tilde{r} = 0, \quad \operatorname{div} \tilde{w} = u \cdot \nabla \varphi$$

in Q_\star with the homogeneous boundary condition $\tilde{w}|_{\partial B(4R) \times]-T, 0]} = 0$. According to the regularity theory for stationary problems and by (3.38), we can state

$$\sup_{z \in Q_\star} (|\nabla^k \partial_t \tilde{w}(z)| + |\nabla^k \tilde{w}(z)| + |\nabla^k \tilde{r}(z)|) \leq c_{0k}^3 < +\infty \quad (3.43)$$

for all $k = 0, 1, \dots$

Setting $U = w - \tilde{w}$ and $P = r - \tilde{r}$, we observe that, by (3.41) and (3.42), U and P meet the Navier-Stokes system with linear lower order terms:

$$\left. \begin{aligned} \partial_t U + \operatorname{div} U \otimes U - \Delta U + \nabla P &= -\operatorname{div}(U \otimes \tilde{w}) \\ &+ \tilde{w} \otimes U + G, \quad \operatorname{div} U = 0 \end{aligned} \right\} \text{in } Q_\star, \quad (3.44)$$

$$U|_{\partial B(4R) \times]-T, 0]} = 0, \quad (3.45)$$

where $G = -\operatorname{div} \tilde{w} \otimes \tilde{w} + g - \partial_t \tilde{w}$, and, taking into account (3.42) and (3.43), we have

$$\sup_{z \in Q_\star} |\nabla^k G(z)| \leq c_{0k}^4 < +\infty \quad (3.46)$$

for all $k = 0, 1, \dots$. Standard regularity results and the differential properties of u and q , described in (3.24), (3.25), and (3.26), lead to the following facts about smoothness of functions U and P :

$$U \in L_{3,\infty}(Q_\star) \cap C([-T, 0]; L_2(B(4R))) \cap L_2(-T, 0; W_2^1(B(4R))),$$

$$\partial_t U, \nabla^2 U, \nabla P \in L_{\frac{4}{3}}(Q_\star).$$

Let $t_0 \in]-T, 0[$ be chosen so that

$$\|\nabla U(\cdot, t_0)\|_{2,B(4R)} < +\infty. \quad (3.47)$$

Then, by the short time unique solvability results for the Navier-Stokes system (see [14, 16]), we can find a number $\delta_0 > 0$ such that

$$\partial_t U, \nabla^2 U, \nabla P \in L_2(B(4R) \times]t_0, t_0 + \delta_0]).$$

In turn, the regularity theory for linear systems implies the bounds

$$\sup_{t_0 + \varepsilon < t < t_0 + \delta_0 - \varepsilon} \sup_{x \in B(4R)} |\nabla^k U(x, t)| \leq c_{0k}^5 < +\infty$$

for all $k = 0, 1, \dots$ and for some nonnegative number $\varepsilon < \delta_0/4$. They immediately imply information about smoothness of the original function u :

$$\sup_{t_0 + \varepsilon < t < t_0 + \delta_0 - \varepsilon} \sup_{x \in B(4R)} |\nabla^k u(x, t)| \leq c_{0k}^6 < +\infty$$

for all $k = 0, 1, \dots$. Hence, we can state that $|\partial_t \omega - \Delta \omega| \leq M(|\omega| + |\nabla \omega|)$ and $|\omega| \leq M_1$ in $B(4R) \times]t_0 + \varepsilon, t_0 + \delta_0 - \varepsilon[$ for some positive constants M and M_1 . But we know that $\omega(z) = 0$ if $z \in (B(4R) \setminus \overline{B}(R)) \times]t_0 + \varepsilon, t_0 + \delta_0 - \varepsilon[$. By the unique continuation theorem of Section 4, see Theorem 4.1, we conclude that:

$$\omega = 0 \quad \text{in} \quad B(4R) \times]t_0 + \varepsilon, t_0 + \delta_0 - \varepsilon[.$$

Since (3.47) holds for a.a. $t_0 \in]-T, 0[$, we find $\omega(\cdot, t) = 0$ in R^3 for a.a. $t \in]-T, 0[$. Repeating the same arguments in the interval $] -T_2, -T_2/2[$, we arrive at (3.36). Theorem 1.3 is proved. \square

We would like to note that the final part of the proof of Theorem 1.3 can be carried out in different ways. For example, we could argue as follows. We should expect that the function $U(\cdot, t)$ and therefore the function $u(\cdot, t)$ are analytic one's in the ball $B(2R)$ for $t_0 + \varepsilon < t < t_0 + \delta_0 - \varepsilon$, see [25]. This means that the vorticity ω is also an analytic function in space variables on the same set. Since $\omega = 0$ outside $B(R)$, we may conclude that $\omega = 0$ in $\mathbb{R}^3 \times]t_0 + \varepsilon, t_0 + \delta_0 - \varepsilon[$ and so on.

4 Unique Continuation Through Spatial Boundaries

In this section, we are going to discuss known facts from the theory of unique continuation for differential inequalities. We restrict ourselves to justification only of those statements which are going to be used in what follows and which can be easily reproved within our unified approach. We hope that this makes our paper more self-contained and more convenient for reading. For advanced theory in this direction, we refer the reader to the paper [4], see also the list of quotations there.

We will work with the backward heat operator $\partial_t + \Delta$ rather than the more usual heat operator $\partial_t - \Delta$ since this will save us writing some minus signs in many formulae. In the space-time cylinder $Q(R, T) \equiv B(R) \times]0, T[\subset \mathbb{R}^3 \times \mathbb{R}^1$, we consider a vector-valued function $u = (u_i) = (u_1, u_2, \dots, u_n)$, satisfying three conditions:

$$u \in W_2^{2,1}(Q(R, T); \mathbb{R}^n); \quad (4.1)$$

$$|\partial_t u + \Delta u| \leq c_1(|u| + |\nabla u|) \quad \text{a.e. in } Q(R, T) \quad (4.2)$$

for some positive constant c_1 ;

$$|u(x, t)| \leq C_k(|x| + \sqrt{t})^k \quad (4.3)$$

for all $k = 0, 1, \dots$, for all $(x, t) \in Q(R, T)$, and for some positive constants C_k . Here,

$$W_2^{2,1}(Q(R, T); \mathbb{R}^n) \equiv \{|u| + |\nabla u| + |\nabla^2 u| + |\partial_t u| \in L_2(Q(R, T))\}.$$

Condition (4.3) means that the origin is zero of infinite order for the function u .

Theorem 4.1 *Assume that a function u satisfies conditions (4.1)–(4.3). Then, $u(x, 0) = 0$ for all $x \in B(R)$.*

Remark 4.2 *For more general results in this direction, we refer the reader to the paper [4] of Escauriaza-Fernández.*

Without loss of generality, we may assume that $T \leq 1$. Theorem 4.1 is an easy consequence of the following lemma.

Lemma 4.3 *Suppose that all conditions of Theorem 4.1 hold. Then, there exist a constant $\gamma = \gamma(c_1) \in]0, 3/16[$ and absolute constants β_1 and β_2 such that*

$$|u(x, t)| \leq c_2(c_1, n)A_0(R, T)e^{-\frac{|x|^2}{4t}} \quad (4.4)$$

for all $(x, t) \in Q(R, T)$ satisfying the following restrictions:

$$0 < t \leq \gamma T, \quad |x| \leq \beta_1 R, \quad \beta_2 t \leq |x|^2.$$

Here,

$$A_0 \equiv \max_{(x,t) \in Q(\frac{3}{4}R, \frac{3}{4}T)} |u(x, t)| + \sqrt{T} |\nabla u(x, t)|.$$

Remark 4.4 *According to the statement of Lemma 4.3, $u(x, 0) = 0$ if $|x| \leq \beta_1 R$.*

Remark 4.5 *From the regularity theory for parabolic equations (see [18]), it follows that*

$$A_0 \leq c_3(R, T) \left(\int_{Q(R, T)} |u|^2 dz \right)^{\frac{1}{2}}.$$

PROOF OF LEMMA 4.3 We let $\lambda = \sqrt{2t}$ and $\varrho = 2|x|/\lambda$. Suppose that $|x| \leq \frac{3}{8}R$ and $8t \leq |x|^2$. Then, as it is easy to verify, we have $\varrho \geq 4$ and

$$\lambda y \in B(3R/4) \quad \text{if } y \in B(\varrho); \quad \lambda^2 s \in]0, 3/4[\quad \text{if } s \in]0, 2[$$

under the condition $0 < \gamma \leq 3/16$. Thus the function $v(y, s) = u(\lambda y, \lambda^2 s)$ is well defined on $Q(\varrho, 2) = B(\varrho) \times]0, 2[$. This function satisfies the conditions:

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|v| + |\nabla v|) \quad (4.5)$$

in $Q(\varrho, 2)$;

$$|v(y, s)| \leq C'_k (|y| + \sqrt{s})^k \quad (4.6)$$

for all $k = 0, 1, \dots$ and for all $(y, s) \in Q(\varrho, 2)$. Here, $C'_k = C_k \lambda^k$.

Given $\varepsilon > 0$, we introduce two smooth cut-off functions with the properties:

$$0 \leq \varphi(y, s) = \begin{cases} 1, & (y, s) \in Q(\varrho - 1, 3/2) \\ 0, & (y, s) \notin Q(\varrho, 2) \end{cases} \leq 1,$$

$$0 \leq \varphi_\varepsilon(s) = \begin{cases} 1, & s \in]2\varepsilon, 2[\\ 0, & s \in]0, \varepsilon[\end{cases} \leq 1.$$

We let $w = \varphi v$ and $w_\varepsilon = \varphi_\varepsilon w$. Obviously, (4.5) implies the following inequality:

$$\begin{aligned} |\partial_s w_\varepsilon + \Delta w_\varepsilon| &\leq c_1 \lambda (|w_\varepsilon| + |\nabla w_\varepsilon|) \\ &+ c_4 (|\nabla \varphi| |\nabla v| + |\nabla \varphi| |v| + |\Delta \varphi| |v| + |\partial_s \varphi| |v|) + c_4 |\varphi'_\varepsilon| |v|. \end{aligned} \quad (4.7)$$

The crucial point is the application of the following Carleman-type inequality, see Section 6 for details, Proposition 6.1, to the function w_ε

$$\begin{aligned} &\int_{Q(\varrho, 2)} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|\nabla w_\varepsilon| + |w_\varepsilon|)^2 dy ds \\ &\leq c_5 \int_{Q(\varrho, 2)} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} |\partial_s w_\varepsilon + \Delta w_\varepsilon|^2 dy ds. \end{aligned} \quad (4.8)$$

Here, c_5 is an absolute positive constant, a is an arbitrary positive number, and $h(t) = te^{\frac{1-t}{3}}$. We let

$$A = \max_{(y, s) \in Q(\varrho, 2) \setminus \overline{Q}(\varrho-1, \frac{3}{2})} |v(y, s)| + |\nabla v(y, s)|$$

and choose γ sufficiently small in order to provide the condition

$$10c_5 c_1^2 \lambda^2 \leq 20c_5 c_1^2 \gamma < \frac{1}{2}. \quad (4.9)$$

Condition (4.9) makes it possible to hide the strongest term in the right hand side of (4.8) into the left hand side of (4.8). So, we derive from (4.7)–(4.9) the following relation

$$\begin{aligned} &\int_{Q(\varrho, 2)} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|\nabla w_\varepsilon| + |w_\varepsilon|)^2 dy ds \\ &\leq c_6 A^2 \int_{Q(\varrho, 2)} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} \chi(y, s) dy ds \\ &\quad + c_6 \frac{1}{\varepsilon^2} \int_{Q(\varrho, 2\varepsilon)} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} |v|^2 dy ds. \end{aligned} \quad (4.10)$$

Here, χ is the characteristic function of the set $Q(\varrho, 2) \setminus \overline{Q}(\varrho-1, 3/2)$. We fix a and take into account (4.6). As a result of the passage to the limit as

$\varepsilon \rightarrow 0$, we find from (4.10)

$$\begin{aligned}
D &\equiv \int_{Q(\varrho^{-1}, 3/2)} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|\nabla v| + |v|)^2 dy ds \\
&\leq c_6 A^2 \int_{Q(\varrho, 2)} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} \chi(y, s) dy ds \\
&\leq c'_6 A^2 \left(h^{-2a}(3/2) + \rho^{n-1} \int_0^2 h^{-2a}(s) e^{-\frac{(\varrho-1)^2}{4s}} ds \right).
\end{aligned} \tag{4.11}$$

Since $\varrho \geq 4$, it follows from (4.11) that:

$$D \leq c_7 A^2 \left(h^{-2a}(3/2) + \rho^{n-1} \int_0^2 h^{-2a}(s) e^{-\frac{\varrho^2}{8s}} ds \right). \tag{4.12}$$

In (4.12), the constant c_7 depends on n and c_1 only.

Given positive number β , we can take a number a in the following way

$$a = \frac{\beta \varrho^2}{2 \ln h(3/2)}. \tag{4.13}$$

This is legal, since $h(3/2) > 1$. Hence, by (4.13), inequality (4.12) can be reduced to the form

$$D \leq c_7 A^2 e^{-\beta \rho^2} \left(1 + \rho^{n-1} e^{-\beta \varrho^2} \int_0^2 h^{-2a}(s) e^{2\beta \varrho^2 - \frac{\varrho^2}{8s}} ds \right).$$

We fix $\beta \in]0, 1/64[$, say, $\beta = 1/100$. Then, the latter relation implies the estimate

$$D \leq c'_7(c_1, n) A^2 e^{-\beta \varrho^2} \left(1 + \int_0^2 h^{-2a}(s) e^{-\frac{\varrho^2}{16s}} ds \right). \tag{4.14}$$

It is easy to check that $\beta \leq \frac{\ln(3/2)}{12}$ and therefore $g'(s) \geq 0$ if $s \in]0, 2[$, where $g(s) = h^{-2a}(s) e^{-\frac{\varrho^2}{16s}}$ and a and ϱ satisfy condition (4.13). So, we have

$$D \leq c_8(c_1, n) A^2 e^{-\beta \varrho^2}, \tag{4.15}$$

where β is an absolute positive constant.

By the choice of ϱ and λ , we have $B(\mu\frac{x}{\lambda}, 1) \subset B(\varrho - 1)$ for any $\mu \in]0, 1[$. Then, setting $\tilde{Q} = B(\mu\frac{x}{\lambda}, 1) \times]1/2, 1[$, we find

$$D \geq \int_{\tilde{Q}} e^{-\frac{|y|^2}{2}} |v|^2 dy ds. \quad (4.16)$$

Observing that $|y|^2 \leq 2\mu^2\frac{|x|^2}{\lambda^2} + 2$ if $y \in B(\mu\frac{x}{\lambda}, 1)$ and letting $\mu = \sqrt{2\beta}$, we derive from (4.15) and (4.16) the following bound

$$\int_{\tilde{Q}} |v|^2 dy ds \leq c'_8 A^2 e^{(-2\beta + \frac{\mu^2}{2})\frac{|x|^2}{t}} = c'_8 A^2 e^{-\beta\frac{|x|^2}{t}}. \quad (4.17)$$

On the other hand, the regularity theory for linear backward parabolic equations give us:

$$|v(\mu x/\lambda, 1/2)|^2 \leq c_9(c_1, n) \int_{\tilde{Q}} |v|^2 dy ds. \quad (4.18)$$

Combining (4.17) and (4.18), we show

$$|u(\sqrt{2\beta}x, t)|^2 = |u(\mu x, t)|^2 = |v(\mu x/\lambda, 1/2)|^2 \leq c'_9 A^2 e^{-\beta\frac{|x|^2}{t}}.$$

Changing variables $\tilde{x} = \sqrt{2\beta}x$, we have

$$|u(\tilde{x}, t)| \leq \sqrt{c'_9} A e^{-\frac{|\tilde{x}|^2}{4t}}$$

for $|\tilde{x}| \leq \beta_1 R$ and $|\tilde{x}|^2 \geq \beta_2 t$ with $\beta_1 = 3/8\sqrt{2\beta}$ and $\beta_2 = 16\beta$. It remains to note that $\lambda \leq \sqrt{2T}$ and

$$A \leq \max_{(x,t) \in Q(\frac{3}{4}R, \frac{3}{4}T)} |u(x, t)| + \lambda |\nabla u(x, t)|.$$

Lemma 4.3 is proved. \square

5 Backward Uniqueness for Heat Operator in Half Space

In this section, we deal with a backward uniqueness problem for the heat operator. Our approach is due to [7], see also [5] and [6].

Let $\mathbb{R}_+^n = \{x = (x_i) \in \mathbb{R}^n \mid x_n > 0\}$ and $Q_+ = \mathbb{R}_+^n \times]0, 1[$. We consider a vector-valued function $u : Q_+ \rightarrow \mathbb{R}^n$, which is "sufficiently regular" and satisfies

$$|\partial_t u + \Delta u| \leq c_1(|\nabla u| + |u|) \quad \text{in } Q_+ \quad (5.1)$$

for some $c_1 > 0$ and

$$u(\cdot, 0) = 0 \quad \text{in } \mathbb{R}_+^n. \quad (5.2)$$

Do (5.1) and (5.2) imply $u \equiv 0$ in Q_+ ? We prove that the answer is positive if we impose natural restrictions on the growth of the function u at infinity. For example, we can consider

$$|u(x, t)| \leq e^{M|x|^2} \quad (5.3)$$

for all $(x, t) \in Q_+$ and for some $M > 0$. Natural regularity assumptions, under which (5.1)–(5.3) can be considered are, for example, as follows:

$$\left. \begin{array}{l} u \text{ and distributional derivatives } \partial_t u, \nabla^2 u \text{ are square} \\ \text{integrable over bounded subdomains of } Q_+. \end{array} \right\} \quad (5.4)$$

We can formulate the main result of this section.

Theorem 5.1 *Using the notation introduced above, assume that u satisfies conditions (5.1)–(5.4). Then $u \equiv 0$ in Q_+ .*

This extends the main result of [5] and [6], where an analogue of Theorem 5.1 was proved for Q_+ replaced with $(\mathbb{R}^n \setminus B(R)) \times]0, T[$. Similarly to those papers, the proof of Theorem 5.1 is based on two Carleman-type inequalities, see (6.1) and (6.12).

Such results are of interest in control theory, see for example [27]. The point is that the boundary conditions are not controlled by our assumptions.

It is an easy exercise for the reader to prove that Theorem 5.1 is true for functions $u : Q_+ \rightarrow \mathbb{R}^m$ with $1 \leq m < +\infty$.

We start with proofs of several lemmas. The first of them plays the crucial role in our approach. It enables us to apply powerful technique of Carleman's inequalities.

Lemma 5.2 *Suppose that conditions (5.1), (5.2), and (5.4) are fulfilled. There exists an absolute positive constant $A_0 < 1/32$ with the following properties. If*

$$|u(x, t)| \leq e^{A|x|^2} \quad (5.5)$$

for all $(x, t) \in Q_+$ and for some $A \in [0, A_0]$, then there are constants $\beta(A) > 0$, $\gamma(c_1) \in]0, 1/12[$, and $c_2(c_1, A) > 0$ such that

$$|u(x, t)| \leq c_2 e^{4A|x|^2} e^{-\beta \frac{x_n^2}{t}} \quad (5.6)$$

for all $(x, t) \in (\mathbb{R}_+^n + 2e_n) \times]0, \gamma[$.

PROOF In what follows, we always assume that the function u is extended by zero to negative values of t .

According to the regularity theory of solutions to parabolic equations, see [18], we may assume

$$|u(x, t)| + |\nabla u(x, t)| \leq c_3 e^{2A|x|^2} \quad (5.7)$$

for all $(x, t) \in (\mathbb{R}_+^n + e_n) \times]0, 1/2[$.

We fix $x_n > 2$ and $t \in]0, \gamma[$ and introduce the new function v by usual parabolic scaling

$$v(y, s) = u(x + \lambda y, \lambda^2 s - t/2).$$

The function v is well defined on the set $Q_\rho = B(\rho) \times]0, 2[$, where $\rho = (x_n - 1)/\lambda$ and $\lambda = \sqrt{3t} \in]0, 1/2[$. Then, relations (5.1), (5.2), and (5.7) take the form:

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|) \quad \text{a.e. in } Q_\rho; \quad (5.8)$$

$$|v(y, s)| + |\nabla v(y, s)| \leq c_3 e^{4A|x|^2} e^{4A\lambda^2|y|^2} \quad (5.9)$$

for $(y, s) \in Q_\rho$;

$$v(y, s) = 0 \quad (5.10)$$

for $y \in B(\rho)$ and for $s \in]0, 1/6[$.

In order to apply inequality (6.1), we choose two smooth cut-off functions:

$$\phi_\rho(y) = \begin{cases} 0 & |y| > \rho - 1/2 \\ 1 & |y| < \rho - 1 \end{cases},$$

$$\phi_t(s) = \begin{cases} 0 & 7/4 < s < 2 \\ 1 & 0 < s < 3/2 \end{cases}.$$

These functions take values in $[0, 1]$. In addition, function ϕ_ρ satisfies the conditions: $|\nabla^k \phi_\rho| < C_k$, $k = 1, 2$. We let $\eta(y, s) = \phi_\rho(y) \phi_t(s)$ and $w = \eta v$. It follows from (5.8) that

$$|\partial_s w + \Delta w| \leq c_1 \lambda (|\nabla w| + |w|) + \chi c_4 (|\nabla v| + |v|). \quad (5.11)$$

Here, c_4 is a positive constant depending on c_1 and C_k only, $\chi(y, s) = 1$ if $(y, s) \in \omega = \{\rho - 1 < |y| < \rho, 0 < s < 2\} \cup \{|y| < \rho - 1, 3/2 < s < 2\}$ and $\chi(y, s) = 0$ if $(y, s) \notin \omega$. Obviously, function w has the compact support in $\mathbb{R}^n \times]0, 2[$ and we may use inequality (6.1), see Proposition 6.1. As a result, we have

$$I \equiv \int_{Q_\rho} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2) dy ds \leq c_0 10 (c_1^2 \lambda^2 I + c_6^2 I_1), \quad (5.12)$$

where

$$I_1 = \int_{Q_\rho} \chi(y, s) h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|v|^2 + |\nabla v|^2) dy ds.$$

Choosing $\gamma = \gamma(c_1)$ sufficiently small, we can assume that the inequality $c_0 10 c_1^2 \lambda^2 \leq 1/2$ holds and then (5.12) implies

$$I \leq c_5(c_1) I_1. \quad (5.13)$$

On the other hand, if $A < 1/32$, then

$$8A\lambda^2 - \frac{1}{4s} < -\frac{1}{8s} \quad (5.14)$$

for $s \in]0, 2]$. By (5.9) and (5.14), we have

$$\begin{aligned} I_1 &\leq c_3^2 e^{8A|x|^2} \int_0^2 \int_{B(\rho)} \chi(y, s) h^{-2a}(s) e^{-\frac{|y|^2}{8s}} dy ds \\ &\leq c_6 e^{8A|x|^2} \left[h^{-2a}(3/2) + \int_0^2 h^{-2a}(s) e^{-\frac{(\rho-1)^2}{8s}} ds \right]. \end{aligned} \quad (5.15)$$

Now, taking into account (5.15), we deduce the bound

$$\begin{aligned} D &\equiv \int_{B(1)} \int_{\frac{1}{2}}^1 |w|^2 dy ds = \int_{B(1)} \int_{\frac{1}{2}}^1 |v|^2 dy ds \\ &\leq c_7 \int_{Q_\rho} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2) dy ds \end{aligned}$$

$$\begin{aligned}
&\leq c_8(c_1)e^{8A|x|^2} \left[h^{-2a}(3/2) + \int_0^2 h^{-2a}(s)e^{-\frac{\rho^2}{32s}} ds \right] \\
&= c_8e^{8A|x|^2-2\beta\rho^2} \left[h^{-2a}(3/2)e^{2\beta\rho^2} + \int_0^2 h^{-2a}(s)e^{2\beta\rho^2-\frac{\rho^2}{32s}} ds \right].
\end{aligned}$$

We can take $\beta = 8A < 1/256$ and then choose

$$a = \beta\rho^2 / \ln h(3/2).$$

Since $\rho \geq x_n$, such a choice leads to the estimate

$$D \leq c_8e^{8A|x'|^2} e^{-\beta\rho^2} \left[1 + \int_0^2 g(s) ds \right],$$

where $g(s) = h^{-2a}(s)e^{-\frac{\rho^2}{64s}}$. It is easy to check that $g'(s) \geq 0$ for $s \in]0, 2[$ if $\beta < \frac{1}{96} \ln h(3/2)$. So, we have

$$D \leq 2c_8e^{8A|x'|^2} e^{-\beta\rho^2} \leq 2c_8e^{8A|x'|^2} e^{-\frac{\beta x_n^2}{12t}}. \quad (5.16)$$

On the other hand, the regularity theory implies

$$|v(0, 1/2)|^2 = |u(x, t)|^2 \leq c'_8 D. \quad (5.17)$$

Combining (5.16) and (5.17), we complete the proof of Lemma 5.2. Lemma 5.2 is proved. \square

Next lemma will be a consequence of Lemma 5.2 and the second Carleman inequality (see (6.12)).

Lemma 5.3 *Suppose that the function u satisfies conditions (5.1), (5.2), (5.4), and (5.5). There exists a number $\gamma_1(c_1, c_*) \in]0, \gamma/2]$ such that*

$$u(x, t) = 0 \quad (5.18)$$

for all $x \in \mathbb{R}_+^n$ and for all $t \in]0, \gamma_1[$.

PROOF As usual, by Lemma 5.2 and by the regularity theory, we may assume

$$|u(x, t)| + |\nabla u(x, t)| \leq c_9(c_1, A)e^{8A|x'|^2} e^{-\beta \frac{x_n^2}{2t}} \quad (5.19)$$

for all $x \in \mathbb{R}_+^n + 3e_n$ and for all $t \in]0, \gamma/2]$.

By scaling, we define function $v(y, s) = u(\lambda y, \lambda^2 s - \gamma_1)$ for $(y, s) \in Q_+$ with $\lambda = \sqrt{2\gamma_1}$. This function satisfies the relations:

$$|\partial_s v + \Delta v| \leq c_1 \lambda (|\nabla v| + |v|) \quad \text{a.e. in } Q_+; \quad (5.20)$$

$$v(y, s) = 0 \quad (5.21)$$

for all $y \in \mathbb{R}_+^n$ and for all $s \in]0, 1/2[$;

$$|\nabla v(y, s)| + |v(y, s)| \leq c_9 e^{8A\lambda^2|y'|^2} e^{-\frac{\beta\lambda^2 y_n^2}{2(\lambda^2 s - \gamma_1)}} \leq c_9 e^{8A\lambda^2|y'|^2} e^{-\beta \frac{y_n^2}{2s}} \quad (5.22)$$

for all $1/2 < s < 1$ and for all $y \in \mathbb{R}_+^n + \frac{3}{\lambda}e_n$. Since $A < 1/32$ and $\lambda \leq \sqrt{\gamma} \leq 1/\sqrt{12}$, (5.22) can be reduced to the form

$$|\nabla v(y, s)| + |v(y, s)| \leq c_{11} e^{\frac{|y'|^2}{48}} e^{-\beta \frac{y_n^2}{2s}} \quad (5.23)$$

for the same y and s as in (5.22).

Let us fix two smooth cut-off functions:

$$\psi_1(y_n) = \begin{cases} 0 & y_n < \frac{3}{\lambda} + 1 \\ 1 & y_n > \frac{3}{\lambda} + \frac{3}{2} \end{cases},$$

and

$$\psi_2(r) = \begin{cases} 1 & r > -1/2 \\ 0 & r < -3/4 \end{cases}.$$

We set (see Proposition 6.2 for the definition of $\phi^{(1)}$ and $\phi^{(2)}$)

$$\phi_B(y_n, s) = \frac{1}{a} \phi^{(2)}(y_n, s) - B = (1-s) \frac{y_n^{2\alpha}}{s^\alpha} - B,$$

where $\alpha \in]1/2, 1[$ is fixed, $B = \frac{2}{a} \phi^{(2)}(\frac{3}{\lambda} + 2, 1/2)$, and

$$\eta(y_n, s) = \psi_1(y_n) \psi_2(\phi_B(y_n, s)/B), \quad w(y, s) = \eta(y_n, s) v(y, s).$$

Although function w is not compactly supported in $Q_+^1 = (\mathbb{R}_+^3 + e_n) \times]0, 1[$, but, by the statement of Lemma 5.2 and by the special structure of the weight in (6.12), we can claim validity of (6.12) for w . As a result, we have

$$\begin{aligned} & \int_{Q_+^1} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} (|w|^2 + |\nabla w|^2) dy ds \\ & \leq c_\star \int_{Q_+^1} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} |\partial_s w + \Delta w|^2 dy ds. \end{aligned}$$

Arguing as in the proof of Lemma 5.2, we can select $\gamma_1(c_1, c_\star)$ so small that

$$\begin{aligned} I & \equiv \int_{Q_+^1} s^2 e^{2a\phi_B} (|w|^2 + |\nabla w|^2) e^{-\frac{|y'|^2}{4s}} dy ds \\ & \leq c_{10}(c_1, c_\star) \int_{(\mathbb{R}_+^n + (\frac{3}{\lambda} + 1)e_n) \times]1/2, 1[} \chi(y_n, s) (s y_n)^2 e^{2a\phi_B} (|v|^2 + |\nabla v|^2) e^{-\frac{|y'|^2}{4s}} dy ds, \end{aligned}$$

where $\chi(y_n, s) = 1$ if $(y_n, s) \in \omega$, $\chi(y_n, s) = 0$ if $(y_n, s) \notin \omega$, and

$$\omega \equiv \{(y_n, s) \mid y_n > 1, \quad 1/2 < s < 1, \quad \phi_B(y_n, s) < -D/2\},$$

where $D = -2\phi_B(\frac{3}{\lambda} + \frac{3}{2}, \frac{1}{2}) > 0$. Now, we wish to estimate the right hand side of the last inequality with the help of (5.23). We find

$$I \leq c_{11} e^{-Da} \int_{\frac{3}{\lambda} + 1}^{+\infty} \int_{1/2}^1 (y_n s)^2 e^{-\beta \frac{y_n^2}{s}} dy_n ds \int_{\mathbb{R}^{n-1}} e^{(\frac{1}{24} - \frac{1}{4s})|y'|^2} dy'.$$

Passing to the limit as $a \rightarrow +\infty$, we see that $v(y, s) = 0$ if $1/2 \leq s < 1$ and $\phi_B(y_n, s) > 0$. Using unique continuation through spatial boundaries, see Section 4, we show that $v(y, s) = 0$ if $y \in \mathbb{R}_+^n$ and $0 < s < 1$. Lemma 5.3 is proved. \square

Now, Theorem 5.1 follows from Lemmas 5.2 and 5.3 with the help of more or less standard arguments. We shall demonstrate them just for completeness.

Lemma 5.4 *Suppose that the function u meets all conditions of Lemma 5.3. Then $u \equiv 0$ in Q_+ .*

PROOF By Lemma 5.3, $u(x, t) = 0$ for $x \in \mathbb{R}_+^n$ and for $t \in]0, \gamma_1[$. By scaling, we introduce the function $u^{(1)}(y, s) = u(\sqrt{1 - \gamma_1}y, (1 - \gamma_1)s + \gamma_1)$. It is easy to check that function $u^{(1)}$ is well-defined in Q_+ and satisfies all conditions of Lemma 5.3 with the same constants c_1 and A . Therefore, $u^{(1)}(y, s) = 0$ for $y_n > 0$ and for $0 < s < \gamma_1$. The latter means that $u(x, t) = 0$ for $x_n > 0$ and for $0 < t < \gamma_2 = \gamma_1 + (1 - \gamma_1)\gamma_1$. Then, we introduce the function

$$u^{(2)}(y, s) = u(\sqrt{1 - \gamma_2}y, (1 - \gamma_2)s + \gamma_2), \quad (y, s) \in Q_+,$$

and apply Lemma 5.3. After k steps we shall see that $u(x, t) = 0$ for $x_n > 0$ and for $0 < t < \gamma_{k+1}$, where $\gamma_{k+1} = \gamma_k + (1 - \gamma_k)\gamma_1 \rightarrow 1$. Lemma 5.4 is proved. \square

PROOF OF THEOREM 5.1 Assume that $A_0 < M$. Then $\lambda^2 \equiv \frac{A_0}{2M} < \frac{1}{2}$. Introducing function $v(y, s) = u(\lambda y, \lambda^2 s)$, $(y, s) \in Q_+$, we see that this function satisfies all conditions of Lemma 5.4 with constants c_1 and $A = \frac{1}{2}A_0$. Therefore, $u(x, t) = 0$ for $x_n > 0$ and for $0 < t < \frac{A_0}{2M}$. Now, we repeat arguments of Lemma 5.4, replacing γ_1 to $\frac{A_0}{2M}$ and A to M , and end up with the proof of the theorem. Theorem 5.1 is proved. \square

6 Carleman-Type Inequalities

The first Carleman-type inequality is essentially the same as the one used in [5] and [6] (see also [3], [8], and [46]).

Proposition 6.1 *For any function $u \in C_0^\infty(\mathbb{R}^n \times]0, 2[; \mathbb{R}^n)$ and for any positive number a , the following inequality is valid:*

$$\begin{aligned} & \int_{\mathbb{R}^n \times]0, 2[} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} \left(\frac{a}{t} |u|^2 + |\nabla u|^2 \right) dx dt \\ & \leq c_0 \int_{\mathbb{R}^n \times]0, 2[} h^{-2a}(t) e^{-\frac{|x|^2}{4t}} |\partial_t u + \Delta u|^2 dx dt. \end{aligned} \tag{6.1}$$

Here, c_0 is an absolute positive constant and $h(t) = t e^{\frac{1-t}{3}}$.

PROOF OF PROPOSITION 6.1 Our proof follows standard techniques used in the L_2 -theory of Carleman inequalities, see for example [13] and [46].

Let u be an arbitrary function from $C_0^\infty(\mathbb{R}^n \times]0, 2[; \mathbb{R}^n)$. We set $\phi(x, t) = -\frac{|x|^2}{8t} - (a+1) \ln h(t)$ and $v = e^\phi u$. Then, we have

$$Lv \equiv e^\phi(\partial_t u + \Delta u) = \partial_t v - \operatorname{div}(v \otimes \nabla \phi) - \nabla v \nabla \phi + \Delta v + (|\nabla \phi|^2 - \partial_t \phi)v.$$

The main trick in the above approach is the decomposition of operator tL into symmetric and skew symmetric parts, i.e.,

$$tL = S + A, \quad (6.2)$$

where

$$Sv \equiv t(\Delta v + (|\nabla \phi|^2 - \partial_t \phi)v) - \frac{1}{2}v \quad (6.3)$$

and

$$Av \equiv \frac{1}{2}(\partial_t(tv) + t\partial_t v) - t(\operatorname{div}(v \otimes \nabla \phi) + \nabla v \nabla \phi). \quad (6.4)$$

Obviously,

$$\begin{aligned} \int t^2 e^{2\phi} |\partial_t u + \Delta u|^2 dx dt &= \int t^2 |Lv|^2 dx dt \\ &= \int |Sv|^2 dx dt + \int |Av|^2 dx dt + \int [S, A]v \cdot v dx dt, \end{aligned} \quad (6.5)$$

where $[S, A] = SA - AS$ is the commutator of S and A . Simple calculations show that

$$\begin{aligned} I &\equiv \int [S, A]v \cdot v dx dt = \\ &= 4 \int t^2 \left[\phi_{,ij} v_{,i} \cdot v_{,j} + \phi_{,ij} \phi_{,i} \phi_{,j} |v|^2 \right] dx dt \\ &\quad + \int t^2 |v|^2 (\partial_t^2 \phi - 2\partial_t |\nabla \phi|^2 - \Delta^2 \phi) dx dt \\ &\quad + \int t |\nabla v|^2 dx dt - \int t |v|^2 (|\nabla \phi|^2 - \partial_t \phi) dx dt. \end{aligned} \quad (6.6)$$

Here and in what follows, we adopt the convention on summation over repeated Latin indices, running from 1 to n . Partial derivatives in spatial variables are denoted by comma in lower indices, i.e., $v_{,i} = \frac{\partial v}{\partial x_i}$, $\nabla v = (v_{,i,j})$, etc. Given choice of function ϕ , we have

$$I = (a+1) \int t^2 \left[-\left(\frac{h'(t)}{h(t)}\right)' - \frac{h'(t)}{th(t)} \right] |v|^2 dx dt = \frac{a+1}{3} \int t |v|^2 dx dt. \quad (6.7)$$

By the simple identity

$$|\nabla v|^2 = \frac{1}{2}(\partial_t + \Delta)|v|^2 - v \cdot (\partial_t v + \Delta v), \quad (6.8)$$

we find

$$\begin{aligned} \int t^2 |\nabla v|^2 dx dt &= - \int t |v|^2 dx dt - \int t^2 v \cdot Lv dx dt \\ &+ \int t^2 |v|^2 (|\nabla \phi|^2 - \partial_t \phi) dx dt. \end{aligned} \quad (6.9)$$

In our case,

$$|\nabla \phi|^2 - \partial_t \phi = -|\nabla \phi|^2 + (a+1) \frac{h'(t)}{h(t)}.$$

The latter relation (together with (6.7)) implies the bound

$$\begin{aligned} &\int t^2 (|\nabla v|^2 + |v|^2 |\nabla \phi|^2) dx dt \\ &\leq 3I - \int t^2 v \cdot Lv dx dt \leq b_1 \int t^2 |Lv|^2 dx dt, \end{aligned} \quad (6.10)$$

where b_1 is an absolute positive constant. Since

$$e^\phi |\nabla u| \leq |\nabla v| + |v| |\nabla \phi|, \quad (6.11)$$

it follows from (6.5)–(6.10) that

$$\begin{aligned} &\int h^{-2a}(t) (th^{-1}(t))^2 \left((a+1) \frac{|u|^2}{t} + |\nabla u|^2 \right) e^{-\frac{|x|^2}{4t}} dx dt \\ &\leq b_2 \int h^{-2a}(t) (th^{-1}(t))^2 |\partial_t u + \Delta u|^2 e^{-\frac{|x|^2}{4t}} dx dt. \end{aligned}$$

Here, b_2 is an absolute positive constant. Inequality (6.1) is proved. \square

The second Carleman-type inequality is, in a sense, an anisotropic one.

Proposition 6.2 *Let*

$$\phi = \phi^{(1)} + \phi^{(2)},$$

where $\phi^{(1)}(x, t) = -\frac{|x'|^2}{8t}$ and $\phi^{(2)}(x, t) = a(1-t) \frac{x_n^{2\alpha}}{t^\alpha}$, $x' = (x_1, x_2, \dots, x_{n-1})$ so that $x = (x', x_n)$, and $e_n = (0, 0, \dots, 0, 1)$. Then, for any function $u \in$

$C_0^\infty((\mathbb{R}_+^n + e_n) \times]0, 1[; \mathbb{R}^n)$ and for any number $a > a_0(\alpha)$, the following inequality is valid:

$$\begin{aligned} & \int_{(\mathbb{R}_+^n + e_n) \times]0, 1[} t^2 e^{2\phi(x,t)} \left(a \frac{|u|^2}{t^2} + \frac{|\nabla u|^2}{t} \right) dx dt \\ & \leq c_\star \int_{(\mathbb{R}_+^n + e_n) \times]0, 1[} t^2 e^{2\phi(x,t)} |\partial_t u + \Delta u|^2 dx dt. \end{aligned} \quad (6.12)$$

Here, $c_\star = c_\star(\alpha)$ is a positive constant and $\alpha \in]1/2, 1[$ is fixed.

PROOF Let $u \in C_0^\infty(Q_+^1; \mathbb{R}^n)$. We are going to use formulae (6.2)–(6.6) for new functions u , v , and ϕ . All integrals in those formulae are taken now over Q_+^1 .

First, we observe that

$$\begin{aligned} \nabla \phi &= \nabla \phi^{(1)} + \nabla \phi^{(2)} \\ \nabla \phi^{(1)}(x, t) &= -\frac{x'}{4t}, \quad \nabla \phi^{(2)}(x, t) = 2\alpha a \frac{1-t}{t^\alpha} x_n^{2\alpha-1} e_n. \end{aligned} \quad (6.13)$$

Therefore,

$$\nabla \phi^{(1)} \cdot \nabla \phi^{(2)} = 0, \quad |\nabla \phi|^2 = |\nabla \phi^{(1)}|^2 + |\nabla \phi^{(2)}|^2. \quad (6.14)$$

Moreover,

$$\begin{aligned} \nabla^2 \phi &= \nabla^2 \phi^{(1)} + \nabla^2 \phi^{(2)}, \\ \phi_{,ij}^{(1)} &= \begin{cases} -\frac{\delta_{ij}}{4t} & \text{if } 1 \leq i, j \leq n-1 \\ 0 & \text{if } i = n \text{ or } j = n \end{cases}, \\ \phi_{,ij}^{(2)} &= \begin{cases} 0 & \text{if } i \neq n \text{ or } j \neq n \\ 2\alpha(2\alpha-1)a \frac{1-t}{t^\alpha} x_n^{2\alpha-2} & \text{if } i = n \text{ and } j = n \end{cases}. \end{aligned} \quad (6.15)$$

In particular, (6.15) implies

$$\phi_{,ij} \phi_{,i} \phi_{,j} = -\frac{1}{4t} |\nabla \phi^{(1)}|^2 + 2\alpha(2\alpha-1)a \frac{1-t}{t^\alpha} x_n^{2\alpha-2} |\nabla \phi^{(2)}|^2 \geq -\frac{1}{4t} \frac{|x'|^2}{16t^3}. \quad (6.16)$$

Using (6.14)–(6.16), we present integral I in (6.6) in the following way:

$$I = I_1 + I_2 + \int t|\nabla v|^2 dxdt, \quad (6.17)$$

where

$$\begin{aligned} I_s &= 4 \int t^2 \left[\phi_{,ij}^{(s)} v_{,i} \cdot v_{,j} + \phi_{,ij}^{(s)} \phi_{,i}^{(s)} \phi_{,j}^{(s)} |v|^2 \right] dxdt \\ &\quad + \int t^2 |v|^2 \left(\partial_t^2 \phi^{(s)} - 2\partial_t |\nabla \phi^{(s)}|^2 - \Delta^2 \phi^{(s)} \right. \\ &\quad \left. - \frac{1}{t} |\nabla \phi^{(s)}|^2 + \frac{1}{t} \partial_t \phi^{(s)} \right) dxdt, \quad s = 1, 2. \end{aligned}$$

Direct calculations give us

$$I_1 = - \int t(|\nabla v|^2 - |v_{,n}|^2) dxdt$$

and, therefore,

$$I = \int t|v_{,n}|^2 dxdt + I_2. \quad (6.18)$$

Now, our aim is to estimate I_2 from below. Since $\alpha \in]1/2, 1[$, we can skip the first integral in the expression for I_2 . As a result, we have

$$I_2 \geq \int t^2 |v|^2 (A_1 + A_2 + A_3) dxdt, \quad (6.19)$$

where

$$\begin{aligned} A_1 &= -\partial_t |\nabla \phi^{(2)}|^2, \\ A_2 &= A_1 - \Delta^2 \phi^{(2)} - \frac{1}{t} |\nabla \phi^{(2)}|^2, \\ A_3 &= \partial_t^2 \phi^{(2)} + \frac{1}{t} \partial_t \phi^{(2)}. \end{aligned}$$

For A_2 , we find

$$A_2 \geq \frac{1-t}{t^\alpha} x_n^{2\alpha-4} a(2\alpha-1) \left[\frac{4\alpha^2 a x_n^{2\alpha+2}}{t^{\alpha+1}} - 2\alpha(2\alpha-2)(2\alpha-3) \right].$$

Since $x_n \geq 1$ and $0 < t < 1$, we see that $A_2 > 0$ for all $a \geq 2$. Hence, it follows from (6.18) and (6.19) that

$$I \geq \int t^2 |v|^2 (A_1 + A_3) dxdt. \quad (6.20)$$

It is not difficult to check the following inequality

$$A_3 \geq a(2\alpha - 1) \frac{x_n^{2\alpha}}{t^{\alpha+2}}. \quad (6.21)$$

On the other hand,

$$-\partial_t |\nabla \phi^{(2)}|^2 - \frac{1}{t} |\nabla \phi^{(2)}|^2 \geq (2\alpha - 1) \frac{1-t}{t^{2\alpha+1}} 4\alpha^2 a^2 x_n^{2(2\alpha-1)} \geq 0$$

and thus

$$A_1 \geq \frac{1}{t} |\nabla \phi^{(2)}|^2. \quad (6.22)$$

Combining (6.20)–(6.22), we deduce from (6.5) the estimate

$$\begin{aligned} \int t^2 |Lv|^2 dxdt &\geq I \\ &\geq a(2\alpha - 1) \int \frac{x_n^{2\alpha}}{t^\alpha} |v|^2 dxdt + \int t|v|^2 |\nabla \phi^{(2)}|^2 dxdt \\ &\geq a(2\alpha - 1) \int |v|^2 dxdt + \int t|v|^2 |\nabla \phi^{(2)}|^2 dxdt. \end{aligned} \quad (6.23)$$

Using (6.8), we can find the following analog of (6.9)

$$\begin{aligned} \int t |\nabla v|^2 dxdt &= -\frac{1}{2} \int |v|^2 dxdt - \int tv \cdot Lv dxdt \\ &\quad + \int t|v|^2 (|\nabla \phi|^2 - \partial_t \phi) dxdt. \end{aligned} \quad (6.24)$$

Due to special structure of ϕ , we have

$$\begin{aligned} |\nabla \phi|^2 - \partial_t \phi &= |\nabla \phi^{(1)}|^2 - \partial_t \phi^{(1)} + |\nabla \phi^{(2)}|^2 - \partial_t \phi^{(2)} \\ &= -|\nabla \phi^{(1)}|^2 + |\nabla \phi^{(2)}|^2 - \partial_t \phi^{(2)} \end{aligned}$$

and, therefore, (6.24) can be reduced to the form

$$\begin{aligned} &\int \left(t |\nabla v|^2 + t|v|^2 (|\nabla \phi^{(1)}|^2 + |\nabla \phi^{(2)}|^2) \right) dxdt \\ &= \int t \left(|\nabla v|^2 + |v|^2 |\nabla \phi|^2 \right) dxdt = -\frac{1}{2} \int |v|^2 dxdt \\ &\quad - \int tv \cdot Lv dxdt + 2 \int t|v|^2 |\nabla \phi^{(2)}|^2 dxdt - \int t|v|^2 \partial_t \phi^{(2)} dxdt. \end{aligned} \quad (6.25)$$

But

$$-t\partial_t\phi^{(2)} \leq a \frac{x_n^{2\alpha}}{t^\alpha}$$

and, by (6.11) and (6.25),

$$\begin{aligned} \frac{1}{2} \int t e^{2\phi} |\nabla u|^2 &\leq - \int v \cdot (tLv) dxdt \\ &+ 2 \int t |v|^2 |\nabla \phi^{(2)}|^2 dxdt + a \int \frac{x_n^{2\alpha}}{t^\alpha} |v|^2 dxdt. \end{aligned} \tag{6.26}$$

The Cauchy-Schwartz inequality, (6.23), and (6.26) imply required inequality (6.12). \square

7 Appendix

HEAT EQUATION

We start with derivation of the known estimates for solutions to the Cauchy problem for the heat equation. So, let us consider the following initial problem

$$\partial_t u - \Delta u = 0 \quad \text{in } Q_T, \tag{7.1}$$

$$u(\cdot, 0) = a(\cdot) \quad \text{in } \mathbb{R}^3. \tag{7.2}$$

Lemma 7.1 *For solutions to problem (7.1) and (7.2), the following bounds are valid:*

$$\|u(\cdot, t)\|_s \leq c_1(s, s_1) t^{-\frac{1}{l}} \|a\|_{s_1}, \quad t > 0, \tag{7.3}$$

for $s \geq s_1$

$$\|u\|_{s,l,Q_T} \leq c_1(s, s_1) \|a\|_{s_1} \tag{7.4}$$

for $s > s_1$. Here,

$$\frac{1}{l} = \frac{3}{2} \left(\frac{1}{s_1} - \frac{1}{s} \right). \tag{7.5}$$

Remark 7.2 *Estimates (7.4) were proved in [18], Chapter 3, Theorem 9.1.*

PROOF We are not going to prove Lemma 7.1 in full generality. Our aim is just to show how it can be done. First, we note that the solution to the Cauchy problem has the form

$$u(\cdot, t) = \Gamma(\cdot, t) * a(\cdot), \tag{7.6}$$

where Γ is the fundamental solution to the heat operator, i.e.,

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \begin{cases} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then, (7.3) can be derived from (7.6) with the help of Hölder inequality and scaling arguments.

Estimate (7.5) is a little bit more delicate and we prove it for the special case $s = 5$ and $s_1 = 3$. We may assume

$$a \in C_0^\infty. \quad (7.7)$$

Then, all further operations will be obviously legal. The required estimate can be achieved by passing to the limit.

Multiplying (7.1) by $|u|u$ and integrating by parts, we find

$$\frac{1}{3} \partial_t \|u(\cdot, t)\|_3^3 + \int_{\mathbb{R}^3} |u(\cdot, t)| |\nabla u(\cdot, t)|^2 dx + \frac{4}{9} \|\nabla |u(\cdot, t)|^{\frac{3}{2}}\|_2^2 = 0. \quad (7.8)$$

Setting $g = |u|^{\frac{3}{2}}$, we observe that (7.8) implies the estimate

$$|g|_{2, Q_T}^2 \equiv \operatorname{ess\,sup}_{0 < t < T} \|g(\cdot, t)\|_2^2 + \|\nabla g\|_{2, Q_T}^2 \leq c_2 \|a\|_3^3, \quad (7.9)$$

where c_2 is an absolute positive constant. Now, by the multiplicative inequality (see [18]),

$$\|g(\cdot, t)\|_{\frac{10}{3}} \leq c'_2 \|g(\cdot, t)\|_2^{\frac{2}{5}} \|\nabla g(\cdot, t)\|_2^{\frac{3}{5}} \quad (7.10)$$

and, therefore,

$$\|u\|_{5, Q_T} = \|g\|_{\frac{10}{3}, Q_T}^{\frac{2}{3}} \leq c''_2 \|a\|_3$$

for some absolute positive constant c''_2 . Lemma 7.1 is proved. \square

STOKES SYSTEM

Simple arguments of the previous subsection also work in the case of the Cauchy problem for the Stokes system:

$$\left. \begin{aligned} \partial_t u - \Delta u &= \operatorname{div} f - \nabla q \\ \operatorname{div} u &= 0 \end{aligned} \right\} \quad \text{in } Q_T, \quad (7.11)$$

$$u(\cdot, 0) = a(\cdot) \quad \text{in } \mathbb{R}^3. \quad (7.12)$$

Theorem 7.3 *Assume that*

$$f \in L_{\frac{5}{2}}(Q_T) \cap L_2(Q_T) \quad (7.13)$$

and

$$a \in L_3 \cap \mathring{J}. \quad (7.14)$$

For any $T > 0$, there exists a pair of functions u and q with the following properties:

$$u \in C([0, T]; L_2) \cap L_2(0, T; \mathring{J}_2^1), \quad \partial_t u \in L_2(0, T; (\mathring{J}_2^1)'); \quad (7.15)$$

$$u \in C([0, T]; L_3) \cap L_5(Q_T) \cap L_4(Q_T); \quad (7.16)$$

$$q \in L_2(Q_T) \cap L_{\frac{5}{2}}(Q_T); \quad (7.17)$$

$$u \text{ and } q \text{ satisfy equations (7.11) in the sense of distributions;} \quad (7.18)$$

$$\text{initial condition holds in the sense } \|u(\cdot, t) - a(\cdot)\|_3 \rightarrow 0 \text{ as } t \rightarrow 0; \quad (7.19)$$

$$\|u\|_{3, \infty, Q_T} + \|u\|_{5, Q_T} \leq c_3(\|f\|_{\frac{5}{2}, Q_T} + \|a\|_3); \quad (7.20)$$

$$\|u\|_{4, Q_T} \leq c_3(\|f\|_{\frac{5}{2}, Q_T} + \|a\|_3 + \|f\|_{2, Q_T} + \|a\|_2), \quad (7.21)$$

where c_3 is an absolute positive constant.

PROOF As usual, we can assume that, in addition,

$$f \in C_0^\infty(Q_T), \quad a \in C_0^\infty(\mathbb{R}^3).$$

The general case is treated with the help of suitable approximations.

L_2 -estimates are obvious:

$$\|u\|_{2, \infty, Q_T} + \|\nabla u\|_{2, Q_T} + \|\partial_t u\|_{L_2(0, T; (\mathring{J}_2^1)')} \leq C(\|a\|_2 + \|f\|_2). \quad (7.22)$$

Here, C is an absolute positive constant.

Next, taking divergence of the first equation in (7.11), we find the equation for the pressure

$$\Delta q = \operatorname{div} \operatorname{div} f.$$

Therefore, (7.17) is proved and, moreover,

$$\|q\|_{\frac{5}{2}} \leq C_0 \|f\|_{\frac{5}{2}}. \quad (7.23)$$

As in the proof of Lemma 7.1, we test our equation with $|u|u$ and, making use of Hölder inequality, arrive at the estimate

$$\begin{aligned} & \frac{1}{3} \partial_t \|u\|_3^3 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 dx + \frac{4}{9} \int_{\mathbb{R}^3} |\nabla v|^2 dx = \int_{\mathbb{R}^3} (p \operatorname{div}(|u|u) \\ & - f : \nabla(|u|u)) dx \leq C_1 \left(\int_{\mathbb{R}^3} (|f|^2 + |q|^2) |u| dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u| |\nabla u|^2 dx \right)^{\frac{1}{2}} \quad (7.24) \\ & \leq (\text{see (7.23)}) \leq C_1 \left(\int_{\mathbb{R}^3} |u| |\nabla u|^2 dx \right)^{\frac{1}{2}} \|v\|_{\frac{10}{3}}^{\frac{1}{3}} \|f\|_{\frac{5}{2}} \equiv A, \end{aligned}$$

where $v = |u|^{\frac{3}{2}}$. The right hand side in (7.24) can be evaluated with the help of the multiplicative inequality (see (7.10)) in the following way:

$$A \leq C_2 \left(\int_{\mathbb{R}^3} |u| |\nabla u|^2 dx \right)^{\frac{1}{2}} (\|v\|_{\frac{3}{2}}^{\frac{1}{3}})^{\frac{2}{5}} (\|\nabla v\|_{\frac{3}{2}}^{\frac{1}{3}})^{\frac{3}{5}} \|f\|_{\frac{5}{2}}.$$

Applying Young's inequality twice and the identity $\|v\|_2^2 = \|u\|_3^3$, we find from (7.24) and from the last bound the basic estimate

$$\partial_t \|u\|_3^3 + \int_{\mathbb{R}^3} |u| |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla v|^2 dx \leq C_3 \|u\|_3^{\frac{1}{2}} \|f\|_{\frac{5}{2}}^{\frac{5}{2}}. \quad (7.25)$$

Obviously, (7.25) implies the inequality

$$\|u\|_{3,\infty,Q_T} \leq C_4 (\|f\|_{\frac{5}{2},Q_T} + \|a\|_3), \quad (7.26)$$

where C_4 is an absolute positive constant. Then, by (7.25) and by (7.26),

$$\int_{Q_T} (|u| |\nabla u|^2 + |\nabla v|^2) dx dt \leq C_5 (\|f\|_{\frac{5}{2},Q_T} + \|a\|_3)^3$$

and thus (see (7.10))

$$\|u\|_{5,Q_T} \leq C_6 \|u\|_{3,\infty,Q_T}^{\frac{2}{5}} \|\nabla v\|_{2,Q_T}^{\frac{2}{5}} \leq C'_6 (\|f\|_{\frac{5}{2},Q_T} + \|a\|_3),$$

where C'_6 is an absolute positive constant.

On the other hand, another multiplicative inequality says that

$$\|u\|_{4,Q_T} \leq C_7 \|u\|_{3,\infty,Q_T}^{\frac{3}{4}} \|\nabla u\|_{2,Q_T}^{\frac{1}{4}} \leq C'_7 (\|u\|_{3,\infty,Q_T} + \|\nabla u\|_{2,Q_T}).$$

So, (7.20) and (7.21) are proved.

It remains to show that

$$u \in C([0, T]; L_3). \quad (7.27)$$

To do this, let us go back to the first identity in (7.24). It gives us:

$$\int_0^T \left| \partial_t \|u\|_3^3 \right| dt \leq C_8 \left[\|f\|_{\frac{5}{2},Q_T}^2 \|u\|_{5,Q_T} + \int_{Q_T} (|u| |\nabla u|^2 + |\nabla v|^2) dx dt \right].$$

Hence, we can claim that the function $t \mapsto \|u(\cdot, t)\|_3$ is continuous. But, by known arguments, the function $t \mapsto \int_{\mathbb{R}^3} u(x, t) \cdot w(x) dx$ is continuous on $[0, T]$ for all $w \in L_{\frac{3}{2}}$. These two facts imply (7.27). Theorem 7.3 is proved. \square

NAVIER-STOKES SYSTEM

Here, we are going to consider the Cauchy problem for the Navier-Stokes equations:

$$\left. \begin{aligned} \partial_t u + \operatorname{div} u \otimes u - \Delta u &= -\nabla q \\ \operatorname{div} u &= 0 \end{aligned} \right\} \quad \text{in } Q_T, \quad (7.28)$$

$$u(\cdot, 0) = a(\cdot) \quad \text{in } \mathbb{R}^3. \quad (7.29)$$

The following statement is due to T. Kato [21].

Theorem 7.4 *Assume that condition (7.14) holds. Then, a positive number T_* , depending on a only, exists and possesses the following property. There exists a unique pair of functions u and q such that:*

$$u \in C([0, T_*]; L_2) \cap L_2(0, T_*; \mathring{J}_2^1), \quad \partial_t u \in L_2(0, T_*; (\mathring{J}_2^1)'); \quad (7.30)$$

$$u \in C([0, T_*]; L_3) \cap L_5(Q_{T_*}) \cap L_4(Q_{T_*}); \quad (7.31)$$

$$q \in C([0, T_*]; L_{\frac{3}{2}}) \cap L_2(Q_{T_*}) \cap L_{\frac{5}{2}}(Q_{T_*}); \quad (7.32)$$

$$u \text{ and } q \text{ meet equations (7.28) in the sense of distributions;} \quad (7.33)$$

$$\text{initial condition holds in the sense } \|u(\cdot, t) - a(\cdot)\|_3 \rightarrow 0 \text{ as } t \rightarrow 0. \quad (7.34)$$

PROOF The proof is more or less standard (see, for instance, [9] and [23]) and based on successive iterations. We let

$$u^1(\cdot, t) = \Gamma(\cdot, t) * a(\cdot), \quad t > 0, \quad (7.35)$$

$$\kappa(T_\star) = \|u^1\|_{5, Q_{T_\star}} + \|u^1\|_{4, Q_{T_\star}}$$

and

$$u^{k+1} = w + u^1,$$

where w is a solution to the following Cauchy problem:

$$\left. \begin{aligned} \partial_t w - \Delta w &= -\operatorname{div} u^k \otimes u^k - \nabla q^k \\ \operatorname{div} w &= 0 \end{aligned} \right\} \quad \text{in } Q_{T_\star}, \quad (7.36)$$

$$w(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^3. \quad (7.37)$$

We also let

$$f = -u^k \otimes u^k.$$

According to Theorem 7.3, we have the estimate (see (7.20) and (7.21))

$$\|u^{k+1} - u^1\|_{5, Q_{T_\star}} + \|u^{k+1} - u^1\|_{4, Q_{T_\star}} \leq 2c_3(\|u^k\|_{5, Q_{T_\star}} + \|u^k\|_{4, Q_{T_\star}})^2.$$

It can be rewritten in the form

$$\begin{aligned} \|u^{k+1}\|_{5, Q_{T_\star}} + \|u^{k+1}\|_{4, Q_{T_\star}} &\leq 2c_3(\|u^k\|_{5, Q_{T_\star}} + \|u^k\|_{4, Q_{T_\star}})^2 \\ &\quad + \|u^1\|_{5, Q_{T_\star}} + \|u^1\|_{4, Q_{T_\star}}. \end{aligned} \quad (7.38)$$

Now, our aim is to show that a number T_\star can be chosen to fulfill the following conditions:

$$\|u^{k+1}\|_{5, Q_{T_\star}} + \|u^{k+1}\|_{4, Q_{T_\star}} \leq 2\kappa(T_\star) \quad (7.39)$$

for $k = 1, 2, \dots$. We argue by induction on k . Then, (7.38) and (7.39) give us:

$$\|u^{k+1}\|_{5, Q_{T_\star}} + \|u^{k+1}\|_{4, Q_{T_\star}} \leq 8c_3\kappa^2(T_\star) + \kappa(T_\star) = \kappa(T_\star)(8c_3\kappa(T_\star) + 1).$$

Obviously, inequalities (7.39) are valid if we choose T_\star so that

$$\kappa(T_\star) < \frac{1}{8c_3}. \quad (7.40)$$

To show that this can be done, we introduce

$$a_\rho = \omega_\rho * a,$$

where ω_ρ is the usual smoothing kernel. We let $u_\rho^1(\cdot, t) = \Gamma(\cdot, t) * a_\rho(\cdot)$ and then

$$\kappa(T_\star) \leq I_\rho^1 + I_\rho^2, \quad (7.41)$$

where

$$I_\rho^1 = \|u_\rho^1\|_{5, Q_{T_\star}} + \|u_\rho^1\|_{4, Q_{T_\star}}, \quad I_\rho^2 = \|u^1 - u_\rho^1\|_{5, Q_{T_\star}} + \|u^1 - u_\rho^1\|_{4, Q_{T_\star}}.$$

Certainly, Theorem 7.3 is valid for the heat equation as well. Therefore,

$$I_\rho^2 \leq C_9(\|a - a_\rho\|_3 + \|a - a_\rho\|_2),$$

where C_9 is an absolute constant. We fix $\rho > 0$ in such a way that

$$C_9(\|a - a_\rho\|_3 + \|a - a_\rho\|_2) < \frac{1}{16c_3}. \quad (7.42)$$

To estimate I_ρ^1 , we apply Lemma 7.1. So, we have

$$\|u_\rho^1(\cdot, t)\|_5 \leq c_1 t^{-\frac{3}{40}} \|a_\rho\|_4, \quad \|u_\rho^1(\cdot, t)\|_4 \leq c_1 t^{-\frac{3}{40}} \|a_\rho\|_3$$

and thus

$$\|u_\rho^1\|_{5, Q_{T_\star}} \leq C_{10} T_\star^{\beta_1} \|a_\rho\|_4, \quad \|u_\rho^1\|_{4, Q_{T_\star}} \leq C_{10} T_\star^{\beta_2} \|a_\rho\|_3$$

for some positive absolute constants C_{10} , β_1 , and β_2 . It remains to choose $T_\star > 0$ so that

$$C_{10}(T_\star^{\beta_1} \|a_\rho\|_4 + T_\star^{\beta_2} \|a_\rho\|_3) < \frac{1}{16c_3}. \quad (7.43)$$

Combining (7.41)–(7.43), we prove (7.40). Then, passing to the limit as $k \rightarrow +\infty$, we establish all statements of Theorem 7.4, except continuity of u in t with values in L_3 and continuity of q in t with values in $L_{\frac{3}{2}}$. Continuity of u immediately follows from Theorem 7.3 and observation that $f = -u \otimes u \in L_{\frac{5}{2}}(Q_{T_\star})$. Continuity of q is a consequence of the pressure equation

$$\Delta q = -\operatorname{div} \operatorname{div} u \otimes u.$$

Theorem 7.4 is proved. \square

Remark 7.5 *It is easy to check that the function u of Theorem 7.4 is in fact the weak Leray-Hopf solution. Since it belongs to $L_5(Q_{T_\star})$, any other weak solution coincide with u (see Theorem 1.2).*

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