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BACKWARD UNIQUENESS FOR THE HEAT OPERATOR IN A HALF-SPACE

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ABSTRACT. A backward uniqueness result is proved for the heat operator with variable lower order terms in a half-space. The main point of the result is that the boundary conditions are not controlled by the assumptions.

§1. INTRODUCTION

In this paper, which can be thought of as a continuation of [3] and [4], we deal with the following backward uniqueness problem for the heat operator. Let $\mathbb{R}^n_+ = \{x = (x_i) \in \mathbb{R}^n \ : \ x_n > 0\}$, and let $Q_+ = \mathbb{R}^n_+ \times]0,1[$. We consider a vector-valued function $u : Q_+ \to \mathbb{R}^n$, assuming that it is "sufficiently regular" and satisfies the conditions

$$|\partial_t u + \Delta u| \le c_1(|\nabla u| + |u|) \quad \text{in } Q_+ \tag{1.1}$$

for some $c_1 > 0$, and

$$u(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^n_+. \tag{1.2}$$

Do (1.1) and (1.2) imply $u \equiv 0$ in Q_+? We prove that the answer is in the positive if we impose natural restrictions on the growth of u at infinity. For instance, we may assume that

$$|u(x,t)| \le e^{M|x|^2} \tag{1.3}$$

for all $(x,t) \in Q_+$ and for some $M > 0$. The natural regularity assumptions under which (1.1)–(1.3) can be considered are, e.g., as follows:

$$\left.\begin{array}{l} u \text{ and distributional derivatives } \partial_t u,\ \nabla^2 u \text{ are square} \\ \text{integrable over the bounded subdomains of } Q_+. \end{array}\right\} \tag{1.4}$$

We can formulate our main result.

Theorem 1.1. *In the notation introduced above, assume that u satisfies conditions (1.1)–(1.4). Then $u \equiv 0$ in Q_+.*

This extends the main result of [3] and [4], where an analog of Theorem 1.1 was proved for $Q_+ = (\mathbb{R}^n \setminus B(R)) \times]0,T[$. Here, as usual, $B(R)$ denotes the n-dimensional ball of radius R with center at the origin.

Such results are of interest in control theory; see, e.g., [8]. Also, as was explained in [10], results of this type are helpful in regularity theory for the Navier–Stokes equations.

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By combining the methods developed in [10] and the boundary regularity results proved in [9], it can be shown that the so-called Leray–Hopf solutions to the Navier–Stokes equations in a regular domain $\Omega \in \mathbb{R}^3$ that satisfy the homogeneous Dirichlet boundary condition and belong to the space $L_\infty(0,1;L_3(\Omega;\mathbb{R}^3))$, are regular up to the boundary of Ω. A detailed proof of this result will be presented elsewhere.

As in [3] and [4], the proof of Theorem 1.1 is based on two Carleman-type inequalities. The first is essentially the same as that used in [3] and [4] (see also [1, 5], and [11]); it is of the form

$$\int\limits_{\mathbb{R}^n\times]0,2[} h^{-2a}(t)e^{-\frac{|x|^2}{4t}}\Big(a\frac{|w|^2}{t}+|\nabla w|^2\Big)\,dxdt \leq c_2 \int\limits_{\mathbb{R}^n\times]0,2[} h^{-2a}(t)e^{-\frac{|x|^2}{4t}}|\partial_t w+\Delta w|^2\,dxdt. \tag{1.5}$$

Here $h(t)=te^{\frac{1-t}{3}}$, and c_2 is a positive universal constant. Inequality (1.5) is fulfilled for any $w\in C_0^\infty(\mathbb{R}^n\times]0,2[)$ and any positive number a.

The second inequality is anisotropic in a sense:

$$\int\limits_{(\mathbb{R}^n_+ + e_n)\times]0,1[} t^2e^{2\phi(x,t)}\Big(a\frac{|w|^2}{t^2}+\frac{|\nabla w|^2}{t}\Big)\,dxdt \leq c_3 \int\limits_{(\mathbb{R}^n_+ + e_n)\times]0,1[} t^2e^{2\phi(x,t)}|\partial_t w+\Delta w|^2\,dxdt. \tag{1.6}$$

Here $\phi=\phi^{(1)}+\phi^{(2)}$, $\phi^{(1)}(x,t)=-\frac{|x'|^2}{8t}$, $\phi^{(2)}(x,t)=a(1-t)\frac{x_n^{2\alpha}}{t^\alpha}$, $\alpha\in]1/2,1[$ is fixed, $x'=(x_1,x_2,...,x_{n-1})$ (so that $x=(x',x_n)$), $c_3(\alpha)>0$, and $e_n=(0,0,...,0,1)$. Inequality (1.6) is fulfilled for any $w\in C_0^\infty((\mathbb{R}^n_+ + e_n)\times]0,1[)$ and any $a>a_0(\alpha)$.

Our concluding remark is that Theorem 1.1 is true for functions $u: Q_+\to\mathbb{R}^m$ with $1\leq m<+\infty$. This is an easy exercise, which is left to the reader.

Our paper is organized as follows. In §2 we prove Theorem 1.1, taking Carleman's inequalities for granted. The proofs of the latter are given in §3 and §4, respectively.

§2. Proof of Theorem 1.1

In what follows, we always assume that the function u is extended by zero to the negative values of t.

We start with several lemmas. The first of them plays the crucial role in our approach. It enables us to apply the powerful technique of Carleman's inequalities.

Lemma 2.1. *Suppose that conditions* (1.1), (1.2), *and* (1.4) *are fulfilled. There exists an absolute positive constant* $A_0<1/32$ *with the following properties. If*

$$|u(x,t)|\leq e^{A|x|^2} \tag{2.1}$$

for all $(x,t)\in Q_+$ *and some* $A\in[0,A_0]$, *then there are constants* $\beta(A)>0$, $\gamma(c_1)\in\,]0,1/12[$, *and* $c_4(c_1,A)>0$ *such that*

$$|u(x,t)|\leq c_4e^{4A|x'|^2}e^{-\beta\frac{x_n^2}{t}} \tag{2.2}$$

for all $(x,t)\in(\mathbb{R}^n_+ + 2e_n)\times]0,\gamma[$.

Proof. Referring to the regularity theory of solutions of parabolic equations (see [7]), we may assume that

$$|u(x,t)|+|\nabla u(x,t)|\leq c_5e^{2A|x|^2} \tag{2.3}$$

for all $(x,t)\in(\mathbb{R}^n_+ + e_n)\times]0,1/2[$.

Fixing $x_n>2$ and $t\in]0,\gamma[$, we define a new function v by the usual parabolic scaling

$$v(y,s)=u(x+\lambda y,\lambda^2s-t/2).$$

The function v is well defined on the set $Q_\rho=B(\rho)\times]0,2[$, where $\rho=(x_n-1)/\lambda$ and $\lambda=\sqrt{3t}\in]0,1/2[$. Then relations (1.1), (1.2), and (2.3) take the following form:

$$|\partial_sv+\Delta v|\leq c_1\lambda(|\nabla v|+|v|)\quad \text{a.e. in } Q_\rho; \tag{2.4}$$

$$|v(y,s)|+|\nabla v(y,s)|\leq c_5e^{4A|x|^2}e^{4A\lambda^2|y|^2} \tag{2.5}$$

for $(y,s)\in Q_\rho$;

$$v(y,s)=0 \tag{2.6}$$

for $y\in B(\rho)$ and $s\in]0,1/6]$.

In order to apply inequality (1.5), we choose two smooth cut-off functions such that

$$\phi_\rho(y)=\begin{cases}0 & \text{if } |y|>\rho-1/2,\\ 1 & \text{if } |y|<\rho-1,\end{cases}$$

and

$$\phi_t(s)=\begin{cases}0 & \text{if } 7/4<s<2,\\ 1 & \text{if } 0<s<3/2.\end{cases}$$

All functions take values in $[0,1]$. Moreover, the function ϕ_ρ satisfies $|\nabla^k\phi_\rho|<C_k$, $k=1,2$. We put $\eta(y,s)=\phi_\rho(y)\phi_t(s)$ and $w=\eta v$. From (2.4) it follows that

$$|\partial_sw+\Delta w|\leq c_1\lambda(|\nabla w|+|w|)+\chi c_6(|\nabla v|+|v|). \tag{2.7}$$

Here c_6 is a positive constant depending on c_1 and C_k only, $\chi(y,s)=1$ if $(y,s)\in\omega=\{\rho-1<|y|<\rho\}\times]3/2,2[$, and $\chi(y,s)=0$ if $(y,s)\notin\omega$. Obviously, the function w has a compact support in $\mathbb{R}^n\times]0,2[$, and we may use inequality (1.5). As a result, we have

$$I\equiv\int\limits_{Q_\rho}h^{-2a}(s)e^{-\frac{|y|^2}{4s}}(|w|^2+|\nabla w|^2)\,dyds\leq c_210(c_1^2\lambda^2I+c_6^2I_1), \tag{2.8}$$

where

$$I_1=\int\limits_{Q_\rho}\chi(y,s)h^{-2a}(s)e^{-\frac{|y|^2}{4s}}(|v|^2+|\nabla v|^2)\,dyds.$$

Choosing $\gamma=\gamma(c_1)$ sufficiently small, we may assume that $c_210c_1^2\lambda^2\leq 1/2$, and then (2.8) implies

$$I\leq c_7(c_1)I_1. \tag{2.9}$$

On the other hand, if $A < 1/32$, then

$$8A\lambda^2 - \frac{1}{4s} < -\frac{1}{8s} \tag{2.10}$$

for $s \in]0,2]$. By (2.5) and (2.10), we have

$$\begin{aligned} I_1 &\leq c_5^2 e^{8A|x|^2} \int\limits_0^2 \int\limits_{B(\rho)} \chi(y,s) h^{-2a}(s) e^{-\frac{|y|^2}{8s}}\, dyds \\ &\leq c_8 e^{8A|x|^2} \Big[h^{-2a}(3/2) + \int\limits_0^2 h^{-2a}(s) e^{-\frac{(\rho-1)^2}{8s}}\, ds \Big]. \end{aligned} \tag{2.11}$$

Now, taking (2.11) into account, we deduce the estimate

$$\begin{aligned} D \equiv \int\limits_{B(1)} \int\limits_{\frac12}^1 |w|^2\, dyds &= \int\limits_{B(1)} \int\limits_{\frac12}^1 |v|^2\, dyds \\ &\leq c_9 \int\limits_{Q_\rho} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2)\, dyds \\ &\leq c_{10}(c_1) e^{8A|x|^2} \Big[h^{-2a}(3/2) + \int\limits_0^2 h^{-2a}(s) e^{-\frac{\rho^2}{32s}}\, ds \Big] \\ &= c_{10} e^{8A|x|^2 - 2\beta\rho^2} \Big[h^{-2a}(3/2) e^{2\beta\rho^2} + \int\limits_0^2 h^{-2a}(s) e^{2\beta\rho^2 - \frac{\rho^2}{32s}}\, ds \Big]. \end{aligned}$$

We can take $\beta = 8A < 1/256$ and then choose

$$a = \beta\rho^2 / \ln h(3/2).$$

Since $\rho \geq x_n$, this leads to the estimate

$$D \leq c_{10} e^{8A|x'|^2} e^{-\beta\rho^2} \Big[1 + \int\limits_0^2 g(s)\, ds \Big],$$

where $g(s) = h^{-2a}(s) e^{-\frac{\rho^2}{64s}}$. It is easy to check that $g'(s) \geq 0$ for $s \in]0,2[$ if $\beta < \frac{1}{96} \ln h(3/2)$. Thus, we obtain

$$D \leq 2c_{10} e^{8A|x'|^2} e^{-\beta\rho^2} \leq 2c_{10} e^{8A|x'|^2} e^{-\frac{\beta x_n^2}{12t}}. \tag{2.12}$$

On the other hand, regularity theory yields

$$|v(0,1/2)|^2 = |u(x,t)|^2 \leq c'_{10} D. \tag{2.13}$$

Combining (2.12) and (2.13), we complete the proof of Lemma 2.1. □

The next lemma will be deduced from consequence of Lemma 2.1 and the second Carleman inequality (see (1.6)).

Lemma 2.2. *Suppose that u satisfies conditions* (1.1), (1.2), (1.4), *and* (2.1). *There exists a number $\gamma_1(c_1, c_3) \in]0, \gamma/2]$ such that*

$$u(x,t) = 0 \tag{2.14}$$

for all $x \in \mathbb{R}^n_+$ and all $t \in]0, \gamma_1[$.

Proof. As usual, Lemma 2.1 and regularity theory allow us to assume that

$$|u(x,t)| + |\nabla u(x,t)| \le c_{11}(c_1, A) e^{8A|x'|^2} e^{-\beta \frac{x_n^2}{2t}} \tag{2.15}$$

for all $x \in \mathbb{R}^n_+ + 3e_n$ and all $t \in]0, \gamma/2]$.

Using scaling, we define $v(y,s) = u(\lambda y, \lambda^2 s - \gamma_1)$ for $(y,s) \in Q_+$; here $\lambda = \sqrt{2\gamma_1}$. This function satisfies the following conditions:

$$|\partial_s v + \Delta v| \le c_1 \lambda (|\nabla v| + |v|) \quad \text{a.e. in } Q_+; \tag{2.16}$$

$$v(y,s) = 0 \tag{2.17}$$

for all $y \in \mathbb{R}^n_+$ and all $s \in]0, 1/2[$;

$$|\nabla v(y,s)| + |v(y,s)| \le c_{11} e^{8A\lambda^2|y'|^2} e^{-\frac{\beta\lambda^2 y_n^2}{2(\lambda^2 s - \gamma_1)}} \le c_{11} e^{8A\lambda^2|y'|^2} e^{-\beta \frac{y_n^2}{2s}} \tag{2.18}$$

for all $1/2 < s < 1$ and all $y \in \mathbb{R}^n_+ + \frac{3}{\lambda} e_n$. Since $A < 1/32$ and $\lambda \le \sqrt{\gamma} \le 1/\sqrt{12}$, inequality (2.18) can be reduced to the form

$$|\nabla v(y,s)| + |v(y,s)| \le c_{11} e^{\frac{|y'|^2}{48}} e^{-\beta \frac{y_n^2}{2s}} \tag{2.19}$$

with the same y and s as in (2.18).

We fix two smooth cut-off functions such that

$$\psi_1(y_n) = \begin{cases} 0 & \text{if } y_n < \frac{3}{\lambda} + 1, \\ 1 & \text{if } y_n > \frac{3}{\lambda} + 2, \end{cases}$$

and

$$\psi_2(r) = \begin{cases} 0 & \text{if } r > -1/2, \\ 1 & \text{if } r < -3/4. \end{cases}$$

Invoking the functions $\phi^{(1)}$ and $\phi^{(2)}$) (see (1.6)), we set

$$\phi_B(y_n, s) = \frac{1}{a}\phi^{(2)}(y_n, s) - B = (1-s)\frac{y_n^{2\alpha}}{s^\alpha} - B,$$

where $\alpha \in]1/2, 1[$ is fixed, $B = \frac{2}{a}\phi^{(2)}(\frac{3}{\lambda} + 2, 1/2)$, and

$$\eta(y_n, s) = \psi_1(y_n)\psi_2(\phi_B(y_n, s)/B), \quad w(y,s) = \eta(y_n, s) v(y,s).$$

Although the function w is not compactly supported in Q^1_+, Lemma 2.1 and the special structure of the weight in (1.6) allow us to claim the validity of (1.6) for w. As a result, we have

$$\int\limits_{Q^1_+} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} (|w|^2 + |\nabla w|^2)\, dyds$$
$$\le c_3 \int\limits_{Q^1_+} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} |\partial_s w + \Delta w|^2\, dyds.$$

Arguing as in the proof of Lemma 2.1, we can choose $\gamma_1(c_1, c_3)$ so small that

$$I \equiv \int\limits_{Q^1_+} s^2 e^{2a\phi_B}(|w|^2 + |\nabla w|^2)e^{-\frac{|y'|^2}{4s}}\,dyds$$

$$\le c_{12}(c_1, c_3) \int\limits_{(\mathbb{R}^n_+ + (\frac{3}{\lambda}+1)e_n)\times]1/2,1[} \chi(y_n, s)(sy_n)^2 e^{2a\phi_B}(|v|^2 + |\nabla v|^2)e^{-\frac{|y'|^2}{4s}}\,dyds,$$

where $\chi(y_n, s) = 1$ if $(y_n, s) \in \omega$, $\chi(y_n, s) = 0$ if $(y_n, s) \notin \omega$, and

$$\omega \equiv \{(y_n, s) : y_n > 1,\ 1/2 < s < 1,\ \phi_B(y_n, s) < -B/2\}.$$

Now, we estimate the right-hand side of the above inequality with the help of (2.19). We find

$$I \le c_{13}e^{-Ba} \int\limits_{\frac{3}{\lambda}+1}^{+\infty} \int\limits_{1/2}^{1} (y_n s)^2 e^{-\beta\frac{y_n^2}{s}}\,dy_n ds \int\limits_{\mathbb{R}^{n-1}} e^{(\frac{1}{24}-\frac{1}{4s})|y'|^2}\,dy'.$$

Passing to the limit as $a \to +\infty$, we see that $v(y, s) = 0$ if $1/2 \le s < 1$ and that $\phi_B(y_n, s) > 0$. Using unique continuation through spatial boundaries (see [2]), we show that $v(y, s) = 0$ if $\in \mathbb{R}^n_+$ and $0 < s < 1$. Lemma 2.2 is proved. □

Now, Theorem 1.1 can be deduced from Lemmas 2.1 and 2.2 with the help of more or less standard arguments. We include them merely for completeness.

Lemma 2.3. *Suppose that u satisfies all conditions of Lemma 2.2. Then $u \equiv 0$ in Q_+.*

Proof. By Lemma 2.2, $u(x, t) = 0$ for all $x \in \mathbb{R}^n_+$ and all $t \in]0, \gamma_1[$. Using scaling, we introduce $u^{(1)}(y, s) = u(\sqrt{1-\gamma_1}y, (1-\gamma_1)s+\gamma_1)$. It is easy to check that the function $u^{(1)}$ is well defined in Q_+ and satisfies all conditions of Lemma 2.2 with the same constants c_1 and A. Therefore, $u^{(1)}(y, s) = 0$ for $y_n > 0$ and $0 < s < \gamma_1$. This means that $u(x, t) = 0$ for all $x_n > 0$ and all $0 < t < \gamma_2 = \gamma_1 + (1 - \gamma_1)\gamma_1$. Then, we introduce the function

$$u^{(2)}(y, s) = u(\sqrt{1-\gamma_2}y, (1-\gamma_2)s + \gamma_2), \quad (y, s) \in Q_+,$$

and apply Lemma 2.2. After k steps, we see that $u(x, t) = 0$ for all $x_n > 0$ and all $0 < t < \gamma_{k+1}$, where $\gamma_{k+1} = \gamma_k + (1 - \gamma_k)\gamma_1 \to 1$. Lemma 2.3 is proved. □

Proof of Theorem 1.1. Assume that $A_0 < M$. Then $\lambda^2 \equiv A_0/2M < 1/2$. Introducing the function $v(y, s) = u(\lambda y, \lambda^2 s)$, $(y, s) \in Q_+$, we see that v satisfies all conditions of Lemma 2.3 with the constants c_1 and $A = \frac{1}{2}A_0$. Therefore, $u(x, t) = 0$ for $x_n > 0$ and all $0 < t < \frac{A_0}{2M}$. Now we can repeat the arguments of Lemma 2.3, replacing γ_1 with $\frac{A_0}{2M}$ and A with M, and finishing the proof of the theorem. □

§3. Proof of the first Carleman inequality

In our proof we employ the standard techniques used in the L_2-theory of Carleman inequalities (see, e.g., [6] and [11]).

Let u be an arbitrary function of class $C^\infty_0(\mathbb{R}^N\times]0, 2[)$. We set $\phi(x, t) = -\frac{|x|^2}{8t} - (a + 1)\ln h(t)$ and $v = e^\phi u$. Then

$$Lv \equiv e^\phi(\partial_t u + \Delta u) = \partial_t v - v\Delta\phi - 2\nabla v \cdot \nabla\phi + \Delta v + (|\nabla\phi|^2 - \partial_t\phi)v.$$

The main trick in the approach mentioned above is the decomposition of the operator tL into the symmetric and skew-symmetric parts. We put

$$tL = S + A, \tag{3.1}$$

where

$$Sv \equiv t(\Delta v + (|\nabla\phi|^2 - \partial_t\phi)v) - \frac{1}{2}v \tag{3.2}$$

and

$$Av \equiv \frac{1}{2}(\partial_t(tv) + t\partial_t v) - t(v\Delta\phi + 2\nabla v\cdot\nabla\phi). \tag{3.3}$$

Obviously,

$$\begin{aligned}\int t^2 e^{2\phi}|\partial_t u + \Delta u|^2\,dxdt &= \int t^2|Lv|^2\,dxdt \\ &= \int |Sv|^2\,dxdt + \int |Av|^2\,dxdt + \int [S,A]v\cdot v\,dxdt,\end{aligned} \tag{3.4}$$

where $[S,A] = SA - AS$ is the commutator of S and A. Simple calculations show that

$$\begin{aligned}I &\equiv \int [S,A]v\cdot v\,dxdt \\ &= 4\int t^2\big[\phi_{,ij}v_{,i}\cdot v_{,j} + \phi_{,ij}\phi_{,i}\phi_{,j}|v|^2\big]\,dxdt \\ &\quad + \int t^2|v|^2(\partial_t^2\phi - 2\partial_t|\nabla\phi|^2 - \Delta^2\phi)\,dxdt \\ &\quad + \int t|\nabla v|^2\,dxdt - \int t|v|^2(|\nabla\phi|^2 - \partial_t\phi)\,dxdt.\end{aligned} \tag{3.5}$$

Here and in what follows, we adopt the convention that summation is done over the repeated Latin indices running from 1 to n. The partial derivatives in spatial variables are denoted by the comma in lower indices, i.e., $v_{,i} = \frac{\partial v}{\partial x_i}$, $\nabla v = (v_{i,j})$, etc. With the function ϕ as above, we have

$$I = (a+1)\int t^2\Big[-\Big(\frac{h'(t)}{h(t)}\Big)' - \frac{h'(t))}{th(t)}\Big]|v|^2\,dxdt = \frac{a+1}{3}\int t|v|^2\,dxdt. \tag{3.6}$$

Using the simple identity

$$|\nabla v|^2 = \frac{1}{2}(\partial_t + \Delta)|v|^2 - v\cdot(\partial_t v + \Delta v), \tag{3.7}$$

we find

$$\int t^2|\nabla v|^2\,dxdt = -\int t|v|^2\,dxdt - \int t^2 v\cdot Lv\,dxdt + \int t^2|v|^2(|\nabla\phi|^2 - \partial_t\phi)\,dxdt. \tag{3.8}$$

In our case,

$$|\nabla\phi|^2 - \partial_t\phi = -|\nabla\phi|^2 + (a+1)\frac{h'(t)}{h(t)}.$$

The latter relation (together with (3.6)) implies the estimate

$$\int t^2(|\nabla v|^2+|v|^2|\nabla\phi|^2)\,dxdt\le 3I-\int t^2 v\cdot Lv\,dxdt\le b_1\int t^2|Lv|^2\,dxdt, \tag{3.9}$$

where b_1 is an positive universal constant. Since

$$e^{\phi}|\nabla u|\le|\nabla v|+|v||\nabla\phi|, \tag{3.10}$$

from (3.4)–(3.10) it follows that

$$\begin{aligned}\int h^{-2a}(t)(th^{-1}(t))^2\Big((a+1)\frac{|u|^2}{t}+|\nabla u|^2\Big)e^{-\frac{|x|^2}{4t}}\,dxdt&\\ \le b_2\int h^{-2a}(t)(th^{-1}(t))^2|\partial_t u+\Delta u|^2 e^{-\frac{|x|^2}{4t}}\,dxdt&,\end{aligned}$$

where b_2 is an positive universal constant. Inequality (1.5) is proved.

§4. Proof of the second Carleman inequality

Let $u\in C_0^\infty(Q^1_+)$, and let

$$\phi=\phi^{(1)}+\phi^{(2)},\quad \phi^{(1)}(x,t)=-\frac{|x'|^2}{8t},\quad \phi^{(2)}(x,t)=a(1-t)\frac{x_n^{2\alpha}}{t^\alpha}, \tag{4.1}$$

where $Q^1_+=(\mathbb{R}^n_+ + e_n)\times]0,1[$, $x=(x',x_n)$, $\alpha\in]1/2,1]$ is fixed, and a is a positive parameter.

We are going to use formulas (3.1)–(3.5) for the new functions u, v, and ϕ. Now, all integrals in those formulas are taken over Q^1_+.

First, we observe that

$$\begin{gathered}\nabla\phi=\nabla\phi^{(1)}+\nabla\phi^{(2)},\\ \nabla\phi^{(1)}(x,t)=-\frac{x'}{4t},\quad \nabla\phi^{(2)}(x,t)=2\alpha a\frac{1-t}{t^\alpha}x_n^{2\alpha-1}e_n.\end{gathered} \tag{4.2}$$

Therefore,

$$\nabla\phi^{(1)}\cdot\nabla\phi^{(2)}=0,\quad |\nabla\phi|^2=|\nabla\phi^{(1)}|^2+|\nabla\phi^{(2)}|^2. \tag{4.3}$$

Moreover,

$$\begin{aligned}&\nabla^2\phi=\nabla^2\phi^{(1)}+\nabla^2\phi^{(2)},\\ &\phi^{(1)}_{,ij}=\begin{cases}-\frac{\delta_{ij}}{4t}, & 1\le i,j\le n-1,\\ 0, & i=n \text{ or } j=n,\end{cases}\\ &\phi^{(2)}_{,ij}=\begin{cases}0, & i\ne n \text{ or } j\ne n,\\ 2\alpha(2\alpha-1)a\frac{1-t}{t^\alpha}x_n^{2\alpha-2}, & i=n \text{ and } j=n.\end{cases}\end{aligned} \tag{4.4}$$

In particular, (4.4) implies

$$\phi_{,ij}\phi_{,i}\phi_{,j}=-\frac{1}{4t}|\nabla\phi^{(1)}|^2+2\alpha(2\alpha-1)a\frac{1-t}{t^\alpha}x_n^{2\alpha-2}|\nabla\phi^{(2)}|^2\ge-\frac{1}{4t}\frac{|x'|^2}{16t^3}. \tag{4.5}$$

Using (4.3)–(4.5), we write the integral I in (3.5) as follows:

$$I = I_1 + I_2 + \int t|\nabla v|^2 \, dxdt, \tag{4.6}$$

where

$$\begin{aligned} I_s &= 4\int t^2\big[\phi^{(s)}_{,ij} v_{,i}\cdot v_{,j} + \phi^{(s)}_{,ij}\phi^{(s)}_{,i}\phi^{(s)}_{,j}|v|^2\big]\, dxdt \\ &\quad + \int t^2|v|^2 \\ &\qquad \times \Big(\partial_t^2\phi^{(s)} - 2\partial_t|\nabla\phi^{(s)}|^2 - \Delta^2\phi^{(s)} - \frac{1}{t}|\nabla\phi^{(s)}|^2 + \frac{1}{t}\partial_t\phi^{(s)}\Big)\, dxdt, \quad s = 1, 2. \end{aligned}$$

Direct calculations yield

$$I_1 = -\int t(|\nabla v|^2 - |v_{,n}|^2)\, dxdt,$$

whence

$$I = \int t|v_{,n}|^2 \, dxdt + I_2. \tag{4.7}$$

Now, our aim is to estimate I_2 from below. Since $\alpha \in]1/2, 1[$, we can suppress the fist integral in the expression for I_2. As a result, we have

$$I_2 \geq \int t^2|v|^2(A_1 + A_2 + A_3)\, dxdt, \tag{4.8}$$

where

$$\begin{aligned} A_1 &= -\partial_t|\nabla\phi^{(2)}|^2, \\ A_2 &= A_1 - \Delta^2\phi^{(2)} - \frac{1}{t}|\nabla\phi^{(2)}|^2, \\ A_3 &= \partial_t^2\phi^{(2)} + \frac{1}{t}\partial_t\phi^{(2)}. \end{aligned}$$

For A_2, we find

$$A_2 \geq \frac{1-t}{t^\alpha} x_n^{2\alpha-4} a(2\alpha - 1)\Big[\frac{4\alpha^2 a x_n^{2\alpha+2}}{t^{\alpha+1}} - 2\alpha(2\alpha-2)(2\alpha-3)\Big].$$

Since $x_n \geq 1$ and $0 < t < 1$, we see that $A_2 > 0$ for all $a \geq 2$. Hence, (4.7) and (4.8) imply the inequality

$$I \geq \int t^2|v|^2(A_1 + A_3)\, dxdt. \tag{4.9}$$

It is not difficult to check that

$$A_3 \geq a(2\alpha - 1)\frac{x_n^{2\alpha}}{t^{\alpha+2}}. \tag{4.10}$$

On the other hand,

$$-\partial_t|\nabla\phi^{(2)}|^2 - \frac{1}{t}|\nabla\phi^{(2)}|^2 \geq (2\alpha - 1)\frac{1-t}{t^{2\alpha+1}} 4\alpha^2 a^2 x_n^{2(2\alpha-1)} \geq 0$$

so that

$$A_1 \geq \frac{1}{t}|\nabla \phi^{(2)}|^2. \tag{4.11}$$

Combining (4.9)–(4.11), from (3.4) we deduce the estimate

$$\begin{aligned} \int t^2|Lv|^2\,dxdt &\geq I \\ &\geq a(2\alpha-1)\int \frac{x_n^{2\alpha}}{t^\alpha}|v|^2\,dxdt + \int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt \\ &\geq a(2\alpha-1)\int |v|^2\,dxdt + \int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt. \end{aligned} \tag{4.12}$$

Recalling (3.7), we can find the following analog of (3.8):

$$\int t|\nabla v|^2\,dxdt = -\frac{1}{2}\int |v|^2\,dxdt - \int tv\cdot Lv\,dxdt + \int t|v|^2(|\nabla\phi|^2 - \partial_t\phi)\,dxdt. \tag{4.13}$$

Due to the special structure of ϕ, we have

$$\begin{aligned} |\nabla\phi|^2 - \partial_t\phi &= |\nabla\phi^{(1)}|^2 - \partial_t\phi^{(1)} + |\nabla\phi^{(2)}|^2 - \partial_t\phi^{(2)} \\ &= -|\nabla\phi^{(1)}|^2 + |\nabla\phi^{(2)}|^2 - \partial_t\phi^{(2)}, \end{aligned}$$

which allows us to reduce (4.13) to the form

$$\begin{aligned} &\int \big(t|\nabla v|^2 + t|v|^2(|\nabla\phi^{(1)}|^2 + |\nabla\phi^{(2)}|^2)\big)\,dxdt = \int t\big(|\nabla v|^2 + |v|^2|\nabla\phi|^2\big)\,dxdt \\ &\quad = -\frac{1}{2}\int |v|^2\,dxdt - \int tv\cdot Lv\,dxdt + 2\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt - \int t|v|^2\partial_t\phi^{(2)}\,dxdt. \end{aligned} \tag{4.14}$$

But

$$-t\partial_t\phi^{(2)} \leq a\frac{x_n^{2\alpha}}{t^\alpha}$$

and, by (3.10) and (4.14),

$$\frac{1}{2}\int te^{2\phi}|\nabla u|^2 \leq -\int v\cdot(tLv)\,dxdt + 2\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt + a\int\frac{x_n^{2\alpha}}{t^\alpha}|v|^2\,dxdt. \tag{4.15}$$

The Cauchy–Schwarz inequality, (4.12), and (4.15) imply the required inequality (1.6).

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