On Smoothness of Suitable Weak Solutions to the Navier-Stokes Equations

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Dedicated to Vsevolod Alexeevich Solonnikov

Abstract *We prove two sufficient conditions for local regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. One of them implies smoothness of $L_{3,\infty}$-solutions as a particular case.*

1 Introduction

In this paper, we address to local properties of weak solutions to the Navier-Stokes equations. To be precise, let us consider the Navier-Stokes equations, describing the motion of a viscous incompressible fluid in Q, i.e.:

$$\left.\begin{array}{c} \partial_t v + \operatorname{div} v\otimes v - \Delta\, v = -\nabla\, p \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in} \quad Q.$$

Here, $Q = B\times]-1,0[$ is the unit space-time cylinder. The question is under what conditions the point $z = 0$ is a regular point of v. The latter means that there exists a nonempty neighborhood of the origin such that v is Hölder continuous in the intersection of this neighborhood and the closure of Q.

Our first assumption on the velocity v and the pressure p is that they form the so-called suitable weak solution to the Navier-Stokes equations in Q. The definition of it is as follows.

Definition 1.1 *Let ω be a open set in $\mathbb{R}^3$. We say that a pair u and q is a suitable weak solution to the Navier-Stokes equations on the set $\omega\times]-T_1,T[$*

if it satisfies the conditions:

$$u \in L_{2,\infty}(\omega\times]-T_1,T[) \cap L_2(-T_1,T;W^1_2(\omega)); \tag{1.1}$$

$$q \in L_{\frac{3}{2}}(\omega\times]-T_1,T[); \tag{1.2}$$

$$\begin{array}{c}\textit{u and q satisfy the Navier-Stokes equations}\\ \textit{in the sense of distributions;}\end{array} \tag{1.3}$$

$$\left.\begin{array}{c}\textit{u and q satisfy the local energy inequality}\\ \\ \int\limits_\omega \varphi(x,t)|u(x,t)|^2\,dx + 2\int\limits_{\omega\times]-T_1,t[}\varphi|\nabla u|^2\,dxdt' \\ \\ \leq \int\limits_{\omega\times]-T_1,t[}(|u|^2(\Delta\varphi+\partial_t\varphi)+u\cdot\nabla\varphi(|u|^2+2q))\,dxdt'\end{array}\right\} \tag{1.4}$$

for a.a. $t \in]-T_1,T[$ *and for all nonnegative functions* $\varphi \in C^\infty_0(\mathbb{R}^3\times\mathbb{R}^1)$, *vanishing in a neighborhood of the parabolic boundary* $\partial' Q \equiv \omega\times\{t=-T_1\}\cup \partial\omega\times[-T_1,T]$ *of* Q.

For discussions about the notion of suitable weak solutions, we refer the reader to papers [8], [1], [7], [6], [10], [11], and [5].

For the reader convenience, we first formulate known results from the the so-called ε-regularity theory of suitable weak solutions. Explanations and proofs of the them can be found in the above cited papers.

Lemma 1.2 *Consider two functions* v *and* p *defined in the space-time cylinder* $Q(z_0,R)=B(x_0,R)\times]t_0-R^2,t_0[$, *where* $B(x_0,R)\subset\mathbb{R}^3$ *stands for the ball of radius* R *with the center at the point* x_0. *Assume that* v *and* p *form a suitable weak solution to the Navier-Stokes equations in* $Q(z_0,R)$. *There exists a universal positive constant* ε *such that if*

$$\frac{1}{R^2}\int\limits_{Q(z_0,R)}\left(|v|^3+|p|^{\frac{3}{2}}\right)dz<\varepsilon, \tag{1.5}$$

then, for any $k=0,1,...$, *the function* $z\mapsto\nabla^k v(z)$ *is Hölder continuous in the closure of the set* $Q(z_0,R/2)$ *and, moreover,*

$$\sup_{z\in Q(z_0,R/2)}|\nabla^k v(z)|<C_k R^{-k-1} \tag{1.6}$$

for some universal positive constant C_k.

Lemma 1.3 *Assume that v and p form a suitable weak solution to the Navier-Stokes equations in $Q(z_0,R)$. There exists an absolute positive constant ε_0 such that if*

$$\sup_{0<r\leq R}\frac{1}{r^2}\int\limits_{Q(z_0,r)}|v|^3\,dz<\varepsilon_0, \tag{1.7}$$

then there exists a nonempty neighborhood $\mathcal{O}_{z_0}$ of the point z_0 such that the function $z\mapsto v(z)$ is Hölder continuous in $\mathcal{O}_{z_0}\cap\overline{Q}(z_0,R)$.

One of the goals of our paper is somehow to weaken condition (1.7). It might seem that it would be enough to replace this condition with just boundedness of the left hand in (1.7). But we cannot prove that. Our result in this direction is as follows.

Theorem 1.4 *Let v and p be a suitable weak solution to the Navier-Stokes equations in $Q\equiv Q(0,1)$. Suppose that v satisfies the condition:*

$$M_0\equiv\sup_{0<R<1}\frac{1}{R^2}\int\limits_{Q(R)}|v|^3\,dz<+\infty. \tag{1.8}$$

Here, $Q(R)\equiv B(R)\times]-R^2,0[$ so that $Q(1)=Q$.

Assume that there exist numbers $\beta_0\geq\alpha_0\geq 1$, $T_0>1$, and $\rho_0>0$ such that the following two conditions hold:

$$\limsup_{R\to 0}\frac{1}{\rho_0^2R^2}\int\limits_{Q(Rz_\star,\rho_0R)}\Big(|v|^3+|p|^{\frac{3}{2}}\Big)\,dz<\varepsilon,\qquad\forall z_\star\in A_0, \tag{1.9}$$

where

$$A_0\equiv\{z=(x,t)\,\|\,|x|\geq\alpha_0,\quad -T_0\leq t\leq 0\},$$

$$\textit{for each}\quad x_\star\in B_0,\quad\textit{there is}\quad \rho_1=\rho_1(x_\star)>0\quad\textit{such that}$$

$$\limsup_{R\to 0}\frac{1}{\rho_1R}\int\limits_{B(Rx_\star,\rho_1R)}|v(x,0)|^2\,dx=0, \tag{1.10}$$

where

$$B_0\equiv\{x\in\mathbb{R}^3\,\|\,x_n\geq\beta_0\}.$$

Then the function v is Hölder continuous in the closure of the set $Q(r)$ for some $r>0$.

However, because of condition (1.9), Theorem 1.4 still looks like another statement of the ε-regularity theory.

Our next result includes smoothness of local $L_{3,\infty}$-solutions proved in [12], [2], [3], and [5] as a particular case.

Theorem 1.5 *Let v and p be a suitable weak solution to the Navier-Stokes equations in Q. Suppose that v satisfies the condition*

$$M_0' \equiv \sup_{z_0 \in \overline{Q}(1/2)} \sup_{0<R\leq 1/4} \frac{1}{R^2} \int\limits_{Q(z_0,R)} |v|^3 \, dz < +\infty. \tag{1.11}$$

Assume also that there are numbers $\beta_0 \geq 1$ and $T_1 > 3$ such that condition (1.10) holds and

$$M_2 \equiv \limsup_{a\to+\infty} \limsup_{R\to 0} \frac{1}{R^2} \int\limits_{-T_1 R^2}^{0} \int\limits_{B(Ra)} |v|^3 \, dz < +\infty. \tag{1.12}$$

Then the function v is Hölder continuous in the closure of the set $Q(r)$ for some $r > 0$.

2 Auxiliary Results

We start with a lemma on a kind of decay estimate for the pressure p.

Lemma 2.1 *We let*

$$C(r) \equiv \frac{1}{r^2} \int\limits_{Q(r)} |v|^3 \, dz, \qquad D(r) \equiv \frac{1}{r^2} \int\limits_{Q(r)} |p|^{\frac{3}{2}} \, dz.$$

Assume that v and p form a suitable weak solutions to the Navier-Stokes equations and satisfy condition (1.8). Then, for each $\alpha \in]0,1[$, a constant $c_1 = c_1(\alpha)$ exists such that

$$D(r) \leq c_1 \Big\{ r^\alpha D(1) + M_0 \Big\} \tag{2.1}$$

for any $r \in]0,1]$. Here, M_0 is the constant in (1.8).

PROOF. In [9], the following estimate was established

$$D(r) \leq c_2\Big\{\Big(\frac{r}{\rho}\Big)D(\rho)+\Big(\frac{\rho}{r}\Big)^2 C(\rho)\Big\}, \tag{2.2}$$

which is valid for all $0<r<\rho\leq 1$ and for some universal constant c_2.

Fix $\alpha\in]0,1[$ and choose $\tau=\tau(\alpha)\in]0,1[$ so that

$$c_2\tau^{1-\alpha}\leq 1. \tag{2.3}$$

By (1.8), (2.2), and (2.3), we have

$$D(\tau\rho)\leq \tau^\alpha D(\rho)+\frac{c_2}{\tau^2}C(\rho)\leq \tau^\alpha D(\rho)+\frac{c_2}{\tau^2}M_0$$

for any $\rho\in]0,1]$. We may iterate the latter estimate. As a result, we find

$$D(\tau^k)\leq \tau^{\alpha k}D(1)+\frac{c_2}{\tau^2}M_0\frac{1}{1-\tau^\alpha} \tag{2.4}$$

for any $k=1,2,....$ Required estimate (2.1) can be easily deduced from (2.4). Lemma 2.1 is proved.

Now, we are in a position to prove the main blow-up lemma.

Lemma 2.2 *Assume that v and p form a suitable weak solutions to the Navier-Stokes equations and satisfy condition (1.8). Assume that $z=0$ is a singular point of v. There exists a pair of functions u and q defined on $Q_-=\mathbb{R}^3\times]-\infty,0[$ with the following properties. For any $a>0$, they form a suitable weak solution to the Navier-Stokes equations in $Q(a)$. In addition,*

$$u\in C([-a^2,0];L_\beta(B(a))) \tag{2.5}$$

for any $a>0$ and for any $\beta\in[1,2[$,

$$\int\limits_{Q(1)}|u|^3\,dz\geq\varepsilon_0>0, \tag{2.6}$$

where ε_0 is the constant of Lemma 1.3,

$$\frac{1}{\rho^2}\int\limits_{Q(z_\star,\rho)}(|u|^3+|q|^{\frac{3}{2}})\,dz\leq\limsup_{R\to 0}\frac{1}{\rho^2R^2}\int\limits_{Q(Rz_\star,\rho R)}(|v|^3+|p|^{\frac{3}{2}})\,dz \tag{2.7}$$

for any $z_\star \in Q_-$ and for any $\rho > 0$, and

$$\Big(\frac{1}{\rho^{7/4}}\int\limits_{B(x_\star,\rho)}|u(x,0)|^{\frac54}\,dx\Big)^{\frac45}\leq c_3\limsup_{R\to 0}\Big(\frac{1}{\rho R}\int\limits_{B(Rx_\star,\rho R)}|v(x,0)|^2\,dz\Big)^{\frac12}\qquad(2.8)$$

for any $x_\star \in \mathbb{R}^3$, for any $\rho > 0$, and for some universal positive constant c_3.

PROOF By Lemma 2.1 and condition (1.8), we show that

$$\frac{1}{R^2}\int\limits_{Q(R)}\Big(|v|^3+|p|^{\frac32}\Big)\,dz\leq M_1<+\infty\qquad(2.9)$$

for any $R \in]0,1]$.

Since the origin is a singular point, there exists a sequence $\{R_k>0\}_{k=1}^{\infty}$, tending to zero as $k\to+\infty$, such that (see Lemma 1.3)

$$\frac{1}{R_k^2}\int\limits_{Q(R_k)}|v|^3\,dz\geq\varepsilon_0>0\qquad(2.10)$$

for any $k = 1, 2, ...$

We extend v and p to zero outside Q and denote these extensions by $\overline{v}$ and $\overline{p}$. For any $e = (y, s) \in \mathbb{R}^3 \times \mathbb{R}^1$, we let

$$u^k(y,s)=R_k\overline{v}(R_ky,R_k^2s),\qquad q^k(y,s)=R_k^2\overline{p}(R_ky,R_k^2s).$$

We can state that the pair u^k and q^k is a suitable weak solution to the Navier-Stokes equations in the cylinder $Q(1/R_k)$. In particular, it satisfies the local energy inequality:

$$\int\limits_{B(1/R_k)}\phi(y,s)|u^k(y,s)|^2\,dy+2\int\limits_{B(1/R_k)\times]-1/R_k^2,s[}\phi|\nabla u^k|^2\,dyds'\leq$$

$$\leq\int\limits_{B(1/R_k)\times]-1/R_k^2,s[}\Big[|u^k|^2(\partial_t\phi+\Delta\phi)+u^k\cdot\nabla\phi(|u^k|^2+2q^k)\Big]\,dyds'\qquad(2.11)$$

for a.a. $s \in]-1/R_k^2, 0[$ and for all non-negative functions $\phi \in C_0^\infty(\mathbb{R}^{3+1})$ vanishing in a neighborhood of the parabolic boundary of the cylinder $Q(1/R_k)$. Moreover, by our scaling, we have

$$\frac{1}{a^2}\int\limits_{Q(a)}\Big(|u^k|^3+|q^k|^{\frac32}\Big)\,de=\frac{1}{a^2R_k^2}\int\limits_{Q(aR_k)}\Big(|v|^3+|p|^{\frac32}\Big)\,dz\leq M_1\qquad(2.12)$$

if $aR_k < 1$. Thanks to (2.11), (2.12) and the known multiplicative inequality, we may find subsequences of u^k and q^k still denoted in the same way such that, for any $a > 0$,

$$\begin{aligned} u^k &\rightharpoonup u \qquad \text{in} \quad L_3(Q(a)), \\ q^k &\rightharpoonup q \qquad \text{in} \quad L_{\frac{3}{2}}(Q(a)), \\ \nabla u^k &\rightharpoonup \nabla u \qquad \text{in} \quad L_2(Q(a)), \\ u^k &\overset{\star}{\rightharpoonup} u \qquad \text{in} \quad L_{2,\infty}(Q(a)), \\ u^k &\rightharpoonup u \qquad \text{in} \quad L_{\frac{10}{3}}(Q(a)). \end{aligned}$$

To evaluate the derivative of v in t, we use the Navier-Stokes equations written in the following form:

$$-\int\limits_{Q(a)} u^k \cdot \partial_t w \, dyds = \int\limits_{Q(a)} \Big[(u^k \otimes u^k - \nabla u^k) : \nabla w + q^k \operatorname{div} w\Big] \, dyds$$

for any $w \in L_3(-a^2, 0; \overset{\circ}{W}{}^1_3(B(a)))$. This gives us

$$\partial_t u^k \rightharpoonup \partial_t u \qquad \text{in} \quad L_{\frac{3}{2}}(-a^2, 0; (\overset{\circ}{W}{}^1_3(B(a)))')$$

and thus, by compactness arguments,

$$u^k \to u \qquad \text{in} \quad L_3(Q(a)). \tag{2.13}$$

Letting $k \to \infty$, we pass to the limit in the Navier-Stokes equations and the local energy inequality and show that the pair of functions u and q, defined on $Q_- = \mathbb{R}^3 \times]-\infty, 0[$, is a suitable weak solution to the Navier-Stokes equations in $Q(a)$ for any $a > 0$. This is a nontrivial solution since the limit function satisfies (2.6).

We know that the convective term $\operatorname{div} u^k \otimes u^k$ is bounded, for example, in $L_{\frac{5}{4}}(Q(a))$ for each $R_k a < 1$. Using an appropriated choice of a cut-off function, the coercive estimates for Stokes system, multiplicative inequalities and duality arguments, we can show that the following norms are bounded:

$$\|\partial_t u^k\|_{\frac{5}{4},Q(a)}, \quad \|\nabla^2 u^k\|_{\frac{5}{4},Q(a)}, \quad \|\nabla q^k\|_{\frac{5}{4},Q(a)}$$

for $R_k a < 1$. This implies that

$$u^k \to u \qquad \text{in} \quad C([-a^2, 0]; L_{\frac{5}{4}}(B(a)))$$

for any $a>0$. Relation (2.5) can be obtained now by the interpolation.

Now, our goal is to prove (2.7) and (2.8). This is an easy task. Indeed, by the scaling, we have the following two relations:

$$\frac{1}{\rho^2}\int\limits_{Q(z_\star,\rho)}(|u^k|^3+|q|^{\frac32})\,dz=\frac{1}{\rho^2R_k^2}\int\limits_{Q(R_kz_\star,\rho R_k)}(|v|^3+|p|^{\frac32})\,dz$$

for any $z_\star\in Q_-$, and

$$\Big(\frac{1}{\rho_0^{7/4}}\int\limits_{B(x_\star,\rho_0)}|u^k(x,0)|^{\frac54}\,dx\Big)^{\frac45}=\Big(\frac{1}{(\rho_0R_k)^{7/4}}\int\limits_{B(R_kx_\star,\rho_0R_k)}|v(x,0)|^{\frac54}\,dx\Big)^{\frac45}$$

$$\leq c_3\Big(\frac{1}{\rho_0R_k}\int\limits_{B(R_kx_\star,\rho_0R_k)}|v(x,0)|^2\,dz\Big)^{\frac12}$$

for any $x_\star\in\mathbb{R}^3$ and for some universal positive constant c_3. Taking the limit as $k\to+\infty$, we find (2.7) and (2.8). Lemma 2.2 is proved.

3 Proof of Theorem 1.4

We see that, by (1.9) and (2.7),

$$\frac{1}{\rho_0^2}\int\limits_{Q(z_\star,\rho_0)}(|u|^3+|q|^{\frac32})\,dz<\varepsilon$$

for any $z_\star\in A_0$. According to Lemma 1.2, we can state that, for any $k=0,1,2,...$, the function ∇^ku is Hölder continuous in the set

$$\widetilde{Q}=(\mathbb{R}^3\setminus B(\alpha_0))\times[-T_0,0]$$

and, moreover,

$$\sup_{z\in\widetilde{Q}}(|u(z)|+|\nabla u(z)|<M \tag{3.1}$$

for some positive constant M. From (3.1) it follows that the vorticity $\omega=\nabla\wedge u$ satisfies the inequality

$$|\partial_t\omega-\Delta\omega|\leq M(|\omega|+|\nabla\omega|) \tag{3.2}$$

at least in $\widetilde{Q}_1 = (\mathbb{R}^3 \setminus \overline{B}(\frac{3}{2}\alpha_0))\times]-\frac{1}{2}(T_0+1), 0[$.

On the hand, condition (1.10) and inequality (2.8) say that $u(\cdot,0)=0$ in B_0 and, therefore, $\omega(\cdot,0)=0$ in B_0. By the backward uniqueness result (see [2] and [3]), we state that

$$\omega(x,t)=0, \qquad x_n>\beta_0, \quad -1<t<0.$$

The unique continuation through spatial boundaries implies (see, for example, [3] and references there)

$$\omega\equiv 0 \qquad \text{in} \quad (\mathbb{R}^3\setminus\overline{B}(\beta_0))\times]-1,0[.$$

The same arguments as in [12] and in [5] allows us to state that in fact

$$\omega\equiv 0 \qquad \text{in} \quad \mathbb{R}^3\times]-1,0[.$$

In turn, this means that, for a.a $t\in]-1,0[$, $u(\cdot,t)$ is bounded harmonic function defined in the whole $\mathbb{R}^3$. So, as a result, we have

$$u(x,t)=a(t), \qquad t\in]-1,0[.$$

Here, the function a can be found by solving the Navier-Stokes equations

$$a'(t)=-\nabla\, q(x,t)$$

and, therefore,

$$q(x,t)=-a'(t)x+b(t).$$

But we know that

$$\frac{1}{\rho_0^2}\int\limits_{Q(z_\star,\rho_0)}|q|^{\frac{3}{2}}\,dz<\varepsilon_1$$

for all $z_\star\in A_0$. Obviously, the last two relations are in a contradiction if $a'\neq 0$. So, we have proved that u is a constant in $R^3\times]-1,0[$. But u is a Hölder continuous function far away from the origin and equal to zero for $x_n>\beta_0$ and $t=0$. So, u is identically equal to zero in $R^3\times]-1,0[$ which contradicts with (2.6). Theorem 1.4 is proved.

It is curious that conditions (1.8) and (1.9) are fulfilled if

$$|v(x,t)|^2+|p(x,t)|^{\frac{3}{2}}\leq\frac{c}{|x|+\sqrt{-t}}$$

for $(x,t)\in Q$ and for some constant c.

However, it is not clear whether $L_{3,\infty}$-solutions satisfy condition (1.9).

4 Proof of Theorem 1.5

Our first observation is that Lemma 2.2 can be strengthened in the following way.

Lemma 4.1 *We let*

$$C(z_0,r;v)\equiv\frac{1}{r^2}\int\limits_{Q(z_0,r)}|v|^3\,dz,\qquad D(z_0,r;p)\equiv\frac{1}{r^2}\int\limits_{Q(z_0,r)}|p|^{\frac{3}{2}}\,dz.$$

Assume that v and p form a suitable weak solutions to the Navier-Stokes equations and satisfy condition (1.11). Then, for each $\alpha\in]0,1[$, a constant $c_1'=c_1'(\alpha)$ exists such that

$$D(z_0,r;p)\leq c_1'\Big\{r^\alpha D(1)+M_0'\Big\}\tag{4.1}$$

for any $r\in]0,1/4]$ and for any $z_0\in\overline{Q}(1/2)$. Here, M_0' is the constant in (1.11).

The proof is the same as the proof of Lemma 2.1.

Now, assume that $z=0$ is a singular point. Since (1.8) holds, we can use arguments of the proof of Lemma 2.2. First of all, we note that all statements of Lemma 2.2 are valid for blow-up functions u and q as well. Moreover, for scaled functions u^k, we have the identity

$$\int\limits_{-T_1}^{0}\int\limits_{B(a)}|u^k|^3\,dz=\frac{1}{R_k^2}\int\limits_{-T_1R_k^2}^{0}\int\limits_{B(R_ka)}|v|^3\,dz$$

for any $a>0$. Thanks to (2.13), passage to the limit as $k\to+\infty$ gives us:

$$\int\limits_{-T_1}^{0}\int\limits_{B(a)}|u|^3\,de\leq\limsup_{k\to+\infty}\frac{1}{R_k^2}\int\limits_{-T_1R_k^2}^{0}\int\limits_{B(R_ka)}|v|^3\,dz$$

$$\leq\limsup_{R\to0}\frac{1}{R^2}\int\limits_{-T_1R^2}^{0}\int\limits_{B(Ra)}|v|^3\,dz.$$

It remains to pass to the limit as $a \to +\infty$ and take into account condition (1.12). As a result, we find

$$\int\limits_{-T_1}^{0} \int\limits_{\mathbb{R}^3} |u|^3 \, de \leq M_2. \tag{4.2}$$

Now, we introduce the notation:

$$E(e_0, r) \equiv \frac{1}{r^2} \int\limits_{Q(e_0,r)} (|u|^3 + |q|^{\frac{3}{2}}) \, de.$$

For $D(e_0, r; q)$, we have the decay estimate similar to (2.2)

$$D(e_0, r; q) \leq c_2 \Big\{ r D(e_0, 1; q) + \Big(\frac{1}{r}\Big)^2 C(e_0, 1; u) \Big\} \tag{4.3}$$

for all $0 < r \leq 1$ and for some absolute constant c_2. From (4.3), it follows that

$$E(e_0, r) \leq (1 + c_2) \Big\{ r D(e_0, 1; q) + \Big(\frac{1}{r}\Big)^2 C(e_0, 1; u) \Big\} \tag{4.4}$$

for the same r as in (4.3).

Now, our goal is to show that

$$D(e_0, 1; q) \leq L < +\infty, \qquad \forall e_0 \in \mathbb{R}^3 \times]-\infty, 0], \tag{4.5}$$

where a positive constant L depends on M_0' and $D(1)$ only. To this end, we first observe that

$$D(e_0, 1; q) \leq \liminf_{k \to +\infty} D(e_0, 1; q^k). \tag{4.6}$$

On the other hand,

$$D(e_0, 1; q^k) = D(z^k, R_k; p), \qquad z^k = (R_k y_0, R_k^2 s_0). \tag{4.7}$$

For the definition of the scaled pressure q^k and its properties, we refer the reader to Lemma 2.2. For sufficiently large k, we have

$$R_k e_0 \in \overline{Q}(1/2), \qquad R_k \leq 1/4.$$

Then, by Lemma 4.1,

$$D(z^k, R_k; p) \leq c_1' \Big\{ R_k^\alpha D(1) + M_0' \Big\}. \tag{4.8}$$

Clearly, that (4.5) follows from (4.6)–(4.8).

Combining (4.4) and (4.5), we find

$$E(e_0,r)\leq(1+c_2)\Big\{rL+\Big(\frac{1}{r}\Big)^2C(e_0,1;u)\Big\} \tag{4.9}$$

for any $0<r\leq 1$ and for any $e_0\in\mathbb{R}^3\times]-\infty,0]$. Now, let ε be the number of Lemma 1.2. We may fix a positive number $r\in]0,1]$ so that

$$(1+c_2)rL<\varepsilon/3. \tag{4.10}$$

For this fixed r, according (4.2), there exists a number $R>10$ such that

$$\int\limits_{-T_1}^{0}\int\limits_{\mathbb{R}^3\setminus B(R/2)}|u|^3\,de\leq\frac{\varepsilon r^2}{3(1+c_2)}. \tag{4.11}$$

Obviously, (4.11) implies the estimate

$$(1+c_2)\Big(\frac{1}{r}\Big)^2C(e_0,1;u)<\varepsilon/3$$

whenever $e_0\in(\mathbb{R}^3\setminus B(R))\times]-T_1/2,0[$ is. Combining the latter inequality with (4.9) and (4.10), we show

$$E(e_0,r)<\varepsilon \tag{4.12}$$

for any $e_0\in(\mathbb{R}^3\setminus B(R))\times]-T_1/2,0[$. It remains to repeat arguments of Theorem 1.4 and complete the proof by getting the same contradiction. Theorem 1.5 is proved.

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