Remarks on regularity of weak solutions to the Navier-Stokes equations near the boundary

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Abstract We give a simple proof of the so-called ε-regularity of suitable weak solutions to the Navier-Stokes equations near the boundary.

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1 Introduction

The partial regularity of suitable weak solutions to the Navier-Stokes near the boundary was studied in the paper [6], where it was proved an analogue of the Caffarelli-Kohn-Nirenberg theorem, see [1], for points belonging to the lateral surface of the space-time cylinder. In this note, we give a new simple proof of the so-called ε-regularity which is one of the main steps to achieve the final statement of the theory of partial regularity for the Navier-Stokes system.

So, let us consider the three-dimensional Navier-Stokes equations

$$\partial_t v + \operatorname{div}(v \otimes v) - \Delta v = f - \nabla p, \quad \operatorname{div} v = 0$$

in a domain $Q_T \equiv \Omega \times]0,T[$ of the space $\mathbb{R}^4$. We assume, in addition, that

$$v = 0 \qquad \text{on} \qquad \Gamma \times [0,T], \qquad \Gamma \subset \partial\Omega.$$

Our aim is to investigate differentiability properties of v in a neighborhood of a point $z_0 = (x_0, t_0) \in \Gamma \times]0,T]$ under minimal assumptions on starting properties of v and p. Now, it is well understood that these minimal assumptions can be expressed in the form that the pair v and p must be a suitable weak solution to the Navier-Stokes equations. For details, we refer the reader to the paper [6].

Definition 1.1 *Let Ω be a domain in $\mathbb{R}^3$, Γ be an open subset of the set $\partial\Omega$, and T be a positive parameter. Suppose that a function f satisfies the condition*

$$f \in L_2(Q_T; \mathbb{R}^3). \tag{1.1}$$

We say that a pair of functions v and p is a suitable weak solution to the Navier-Stokes equations in Q_T near the boundary $\Gamma \times]0,T]$ if the following four conditions hold. The functions v and p have the properties

$$\left.\begin{array}{c} v \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T), \quad p \in L_{\frac32}(Q_T), \\ \\ \nabla^2 v \in L_{\frac98,\frac32}(Q_T), \quad \nabla p \in L_{\frac98,\frac32}(Q_T). \end{array}\right\} \tag{1.2}$$

They meet the Navier-Stokes equations

$$\left.\begin{array}{c} \partial_t v + \operatorname{div}(v \otimes v) - \Delta v = f - \nabla p, \\ \\ \operatorname{div} v = 0 \end{array}\right\} \tag{1.3}$$

a.e. in Q_T. The function v satisfies the boundary condition

$$v = 0 \qquad \text{on} \qquad \Gamma \times]0,T]. \tag{1.4}$$

The functions v and p satisfy the local energy inequality

$$\left.\begin{array}{l} \int\limits_\Omega |v(x,t)|^2 \phi(x,t)\,dx + 2 \int\limits_{Q_t \equiv \Omega \times]0,t[} |\nabla v|^2 \phi \,dx\,dt' \le \\ \\ \le \int\limits_{Q_t} \{ |v|^2 (\partial_t \phi + \Delta \phi) + (|v|^2 + 2p)\, v \cdot \nabla \phi + 2 f \cdot v \phi \}\,dx\,dt' \end{array}\right\} \tag{1.5}$$

for a. a. $t \in]0,T]$ *and for all non-negative functions* $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$, *vanishing in a neighborhood of the set* $\Omega \times \{t = 0\} \cup (\partial\Omega \setminus \Gamma) \times [0,T]$.

To formulate our results, we introduce the following notion:

$$B(x_0,R) \equiv \{x \in \mathbb{R}^3 \parallel |x - x_0| < R\}, \quad x' = (x_1,x_2) \in \mathbb{R}^2,$$
$$B^+(x_0,R) \equiv \{x \in B(x_0,R) \parallel x = (x',x_3), \quad x_3 > x_{03}\},$$
$$B(\theta) \equiv B(0,\theta), \quad B \equiv B(1), \quad B^+(\theta) \equiv B^+(0,\theta), \quad B^+ \equiv B^+(1);$$
$$\Gamma^+(x_0,R) \equiv \{x \in B(x_0,R) \parallel x = (x',x_{03})\},$$
$$\Gamma^+(\theta) \equiv \Gamma^+(0,\theta), \quad \Gamma^+ \equiv \Gamma^+(1);$$
$$Q(z_0,R) \equiv B(x_0,R) \times]t_0 - R^2, t_0[, \quad z_0 = (x_0,t_0),$$
$$Q^+(z_0,R) \equiv B^+(x_0,R) \times]t_0 - R^2, t_0[,$$
$$Q(\theta) \equiv Q(0,\theta), \quad Q \equiv Q(1), \quad Q^+(\theta) \equiv Q^+(0,\theta), \quad Q^+ \equiv Q^+(1);$$
$$(v)_\omega \equiv \frac{1}{|\omega|} \int\limits_\omega v\, dz, \qquad [p]_\Omega(t) \equiv \frac{1}{|\Omega|} \int\limits_\Omega p(x,t)\, dx.$$

Just for simplicity we assume that

$$f = 0.$$

Theorem 1.2 *Given* $\tau \in]0,1[$, *there are positive numbers* $\varepsilon_0 = \varepsilon_0(\tau)$ *and* $c_0 = c_0(\tau)$ *with the following property. Let a pair* v *and* p *be an arbitrary weak solution to the Navier-Stokes equations in* $Q^+(z_0,R)$ *near* $\Gamma^+(z_0,R) \times]t_0 - R^2, t_0]$. *Suppose that* v *and* p *satisfy the additional condition*

$$U(z_0,R;v,p) \equiv \frac{1}{R^2} \int\limits_{Q^+(z_0,R)} \left(|v|^3 + |p|^{\frac{3}{2}}\right) dz < \varepsilon_0. \tag{1.6}$$

Then, the function v *is Hölder continuous in the closure of the set* $Q^+(z_0,\tau R)$ *and the estimate*

$$\sup_{z \in Q^+(z_0,\tau R)} |v(z)| \le \frac{c_0}{R} \tag{1.7}$$

is valid.

2 Main Lemma

In this section, we are going to prove the following key statement.

Lemma 2.1 *Given* $\theta\in]0,1/2[$*, there exists a positive constant* ε*, depending on* θ *only, with the following property. For any suitable weak solution* v *and* p *to the Navier-Stokes equations in* Q^+ *near* $\Gamma^+\times]-1,0]$*, satisfying the additional condition*

$$Y_1^+(v,p)<\varepsilon, \tag{2.1}$$

the following estimate is valid:

$$Y_\theta^+(v,p)\leq c\theta^{\frac{1}{2}}Y_1^+(v,p). \tag{2.2}$$

Here, c *is an absolute positive constant and*

$$Y_\theta^+(v,p)=Y_\theta^{+1}(v)+Y_\theta^{+2}(p),\quad Y_\theta^{+1}(v)=\Big(\frac{1}{|Q^+(\theta)|}\int\limits_{Q^+(\theta)}|v|^3dz\Big)^{\frac{1}{3}},$$

$$Y_\theta^{+2}(p)=\theta\Big(\frac{1}{|Q^+(\theta)|}\int\limits_{Q^+(\theta)}|p-[p]_{B^+(\theta)}|^{\frac{3}{2}}dz\Big)^{\frac{2}{3}}$$

Proof Assume that the statement of the lemma is false. This means that a number $\theta\in]0,1/2[$ and a sequence v^k and p^k of a suitable weak solutions in Q^+ near $\Gamma^+\times]-1,0]$ exist such that

$$Y_1^+(v^k,p^k)=\varepsilon_k\to 0 \tag{2.3}$$

as $k\to+\infty$, and

$$Y_\theta^+(v^k,p^k)>c\theta^{\frac{1}{2}}\varepsilon_k. \tag{2.4}$$

The constant c will be chosen later in order to get a contradiction. We introduce new functions

$$u^k=v^k/\varepsilon_k,\qquad q^k=(p^k-[p^k]_{B^+})/\varepsilon_k.$$

By (2.3) and (2.4), they satisfy the relations

$$Y_1^+(u^k,q^k)=1, \tag{2.5}$$

$$Y_\theta^+(u^k,q^k)>c\theta^{\frac{1}{2}}, \tag{2.6}$$

the system, see (1.3),

$$\left.\begin{aligned} \partial_t u^k + \varepsilon_k \mathrm{div}\,(u^k \otimes u^k) - \Delta\, u^k = -\nabla\, q^k, \\ \mathrm{div}\, u^k = 0 \end{aligned}\right\} \qquad \text{in } Q^+, \tag{2.7}$$

and the boundary condition, see (1.4),

$$u^k = 0 \qquad \text{on} \qquad \Gamma^+ \times]-1,0]. \tag{2.8}$$

In addition, functions u^k and q^k have the following differentiability properties, see (1.2),

$$\left.\begin{aligned} u^k \in L_{2,\infty}(Q^+) \cap W^{1,0}_2(Q^+), \quad q^k \in L_{\frac{3}{2}}(Q^+), \\ \nabla^2 u^k \in L_{\frac{9}{8},\frac{3}{2}}(Q^+), \quad \nabla q^k \in L_{\frac{9}{8},\frac{3}{2}}(Q^+), \end{aligned}\right\} \tag{2.9}$$

and, by (1.5), the local energy inequality

$$\left.\begin{aligned} \int\limits_{B^+} |u^k(x,t)|^2 \phi(x,t)\, dx + 2 \int\limits_{B^+ \times]-1,t[} |\nabla\, u^k|^2 \phi\, dx\, dt' \leq \\ \leq \int\limits_{B^+ \times]-1,t[} \{ |u^k|^2 (\partial_t \phi + \Delta\, \phi) + (|u^k|^2 \varepsilon_k + 2q^k)\, u^k \cdot \nabla\, \phi \}\, dx\, dt' \end{aligned}\right\} \tag{2.10}$$

holds for a. a. $t \in]-1,0]$ and for all non-negative functions $\phi \in C^\infty_0(\mathbb{R}^3 \times \mathbb{R})$, vanishing in a neighborhood of the set $B^+ \times \{t = -1\} \cup (\partial B^+ \setminus \Gamma^+) \times [-1,0]$.

Choosing a cut-off function ϕ in an appropriate way, we easily deduce from (2.10) the estimate

$$\|u^k\|_{2,\infty,Q^+(\tau)} + \|\nabla\, u^k\|_{2,Q^+(\tau)} \leq c_1(\tau), \tag{2.11}$$

which is valid for any $\tau \in]0,1[$. But then Hölder inequality and the known multiplicative inequality together with (2.11) imply two bounds

$$\|u^k\|_{\frac{10}{3},Q^+(\tau)} \leq c_2(\tau) \tag{2.12}$$

and

$$\|\mathrm{div}\, u^k \otimes u^k\|_{\frac{9}{8},\frac{3}{2},Q^+(\tau)} \leq c_2(\tau). \tag{2.13}$$

Both holds for any $\tau \in]0,1[$.

Now, we are in a position to apply local estimates of solutions to the Stokes equations near the boundary, see [5], Proposition 1. As a result, we have

$$\|\partial_t u^k\|_{\frac98,\frac32,Q^+(\tau_1)}+\|\nabla^2 u^k\|_{\frac98,\frac32,Q^+(\tau_1)}+\|\nabla\, q^k\|_{\frac98,\frac32,Q^+(\tau_1)}$$
$$\leq c_3(\tau_1,\tau)\Big(\|u^k\|_{\frac98,\frac32,Q^+(\tau)}+\|\nabla\, u^k\|_{\frac98,\frac32,Q^+(\tau)} \tag{2.14}$$
$$+\|q^k\|_{\frac98,\frac32,Q^+(\tau)}+\|\mathrm{div}\, u^k\otimes u^k\|_{\frac98,\frac32,Q^+(\tau)}\Big)\leq c_3'(\tau_1,\tau).$$

The latter estimate is valid for any $0<\tau_1<\tau<1$.

Without loss of generality, we claim that, by standard compactness arguments,

$$\left\{\begin{array}{ccc} u^k\rightharpoonup u & \text{in} & L_3(Q^+)\\ q^k\rightharpoonup q & \text{in} & L_{\frac32}(Q^+)\\ u^k\to u & \text{in} & \mathbb{L}_3(Q^+(5/6)).\end{array}\right. \tag{2.15}$$

Moreover, for these limit functions, the following relations are valid:

$$Y_1^+(u,q)\leq 1,\qquad [q]_{B^+}(t)=0\qquad \text{for } t\in]-1,0[, \tag{2.16}$$

$$\left.\begin{array}{c}\|u\|_{2,\infty,Q^+(\tau)}+\|\nabla\, u\|_{2,Q^+(\tau)}\leq c_1(\tau)\\[2ex] \|u\|_{\frac{10}{3},Q^+(\tau)}\leq c_2(\tau)\\[2ex] \|\mathrm{div}\, u\otimes u\|_{\frac98,\frac32,Q^+(\tau)}\leq c_2(\tau)\\[2ex] \|\partial_t u\|_{\frac98,\frac32,Q^+(\tau_1)}+\|\nabla^2 u\|_{\frac98,\frac32,Q^+(\tau_1)}+\|\nabla\, q\|_{\frac98,\frac32,Q^+(\tau_1)}\leq c_3'(\tau_1,\tau).\end{array}\right\} \tag{2.17}$$

Here, τ_1 and τ are arbitrary numbers satisfying $0<\tau_1<\tau<1$. It remains to note that the pair u and q solves the following boundary value problem:

$$\left.\begin{array}{c}\partial_t u-\Delta\, u=-\nabla\, q\\ \mathrm{div}\, u=0\end{array}\right\}\qquad \text{in } Q^+ \tag{2.18}$$

and

$$u=0\qquad \text{on}\qquad \Gamma^+\times]-1,0]. \tag{2.19}$$

From the theory of regularity up to the boundary for the Stokes system, see Propositions 3.2 and 3.3 in [7] for $s=18$, it follows that the function u is Hölder continuous in $\overline{Q}(\tau)$ and

$$\sup_{z\in\overline{Q}(\tau)}|u(z)|\leq c_4\tau^{\frac12} \tag{2.20}$$

for any $0<\tau<\tau_0<1$. In (2.20), c_4 depends on τ_0 only.

Now, we pass to the limit in (2.6) and find with the help of (2.15) and (2.20) the following bound

$$c\theta^{\frac{1}{2}}\leq c_4\theta^{\frac{1}{2}}+\limsup_{k\to+\infty}Y^{+2}_\theta(q^k). \tag{2.21}$$

Let us fix a domain $\widetilde{B}$ with sufficiently smooth boundary so that

$$B^+(4/5)\subset\widetilde{B}\subset B^+(5/6)$$

and consider the following initial boundary value problem

$$\left.\begin{array}{c}\partial_t u^{1k}-\Delta\,u^{1k}+\nabla\,q^{1k}=-\varepsilon_k\mathrm{div}\,u^k\otimes u^k\\ \mathrm{div}\,u^{1k}=0\end{array}\right\}\quad\text{in }\widetilde{Q}=\widetilde{B}\times]-(5/6)^2,0[,$$

$$[q^{1k}]_{\widetilde{B}}(s)=0,\qquad s\in]-(5/6)^2,0[,$$
$$u^{1k}=0\qquad\text{on}\qquad\partial\widetilde{B}\times[-(5/6)^2,0],$$
$$u^{1k}=0\qquad\text{on}\qquad\widetilde{B}\times\{s=-(5/6)^2\}.$$

It is well known that, for the solution to this problem, we have the coercive estimate

$$\|\nabla\,q^{1k}\|_{\frac{9}{8},\frac{3}{2},\widetilde{Q}}+\|q^{1k}\|_{\frac{9}{8},\frac{3}{2},\widetilde{Q}}\leq c_5\varepsilon_k\|\mathrm{div}\,u^k\otimes u^k\|_{\frac{9}{8},\frac{3}{2},\widetilde{Q}}.$$

By the imbedding theorem and by (2.17), we find

$$\|q^{1k}\|_{\frac{3}{2},Q(\frac{4}{5})}\leq c_5'\varepsilon_k. \tag{2.22}$$

If we introduce functions

$$u^{2k}=u^k-u^{1k},\qquad q^{2k}=q^k-q^{1k},$$

we see that they satisfy the relations

$$\begin{gathered}\|\nabla^2u^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})}+\|\nabla\,u^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})}+\|\nabla\,q^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})}\\ +\|q^{2k}\|_{\frac{9}{8},\frac{3}{2},Q(\frac{4}{5})}\leq c_6,\end{gathered} \tag{2.23}$$

$$\left.\begin{array}{c}\partial_t u^{2k}-\Delta\,u^{2k}=-\nabla\,q^{2k}\\ \mathrm{div}\,u^{2k}=0\end{array}\right\}\qquad\text{in }Q^+(4/5), \tag{2.24}$$

$$u^{2k} = 0 \qquad \text{on} \qquad \Gamma^+ \times]-1, 0]. \tag{2.25}$$

By Proposition 2, see [5], for solutions to problem (2.24), (2.25), subjected to inequality (2.23), the following estimate is valid:

$$\| \nabla q^{2k} \|_{18, \frac{3}{2}, Q^+(\frac{3}{4})} \leq c'_6. \tag{2.26}$$

Next, using (2.22), we can deduce from (2.21)

$$c\theta^{\frac{1}{2}} \leq c_4 \theta^{\frac{1}{2}} + \limsup_{k \to +\infty} Y^{+2}_\theta(q^{2k}). \tag{2.27}$$

In turn, Poincaré's inequality implies

$$c\theta^{\frac{1}{2}} \leq c_4 \theta^{\frac{1}{2}} + c_7 \theta \limsup_{k \to +\infty} \Big(\frac{\theta^{\frac{3}{2}}}{\theta^5} \int\limits_{-\theta^2}^{0} ds \int\limits_{B^+(\theta)} |\nabla q^{2k}|^{\frac{3}{2}} dy \Big)^{\frac{2}{3}}.$$

Now, (2.26) together with Hölder inequality with respect to the spatial variables leads to the estimate

$$c\theta^{\frac{1}{2}} \leq c_4 \theta^{\frac{1}{2}} + c'_7 \theta \limsup_{k \to +\infty} \Big(\frac{1}{\theta^{\frac{7}{2}}} \int\limits_{-\theta^2}^{0} ds \Big(\int\limits_{B^+(\theta)} |\nabla q^{2k}|^{18} dy \Big)^{\frac{1}{12}} |B^+(\theta)|^{\frac{11}{12}} \Big)^{\frac{2}{3}}$$

$$\leq c_4 \theta^{\frac{1}{2}} + c''_7 \theta^{\frac{1}{2}} = c_8 \theta^{\frac{1}{2}},$$

where c_8 is a known absolute constant. In order to get a contradiction, it is sufficient to let $c = 2c_8$. Lemma 2.1 is proved. $\square$

Lemma 2.1 can be iterated in the following way.

Lemma 2.2 *Let* $\theta \in]0, 1/2[$ *satisfy the condition*

$$c\theta^{\frac{1}{6}} < 1. \tag{2.28}$$

Assume that

$$Y^+_1(v, p) < \varepsilon. \tag{2.29}$$

Then, for any $k = 1, 2, ...,$

$$Y^+_{\theta^{k-1}}(v, p) < \varepsilon,$$

$$Y^+_{\theta^k}(v, p) \leq \theta^{\frac{k}{3}} Y^+_1(v, p). \tag{2.30}$$

Proof We argue by induction on k. For $k = 1$, this is nothing but the statement of Lemma 2.1.

Assume now that statements (2.30) are valid for $s = 1, 2, ..., k \geq 2$. Our goal is to prove that they are valid for $s = k + 1$ as well.

We make natural scaling

$$v^k(y, s) = \theta^k v(\theta^k y, \theta^{2k} s), \qquad p^k(y, s) = \theta^{2k} v(\theta^k y, \theta^{2k} s)$$

for $(y, s) \in Q^+$. It is easy to check that v^k and p^k form suitable weak solution in Q^+. Since

$$Y_1^+(v^k, p^k) = \theta^k Y_{\theta^k}^+(v, p) < \varepsilon,$$

we can derive from Lemma 2.1 and from (2.28)

$$Y_\theta^+(v^k, p^k) \leq c\theta^{\frac{1}{2}} Y_1^+(v^k, p^k) < \theta^{\frac{1}{3}} Y_1^+(v^k, p^k)$$

and therefore

$$Y_{\theta^{k+1}}^+(v, p) \leq \theta^{\frac{1}{3}} Y_{\theta^k}^+(v, p).$$

The latter immediately implies statements (2.30) for $s = k + 1$. Lemma 2.2 is proved. $\Box$

A direct consequence of Lemma 2.2 and scaling

$$v^R(y, s) = Rv(x_0 + Ry, t_0 + R^2 s), \qquad p^R(y, s) = R^2 p(x_0 + Ry, t_0 + R^2 s)$$

is the following statement.

Lemma 2.3 *Let θ and ε be as in Lemma 2.2. Let v and p be an arbitrary suitable weak solution to the Navier-Stokes equations in $Q^+(z_0, R)$ near $\Gamma(x_0, R)\times]t_0 - R^2, t_0]$. Assume that v and p satisfy the additional condition*

$$RY^+(z_0, R; v, p) < \varepsilon. \tag{2.31}$$

Then, for $k = 1, 2, ...,$ we have

$$Y^+(z_0, \theta^k R; v, p) \leq \theta^{\frac{k}{3}} Y^+(z_0, R; v, p). \tag{2.32}$$

3 Interior Regularity

In this section, we are going to formulate results on the interior (in the parabolic sense) regularity. The detail proof in the style of Section 2 can be found in the forthcoming paper [2], see [3] for an alternative approach.

To formulate results for the interior case, we need the following notation

$$Y(z_0,R;v,p)=Y^1(z_0,R;v)+Y^2(z_0,R;p),$$

$$Y^1(z_0,R;v)=\Big(\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)}|v-(v)_{z_0,R}|^3\,dz\Big)^{\frac13},$$

$$Y^2(z_0,R;p)=R\Big(\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)}|p-[p]_{z_0,R}|^{\frac32}\,dz\Big)^{\frac23},$$

$$(v)_{z_0,R}=(v)_{Q(z_0,R)},\qquad [p]_{x_0,R}=[p]_{B(x_0,R)},$$

$$Y^1_\theta(v)=Y^1(0,\theta;v),\qquad Y^2_\theta(p)=Y^2(0,\theta;p),$$

$$Y_\theta(v,p)=Y(0,\theta;v,p),\qquad (v)_{,\theta}=(v)_{0,\theta},\qquad [p]_{,\theta}=[p]_{0,\theta}.$$

Lemma 3.1 *Given numbers $\theta\in]0,1/2[$ and $M>3$, there exist two positive constants $\varepsilon_1(\theta,M)$ and $c_1(M)$ such that, for any suitable weak solution v and p to the Navier-Stokes equations in Q, satisfying the additional conditions*

$$|(v)_{,\theta}|<M,\qquad Y_1(v,p)<\varepsilon_1, \tag{3.1}$$

the following estimate is valid:

$$Y_\theta(v,p)\le c_1\theta^{\frac23}Y_1(v,p). \tag{3.2}$$

Lemma 3.2 *Given numbers $M>3$ and $\beta\in]0,2/3[$, we choose $\theta\in]0,1/2[$ so that*

$$c_1(M)\theta^{\frac{2-3\beta}{6}}<1. \tag{3.3}$$

Let $\overline{\varepsilon}_1(\theta,M)=\min\{\varepsilon_1(\theta,M),\theta^5M/2\}$. If

$$|(v)_{,1}|<M,\qquad Y_1(v,p)<\overline{\varepsilon}_1, \tag{3.4}$$

then, for any $k=1,2,...$,

$$\begin{gathered}\theta^{k-1}|(v)_{,\theta^{k-1}}|<M,\qquad Y_{\theta^{k-1}}(v,p)<\overline{\varepsilon}_1\le\varepsilon_1,\\ Y_{\theta^k}(v,p)\le\theta^{\frac{2+3\beta}{6}}Y_{\theta^{k-1}}(v,p).\end{gathered} \tag{3.5}$$

Lemma 3.3 . *Let M, β, θ, and $\overline{\varepsilon}_1$ be as in Lemma 3.2. Let a pair v and p be an arbitrary suitable weak solution to the Navier-Stokes equations in the parabolic cylinder $Q(z_0,R)$, satisfying the additional conditions*

$$R|(v)_{z_0,R}|<M, \qquad RY(z_0,R;v,p)<\overline{\varepsilon}_1. \tag{3.6}$$

Then, for any $k=1,2,...$, we have

$$Y(z_0,\theta^kR;v,p)\leq\theta^{\frac{2+3\beta}{6}k}Y(z_0,R;v,p). \tag{3.7}$$

4 Proof of Theorem 1.2

Obviously, by natural scaling, it is sufficient to prove Theorem 1.2 for $z_0=0$ and $R=1$. For simplicity, we consider the case $\tau=1/2$. The general case differs from particular one just by technical details.

From Lemmata 2.3 and 3.3, we can easily deduce the following statements.

Lemma 4.1 *There are two absolute positive constants ε_+ and c_+ such that, for any suitable weak solution v and p to the Navier-Stokes equations in $Q^+(z_0,R)$ near $\Gamma(x_0,R)\times]t_0-R^2,t_0]$, satisfying the condition*

$$RY^+(z_0,R;v,p)<\varepsilon_+,$$

the following estimate is valid:

$$Y^+(z_0,\rho;v,p)\leq c_+\Big(\frac{\rho}{R}\Big)^{\frac13}Y^+(z_0,R;v,p), \qquad \forall\rho\in]0,R].$$

Lemma 4.2 *There are two absolute positive constants $\varepsilon_\times$ and $c_\times$ such that, for any suitable weak solution v and p to the Navier-Stokes equations in $Q(z_0,R)$, satisfying the condition*

$$R\widetilde{Y}(z_0,R;v,p)<\varepsilon_\times,$$

where

$$\widetilde{Y}(z_0,R;v,p)\equiv\Big(\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)}|v|^3\,dz\Big)^{\frac13}$$

$$+R\Big(\frac{1}{|Q(R)|}\int\limits_{Q(z_0,R)}|p-[p]_{x_0,R}|^{\frac32}\,dz\Big)^{\frac23},$$

the following estimate is valid:

$$Y(z_0,\rho;v,p)\leq c_\times\Big(\frac{\rho}{R}\Big)^{\frac13}Y(z_0,R;v,p),\qquad\forall\rho\in]0,R].$$

To prove Lemma 4.2, we may take $M=2002$ and use two simple inequalities:

$$|(v)_{z_0,R}|\leq\widetilde{Y}(z_0,R;v,p),\qquad Y(z_0,R;v,p)\leq 2\widetilde{Y}(z_0,R;v,p).$$

In turn, Lemmata 4.1 and 4.2 imply the following result.

Proposition 4.3 *There are two absolute positive constants $\overline{\varepsilon}$ and $\overline{c}$ such that, for any suitable weak v and p to the Navier-Stokes equations in Q^+ near $\Gamma^+\times]-1,0]$, satisfying the additional condition*

$$RV^+(z_0,R;v,p)<\overline{\varepsilon},\qquad z_0\in\overline{Q^+}(2/3),\quad 0<R\leq 1/4,$$

we have the estimate

$$V(z_0,\rho;v,p)\leq\overline{c}\Big(\frac{\rho}{R}\Big)^{\frac13}V^+(z_0,R:v,p),$$

which is valid for all $0<\rho\leq R$.

To formulate Proposition 4.3, we used the following notation. For $z_0\in\overline{Q^+}$ and $0<R\leq 1$, satisfying in addition

$$B(x_0,R)\subset B,\qquad t_0>R^2-1,$$

we let

$$V^+(z_0,R;v,p)=\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|v|^3\,dz\Big)^{\frac13}$$

$$+\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|p-[p]_{\Omega(x_0,R)}|^{\frac32}\,dz\Big)^{\frac23},$$

$$V(z_0,R;v,p)=\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|v-(v)_{\omega(z_0,R)}|^3\,dz\Big)^{\frac13}$$

$$+R\Big(\frac{1}{|\omega(z_0,R)|}\int\limits_{\omega(z_0,R)}|p-[p]_{\Omega(x_0,R)}|^{\frac32}\,dz\Big)^{\frac23},$$

where $\Omega(x_0,R)=B(x_0,R)\cap B^+$ and $\omega(x_0,R)=\Omega(x_0,R)\times]t_0-R^2,t_0[$.

The proof of Proposition 4.3 is quite standard, see [6], Section 5, or [7], Section 6, and we omit it.

Proof of Theorem 1.2. First we let

$$A=U(0,1;v,p).$$

Obviously,

$$\frac{1}{4}V^+(z_0,\frac{1}{4};v,p)\leq C(A^{\frac{1}{3}}+A^{\frac{2}{3}})\leq C(\varepsilon_0^{\frac{1}{3}}+\varepsilon_0^{\frac{2}{3}})<\overline{\varepsilon}$$

for sufficiently small ε_0. Here, C is an absolute positive constant. Then,

$$V(z_0,\rho;v,p)\leq\overline{c}(4\rho)^{\frac{1}{3}}V^+(z_0,\frac{1}{4};v,p)\leq\overline{c}_1\rho^{\frac{1}{3}}\overline{\varepsilon}$$

for all $z_0\in\overline{Q^+}$ and for all $0<\rho\leq\frac{1}{4}$. By Campanato's condition, v is Hölder continuous at least in $\overline{Q^+}(1/2)$. Theorem 4.3 is proved. $\Box$

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