The problem of global unique solvability of the Navier-Stokes equations

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Contents

1 Lecture 1: Physical background

1.1 Derivation of the equations

1. **Navier-Stokes equations**

   Assume $\Omega \subset \mathbb{R}^3$, $Q_T := \Omega \times (0,T)$

   - We are given: $a : \Omega \to \mathbb{R}^3$, $\nu > 0$
   - We want to find: $u : Q_T \to \mathbb{R}^3$, $p : Q_T \to \mathbb{R}$, such that

   $$\left\{ \begin{array}{rcl} \partial_t u - \nu\,\Delta u + (u\cdot\nabla)u + \nabla p &=& 0 \quad \text{in} \quad Q_T \\ \operatorname{div} u &=& 0 \quad \text{in} \quad Q_T \\ u|_{\partial\Omega\times(0,T)} = 0, \quad u|_{t=0} &=& a \end{array} \right. \tag{NS}$$

   $$(u\cdot\nabla)u = u_1\tfrac{\partial u}{\partial x_1} + u_2\tfrac{\partial u}{\partial x_2} + u_3\tfrac{\partial u}{\partial x_3}, \qquad \operatorname{div} u = 0 \quad \Longrightarrow \quad \operatorname{div}(u\otimes u) = (u\cdot\nabla)u$$

2. **Where the NSE comes from?**

   - Motion of continuum:

   $$\Phi_t : \mathbb{R}^3 \times \to \mathbb{R}^3 \quad \text{is a diffeomorphism,} \qquad \Omega_t := \Phi_t(\Omega_0), \qquad \Phi_0 = \mathrm{Id}$$

   - Conservation of mass:

   $$0 \;=\; \frac{d}{dt}\int\limits_{\Omega_t} \rho\,dx \;=\; \int\limits_{\Omega_t} \Big(\frac{\partial\rho}{\partial t} + \operatorname{div}(u\rho)\Big)\,dx, \qquad \rho = const \quad \Longrightarrow \quad \operatorname{div} u = 0$$

   - Conservation of momentum:

   $$F_{\Omega_t} \;=\; \frac{d}{dt}\int\limits_{\Omega_t} \rho u\,dx \;=\; \int\limits_{\Omega_t} \rho\Big(\frac{\partial u}{\partial t} + \operatorname{div}(u\otimes u)\Big)\,dx, \qquad \rho\big(\partial_t u + \operatorname{div}(u\otimes u)\big) = f + \operatorname{div}\mathcal{T}$$

3. **Technical tools**

   - Lagrangian and Euler coordinates:

   $$\left.\begin{array}{r} y\in\Omega_0 \quad \text{— Lagrangian} \\ x = \Phi_t(y)\in\Omega_t \quad \text{— Euler} \end{array}\right\} \qquad \hat u(y,t) = \frac{\partial\Phi}{\partial t}(y,t), \quad \hat a(y,t) = \frac{\partial^2\Phi}{\partial t^2}(y,t)$$

   - Material derivative:

   $$\left.\begin{array}{l} u(x,t) = \hat u(y,t)\big|_{y=\Phi_t^{-1}(x)} \\ a(x,t) = \hat a(y,t)\big|_{y=\Phi_t^{-1}(x)} \end{array}\right\} \qquad \Longrightarrow \qquad a(x,t) \;=\; \frac{Du}{Dt}(x,t) = \frac{\partial u}{\partial t} + (u\cdot\nabla)u$$

- Euler's identity for Jacobians:

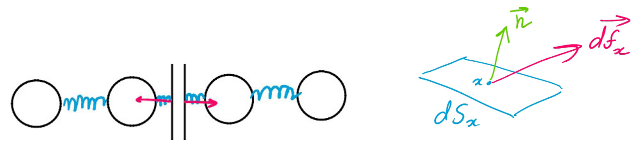
$$J(y,t) = \det \nabla\Phi(y,t) \qquad \Longrightarrow \qquad \frac{\partial J}{\partial t}(y,t) \;=\; J(y,t)\,\mathrm{div}_x u(x,t)\,\Big|_{x=\Phi_t(y)}$$

- Transport theorem:

$$\frac{d}{dt}\int\limits_{\Omega_t} \rho(x,t)\;dx \;=\; \int\limits_{\Omega_t}\Big(\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho u)\Big)\;dx,$$

4. **Forces**

- Internal surface forces — the concept of stresses



- Cauchy hypothesis (Axiom)

$$\exists\,\sigma : S^2\times\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3, \qquad S^2 := \{\,x\in\mathbb{R}^3 : \,|x| = 1\,\}$$

$$df_x \;=\; \sigma(n_x,x,t)\;dS_x \quad — \quad \begin{array}{c}\text{the force acting on the area } dS_x\\ \text{with normal } n_x \text{ at time } t\end{array}$$

- The resultant force and balance of momentum (Axiom)

$$F_{\Omega_t} \;:=\; \int\limits_{\Omega_t} \underbrace{f(x,t)}_{\rho(x,t)g(x,t)}\;\;dx \;+\; \int\limits_{\partial\Omega_t} \sigma(n_x,x,t)\;dS_x, \qquad \frac{d}{dt}\int\limits_{\Omega_t}\rho u\,dx \;=\; F_{\Omega_t}$$

- Balance of angular momentum (Axiom)

$$M_{\Omega_t} \;:=\; \int\limits_{\Omega_t} x\times f\;dx \;+\; \int\limits_{\partial\Omega_t} x\times\sigma\;dS_x, \qquad \frac{d}{dt}\int\limits_{\Omega_t} x\times\rho u\,dx \;=\; M_{\Omega_t}$$

- Cauchy stress tensor (Theorem)

$$\sigma(n,x,t) \;=\; T(x,t)n, \qquad T_{ij}(x,t) \;=\; T_{ji}(x,t)$$

5. **Classification of materials**

- material without internal constraints

$$\text{admissible deformations} \quad = \quad \text{all diffeomorphisms of } \Phi_t:\mathbb{R}^3\to\mathbb{R}^3$$

- incompressible materials

$$\text{admissible deformations} \quad = \quad \text{volume-preserving diffeomorphisms}$$

- incompressibility condition

$$|\Phi_t(E)| \;=\; |E|, \qquad \forall\, t\in\mathbb{R} \qquad \Longrightarrow \qquad \mathrm{div}\,u \;=\; 0$$

6. **Rheological relations**

- fluid at rest: $df_x = -p\, n_x\, dS_x \implies T(x) = -p(x)\,\mathbb{I}$ — hydrostatical pressure
- $T(x,t) = -p(x,t)\,\mathbb{I}$ — perfect fluid, $\nu = 0$,
- $T(x,t) = -p(x,t)\,\mathbb{I} + \sigma(x,t), \quad \sigma = 2\nu Du$ — Newtonian fluid
- $T(x,t) = -p(x,t)\,\mathbb{I} + \sigma(x,t), \quad \sigma = \sigma(Du, \ldots)$ — non-Newtonian fluid

$$Du = \tfrac{1}{2}(\nabla u + (\nabla u)^T), \qquad \operatorname{div} u = 0 \quad \Longrightarrow \quad \operatorname{div} Du = \tfrac{1}{2}\Delta u$$

7. **Pressure in fluids and gases**

- gas: $\rho = \rho(x,t)$, $p = c\rho^{\gamma}$ $\quad pV = const \quad \gamma = 1$
- fluid: $\rho = const$, $p(x,t)$ — unknown function

8. **Problem of the modelling of boundary conditions**

- $\nu > 0 \implies u|_{\partial\Omega} = 0$
- $\nu = 0 \implies u \cdot n|_{\partial\Omega} = 0$, $\quad n$ is a unit normal to $\partial\Omega$

1.2 Some basic properties of the Navier-Stokes equations

1. **Energy balance equation** $\quad f \equiv 0, \;\; \mathrm{NSE} \cdot u$

$$\tfrac{1}{2} \int\limits_{\Omega} |u(x,t)|^2 \, dx \; + \; \nu \int\limits_0^t \int\limits_{\Omega} |\nabla u(x,\tau)|^2 \, dx d\tau \;\; = \;\; \tfrac{1}{2} \int\limits_{\Omega} |a(x)|^2 \, dx$$

$$\operatorname{div} v = 0, \quad (u \cdot \nabla u) = \operatorname{div}(u \otimes u) \quad \Longrightarrow \quad \int\limits_{\Omega} u \otimes u : \nabla u \, dx \; = \; \int\limits_{\Omega} u \cdot \nabla \tfrac{1}{2} |u|^2 \, dx \; = \; 0$$

2. **NSE in the vorticity form**

$$\omega = \operatorname{rot} u, \quad (u \cdot \nabla) u = \omega \times u + \tfrac{1}{2} \nabla |u|^2, \quad \nabla \times (\omega \times u) = \mathrm{BAC}{-}\mathrm{CAB}$$

$$\partial_t \omega + (u \cdot \nabla)\omega - \nu \, \Delta \omega \;\; = \;\; (\omega \cdot \nabla) u \quad \text{in} \quad Q_T$$

$u = u(x_1, x_2, t), \;\; u_3 = 0 \;\; \Longrightarrow \;\; (\omega \cdot \nabla) u = 0$

3. **What can we expect from a quadratic non-linearity?**

$$\left. \begin{array}{c} y'(t) = y^2(t) \\ y(0) = a \end{array} \right\} \quad \Longrightarrow \quad \left. \begin{array}{c} y(t) = \dfrac{1}{T_a - t} \\ T_a = 1/a \end{array} \right\} \quad \Longrightarrow \quad \begin{array}{c} y(t) \underset{t \to T_a}{\longrightarrow} \infty \\ \text{blow up!} \end{array}$$

But in the case of the NSE we have the conservation law!

$$\tfrac{1}{2} \, \|u(t)\|^2_{L_2(\Omega)} \;\; + \;\; \nu \int\limits_0^t \|\nabla u(\tau)\|^2_{L_2(\Omega)} \, d\tau \;\; = \;\; \tfrac{1}{2} \, \|a\|^2_{L_2(\Omega)}$$

4. **Main question**

Are the Navier-Stokes equations adequate for describing viscous fluid dynamics?

2 Lecture 2: Weak solutions

2.1 Tools of analysis

1. **Helmholtz decomposition**

$$L_2(\Omega;\mathbb{R}^3) \ = \ H \oplus G, \quad \forall\, f \in L_2(\Omega;\mathbb{R}^3) \quad f = u + \nabla\varphi$$

$$H := \mathrm{Closure}_{L_2(\Omega)} \left\{ v \in C_0^\infty(\Omega) \,|\, \mathrm{div}\, v = 0 \right\}, \qquad u \in H, \qquad \nabla\varphi \in L_2(\Omega)$$

2. **Stokes operator**

$$\tilde\Delta : D_{\tilde\Delta} \subset H \to H, \qquad \tilde\Delta = P_H \Delta \quad \text{is self-adjoint}$$

P_H is an orthogonal projection onto H in $L_2(\Omega;\mathbb{R}^3)$

3. **Eigenfunctions of the Stokes operator**

$$\begin{matrix} 0 < \lambda_1 \le \lambda_2 \le \ldots \lambda_k \le \ldots \ \to \ +\infty \\ \varphi_k \in D_{\tilde\Delta} \end{matrix} \qquad \left\{ \begin{aligned} -\tilde\Delta\varphi_k \ &= \ \lambda_k\,\varphi_k \quad \text{in} \quad \Omega \\ \mathrm{div}\,\varphi_k \ &= \ 0 \quad \text{in} \quad \Omega \\ \varphi_k|_{\partial\Omega} \ &= \ 0 \end{aligned} \right.$$

4. **Orthonormal basis in H**

Let $\Omega \subset \mathbb{R}^n$ be bounded and $\partial\Omega$ be smooth. Then

- φ_k are **smooth**, $\mathrm{div}\,\varphi_k = 0$ in Ω
- $\{\varphi_k\}_{k=1}^\infty$ is an **orthonormal basis** in H $\quad (\varphi_k,\varphi_j)_{L_2(\Omega)} = 1, \quad \|\varphi_k\|_{L_2(\Omega)} = 1$
- $(\nabla\varphi_k, \nabla\varphi_j) = \lambda_k\delta_{kj}$ where $(\nabla u, \nabla v) \ := \ \int_\Omega \nabla u : \nabla v\, dx$

5. **Fourier series with respect to the eigenbasis of the Stokes operator**

For any $u \in H$ such that $\nabla u \in L_2(\Omega)$

$$\|u\|^2_{L_2(\Omega)} = \sum_{k=1}^\infty |c_k|^2, \qquad \|\nabla u\|^2_{L_2(\Omega)} = \sum_{k=1}^\infty \lambda_k |c_k|^2, \qquad c_k := (u,\varphi_k)_{L_2(\Omega)}$$

6. **Gagliardo–Nirenberg inequality and Sobolev embedding theorem**

Assume $n \ge 3$. Then for any $\forall\, u, v \in C_0^\infty(\mathbb{R}^n)$

$$\|u\|_{L_{\frac{n}{n-1}}(\mathbb{R}^n)} \ \le \ c\,\|\nabla u\|_{L_1(\mathbb{R}^n)}, \qquad \|v\|_{L_{\frac{2n}{n-2}}(\mathbb{R}^n)} \ \le \ c\,\|\nabla v\|_{L_2(\mathbb{R}^n)}$$

PROOF. **1.** Assume $n = 2$, $u \in C_0^1(\mathbb{R}^2)$. Then

$$|u(x_1, x_2)| \ \le\ 2 \int_{-\infty}^{+\infty} |u_{,2}(x_1, s)|\, ds \ \le\ 2\, \underbrace{\|u_{,2}(x_1, \cdot)\|_{L_2(\mathbb{R})}}_{=: G_1(x_1)}$$

$$|u(x_1, x_2)| \ \le\ 2 \int_{-\infty}^{+\infty} |u_{,1}(s, x_2)|\, ds \ \le\ 2\, \underbrace{\|u_{,1}(\cdot, x_2)\|_{L_2(\mathbb{R})}}_{=: G_2(x_2)}$$

Multiplying these inequalities and integrating over $\mathbb{R}^2$ we obtain:

$$\int_{\mathbb{R}^2} |u(x_1, x_2)|^2\, dx_1 dx_2 \ = \ 4 \int_{-\infty}^{+\infty} G_1(x_1)\, dx_1 \cdot \int_{-\infty}^{+\infty} G_2(x_2)\, dx_2 \ \le$$

$$\le\ 4\, \underbrace{\Big(\int_{-\infty}^{+\infty} G_1^2(x_1)\, dx_1 \Big)^{1/2}}_{= \|u_{,2}\|_{L_2(\mathbb{R}^2)}} \underbrace{\Big(\int_{-\infty}^{+\infty} G_2^2(x_2)\, dx_2 \Big)^{1/2}}_{= \|u_{,1}\|_{L_2(\mathbb{R}^2)}} \ =$$

$$= \ 4\, \|u_{,1}\|_{L_2(\mathbb{R}^2)} \|u_{,2}\|_{L_2(\mathbb{R}^2)} \ \le\ 4\, \|\nabla u\|^2_{L_2(\mathbb{R}^2)}$$

2. Assume $n = 3$, $u \in C_0^1(\mathbb{R}^3)$. From the Hölder inequality we get

$$\int_{\mathbb{R}^3} |u|^{\frac{3}{2}}\, dx \ = \ \int_{-\infty}^{+\infty} dx_3 \int_{\mathbb{R}^2} |u|\, |u|^{\frac{1}{2}}\, dx_1 dx_2 \ \le \int_{-\infty}^{+\infty} \|u(\cdot, x_3)\|_{L_2(\mathbb{R}^2)} \|u(\cdot, x_3)\|^{\frac{1}{2}}_{L_1(\mathbb{R}^2)}\, dx_3$$

For any x_3 by the Newton-Leibniz formula we obtain

$$|u(x_1, x_2, x_3)| \ \le\ \int_{-\infty}^{+\infty} |u_{,3}(x_1, x_2, s)|\, ds$$

Integrating this identity with respect to $(x_1, x_2) \in \mathbb{R}^2$ for any x_3 we obtain

$$\|u(\cdot, x_3)\|_{L_1(\mathbb{R}^2)} \ \le\ \|u_{,3}\|_{L_1(\mathbb{R}^3)}$$

On the other hand, for any x_3 we have $u(\cdot, x_3) \in C_0^\infty(\mathbb{R}^2)$ and hence by the 2D Gagliardo–Nirenberg inequality we obtain

$$\|u(\cdot, x_3)\|_{L_2(\mathbb{R}^2)} \ \le\ c\, \|u_{,1}(\cdot, x_3)\|^{\frac{1}{2}}_{L_2(\mathbb{R}^2)} \|u_{,2}(\cdot, x_3)\|^{\frac{1}{2}}_{L_2(\mathbb{R}^2)}$$

Gathering these estimates together and using the Hölder inequality we arrive at

$$\int_{-\infty}^{+\infty} \|u(\cdot, x_3)\|_{L_2(\mathbb{R}^2)} \|u(\cdot, x_3)\|^{\frac{1}{2}}_{L_1(\mathbb{R}^2)}\, dx_3 \ \le\ c\, \|u_{,3}\|^{\frac{1}{2}}_{L_1(\mathbb{R}^3)} \int_{-\infty}^{+\infty} \|u_{,1}(\cdot, x_3)\|^{\frac{1}{2}}_{L_2(\mathbb{R}^2)} \|u_{,2}(\cdot, x_3)\|^{\frac{1}{2}}_{L_2(\mathbb{R}^2)}\, dx_3$$

$$\le\ c\, \|u_{,3}\|^{\frac{1}{2}}_{L_1(\mathbb{R}^3)}\, \|u_{,1}\|^{\frac{1}{2}}_{L_1(\mathbb{R}^3)} \|u_{,2}\|^{\frac{1}{2}}_{L_1(\mathbb{R}^3)} \ \le\ c\, \|\nabla u\|^{\frac{3}{2}}_{L_1(\mathbb{R}^3)}$$

3. We prove Sobolev inequality for $n=3$. Assume $v\in C_0^\infty(\mathbb{R}^3)$ and take in the 3D Gagliardo–Nirenberg inequality

$$u=v^4,\qquad \nabla u=4\,v^3\,\nabla v$$

Then we obtain

$$\|v\|^6_{L_6(\mathbb{R}^3)} \;=\; \|u\|^{\frac32}_{L_{\frac32}(\mathbb{R}^3)} \;\le\; c\,\|\nabla u\|^{\frac32}_{L_1(\mathbb{R}^3)} \;\le\; c\,\|v^3\,\nabla v\|^{\frac32}_{L_1(\mathbb{R}^3)} \;\le\; c\,\|v\|^{\frac92}_{L_2(\mathbb{R}^3)}\,\|\nabla v\|^{\frac32}_{L_2(\mathbb{R}^3)}$$

As $6-\frac92=\frac32$ we obtain

$$\|v\|^{\frac32}_{L_6(\mathbb{R}^3)} \;\le\; c\,\|\nabla v\|^{\frac32}_{L_2(\mathbb{R}^3)}$$

$\square$

7. **Ladyzhenskaya inequality**

$$\forall\, w\in C_0^\infty(\Omega)\qquad \|w\|^{4}_{L_4(\Omega)} \;\le\; \begin{cases} 2\,\|w\|^{2}_{L_2(\Omega)}\|\nabla w\|^{2}_{L_2(\Omega)}, & n=2\\ 4\,\|w\|_{L_2(\Omega)}\|\nabla w\|^{3}_{L_2(\Omega)}, & n=3\end{cases}$$

Proof. **1.** Assume $n=2$, $u\in C_0^1(\mathbb{R}^2)$. Then

$$|u(x_1,x_2)|^2 \;\le\; 2\int\limits_{-\infty}^{+\infty}|u(x_1,s)|\,|u_{,2}(x_1,s)|\,ds \;\le\; 2\,\underbrace{\|u(x_1,\cdot)\|_{L_2(\mathbb{R})}}_{=:\,F_1(x_1)}\,\underbrace{\|u_{,2}(x_1,\cdot)\|_{L_2(\mathbb{R})}}_{=:\,G_1(x_1)}$$

$$|u(x_1,x_2)|^2 \;\le\; 2\int\limits_{-\infty}^{+\infty}|u(s,x_2)|\,|u_{,1}(s,x_2)|\,ds \;\le\; 2\,\underbrace{\|u(\cdot,x_2)\|_{L_2(\mathbb{R})}}_{=:\,F_2(x_2)}\,\underbrace{\|u_{,1}(\cdot,x_2)\|_{L_2(\mathbb{R})}}_{=:\,G_2(x_2)}$$

Multiplying these inequalities and integrating over $\mathbb{R}^2$ we obtain:

$$\int\limits_{\mathbb{R}^2}|u(x_1,x_2)|^4\,dx_1dx_2 \;=\; 4\int\limits_{\mathbb{R}^2}F_1(x_1)G_1(x_1)F_2(x_2)G_2(x_2)\,dx_1dx_2 \;=$$

$$=\;4\int\limits_{-\infty}^{+\infty}F_1(x_1)G_1(x_1)\,dx_1\cdot\int\limits_{-\infty}^{+\infty}F_2(x_2)G_2(x_2)\,dx_2\;\le$$

$$\le\;4\,\underbrace{\Big(\int\limits_{-\infty}^{+\infty}F_1^2(x_1)\,dx_1\Big)^{1/2}}_{=\|u\|_{L_2(\mathbb{R}^2)}}\underbrace{\Big(\int\limits_{-\infty}^{+\infty}G_1^2(x_1)\,dx_1\Big)^{1/2}}_{=\|u_{,2}\|_{L_2(\mathbb{R}^2)}}\underbrace{\Big(\int\limits_{-\infty}^{+\infty}F_2^2(x_2)\,dx_2\Big)^{1/2}}_{=\|u\|_{L_2(\mathbb{R}^2)}}\underbrace{\Big(\int\limits_{-\infty}^{+\infty}G_2^2(x_2)\,dx_2\Big)^{1/2}}_{=\|u_{,1}\|_{L_2(\mathbb{R}^2)}}\;=$$

$$=\;4\,\|u\|^2_{L_2(\mathbb{R}^2)}\|u_{,1}\|_{L_2(\mathbb{R}^2)}\|u_{,2}\|_{L_2(\mathbb{R}^2)}\;\le\;2\,\|u\|^2_{L_2(\mathbb{R}^2)}\Big(\|u_{,1}\|^2_{L_2(\mathbb{R}^2)}+\|u_{,2}\|^2_{L_2(\mathbb{R}^2)}\Big)\;=$$

$$=\;2\,\|u\|^2_{L_2(\mathbb{R}^2)}\,\|\nabla u\|^2_{L_2(\mathbb{R}^2)}$$

2. Assume $n = 3$, $u \in C_0^1(\mathbb{R}^3)$. From the Hölder inequality we get

$$\int\limits_{\mathbb{R}^3} |u|^4\,dx \;=\; \int\limits_{\mathbb{R}^3} |u|\,|u|^3\,dx \;\le \Big(\int\limits_{\mathbb{R}^3} |u|^2\,dx\Big)^{1/2}\Big(\int\limits_{\mathbb{R}^3} |u|^6\,dx\Big)^{1/2} \;=\; \|u\|_{L_2(\mathbb{R}^3)}\|u\|^3_{L_6(\mathbb{R}^3)}$$

From the Gagliardo–Nirenberg inequality for $n = 3$ we obtain:

$$\|u\|_{L_6(\mathbb{R}^3)} \;\le\; 4\,\|\nabla u\|_{L_2(\mathbb{R}^3)}, \qquad \forall\; u \in C_0^1(\mathbb{R}^3)$$

Then

$$\|u\|^4_{L_4(\mathbb{R}^3)} \;\le\; \|u\|_{L_2(\mathbb{R}^3)}\|u\|^3_{L_6(\mathbb{R}^3)} \;\le\; 4\,\|u\|_{L_2(\mathbb{R}^3)}\|\nabla u\|^3_{L_6(\mathbb{R}^3)}$$

$\square$

8. **Gronwall lemma**

$$y'(t) \;\le\; a(t)y(t) \qquad \Longrightarrow \qquad y(t) \;\le\; y(0)\,\exp\left(\int\limits_0^t a(\tau)\,d\tau\right), \qquad a \in L_1(0,T)$$

2.2 Leray-Hopf solutions

1. **Galerkin approximations**

$$u^m(x,t) \;=\; \sum_{j=1}^{m} C_j^m(t)\varphi_j(x), \qquad C_j^m \quad \text{satisfy the system of non-linear ODEs}$$

$$\frac{d}{dt}\big(u^m(t),\varphi_k\big) + \big(\nabla u^m(t),\varphi_k\big) - \big(u^m(t)\otimes u^m(t),\nabla\varphi_k\big) \;=\; 0, \qquad k=1,\dots,m$$

$$\begin{cases} \dfrac{dC_k^m}{dt}(t) \;+\; \lambda_k C_k^m(t) \;-\; \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{m} a_{kij}C_i^m(t)C_j^m(t) \;=\; 0 \\ C_k^m(0) \;=\; a_k, \qquad k=1,2,\dots,m. \end{cases}$$

$$a_k = (a,\varphi_k), \qquad a_{kij} = (\varphi_i\otimes\varphi_j,\nabla\varphi_k)$$

2. **A priori estimates**

$a\in H,\;\; u^m\in C^1(\bar Q_T)$ are Galerkin's approximations

$$\sup_{t\in(0,T)}\|u^m(t)\|_{L_2(\Omega)} \;+\; \|\nabla u^m\|_{L_2(Q_T)} \;\le\; c\,\|a\|_{L_2(\Omega)}$$

$$\left.\begin{aligned} u^m \rightharpoonup u \;\;\text{in}\;\; L_2(Q_T) \\ \nabla u^m \rightharpoonup \nabla u \;\;\text{in}\;\; L_2(Q_T)\end{aligned}\right\} \quad\Longrightarrow\quad \begin{gathered} u\in L_\infty(0,T;L_2(\Omega)), \quad \nabla u\in L_2(Q_T) \\ \text{Is it true that } u \text{ satisfies (NS) ?}\end{gathered}$$

3. **How to pass to the limit in the non-linear term $u^m\otimes u^m$?**

Problem: nonlinear quantities are not continuous with respect to the weak convergence.

$$\text{We need:}\quad u^m\to u \;\;\text{in}\;\; L_2(Q_T) \qquad\Longrightarrow\qquad u \text{ is a solution to (NS)}$$

4. **Compactness of u^m in $L_2(Q_T)$**

- Equicontinuity of Fourier coefficients

$$|C_k^m(t)| \;\le\; \|u^m(t)\|_{L_2(\Omega)} \;\le\; \|a\|_{L_2(\Omega)} \quad\Longrightarrow\quad \sup_{m\ge k}\|C_k^m\|_{C([0,T])} < +\infty$$

$$\frac{dC_k^m}{dt}(t) = -(\nabla u^m(t),\nabla\varphi_k) + (u^m(t)\otimes u^m(t),\nabla\varphi_k)$$

$$|C_k^m(t+\Delta t) - C_k^m(t)| \;\le\; c_\Omega\,\|\nabla\varphi_k\|_{L_\infty(\Omega)}\int_t^{t+\Delta t}\Big(\|\nabla u^m(\tau)\|_{L_2(\Omega)} + \|u^m(\tau)\|^2_{L_2(\Omega)}\Big)\,dt \;\le$$

$$\le c_\Omega\,\|\nabla\varphi_k\|_{L_\infty(\Omega)}\Big(\sqrt{\Delta t}\,\|\nabla u^m\|_{L_2(Q_T)} + \Delta t \sup_{\tau\in(0,T)}\|u^m(\tau)\|^2_{L_2(\Omega)}\Big)$$

Hence $\{C_k^m\}_{m=k}^\infty$ is equibounded and equicontinuous.

- For any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$, such that

$$\|w\|^2_{L_2(\Omega)} \ \le \ \sum_{k=1}^{N_\varepsilon} \underbrace{|(w,\varphi_k)|^2}_{=|c_k|^2} \ + \varepsilon \, \|\nabla w\|^2_{L_2(\Omega)}$$

- Take $w = u^m(t) - u(t)$ and integrate over $t \in (0,T)$

$$\|u^m - u\|^2_{L_2(Q_T)} \ \le \ \sum_{k=1}^{N_\varepsilon} \int\limits_0^T \underbrace{|(u^m(t)-u(t),\varphi_k)|^2}_{=\ |C_k^m(t)-C_k(t)|^2} \, dt \ + \ \varepsilon \ \underbrace{\|\nabla u^m - \nabla u\|^2_{L_2(Q_T)}}_{\le\ \frac{2\varepsilon}{\nu}\,\|a\|^2_{L_2(\Omega)}}$$

5. **Global existence of Leray-Hopf solutions**

Theorem. $a \in H - \forall$, $T > 0 - \forall \implies \exists\, u : Q_T \to \mathbb{R}^3$:

- $u \in L_\infty(0,T;H)$, $\nabla u \in L_2(Q_T)$ — energy class
- integral identity for test functions $\eta \in C_0^\infty(Q_T)$: $\operatorname{div} \eta = 0$
- $\int\limits_\Omega |u(x,t)|^2\,dx \ + \ 2\nu \int\limits_0^t \int\limits_\Omega |\nabla u(x,\tau)|^2\,dxd\tau \ \le \ \int\limits_\Omega |a(x)|^2\,dx, \ \ \forall\, t \in (0,T)$

6. **Embedding of the energy class**

$$n=2 \quad \implies \quad \|u(\cdot,t)\|^4_{L_4(\Omega)} \ \le \ c\ \underbrace{\|u(\cdot,t)\|^2_{L_2(\Omega)}}_{\in L_\infty(0,T)} \ \underbrace{\|\nabla u(\cdot,t)\|^2_{L_2(\Omega)}}_{\in L_1(0,T)} \quad \implies \quad u \in L_4(Q_T)$$

$$n=3 \quad \implies \quad \|u(\cdot,t)\|^{\frac{10}{3}}_{L_{\frac{10}{3}}(\Omega)} \ \le \ c\ \underbrace{\|u(\cdot,t)\|^{\frac{4}{3}}_{L_2(\Omega)}}_{\in L_\infty(0,T)} \ \underbrace{\|\nabla u(\cdot,t)\|^2_{L_2(\Omega)}}_{\in L_1(0,T)} \quad \implies \quad u \in L_{\frac{10}{3}}(Q_T)$$

2.3 Uniqueness in 2D and the weak-strong uniqueness

1. **Estimate of the difference of two solutions**

   u_1 and u_2 are two solutions corresponding to $a \in H$, $u = u_2 - u_1$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L_2(\Omega)} + \|\nabla u(t)\|^2_{L_2(\Omega)} = \big(u_2(t) \otimes u_2(t) - u_1(t) \otimes u_1(t), \nabla u(t)\big)$$
$$= \underbrace{\big(u(t) \otimes u_2(t), \nabla u(t)\big)}_{=\,0} + \big(u_1(t) \otimes u(t), \nabla u(t)\big)$$

2. **Uniqueness theorem in 2D**

$$\big|\big(u_1(t) \otimes u(t), \nabla u(t)\big)\big| \le \|u_1(t)\|_{L_4(\Omega)} \|u(t)\|^{\frac{1}{2}}_{L_2(\Omega)} \|\nabla u(t)\|^{\frac{3}{2}}_{L_2(\Omega)}, \quad n = 2$$

$$y(t) := \|u_2(t) - u_1(t)\|^2_{L_2(\Omega)} \implies y'(t) \le c\, y(t)\, \|u_1(t)\|^4_{L_4(\Omega)}$$

3. **Why the same does not work in 3D?**

$$\big|\big(u_1(t) \otimes u(t), \nabla u(t)\big)\big| \le \|u_1(t)\|_{L_4(\Omega)} \|u(t)\|^{\frac{1}{4}}_{L_2(\Omega)} \|\nabla u(t)\|^{\frac{7}{4}}_{L_2(\Omega)}, \quad n = 3$$

$$\underbrace{\tfrac{1}{2}\,\frac{d}{dt}\|u(t)\|^2_{L_2(\Omega)}}_{=:\ y'(t)} + \tfrac{\nu}{2}\,\|\nabla u(t)\|^2_{L_2(\Omega)} \le c\, \underbrace{\|u_1(t)\|^8_{L_4(\Omega)}}_{=:\ a(t)}\ \underbrace{\|u(t)\|^2_{L_2(\Omega)}}_{=:\ y(t)}$$

$$\int_0^T a(t)\,dt = \int_0^T \|u_1(t)\|^8_{L_4(\Omega)}\,dt \le \underbrace{\sup_{t\in(0,T)} \|u_1(t)\|^2_{L_2(\Omega)}}_{<+\infty}\ \underbrace{\int_0^T \|\nabla u_1(t)\|^6_{L_2(\Omega)}\,dt}_{=+\infty\,???} = +\infty$$

4. **Strong solutions and the "weak-strong" uniqueness in 3D**

$$\nabla u_1 \in L_\infty(0,T; L_2(\Omega)) \quad \text{is a strong solution in } Q_T$$

$$u_2 \text{ is a weak Leray-Hopf solution, } u_2|_{t=0} = u_1|_{t=0} \implies u_2 \equiv u_1 \text{ in } Q_T$$

5. **Ladyzhenskaya-Prodi-Serrin condition**

$$u \in L_{s,l}(Q_T) \equiv L_l(0,T; L_s(\Omega)), \quad \frac{3}{s} + \frac{2}{l} = 1 \qquad \text{(LPS)}$$

   Theorem. $a \in H^1$, u is a Leray-Hopf solution satisfying (LPS)

$$y(t) := \|\nabla u(t)\|^2_{L_2(\Omega)} \implies y(t) \le y(0)\, \exp\left(c\, \|u\|^l_{L_{s,l}(Q_T)}\right)$$

Proof.

$$\frac{d}{dt}\underbrace{\|\nabla u\|^2_{L_2(\Omega)}}_{=y(t)}+\|\tilde{\Delta} u\|^2_{L_2(\Omega)} \ \le\ c\,\|(u\cdot\nabla)u\|^2_{L_2(\Omega)} \ \le\ c\,\|u\|_{L_s(\Omega)}\quad \underbrace{\|\nabla u\|_{L_{\frac{2s}{s-2}}(\Omega)}}_{\|\nabla u\|^{1-\lambda}_{L_2(\Omega)}\|\tilde{\Delta} u\|^{\lambda}_{L_2(\Omega)}}$$

$$y'(t)\ \le\ c\,\|u(\cdot,t)\|^l_{L_s(\Omega)}\,y(t)$$

6. A gap between the classes of existence and uniqueness

we know: $\quad u\in L_\infty(0,T;L_2(\Omega)),\quad \nabla u\in L_2(Q_T)\quad\Longrightarrow\quad u\in L_{\frac{10}{3}}(Q_T)$

we want: $\quad u\in L_{s,l}(Q_T),\ \frac{3}{s}+\frac{2}{l}=1,\ s=l=5\quad\Longrightarrow\quad u\in L_5(Q_T)$

3 Lecture 3: Millennium problem. In search of blow up

3.1 Local-in-time well-posedness and the blow up time

1. **Existence of strong solutions**

   Theorem. $a \in H^1 := \{\, v, \nabla v \in L_2(\Omega),\ \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0 \,\}$

   - $\|a\|_{H^1} \le \varepsilon_0 \quad \Longrightarrow \quad \forall\, T > 0 \quad \exists$ a strong solution in $Q_T = \Omega \times (0, T)$
   - $T_a = C(\Omega)\, \|\nabla a\|_{L_2(\Omega)}^{-4} \quad \Longrightarrow \ \exists$ a strong solutions in $Q_{T_a} = \Omega \times (0, T_a)$

2. **Ingredients of the proof**

   - Cattabriga-Solonnikov inequality for the Stokes operator $\tilde{\Delta} = P_J \Delta$

   $$\|\nabla u\|_{L_2(\Omega)} + \|\nabla^2 u\|_{L_2(\Omega)} \ \le\ c\, \|\tilde{\Delta} u\|_{L_2(\Omega)}, \qquad \forall\, u \in H^1 \cap C^2(\bar{\Omega})$$

   - Testing the NSE by $\tilde{\Delta} u$

   $$\frac{d}{dt} \|\nabla u(t)\|^2_{L_2(\Omega)} + \|\nabla^2 u(t)\|^2_{L_2(\Omega)} \ \le\ c_\Omega\, \|\nabla u(t)\|^4_{L_2(\Omega)} \Big(1 + \|\nabla u(t)\|^2_{L_2(\Omega)}\Big)$$

   - Gronwall's inequality for $y(t) := \|\nabla u\|^2_{L_2(\Omega)}$

   $$y'(t) + \lambda_1\, y(t) \ \le\ c_\Omega\, y^2(t)\, \big(1 + y(t)\big) \qquad \Longrightarrow \qquad \left\{ \begin{array}{l} \operatorname{arctg} y(t) \ \le\ \gamma_a \ <\ \frac{\pi}{2} \\ \frac{1}{y^2(0)} - \frac{1}{y^2(t)} \ \le\ 2 c_\Omega\, t \end{array} \right.$$

3. **Blow up time**

   Assume $a \in H^1$, define $T_* \in (0, +\infty]$

   $$T_* := \sup \big\{\, T > 0 \ \big|\ \exists \text{ a strong solution } u \text{ of (NS) in } Q_T = \Omega \times (0, T) \,\big\}$$

   Theorem. $T_* < +\infty \ \Longrightarrow\ \|\nabla u(t)\|_{L_2(\Omega)} \ \ge\ \dfrac{C(\Omega)^{1/4}}{(T_* - t)^{1/4}}, \quad \forall\, t \in (0, T)$

4. **NSE Millennium Problem**

   In terms of blow up time $T_* = T_*(a)$:

   - either to prove that for any $a \in H^1$, $T_* = +\infty$
     i.e. $\forall\, T < +\infty\ \exists$ a strong solution u to (NS) in Q_T
   - or to prove that there exists $a \in H^1$ such that $T_* < +\infty$,
     i.e. $\|\nabla u(t)\|_{L_2(\Omega)} \to +\infty$ as $t \to T_* - 0$

3.2 Critical spaces and classification of blow up types

1. **Critical and supercritical models of mathematical physics**

$$\boxed{\text{Physical model}} \quad \begin{array}{l} \Longrightarrow \ \boxed{\text{Conservation laws}} \ \Longrightarrow \ \boxed{\text{A priori estimates}} \\[2mm] \Longrightarrow \ \boxed{\text{Group of invariance}} \ \Longrightarrow \ \boxed{\text{Scaling transformations}} \end{array}$$

What happens to the estimates when switching to small scales?

$$u(x) \quad \mapsto \quad u^R(y) := R^\alpha u(Ry), \qquad R \to 0$$

Models:

$$\begin{array}{ccccc} \text{supercritical} & \Leftrightarrow & \text{estimates "blow up"} & \Leftrightarrow & \|u^R\|_X \to +\infty \\ \text{critical} & \Leftrightarrow & \text{estimates are invariant} & \Leftrightarrow & \|u^R\|_X = \|u\|_X \\ \text{subcritical} & \Leftrightarrow & \text{provide "smallness condition"} & \Leftrightarrow & \|u^R\|_X \to 0 \end{array}$$

2. **Small scales from the point of view of harmonic analysis**

$$\Omega = \mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3, \quad u(x,t) = \sum_{k\in\mathbb{Z}^3} \vec{c}_k(t)e^{ik\cdot x}, \quad \vec{c}_k(t)\cdot k = 0$$

$$\frac{d\vec{c}_k}{dt}(t) + |k|^2\vec{c}_k(t) + \sum_{m\in\mathbb{Z}^3} \big(im\cdot\vec{c}_{k-m}(t)\big)\vec{c}_m(t) + ikp_k \ = \ 0 \qquad \text{(NS)}$$

$$\|u(t)\|^2_{L_2(\Omega)} = \sum_k |\vec{c}_k(t)|^2 \ \le \ const, \quad t\in(0,T_*)$$

$$\|\nabla u(t)\|^2_{L_2(\Omega)} = \sum_k |k|^2|\vec{c}_k(t)|^2 \ \to \ +\infty, \quad t\to T_*-0$$

$$E_{\le N}(t) := \sum_{|k|\le N} |\vec{c}_k(t)|^2, \qquad E_{\ge N}(t) := \sum_{|k|\ge N} |\vec{c}_k(t)|^2$$

$$E(t) = E_{\le N}(t) + E_{\ge N}(t) \ \le \ const, \quad E_{\le N}(t) - \downarrow \quad E_{\ge N}(t) - \uparrow$$

3. **Scale invariance of the Navier-Stokes equations**

$$u^R(x,t) := R\,u(Rx,R^2t), \qquad p^R(x,t) := R^2\,p(Rx,R^2t)$$

u and p is a solution to (NS) in Q_R $\Longleftrightarrow$ u^R and p^R is a solution to (NS) in Q

4. **3D NSE are supercritical**

Navier-Stokes equations: a priori estimate = energy estimate, $\|\cdot\|_X$ = energy norm

$$\|u\|_X \ = \ \sup_{t\in(-R^2,0)} \|u(\cdot,t)\|_{L_2(B_R)} + \|\nabla u\|_{L_2(Q_R)}$$

- $n = 2$ — critical model
- $n = 3$ — supercritical model

5. **Integrability vs control of behaviour of local integrals**

Denote $Q_T := \Omega \times (0,T)$, $z_0 = (x_0, t_0)$, $Q_R(z_0) := B_R(x_0) \times (t_0 - R^2, t_0)$

$$u \in L^{p,\lambda}(Q_T) \quad \Longleftrightarrow \quad \exists\, K > 0 \quad \int\limits_{Q_R(z_0)} |u|^p\, dx \;\le\; K\, R^\lambda, \qquad \forall\, z_0 \in Q_T,\ \forall\, R < 1$$

$$\Omega \subset \mathbb{R}^n \qquad L^{1,n+1}(Q_T) \;\supset\; L^{p,n+2-p}(Q_T) \;\supset\; L_{n+2}(Q_T)$$

6. **Examples of scale invariant functionals**

kinetic energy	$A(u,r) = \sup\limits_{t\in(-r^2,0)} \frac{1}{r^{1/2}}\, \\|u(\cdot,t)\\|_{L_2(B_r)}$	$u \in L_\infty(0,T; L^{2,1}(\Omega))$
viscous dissipation	$E(u,r) = \frac{1}{r^{1/2}}\, \\|\nabla u\\|_{L_2(Q_r)}$	$\nabla u \in L^{2,1}(Q_T)$
LPS norm	$M(u,r) = \\|u\\|_{L_5(Q_r)}$	$u \in L^5(Q_T)$
Morrey norm	$C(u,r) = \frac{1}{r^{2/3}}\, \\|u\\|_{L_3(Q_r)}$	$u \in L^{3,2}(Q_T)$
pressure	$D(p,r) = \frac{1}{r^{4/3}}\, \\|p\\|_{L_{\frac32}(Q_r)}$	$p \in L^{\frac32,2}(Q_T)$

All these functional are scale invariant:

$$F(u^R, 1) = F(u,R), \quad F(u,R) = \text{ any of } A(u,R),\ E(u,R),\ \dots$$

7. **Properties of scale invariant functionals**

if at least one of $F(u,r)$ is uniformly bounded $\sup\limits_{r<R} F(u,r) < +\infty$ $\Longrightarrow$ then all others are also uniformly bounded: $\sup\limits_{r<R} A(u,r) < +\infty$ etc

if at least one of $F(u,r)$ is uniformly small $\sup\limits_{r<R} F(u,r) < \varepsilon$ $\Longrightarrow$ then all others are also uniformly small: $\sup\limits_{r<R} A(u,r) < \varepsilon'$ etc

8. **ε–regularity theory**

$$\sup_{r<R} \frac{1}{r} \int\limits_{Q_r(z_0)} |\nabla u|^2\, dz < \varepsilon_* \quad \Longrightarrow \quad \exists\, r_* > 0 : \quad \sup_{Q_{r_*}(z_0)} |u| < +\infty$$

singularity at $z_0 = (x_0, t_0)$ $\longleftrightarrow$ local concentration of energy

9. **Caffarelli-Kohn-Nirenberg theorem**

$$\Sigma := \left\{ z_0 \in Q_T : \ u \notin L_\infty(Q_r(z_0)), \ \forall\, r > 0 \right\} \quad \Longrightarrow \quad \mathcal{P}^1(\Sigma) = 0$$

Singular points can not be too numerous
(less than a 1D space-curve)

10. **Classification of singular points**

Type I singularity at $z_0 = 0$	$\Longleftrightarrow$	all scale invariant quantities are finite: $\sup_{r<R} \big(A(u,r) + E(u,r) + \dots \big) < +\infty$
Type II singularity	$\Longleftrightarrow$	all other singularities

11. **Examples of Type I singularities**

$$|u(x,t)| \le \frac{M}{|x| + \sqrt{-t}}, \qquad u(x,t) = \frac{1}{\sqrt{-t}}\, U\left(\frac{x}{\sqrt{-t}}\right) \quad \text{etc}$$

3.3 Solutions with symmetries

1. **NSE in the vorticity form (reminder)**

$$\partial_t v - \Delta v + \operatorname{rot} v \times v + \nabla\Big(p + \frac{|v|^2}{2}\Big) \;=\; 0$$

$$\omega = \operatorname{rot} u, \qquad (u\cdot\nabla)u = \omega\times u + \tfrac{1}{2}\nabla|u|^2, \qquad \nabla\times(\omega\times u) = \text{BAC}-\text{CAB}$$

$$\partial_t\omega + (u\cdot\nabla)\omega - \nu\,\Delta\omega \;=\; (\omega\cdot\nabla)u \quad \text{in} \quad Q_T$$

2. **Hidden scalar specific of symmetric solutions**

- 2D case

$$u(x,t) = u_1(x_1,x_2,t)\,\mathbf{e}_1 + u_2(x_1,x_2,t)\,\mathbf{e}_2$$

$$\operatorname{rot} u = \omega(x_1,x_2,t)\,\mathbf{e}_3, \qquad \omega := u_{2,1} - u_{1,2}$$

$$u = u(x_1,x_2,t), \quad u_3 = 0 \quad\Longrightarrow\quad (\omega\cdot\nabla)u = 0$$

$$\boxed{\partial_t\omega - \Delta\omega + u\cdot\nabla\omega \;=\; 0} \quad \text{— scalar parabolic equation}$$

- Axially symmetric without swirl

$$u(x,t) = u_r(r,z,t)\,\mathbf{e}_r + u_z(r,z,t)\,\mathbf{e}_z$$

$$\operatorname{rot} u = \omega(r,z,t)\,\mathbf{e}_\varphi, \qquad \omega := u_{r,z} - u_{z,r}$$

$$w(r,z,t) \;=\; \frac{\omega(r,z,t)}{r}, \qquad r = |x'|, \qquad x' := (x_1,x_2,0)^T$$

$$\boxed{\partial_t w - \Delta w + \Big(u - \frac{x'}{r^2}\Big)\cdot\nabla w \;=\; 0} \quad \text{— scalar parabolic equation}$$

3. **Maximum principle for scalar parabolic equations**

$$\begin{cases} \partial_t w - \Delta w + b(x,t)\cdot\nabla w \;=\; 0 \quad \text{in} \quad \Pi_T := \mathbb{R}^n\times(0,T) \\ w|_{t=0} \;=\; w_0 \end{cases}$$

$$b\in L_2(\Pi_T), \qquad \operatorname{div} b \le 0 \quad\Longrightarrow\quad \|w\|_{L_\infty(\Pi_T)} \;\le\; \|w_0\|_{L_\infty(\mathbb{R}^n)}$$

4. **What new brings the hidden scalar specific in the theory?**

$$\left.\begin{array}{c} \text{Maximum principle} \;\approx\; \text{additional "conservation law"} \\ \text{provides an extra control of some subcritical norms} \end{array}\right\} \quad\Longrightarrow\quad \text{regularity of } u$$

5. **Backward self-similar solutions**

$$u(x,t) = \frac{1}{\sqrt{-t}}\, U\Big(\frac{x}{\sqrt{-t}}\Big), \quad u(0,t) \to \infty \quad t \to 0- \ ???$$

$$\left\{ \begin{array}{rcl} -\Delta U + \big(U + \frac{x}{2}\big)\cdot \nabla U + \frac{1}{2}\, U + \nabla P &=& 0 \\ \operatorname{div} U &=& 0 \end{array} \right. \quad \text{in} \quad \mathbb{R}^3$$

Note that the drift term $\frac{x}{2}\cdot\nabla U$ "shifts" the operator to the "spectral area"

$$\mathcal{B}[U,U] \ := \ \int\limits_{\mathbb{R}^3} \Big(\big(U + \tfrac{x}{2}\big)\cdot \nabla U + \tfrac{1}{2}\, U \Big)\cdot U \, dx \ = \ -\int\limits_{\mathbb{R}^3} |U|^2 \, dx$$

and hence the non-trivial U can not be exclude by the energy estimate. Nevertheless,

Theorem. (Necas, Ruzicka, Sverak, 1996, Tsai, 1998)

$$u \ \in \ \text{energy class in } Q = B\times(-1,0) \quad \Longrightarrow \quad U \ \equiv \ 0$$

The proof again is based on the hidden scalar specific for $\Pi = \frac{1}{2}\,|U|^2 + P + \frac{x}{2}\cdot U$

$$\boxed{-\Delta \Pi + \big(U + \tfrac{x}{2}\big)\cdot\nabla\Pi \ \le \ 0}$$

4 Lecture 4: Towards turbulence

4.1 Hydrodynamical similarity and zero viscosity limit

1. **The "dimensional" Navier-Stokes equations**

$$\begin{cases} \rho\big(\partial_t v + (v\cdot\nabla)v\big) - \mu\Delta v + \nabla q = 0 \\ \operatorname{div} v \ = \ 0 \end{cases}$$

   $[\mu] = \mathrm{N}\cdot\mathrm{c}/\mathrm{m}^2$ is the **dynamic viscosity**.

2. **The "dimensionless" Navier-Stokes equations**

   Let us introduce dimensionless parameters

$$u = \frac{v}{U}, \qquad y = \frac{x}{L}, \qquad \tau = \frac{t}{T}, \qquad T = \frac{L}{U}, \qquad p = \frac{q}{\rho U^2}$$

   Then the "dimensional" Navier-Stokes equations transfer to

$$\begin{cases} \partial_\tau u + (u\cdot\nabla)u - \frac{1}{Re}\Delta u + \nabla p = 0 \\ \operatorname{div} u \ = \ 0 \end{cases}$$

$$Re \ = \ \frac{UL}{\nu}, \qquad \nu = \frac{\mu}{\rho}$$

   $[\nu] = \mathrm{m}^2/\mathrm{c}$ is the **kinematic viscosity**.

3. **Reynolds number**

$$\boxed{Re \ = \ \frac{UL}{\nu}} \qquad Re \ \approx \ \frac{\|\,\rho(v\cdot\nabla)v\,\|}{\|\,\mu\Delta v\,\|}$$

4. **The value of Re in real processes**

   - bike: $Re \ \approx \ 10^6$
   - car: $Re \ \approx \ 10^7$
   - meteorology: $Re \ \approx \ 10^{12}$

5. **Turbulence**

   Picture

4.2 Basic facts about the Euler equations

1. **The Euler equations**

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = a, \quad \operatorname{div} a = 0 \end{cases} \quad \text{in} \quad \mathbb{R}^3 \times (0,T)$$

2. **The Euler equation in vorticity form in 3D**

$$\omega := \operatorname{rot} u, \qquad \omega_0 = \operatorname{rot} a$$

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u \qquad \text{in} \quad \mathbb{R}^3 \times (0,T) \\ \omega|_{t=0} = \omega_0 \end{cases}$$

3. **The Euler equation in vorticity form in 2D**

$$u = u(x_1, x_2, t), \quad u_3 = 0, \quad \omega := u_{2,1} - u_{1,2}, \quad \omega_0 = a_{2,1} - a_{1,2}$$

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = 0 \qquad \text{in} \quad \mathbb{R}^2 \times (0,T) \\ \omega|_{t=0} = \omega_0 \end{cases}$$

4. **The Hodge system**

$$\begin{cases} \operatorname{rot} u = \omega \\ \operatorname{div} u = 0 \end{cases} \quad \text{in} \quad \mathbb{R}^3$$

$$\operatorname{rot} \omega = \operatorname{rot} \operatorname{rot} u = -\Delta u + \nabla \underbrace{\operatorname{div} u}_{=0} \quad \Longrightarrow \quad u = (-\Delta)^{-1} \operatorname{rot} \omega$$

$$u(x,t) = \int_{\mathbb{R}^3} \tfrac{1}{|x-y|} \operatorname{rot} \omega(y,t)\, dy = -\int_{\mathbb{R}^3} \nabla_y \tfrac{1}{|x-y|} \times \omega(y,t)\, dy$$

5. **Bio-Savart law**

$$n = 3 \qquad u(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times \omega(y,t)\; dy,$$

$$n = 2 \qquad u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y,t)\; dy, \quad x^\perp := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

6. **Conservation of the kinetic energy**

Theorem. Assume $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is a deformation function, $\Omega_0 \subset \mathbb{R}^3$ is a smooth bounded domain and $\Omega_t = \psi_t(\Omega_0)$. Then for any smooth solution u to the Euler equations

$$\frac{d}{dt} \int\limits_{\Omega_t} |u(x,t)|^2 \, dx \;=\; 2 \int\limits_{\Omega_t} u \cdot \frac{Du}{Dt} \, dx$$

Proof. By the transport theorem we have

$$\frac{d}{dt} \int\limits_{\Omega_t} |u(x,t)|^2 \, dx \;=\; \int\limits_{\Omega_t} \Big(\underbrace{\partial_t |u|^2}_{2u_j \partial_t u_j} + \underbrace{(u \cdot \nabla)|u|^2}_{u_k (u_j u_j)_{,k}} \Big) \, dx \;=$$

$$= \; 2 \int\limits_{\Omega_t} u \cdot \underbrace{\big(\partial_t u + (u \cdot \nabla) u\big)}_{= \frac{Du}{Dt}} \, dx \;=\; -2 \int\limits_{\Omega_t} u \cdot \nabla p \, dx$$

Corollary. The total kinetic energy of the flow is conserved:

$$\frac{1}{2} \frac{d}{dt} \int\limits_{\mathbb{R}^3} |u(x,t)|^2 \, dx \;=\; -\int\limits_{\mathbb{R}^3} u \cdot \nabla p \, dx \;=\; \int\limits_{\mathbb{R}^3} \underbrace{\operatorname{div} u}_{=\,0} \, p \, dx \;=\; 0$$

(we assume that u and p decay properly as $|x| \to \infty$)

7. **Conservation of helicity**

Theorem. Assume $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is a deformation function, $\Omega_0 \subset \mathbb{R}^3$ is a smooth bounded domain and $\Omega_t = \psi_t(\Omega_0)$. Then for any smooth solution u to the Euler equations with $\omega := \operatorname{rot} u$

$$\frac{d}{dt} \int\limits_{\Omega_t} u(x,t) \cdot \omega(x,t) \, dx \;=\; 2 \int\limits_{\Omega_t} \Big(u \cdot \frac{D\omega}{Dt} + \omega \cdot \frac{Du}{Dt} \Big) \, dx$$

Corollary. The total helicity is conserved

$$\frac{1}{2} \frac{d}{dt} \int\limits_{\mathbb{R}^3} u(x,t) \cdot \omega(x,t) \, dx \;=\; \int\limits_{\mathbb{R}^3} \Big(u \cdot \frac{D\omega}{Dt} + \omega \cdot \frac{Du}{Dt} \Big) \, dx \;=$$

$$= \int\limits_{\mathbb{R}^3} \Big(\underbrace{u \cdot (\omega \cdot \nabla) u}_{=\; \omega \cdot \nabla \frac{1}{2}|u|^2} - \omega \cdot \nabla p \Big) \, dx \;=\; -\int\limits_{\mathbb{R}^3} \underbrace{\operatorname{div} \omega}_{\operatorname{div} \operatorname{rot} u = 0} \Big(\tfrac{|u|^2}{2} - p \Big) \, dx \;=\; 0$$

8. **Kelvin's circulation theorem**

Theorem. Assume $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is a deformation function, $\mathcal{C}_0$ is a smooth closed curve in $\mathbb{R}^3$ and

$$C_t = \psi_t(C_0)$$

Then for any smooth solution u to the Euler equations

$$\frac{d}{dt}\oint_{\mathcal{C}_t} u(x,t)\cdot\vec{dl} \;=\; 0$$

Proof. Assume $\mathcal{C}_0 = \{\, y(s)\,|\, s\in[0,1]\,\}$ is a regular parametrization of $\mathcal{C}_0$. Then $\mathcal{C}_t = \{\, \psi_t(y(s))\,|\, s\in[0,1]\,\}$ is a regular parametrization of $\mathcal{C}_t$ and

$$\oint_{\mathcal{C}_t} u(x,t)\cdot\vec{dl} \;=\; \int_0^1 u(\psi_t(y(s)),t)\cdot\frac{d}{ds}\psi_t(y(s))\,ds$$

$$\frac{d}{dt}\oint_{\mathcal{C}_t} u(x,t)\cdot\vec{dl} \;=\; \int_0^1 \Big(\underbrace{\frac{Du}{Dt}(\psi_t(y(s)),t)}_{=-\nabla p(\psi_t(y),t)}\cdot\frac{d}{ds}\psi_t(y(s)) + u(\psi_t(y(s)),t)\cdot\frac{d}{ds}u(\psi_t(y(s)),t)\Big)\,ds$$

$$\frac{d}{dt}\oint_{\mathcal{C}_t} u(x,t)\cdot\vec{dl} \;=\; \int_0^1 \frac{d}{ds}\Big(-p(\psi_t(y(s)),t)+\tfrac12\,|u(\psi_t(y(s)),t)|^2\Big)\,ds \;=\; 0$$

9. **Uniqueness in the class of smooth solutions**

$$u \;:=\; u_1-u_2 \qquad u(0)=0$$

$$\begin{cases} \partial_t u + (u_1\cdot\nabla)u_1 - (u_2\cdot\nabla)u_2 + \nabla p \;=\; 0 & \cdot\, u,\quad \displaystyle\int_{\mathbb{R}^n}\ldots\,dx \\ \operatorname{div} u \;=\; 0 \\ u|_{t=0} \;=\; 0 \end{cases}$$

$$\tfrac12\frac{d}{dt}\|u(t)\|^2_{L_2(\mathbb{R}^n)} \;=\; \big((u_1\cdot\nabla)u_1-(u_2\cdot\nabla)u_2,u\big) \;=$$
$$=\; \underbrace{\big((u_1\cdot\nabla)u,u\big)}_{(u_1,\nabla\frac12|u|^2)=0} \;+\; \big((u\cdot\nabla)u_2,u\big)$$

$$\frac{d}{dt}\|u(t)\|^2_{L_2(\mathbb{R}^n)} \;\le\; 2\,\|\nabla u_2(t)\|_{L_\infty(\mathbb{R}^n)}\,\|u(t)\|^2_{L_2(\mathbb{R}^n)}$$

Theorem. Assume $a\in C^1(\mathbb{R}^n)$, $\operatorname{div} a = 0$ and let u_1, u_2 be two smooth solutions to the Euler equations in $\mathbb{R}^n\times(0,T)$ corresponding to the same initial data a. Then $u_1=u_2$ in $\mathbb{R}^n\times(0,T)$.

10. **Well-posedness of the Euler equations**

Theorem. $a\in C_0^\infty(\mathbb{R}^n)$, $\operatorname{div} a = 0$

- $n=2 \implies \forall\, T>0\quad \exists$ a unique smooth solution in $\mathbb{R}^2\times(0,T)$
- $n=3 \implies \exists\, T_a>0$ s.t. $\exists$ a unique smooth solution in $\mathbb{R}^3\times(0,T_a)$

4.3 Further interesting topics

1. **Method of characteristics for the 2D Euler equations**

   The **trajectory** $X_y(t)$ of a particle y is a solution of ODEs

$$\begin{cases} \dfrac{dX_y}{dt}(t) = u(X_y(t), t) \\ X_y(0) = y \end{cases} \qquad \begin{array}{c} \Phi_t(y) := X_y(t) \\ \Phi_t : \mathbb{R}^3 \to \mathbb{R}^3 \ \text{ is one-to-one} \end{array}$$

   In 2D case vorticity is preserved along the trajectories:

$$\left. \begin{array}{c} \partial_t \omega + u \cdot \nabla \omega = 0 \\ \omega|_{t=0} = \omega_0 \end{array} \right\} \implies \omega(X_y(t), t) = \omega_0(y), \quad \forall t \in \mathbb{R}$$

   Hence at any $t > 0$ the vorticity is determined by the initial data:

$$\omega(x,t) = \omega_0(\Phi_t^{-1}(x)), \quad \forall\, x \in \mathbb{R}^2$$

2. **Zero viscosity limit**

   Assume $n = 2, 3$. Denote by u^ν a solution to the Navier-Stokes equations

$$\begin{cases} \partial_t u^\nu - \nu \Delta u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla p^\nu = 0 \\ \operatorname{div} u^\nu = 0 \\ u^\nu|_{t=0} = a, \quad a \in C_0^\infty(\mathbb{R}^n), \quad \operatorname{div} a = 0 \end{cases} \quad \text{in} \quad \mathbb{R}^n \times (0,T)$$

$$\int_{\mathbb{R}^n} |u^\nu(x,t)|^2 \, dx + 2\nu \int_0^t \int_{\mathbb{R}^n} |\nabla u^\nu|^2 \, dx d\tau = \int_{\mathbb{R}^n} |a|^2 \, dx, \quad \forall t > 0$$

   The kinetic energy is estimated uniformly with respect to $\nu > 0$

$$\|u^\nu\|^2_{L_\infty(0,T;L_2(\mathbb{R}^n))} := \sup_{t \in (0,T)} \|u^\nu(\cdot,t)\|^2_{L_2(\mathbb{R}^n)}$$

   Hence $\exists$ a subsequence $u^{\nu_k} =: u^\nu$ and $\bar{u} \in L_\infty(0,T; L_2(\mathbb{R}^n))$ such that

$$u^\nu \overset{*}{\rightharpoonup} \bar{u} \quad \text{in} \quad L_\infty(0,T; L_2(\mathbb{R}^n))$$

   Is it true that $\bar{u}$ is a solution to the Euler equations?

3. **Weak solutions to the Euler equations**

   We say u is a **weak solution** to the Euler equations in $\mathbb{R}^n \times (0,T)$ if

   1) $u \in L_\infty(0,T; L_2(\mathbb{R}^n))$

2) $\operatorname{div} u = 0$ in the sense of distributions

3) $\forall\, \eta \in C_0^\infty(\mathbb{R}^n \times [0,T))$ such that $\operatorname{div} \eta = 0$

$$-\int\limits_{Q_T} \Big(u \cdot \partial_t \eta + u \otimes u : \nabla \eta \Big)\, dxdt \;=\; \int\limits_{\Omega} a(x) \cdot \eta(x,0)\, dx$$

4. **Non-uniqueness of weak solutions**

Theorem. There exists $u \in L_\infty(0,T; L_2(\mathbb{R}^n))$ such that u is compactly supported in $\mathbb{R}^n \times (0,T)$, $u \not\equiv 0$, and u is weak solution of the Euler equations corresponding to the initial given $a \equiv 0$.

Scheffer 1993, Shnirelman 1997, De Lellis & Szekelyhidi 2010 – ...

5. **Normal and anomalous energy dissipation**

Let u^ν be Leray–Hopf solutions to the Navier-Stokes equations. We say that the energy dissipation is **normal** if

$$\limsup_{\nu \to 0} \Big(\nu \int\limits_0^T \int\limits_{\mathbb{R}^n} |\nabla u^\nu|^2\, dxdt \Big) \;=\; 0$$

Otherwise, we will say that there is an **anomalous energy dissipation**.

6. **Conservative and dissipative solutions**

A weak solution u of the Euler equations is called **conservative** if it satisfies the energy equation balance, and **dissipative** if its kinetic energy is not increasing.

$$\begin{array}{lcl} \|u(t)\|_{L_2(\mathbb{R}^n)} \;=\; \|a\|_{L_2(\mathbb{R}^n)} & — & \textbf{conservative} \\ \|u(t)\|_{L_2(\mathbb{R}^n)} \;\le\; \|a\|_{L_2(\mathbb{R}^n)} & — & \textbf{dissipative} \end{array}$$

Theorem. Leray–Hopf solutions on the interval $(0,T)$ converge in $L_\infty(0,T; L_2(\mathbb{R}^n))$ weakly$-*$ to a conservative solution of the Euler equation if and only if the energy dissipation is normal.

7. **Onsager's conjecture (1949)**

Motivation:

$$0 \;=\; \int\limits_{\mathbb{R}^n} (u \cdot \nabla) u \cdot u\, dx \;\approx\; \int\limits_{\mathbb{R}^n} (\nabla^{\frac13} u \cdot \nabla^{\frac13} u \cdot \nabla^{\frac13} u)\, dx \;\approx\; \int\limits_{\mathbb{R}^n} |\nabla^{\frac13} u|^3\, dx$$

Onsager's conjecture:

$$u \in C([0,T]; C^{1/3}(\mathbb{R}^n)) \quad \Longrightarrow \quad \|u(t)\|_{L_2(\mathbb{R}^n)} = \|a\|_{L_2(\mathbb{R}^n)}, \quad \forall\, t \in [0,T]$$

8. **Modeling turbulent flows**

   There are some physical evidence that energy dissipation in a turbulent regime could be anomalous. Therefore, mathematically such regimes should be modeled using non-smooth dissipative solutions of the Euler equations. Since there is no uniqueness in the class of dissipative solutions, an additional principle of choice is needed among all dissipative solutions of the physically meaningful one.

9. **Boundary conditions for NSE and the Euler equations**

   Assume Ω is a domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and the external unit normal n. Then the boundary conditions we impose in the cases of the Navier-Stokes and the Euler equations are different:

   - $\nu > 0 \implies u|_{\partial\Omega} = 0$ — **non-slip** boundary conditions
   - $\nu = 0 \implies u \cdot n|_{\partial\Omega} = 0$ — **non-penetration** boundary conditions

   So, in the presence of a boundary one can not expect a uniform convergence of u^ν to u

   $$\|u^\nu - u\|_{L_\infty(Q_T)} \not\to 0 \quad \text{as} \quad \nu \to +\infty$$

10. **Boundary layer theory**

    In fluid mechanics it is assumed that in a thin layer near the boundary the velocity field satisfies some specific equations called the equations of boundary layer.