Higher Integrability Technique for Elliptic Equations

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Abstract

This mini course is (in a sense) a complement/continuation of my mini-course "The De Giorgi and Moser Techniques for Elliptic Equations".

In this lectures we develop a technique which allows to prove the *higher integrability* for gradients of weak solutions to elliptic equations with non-smooth coefficients. By "higher integrability" we mean the following property: the gradient of weak solutions is actually integrable with some bigger power than the exponent of integrability in the definition of the energy space.

In the first part of the course we consider the simplest uniformly elliptic equation in the divergence form with a bounded measurable coefficients. First, we develop "global" methods which allows us to prove so called Meyers-type estimates for weak solutions to the Dirichlet problem. Then we develop a local technique which is based on the reverse Hölder inequality and the results of Gehring and Giaquinta-Modica.

In the second part of the course we consider the Dirichlet problem for elliptic equations with a singular drift term. This problem is motivated by some questions arising in the theory of axially symmetric solutions to the Navier-Stokes equation. We assume that the drift belongs to some scale invariant weak Morrey space which includes in the 3D case, in particular, drifts having a singularity along the axis x_3 with the asymptotic $b(x) \sim c/r$, where $r = \sqrt{x_1^2 + x_2^2}$. The singularity of the drift is so strong that weak solutions from the standard energy space are not well defined. Nevertheless, one can consider so called p-weak solutions, i.e. weak solutions which possess higher integrability property.

For the described problem we investigate the issues of boundedness of a bilinear form corresponding to the drift term. Using the tools of harmonic analysis we present a proof of the Fefferman–Chiarenza–Frasca estimate for the bilinear form with a weight from a Morrey space. This estimate includes many other well-known results (such as the Hardy inequality and the Hardy–Littlewood-Sobolev inequality) as a particular case. Then we apply the developed theory to the investigation of existence of weak solutions to elliptic equations with a singular drift. Under the assumption that the divergence of the drift is sign-defined we prove existence and uniqueness of p-weak solutions to the Dirichlet problem.

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