Order of a homotopy invariant in the stable case

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Abstract

Let X and Y be CW-complexes, U be an abelian group, and $f\colon [X,Y]\to U$ be a map (a homotopy invariant). We say that f has *order* at most r if the characteristic function of the rth Cartesian power of the graph of a continuous map $a\colon X\to Y$ $\mathbf{Z}$-linearly determines $f([a])$. Suppose that the CW-complex X is finite and we are in the stable case: $\dim X < 2n-1$ and Y is $(n-1)$-connected. We prove that then the order of f equals its degree with respect to the Curtis filtration of the group $[X,Y]$.

1. Introduction

Order of a homotopy invariant. Let X and Y be (topological) spaces. For $r\in\mathbf{N}$ $(=\{0,1,\dots\})$, let E_r be the group of all functions $(X\times Y)^r\to\mathbf{Z}$. For a map $a\in C(X,Y)$, let $\Gamma_a\subset X\times Y$ be its graph and $I_r(a)\in E_r$ be the characteristic function of the set $\Gamma_a^r\subset(X\times Y)^r$. Let $D_r\subset E_r$ be the subgroup generated by the functions $I_r(a)$, $a\in C(X,Y)$.

Let U be an abelian group and $f\colon [X,Y]\to U$ be a map. Define the *order* of f, $\operatorname{ord} f\in\hat{\mathbf{N}}$ $(=\mathbf{N}\cup\{\infty\})$, to be the infimum of those $r\in\mathbf{N}$ for which there exists a homomorphism $l\colon D_r\to U$ such that $f([a])=l(I_r(a))$ for all $a\in C(X,Y)$. As one easily sees, the existence of such l for some r implies that for all greater r.

Main result. Suppose that X is a finite CW-complex, Y is a CW-complex, and we are in the *stable case*: $\dim X\le m$, Y is $(n-1)$-connected, and $m<2n-1$. The set $[X,Y]$ becomes an abelian group canonically. There is the Curtis filtration $B=(B_s)_{s=1}^{\infty}$, $[X,Y]=B_1\supset B_2\supset\dots$, see § 3. It is known [1] that $B_s=0$ for $s>2^{m-n}$. The degree of f with respect to B, $\deg_B f\in\hat{\mathbf{N}}$, is defined, see below.

(1.1) Theorem. $\operatorname{ord} f=\deg_B f$.

Example: if f is a homomorphism, its order equals the greatest s for which $f|B_s\ne 0$. (If $f=0$, then $\operatorname{ord} f=0$).

Degree of a map between abelian groups with respect to a filtration. Let T and U be abelian groups, $f\colon T\to U$ be a map, and $P=(P_s)_{s=1}^{\infty}$ be a filtration of the group T: $T=P_1\supset P_2\supset\dots$. Define the *degree* of f with respect to P, $\deg_P f\in\hat{\mathbf{N}}$, to be the infimum of those $r\in\mathbf{N}$ for which

$$\sum_{e_1,\dots,e_k=0,1}(-1)^{e_1+\dots+e_k}f(e_1t_1+\dots+e_kt_k)=0$$

whenever $k\in\mathbf{N}$, $t_l\in P_{s_l}$, $l=1,\dots,k$, and $s_1+\dots+s_k>r$.

2. Preliminaries

Polyhedra. A *polyhedron* L is a finite set of affine simplices in $\mathbf{R}^\infty$ satisfying the "axioms of a simplicial complex" and equipped with a linear order of the vertices of each simplex in such a way that the order of the vertices of a simplex induces the order of the vertices of each of its faces. The *body* $|L|$ of L is the union of its simplices. A *polyhedral body* is the body of some polyhedron.

Morphisms of polyhedra. For polyhedra K and L, a map $f\colon K \to L$ is called a *morphism* if a vertex is sent to a vertex, the image of a simplex is spanned by the images of its vertices, and the non-strict order of vertices is preserved. A morphism $f\colon K \to L$ induces a continuous map $|f|\colon |K| \to |L|$.

Generation. A simplex $y \in L$ generates a subpolyhedron $\bar{\bar{y}} \subset L$. A set $T \subset L$ generates a subpolyhedron $\bar{T} \subset L$.

Small sets. A set $T \subset L$ is *small* if there exists a simplex $y \in L$ with $\bar{\bar{y}} \supset T$; the least of such simplices is *spanned* by T.

The distance ρ_L. For $x, y \in L$, let $\rho_L(x, y) \in \hat{\mathbf{N}}$ be the infimum of lengths of edge chains connecting x and y. (The orientation of edges is disregarded; the length of a chain is the number of its edges.) If $\rho_L(x, y) < a$, $\rho_L(y, z) < b$ $(x, y, z \in L$, $a, b \in \mathbf{N})$, then $\rho_L(x, z) < a + b$.

Neighbourhoods O_L. For $y \in L$ and $d \in \mathbf{N}$, put $O_L(y, d) = \{z \in L : \rho_L(y, z) < d\}$. For $T \subset L$, let $O_L(T, d)$ be the union of the sets $O_L(y, d)$, $y \in T$.

Separation ϵ_L. For $T \subset L$, put $\epsilon_L(T) = \inf\{\rho_L(x, y) : x, y \in T, x \neq y\} \in \hat{\mathbf{N}}$.

Subdivisions. Equip the barycentric subdivision of L with the following order: the greater dimension of a simplex is, the higher its barycentre is. Let δL denote the resulting polyhedron. Let $\phi_L\colon \delta L \to L$ be the morphism taking the barycentre of a simplex to the highest of its vertices. Equip the barycentric subdivision of L with the opposite order. Let $\delta' L$ denote the resulting polyhedron. Let $\phi'_L\colon \delta' L \to L$ be the morphism taking the barycentre of a simplex to the lowest of its vertices. Put $\Delta L = \delta' \delta L$ and $\Phi_L = \phi_L \circ \phi'_{\delta L}\colon \Delta L \to L$. The map $|\Phi_L|\colon |L| = |\Delta L| \to |L|$ is homotopic to the identity. The image of the star of each simplex of ΔL under Φ_L is small. Thus, if $\rho_{\Delta L}(x, y) \le 2d$ $(x, y \in \Delta L$, $d \in \mathbf{N})$, then $\rho_L(\Phi_L(x), \Phi_L(y)) \le d$.

The empty simplex. Put $L^\circ = L \cup \{\varnothing\}$. Let the empty simplex generate the empty subpolyhedron: $\bar{\bar{\varnothing}} = \varnothing$. For $x, y \in L^\circ$, we have $x \cap y \in L^\circ$.

Completion. Adding degenerate simplices to L, we get a simplicial set $\hat{L}$. We have $L \subset \hat{L}_0 \cup \hat{L}_1 \cup \ldots$. The spaces $|L|$ and $|\hat{L}|$ are canonically homeomorphic. A moprhism $f\colon K \to L$ of polyhedra induces a simplicial map $\hat{f}\colon \hat{K} \to \hat{L}$. The correspondence $f \mapsto \hat{f}$ is bijective.

Sections. For a simplicial set E, let $E(L)$ be the set of simplicial maps $v\colon \hat{L} \to E$, *sections.* A section $v \in E(L)$ induces a map $|v| \in C(|L|, |E|)$. For a subpolyhedron $K \subset L$, we have the restriction $v|_K \in E(K)$. For a morphism $f\colon K \to L$ of polyhedra, we have the composition $v \circ f \in E(K)$. A simplicial map $t\colon D \to E$ induces a map $t_\#\colon D(L) \to E(L)$. For a simplicial group G and a section $v \in G(L)$, put $\sigma(v) = \{y \in L : v|_{\bar{\bar{y}}} \neq 1\}$.

Quasisections. For a set $T \subset L$ and a simplicial set E, put

$$E_T = \prod_{y \in T} E(\bar{\bar{y}}).$$

For $v \in E(L)$, put $v\|_T = (v|_{\bar{\bar{y}}})_{y \in T} \in E_T$. For a *quasisection* $w \in E_L$ and a morphism $f\colon K \to L$ of polyhedra, define the composition $w \circ f \in E_K$ by $(w \circ f)_x = w_{f(x)} \circ f'_x$, $x \in K$, where $f'_x\colon \bar{\bar{x}} \to \overline{\overline{f(x)}}$ are the restrictions of f. We have the map $f^\#\colon E_L \to E_K$, $f^\#(w) = w \circ f$. For a simplicial map $t\colon D \to E$ and a quasisection $v \in D_L$, we have the composition $t \circ v \in E_L$.

Free groups. For a set E with a marked element $*$, we have the group FE given by the generators $\underline{e}$, $e \in E$, and the relation $\underline{*} = 1$. The map $i\colon E \to FE$, $i(e) = \underline{e}$, is called *canonical*.

The lower central series and the abelianization. For a group G, let $(\gamma_s G)_{s=1}^\infty$ be its lower central series. Put $G^+ = G/\gamma_2 G$.

Free abelian groups. For a set E, we have the abelian group $\langle E \rangle$ with the base $(`e\text{'})_{e \in E}$. The map $j\colon E \to \langle E \rangle$, $j(e) = `e\text{'}$, is called *canonical*. Let $\langle E \rangle_\vartriangle$ be the kernel of the homomorphism $\langle E \rangle \to \mathbf{Z}$, $`e\text{'} \mapsto 1$. A map $t\colon D \to E$ induces a homomorphism $\langle t \rangle\colon \langle D \rangle \to \langle E \rangle$.

Let L be a polyhedron, E be a simplicial set, and $V \in \langle E(L) \rangle$ be an element (an *ensemble*). Let $|V| \in \langle C(|L|, |E|) \rangle$ denote the image of V under the homomorphism induced by the map $|?|\colon E(L) \to C(|L|, |E|)$. For a subpolyhedron $K \subset L$, the ensemble $V|_K \in \langle E(K) \rangle$ is defined similarly; for a set $T \subset L$, we have the element $V\|_T \in \langle E_T \rangle$. For spaces X and Y and an ensemble $A \in \langle C(X, Y) \rangle$, we have the element $[A] \in \langle [X, Y] \rangle$. For a set $Z \subset X$, we have the ensemble $A|_Z \in \langle C(Z, Y) \rangle$.

For a simplicial group G and an ensemble $V \in \langle G(L) \rangle$,

$$V = \sum_{v \in G(L)} m_v `v\text{'}$$

($m_v \in \mathbf{Z}$), put

$$\Sigma(V) = \bigcup_{v \in G(L)\,:\, m_v \neq 0} \sigma(v).$$

Group rings. For a group G, $\langle G \rangle$ is the group ring, $\langle G \rangle_\vartriangle$ is its (two-sided) ideal. For $s \in \mathbf{N}_+$ $(= \mathbf{N} \setminus \{0\})$, the ideal $\langle G \rangle_\vartriangle^s$ is additively generated by all elements of the form $(`g_1\text{'} - 1) \ldots (`g_s\text{'} - 1)$, $g_1, \ldots, g_s \in G$.

Simplicial application. Natural constructions can be applied to simplicial objects dimension-wise. For a pointed simplicial set E, we have the simplicial group FE and the canonical simplicial map $i\colon E \to FE$. The map i is a model of the canonical map of a pointed space to the loop space of its suspension (*Milnor's model*, see [2]). For a simplicial group G, we have the simplicial abelian group G^+, the simplicial ring $\langle G \rangle$, the canonical simplicial map $j\colon G \to \langle G \rangle$, and the simplicial subgroups $\gamma_s G \subset G$, $s \in \mathbf{N}_+$, and $\langle G \rangle_\vartriangle^s \subset \langle G \rangle$, $s \in \mathbf{N}$.

Simplicial trifles. A simplicial map between pointed simplicial sets is called *bound* if it preserves the pointing. A simplicial abelian group D is called *free* if the abelian groups D_n, $n \in \mathbf{N}$, are free. For a simplicial set E, let $E_{(m)} \subset E$ ($m \in \mathbf{N}$) denote its m-skeleton.

Fusion. Let L be a polyhedron and G be a simplicial group. Let $j\colon G \to \langle G \rangle$ be the canonical map. The ring homomorphism $J\colon \langle G(L) \rangle \to \langle G \rangle(L)$, $J(`v') = j \circ v$, is called *fusion.*

3. The Curtis filtration in the stable case

Let X and Y be CW-complexes. Suppose that $\dim X \leq m$, Y is $(n-1)$-connected, and $m < 2n-1$. We shall construct a filtration $B = (B_s)_{s=1}^{\infty}$ of the abelian group $[X, Y]$, $[X, Y] = B_1 \supset B_2 \supset \dots$, the *Curtis filtration.* There are a simplicial set E and a homotopy equivalence $k\colon Y \to |E|$. Let us point E. We have the simplicial group $G = FE$. By the Freudenthal theorem, the canonical simplicial map $i\colon E \to G$ is $(2n-1)$-connected. The map $h = |i| \circ k\colon Y \to |G|$ is also $(2n-1)$-connected. Let $j_s\colon \gamma_s G \to G$, $s \in \mathbf{N}_+$, be the inclusions. For $s \in \mathbf{N}_+$, we have the chain of groups and homomorphisms

$$[X, Y] \xrightarrow{h_*} [X, |G|] \xleftarrow{|j_s|_*} [X, |\gamma_s G|].$$

Since $m < 2n-1$, h_* is an isomorphism. Put $B_s = h_*^{-1}(\operatorname{im} |j_s|_*)$. (The result does not depend on the choice of E etc.)

4. A claim on Lie rings

Here U denotes the universal enveloping ring functor.

(4.1) *Let L and M be Lie rings, free as abelian groups, and $k\colon L \to M$ be an injective homomorphism. Then the homomorphism $Uk\colon UL \to UM$ is injective.*

This follows easily from the Poincaré–Birkhoff–Witt theorem. □

5. A claim on group rings

Let V and W be groups and $t\colon V \to W$ be a homomorphism. We have the ring homomorphism $\langle t \rangle\colon \langle V \rangle \to \langle W \rangle$. For $s \in \mathbf{N}$, let $I_s \subset \langle V \rangle$ be the subgroup generated by all elements of the form $(`v_1' - 1) \dots (`v_k' - 1)$, where $k \in \mathbf{N}$, $v_l \in t^{-1}(\gamma_{s_l} W)$, and $s_1 + \dots + s_k \geq s$. It is easy to see that I_s are ideals, $I_s \supset I_{s+1}$, and $I_s I_t \subset I_{s+t}$.

(5.1) *Suppose that W is a product of a finite number of free groups. Then $\langle t \rangle^{-1}(\langle W \rangle_{\Delta}^{s}) = I_s$, $s \in \mathbf{N}$.*

Proof. If $w \in \gamma_s W$, then $`w' - 1 \in \langle W \rangle_{\Delta}^{s}$ (this holds for arbitrary W [3, III.1.3]). This yields the inclusion $\langle t \rangle^{-1}(\langle W \rangle_{\Delta}^{s}) \supset I_s$.

We have the graded rings P, $P_s = I_s/I_{s+1}$, and Q, $Q_s = \langle W\rangle^s_\Delta/\langle W\rangle^{s+1}_\Delta$. Since $\langle t\rangle(I_s) \subset \langle W\rangle^s_\Delta$, the homomorphism $\langle t\rangle$ induces a graded ring homomorphism $l\colon P \to Q$. We shall show that l is injective. Then induction on s with application of the 5-lemma shows that the induced homomorphism $\langle V\rangle/I_s \to \langle W\rangle/\langle W\rangle^s_\Delta$ is injective, which is the desired equality.

We have the graded Lie rings L, $L_s = t^{-1}(\gamma_s W)/t^{-1}(\gamma_{s+1}W)$, and M, $M_s = \gamma_s W/\gamma_{s+1}W$ (the product is induced by the group commutator, see [3, VIII.2]). The homomorphism t induces a graded Lie ring homomorphism $k\colon L \to M$, which is obviously injective.

We have the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{k} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ P & \xrightarrow{l} & Q, \end{array}$$

where f and g are the representations with the components $f_s\colon L_s \to P_s$, $f_s(v) =$ 'v' $- 1$, $v \in t^{-1}(\gamma_s W)$, and $g_s\colon M_s \to Q_s$, $g_s(w) =$ 'w' $- 1$, $w \in \gamma_s W$. Extending the representations f and g to homomorphisms of the universal enveloping rings, we get the commutative diagram

$$\begin{array}{ccc} UL & \xrightarrow{Uk} & UM \\ {\scriptstyle \tilde f}\downarrow & & \downarrow{\scriptstyle \tilde g} \\ P & \xrightarrow{l} & Q. \end{array}$$

By Magnus' method, one easily shows that $\tilde g$ is an isomorphism, and M is free as an abelian group (cf. [3, VIII.6]). By (4.1), the homomorphism Uk is injective. The ring P is generated by elements of the form 'v' $- 1 \in P_s$, where $s \in \mathbf{N}_+$, $v \in t^{-1}(\gamma_s W)$. They belong to the image of the representation f and, consequently, of the homomorphism $\tilde f$, which is thus surjective. Therefore, the homomorphism l is injective (and $\tilde f$ is an isomorphism.) □

6. Some ideals of the group ring of a product of groups

Let $(G_i)_{i\in I}$ be a finite collection of groups. For $J \subset I$, put

$$G_J = \prod_{i\in J} G_i,$$

and let $p_J\colon G_I \to G_J$ be the projection homomorphism. We have the ring homomorphisms $\langle p_J\rangle\colon \langle G_I\rangle \to \langle G_J\rangle$.

(6.1) *For $s \in \mathbf{N}$, we have*

$$\bigcap_{\#J<s} \ker\langle p_J\rangle \subset \langle G_I\rangle^s_\Delta.$$

Proof. We have

$$\langle G_I \rangle = \bigotimes_{i\in I} \langle G_i \rangle.$$

Since $\langle G_i \rangle = \langle G_i \rangle_\Delta \oplus \langle 1 \rangle$,

$$\langle G_I \rangle = \bigoplus_{J\subset I} S(J), \qquad S(J) = \bigotimes_{i\in I} T_i(J),$$

where the subgroup $T_i(J) \subset \langle G_i \rangle$ is: $\langle G_i \rangle_\Delta$ if $i \in J$, and $\langle 1 \rangle$ otherwise. Obviously, $\langle p_J \rangle | S(J')$ is: a monomorphism if $J' \subset J$, and zero otherwise. Therefore,

$$\bigcap_{\#J<s} \ker\langle p_J \rangle = \bigoplus_{\#J\geq s} S(J).$$

Now it suffices to note that $S(J) \subset \langle G_I \rangle_\Delta^{\#J}$. □

7. The functions η and θ

Let L be a polyhedron and G be a simplicial group. We have the homomorphism $?\|_L \colon G(L) \to G_L$. For $V \in \langle G(L) \rangle$, put $\eta(V) = \sup\{s \in \mathbf{N} : V\|_L \in \langle G_L \rangle_\Delta^s\} \in \hat{\mathbf{N}}$. For $s \in \mathbf{N}$, we have the subgroup $I_s \subset \langle G(L) \rangle$ generated by all elements of the form $(‘v_1’ - 1) \ldots (‘v_k’ - 1)$, where $k \in \mathbf{N}$, $v_l \in (\gamma_{s_l} G)(L) \subset G(L)$, and $s_1 + \ldots + s_k \geq s$. (It is an ideal.)

(7.1) *Suppose that the groups G_n, $n \in \mathbf{N}$, are free. Then $\{V \in \langle G(L) \rangle : \eta(V) \geq s\} = I_s$, $s \in \mathbf{N}$.*

This follows from (5.1). □

For a simplicial set E and an ensemble $V \in \langle E(L) \rangle$, put $\theta(V) = \inf\{\#T : T \subset L, V\|_T \neq 0\} \in \hat{\mathbf{N}}$.

(7.2) *For $V \in \langle G(L) \rangle$, we have $\theta(V) \leq \eta(V)$.*

This follows from (6.1). □

8. Product of affine functions

(8.1) *Let V be a group, H be a ring, and $a_1, \ldots, a_r \colon V \to H$ be homomorphisms (to the additive group; $r \in \mathbf{N}$). We have the additive homomorphism $Q\colon \langle V \rangle \to H$,*

$$Q(‘v’) = \prod_{s=1}^{r} (1 + a_s(v)).$$

Then $Q|\langle V \rangle_\Delta^{r+1} = 0$.

This follows from [3, V.2.1]. □

9. Strict and r-strict homomorphisms

Let V and W be groups. An additive homomorphism $h\colon \langle V\rangle \to \langle W\rangle$ is called *strict* if $h(\langle V\rangle_\vartriangle^s) \subset \langle W\rangle_\vartriangle^s$ for all $s \in \mathbf{N}$ and *r-strict* ($r \in \mathbf{N}$) if this holds for $s \le r$.

(9.1) *Let $t\colon V \to W$ be a homomorphism. Then the homomorphism $\langle t\rangle\colon \langle V\rangle \to \langle W\rangle$ is strict.* $\square$

(9.2) *Let $f, g\colon \langle V\rangle \to \langle W\rangle$ be r-strict ($r \in \mathbf{N}$) homomorphisms. Then the homomorphism $h\colon \langle V\rangle \to \langle W\rangle$, $h(`v') = f(`v')g(`v')$, is r-strict.*

Proof. Take $s \in \mathbf{N}_+$, $s \le r$, and $v_1, \ldots, v_s \in V$. Put $x_t = `v_t' - 1 \in \langle V\rangle_\vartriangle$. Let us show that $h(x_1 \ldots x_s) \in \langle W\rangle_\vartriangle^s$. We have

$$
\begin{aligned}
(-1)^s h(x_1 \ldots x_s) &= \sum_{e_1,\ldots,e_s=0,1} (-1)^{e_1+\ldots+e_s} h(`v_1^{e_1} \ldots v_s^{e_s}') = \\
&= \sum_{e_1,\ldots,e_s=0,1} (-1)^{e_1+\ldots+e_s} f(`v_1^{e_1} \ldots v_s^{e_s}') g(`v_1^{e_1} \ldots v_s^{e_s}') = \\
&= \sum_{e_1,\ldots,e_s=0,1} (-1)^{e_1+\ldots+e_s} f\Big(\prod_{t=1}^{s}(1+e_t x_t)\Big) g\Big(\prod_{t=1}^{s}(1+e_t x_t)\Big) = \\
&= \sum_{e_1,\ldots,e_s=0,1} (-1)^{e_1+\ldots+e_s} \Big(\sum_{a_1,\ldots,a_s=0,1} e_1^{a_1} \ldots e_s^{a_s} f(x_1^{a_1} \ldots x_s^{a_s})\Big)\cdot \\
&\qquad \cdot \Big(\sum_{b_1,\ldots,b_s=0,1} e_1^{b_1} \ldots e_s^{b_s} g(x_1^{b_1} \ldots x_s^{b_s})\Big) = \\
&= \sum_{a_1,b_1,\ldots,a_s,b_s=0,1} \Big(\sum_{e_1,\ldots,e_s=0,1} (-1)^{e_1+\ldots+e_s} e_1^{a_1+b_1} \ldots e_s^{a_s+b_s}\Big) f(x_1^{a_1} \ldots x_s^{a_s})\cdot \\
&\qquad \cdot g(x_1^{b_1} \ldots x_s^{b_s}).
\end{aligned}
$$

Fix $a_1, b_1, \ldots, a_s, b_s$. We show that the corresponding summand of the outer sum belongs to $\langle W\rangle_\vartriangle^s$. Put $a = a_1 + \ldots + a_s$, $b = b_1 + \ldots + b_s$. Since $a, b \le s \le r$ and the homomorphisms f and g are r-strict, we have

$$f(x_1^{a_1} \ldots x_s^{a_s}) g(x_1^{b_1} \ldots x_s^{b_s}) \in \langle W\rangle_\vartriangle^{a+b}.$$

If $a + b \ge s$, this suffices. Otherwise, there is t such that $a_t = b_t = 0$. Then the quantity $e_1^{a_1+b_1} \ldots e_s^{a_s+b_s}$ does not depend on e_t, and thus the inner sum equals zero. $\square$

10. Group ring of a free group

Let E be a pointed set. Put $G = FE$. Let $i\colon E \to G$ be the canonical map. For $s \in \mathbf{N}$, we have the ponted set $E^{\wedge s} = E \wedge \ldots \wedge E$ ($E^{\wedge 0}$ is the 0-sphere) and the homomorphism $k_s\colon \langle E^{\wedge s}\rangle_\vartriangle \to \langle G\rangle_\vartriangle^s$,

$$k_s(`(e_1, \ldots, e_s)' - `*') = \prod_{t=1}^{s} (`\underline{e_t}' - 1),$$

where $* \in E^{\wedge s}$ is the marked element. By [3, VIII.6.2], the composition

$$\langle E^{\wedge s}\rangle_\vartriangle \xrightarrow{k_s} \langle G\rangle^s_\vartriangle \xrightarrow{\text{projection}} \langle G\rangle^s_\vartriangle / \langle G\rangle^{s+1}_\vartriangle$$

is an isomorphism. Therefore, $\langle G\rangle^s_\vartriangle = D^s \oplus \langle G\rangle^{s+1}_\vartriangle$, where $D^s \cong \langle E^{\wedge s}\rangle_\vartriangle$.

11. Lift of a simplicial homomorphism

(11.1) *Consider the diagram*

$$\begin{array}{ccc} & & Q \\ & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{s} & P \end{array}$$

of simplicial abelian groups and homomorphisms. Suppose that D is free and m-connected ($m \in \mathbf{N}$) and f is surjective. Then there exists a simplicial homomorphism $t\colon D \to Q$ such that $f \circ t|D_{(m)} = s|D_{(m)}$.

Proof. Let $\heartsuit$ denote the normalization functor. The complex $D^\heartsuit$ is free. Thus $D^\heartsuit = C^0 \oplus C^1 \oplus \dots$, where C^n is a free complex with $C^n_i = 0$ for $i \neq n, n+1$ and the differential $\partial\colon C^n_{n+1} \to C^n_n$ injective. The complex $D^\heartsuit$ is m-connected. Thus, for $n \le m$, the differential $\partial\colon C^n_{n+1} \to C^n_n$ is an isomorphism. The morphism $f^\heartsuit\colon Q^\heartsuit \to P^\heartsuit$ is surjective. Thus, for $n \le m$, there is a morphism $g^n\colon C^n \to Q^\heartsuit$ such that $f^\heartsuit \circ g^n = s^\heartsuit|C^n$. We have the morphism $h\colon D^\heartsuit \to Q^\heartsuit$ with $h|C^n$ equal to: g^n if $n \le m$, and zero otherwise. Obviously, $(f^\heartsuit \circ h)_n = s^\heartsuit_n$ for $n \le m$. The Dold–Kan correspondence yields the simplicial homomorphism $t\colon D \to Q$ with $t^\heartsuit = h$. It has the desired property. □

12. The function μ_L

Let L be a polyhedron. For $x \in L^\circ$, put $\mu_L(x) = 1 - \chi(\operatorname{lk}_L x)$ (χ is the Euler characteristic; lk is the link; convention: $\operatorname{lk}_L \varnothing = L$).

(12.1) *For $y, z \in L^\circ$, we have*

$$\sum_{x \in L^\circ\,:\, x \cap y = z} \mu_L(x) = \begin{cases} 1 & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $t \in L^\circ$, we have

$$\sum_{x \in L^\circ\,:\, x \subset t,\, x \cap y = z} (-1)^{\dim x} = \begin{cases} (-1)^{\dim z} & \text{if } z \subset t \subset y, \\ 0 & \text{otherwise} \end{cases}$$

(convention: $\dim \varnothing = -1$). For $x \in L^\circ$, we have

$$\chi(\operatorname{lk}_L x) = \sum_{t \in L^\circ\,:\, x \subsetneq t} (-1)^{\dim t - \dim x - 1},$$

and thus

$$\mu_L(x) = \sum_{t\in L^\circ\,:\,x\subset t} (-1)^{\dim t - \dim x}.$$

We have

$$\sum_{x\in L^\circ\,:\,x\cap y=z} \mu_L(x) = \sum_{x,t\in L^\circ\,:\,x\subset t,\,x\cap y=z} (-1)^{\dim t-\dim x} =$$

$$= \sum_{t\in L^\circ} (-1)^{\dim t} \sum_{x\in L^\circ\,:\,x\subset t,\,x\cap y=z} (-1)^{\dim x} =$$

$$= \sum_{t\in L^\circ\,:\,z\subset t\subset y} (-1)^{\dim t+\dim z} = \begin{cases} 1 & \text{if } y = z, \\ 0 & \text{otherwise.} \end{cases}$$

□

13. Dummy of a simplicial group

A model of the path fibration. Let B be the cosimplicial simplicial pointed set where B^n_m is the set of non-strictly increasing partial maps $b\colon [m] \dashrightarrow [n]$ (we have $\operatorname{dom} b \subset [m]$) with the marked element o^n_m, $\operatorname{dom} o^n_m = \varnothing$, and the structure maps are obvious. For $n \in \mathbf{N}$, we have the pointed simplicial set B^n.

Let G be a simplicial group. Let $\tilde{G}$, the *dummy*, be the simplicial group where $\tilde{G}_n$ is the group of bound simplicial maps $B^n \to G$ and the structure homomorphisms are induced by the cosimplicial structure.

(13.1) *The space $|\tilde{G}|$ is contractible.*

Proof. Let I be the simplicial set that is the standard 1-simplex: I_n is the set of non-strictly increasing maps $s\colon [n] \to [1]$. The collection of maps $I_n \times B^n_m \to B^n_m$, $(s, b) \mapsto b|(s\circ b)^{-1}(1)$, $m, n \in \mathbf{N}$, induces a contracting homotopy $I \times \tilde{G} \to \tilde{G}$. □

Evaluation at the elements $i_n \in B^n_n$, $i_n = \mathrm{id}\colon [n] \to [n]$, yields the simplicial homomorphism $p\colon \tilde{G} \to G$, the *projection*.

(13.2) *Suppose that $G_0 = 1$. Then p is surjective.*

Proof. Take an element $g \in G_n$ ($n \in \mathbf{N}$). We seek an element $\tilde{g} \in \tilde{G}_n$ with $p_n(\tilde{g}) = g$, that is, a bound simplicial map $\tilde{g}\colon B^n \to G$ with $\tilde{g}_n(i_n) = g$. Let $V \subset B^n$ be the simplicial subset generated by the elements i_n and $l_n \in B^n_1$, $\operatorname{dom} l_n = \{0\}$, $l_n(0) = 0$. It is the wedge of the standard n-simplex and 1-simplex. We have the simplicial map $f\colon V \to G$, $f_n(i_n) = g$, $f_1(l_n) = 1$. Since V is contractible and G is a Kan set, f extends to B^n, which yields the desired $\tilde{g}$. □

Extension of sections. Let L be a polyhedron. Take simplices $x, y \in L$ of dimensions r, s, respectively. Let $i\colon [r] \to L$ and $j\colon [s] \to L$ be the increasing enumerations of their vertices. We have the partial map $t = i^{-1} \circ j\colon [s] \dashrightarrow [r]$.

For a bound simplicial map $\tilde{g}\colon B^r \to G$, let $e_{xy}(\tilde{g})\colon B^s \to G$ be the bound simplicial map such that $e_{xy}(\tilde{g})_m(b) = \tilde{g}_m(t \circ b)$ for $b\colon [m] \dashrightarrow [s]$ ($m \in \mathbf{N}$). Thus we have the homomorphism $e_{xy}\colon \tilde{G}_r \to \tilde{G}_s$.

For $x \in L$, $\dim x = r$, let the homomorphism $E_x\colon \tilde{G}(\bar{\bar{x}}) \to \tilde{G}(L)$ be given by $E_x(v)_s(y) = e_{xy}(v_r(x))$ ($y \in L$, $\dim y = s$). Extend this construction to the case $x \in L^\circ$: put $E_\varnothing(1) = 1$ (we have $\tilde{G}(\bar{\bar{\varnothing}}) = 1$).

(13.3) *For $x \in L^\circ$ and $v \in \tilde{G}(\bar{\bar{x}})$, we have*

(a) $E_x(v)|_{\bar{\bar{x}}} = v$;

(b) $E_x(v)|_{\bar{\bar{y}}} = E_{x\cap y}(v|_{\bar{\bar{x}}\cap\bar{\bar{y}}})|_{\bar{\bar{y}}}$ *($y \in L^\circ$);*

(c) $\sigma(E_x(v)) \subset O_L(x, 1)$ *if* $x \neq \varnothing$. □

Realization. Let $\tilde{J}\colon \langle\tilde{G}(L)\rangle \to \langle\tilde{G}\rangle(L)$ and $\tilde{J}_x\colon \langle\tilde{G}(\bar{\bar{x}})\rangle \to \langle\tilde{G}\rangle(\bar{\bar{x}})$, $x \in L$, be fusions. Obviously, $\tilde{J}_x$ are isomorphisms. We have the additive homomorphism, the *realization*, $R\colon \langle\tilde{G}\rangle(L) \to \langle\tilde{G}(L)\rangle$,

$$R(w) = \sum_{x\in L} \mu_L(x)(\langle E_x\rangle \circ \tilde{J}_x^{-1})(w|_{\bar{\bar{x}}}).$$

We have $R(\langle\tilde{G}\rangle_\vartriangle(L)) \subset \langle\tilde{G}(L)\rangle_\vartriangle$.

(13.4) *For $w \in \langle\tilde{G}\rangle_\vartriangle(L)$, we have $\tilde{J}(R(w)) = w$.*

Proof. For $z \in L^\circ$, we have the homomorphism $H_z\colon \langle\tilde{G}\rangle(\bar{\bar{z}}) \to \langle\tilde{G}(L)\rangle$ with $H_z = \langle E_z\rangle \circ \tilde{J}_z^{-1}$, $z \neq \varnothing$, and $H_\varnothing = 0$. It follows from (13.3 b) that for $x, y \in L^\circ$ and $u \in \langle\tilde{G}\rangle_\vartriangle(\bar{\bar{x}})$, we have $H_x(u)|_{\bar{\bar{y}}} = H_{x\cap y}(u|_{\bar{\bar{x}}\cap\bar{\bar{y}}})|_{\bar{\bar{y}}}$. For $y \in L$, we have

$$\begin{aligned}
\tilde{J}(R(w))|_{\bar{\bar{y}}} &= \tilde{J}_y(R(w)|_{\bar{\bar{y}}}) = \\
&= \sum_{x\in L^\circ} \mu_L(x)\tilde{J}_y(H_x(w|_{\bar{\bar{x}}})|_{\bar{\bar{y}}}) = \sum_{x\in L^\circ} \mu_L(x)\tilde{J}_y(H_{x\cap y}(w|_{\bar{\bar{x}}\cap\bar{\bar{y}}})|_{\bar{\bar{y}}}) = \\
&= \sum_{z\in L^\circ}\Big(\sum_{x\in L^\circ\,:\,x\cap y=z} \mu_L(x)\Big)\tilde{J}_y(H_z(w|_{\bar{\bar{z}}})|_{\bar{\bar{y}}}) \overset{\text{by (12.1)}}{=} \tilde{J}_y(H_y(w|_{\bar{\bar{y}}})|_{\bar{\bar{y}}}) = \\
&= \tilde{J}_y(\langle E_y\rangle(\tilde{J}_y^{-1}(w|_{\bar{\bar{y}}}))|_{\bar{\bar{y}}}) \overset{\text{by (13.3 a)}}{=} \tilde{J}_y(\tilde{J}_y^{-1}(w|_{\bar{\bar{y}}})) = w|_{\bar{\bar{y}}}. \quad \square
\end{aligned}$$

(13.5) *For $w \in \langle\tilde{G}\rangle(L)$, we have $\Sigma(R(w)) \subset O_L(\sigma(w), 1)$.*

This follows from (13.3 c). □

(13.6) *We have $R(\langle\tilde{G}\rangle^s_\vartriangle(L)) \subset \langle\tilde{G}(L)\rangle^s_\vartriangle$, $s \in \mathbf{N}$.*

Proof. For $w \in \langle\tilde{G}\rangle^s_\vartriangle(L)$ and $x \in L$, we have $w|_{\bar{\bar{x}}} \in \langle\tilde{G}\rangle^s_\vartriangle(\bar{\bar{x}})$, $\tilde{J}_x^{-1}(w|_{\bar{\bar{x}}}) \in \langle\tilde{G}(\bar{\bar{x}})\rangle^s_\vartriangle$, and, by (9.1), $\langle E_x\rangle(\tilde{J}_x^{-1}(w|_{\bar{\bar{x}}})) \in \langle\tilde{G}(L)\rangle^s_\vartriangle$. Summing over $x \in L$, we get $R(w) \in \langle\tilde{G}(L)\rangle^s_\vartriangle$. □

14. Partitions

Let L be a polyhedron and D be a simplicial abelian group. A collection $(h_z\colon D(\bar{\bar{z}}) \to D(L))_{z\in L}$ of homomorphisms is called a *partition* if for $w \in D(L)$, we have

$$\sum_{z\in L} h_z(w|_{\bar{\bar{z}}}) = w$$

and $\sigma(h_z(w)) \subset O_L(z,1)$ for all $z \in L$.

(14.1) *Suppose that* $\dim L \le m$ *(*$m \in \mathbf{N}$*) and* D *is free and* m*-connected. Then there exists a partition* $(h_z\colon D(\bar{\bar{z}}) \to D(L))_{z\in L}$.

Proof. We shall use the Dold–Kan correspondence. There is a decomposition $D = D^0 \oplus D^1 \oplus \dots$, where D^n is a simplicial abelian group such that its normalization C^n is concentrated in dimensions n and $n+1$ and the differential $\partial\colon C^n_{n+1} \to C^n_n$ is injective (cf. proof of (11.1)). It suffices to construct a partition $(h^n_z\colon D^n(\bar{\bar{z}}) \to D^n(L))_{z\in L}$ for each n. Take $n \le m$. Then $\partial\colon C^n_{n+1} \to C^n_n$ is an isomorphism, since D is m-connected. Thus a section on a polyhedron with values in D^n is the same as an n-cochain on it with coefficients in C^n_n. Let h^n_z be: the extension of a cochain by zero if $\dim z = n$, and zero otherwise. Take $n > m$. Then $D^n(L) = 0$ since $\dim L \le m$. Thus there is the zero partition. $\square$

15. Modification of an ensemble of sections

Fix numbers $b_1, \dots, b_5, c \in \mathbf{N}$ such that each is sufficiently great with respect to the previous, namely: $b_1 \ge 2$, $b_2 \ge b_1 + 2$, $b_3 \ge 2b_2$, $b_4 \ge 2b_1 + b_3$, $b_5 \ge 2b_2 + b_4$, $2^{c-1} \ge 2b_5 + 1$.

The morphism $e\colon L \to K$. Let K be a polyhedron with $\dim K \le m$ ($m \in \mathbf{N}$). Put $L = \Delta^c K$ and $e = \Phi_K \circ \dots \circ \Phi_{\Delta^{c-1}K}\colon L \to K$. For $z \in L$, the set $e(O_L(z, b_5)) \subset K$ is small (this follows from the properties of the operation Δ and the inequaity $2^{c-1} \ge 2b_5 + 1$).

The morphisms e_z. Take a simplex $z \in L$. Since $b_2 \le b_5$, the set $e(O_L(z, b_2))$ is small. It spans a simplex $x \in K$. Let $u \in K$ be the highest vertex of x. We shall construct a morphism $e_z\colon L \to K$ with the following properties:

(1) $e_z(O_L(z, b_1)) = \{u\}$;

(2) $e_z(O_L(z, b_2)) \subset \bar{\bar{x}}$;

(3) e_z agrees with e outside $O_L(z, b_2)$.

Put $L_1 = \delta\Delta^{c-1}K$. We have $L = \delta' L_1$. Let $B_1 \subset L_1$ be the subpolyhedron generated by the simplices whose centres (which are vertices of L) belong to $O_L(z, b_1 + 1)$. Put $B = \delta' B_1$. We have $B \subset L$ (a subpolyhedron). We have $O_L(z, b_1) \subset B$ and (since $b_2 \ge b_1 + 2$) $O_L(B, 1) \subset O_L(z, b_2)$. The polyhedron L has no edges outcoming from B. Let e_z take a vertex $t \in L$ to: u if $t \in B$, and $e(t)$ otherwise. One easily checks that e_z is well-defined and has the desired properties.

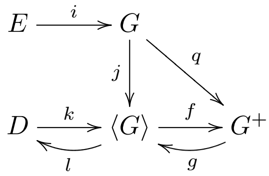
The morphisms e_Z. Take a set $Z \subset L$ with $\epsilon_L(Z) \geq b_3$. Define a morphism $e_Z\colon L \to K$ by the following conditions:

(1) for $z \in Z$, the morphisms e_Z and e_z agree on $O_L(z, b_2)$;

(2) the morphisms e_Z and e agree outside $O_L(Z, b_2)$.

Since $b_3 \geq 2b_2$, e_Z is well-defined.

The simplicial groups G and D. Let E be an $(n-1)$-connected ($n \in \mathbf{N}$) simplicial set with a single vertex. Suppose that $m \leq 2n-1$. Put $G = FE$. Let $i\colon E \to G$ and $j\colon G \to \langle G \rangle$ be the canonical simplicial maps and $q\colon G \to G^+$ be the simplicial homomorphism that is the projection. We shall need a decomposition $\langle G \rangle \cong \langle 1 \rangle \oplus G^+ \oplus D$ (cf. § 10) and some related simplicial homomorphisms. Let $d\colon \langle G \rangle \to \langle G \rangle$ be the simplicial homomorphism that is the identity on $\langle G \rangle_\vartriangle$ and zero on $\langle 1 \rangle$. We have the simplicial homomorphisms $f\colon \langle G \rangle \to G^+$ with $f \circ j = q$ and $g\colon G^+ \to \langle G \rangle$ with $g \circ q \circ i = d \circ j \circ i$. We have $f \circ g = \mathrm{id}$. Put $D = \langle G \rangle^2_\vartriangle \subset \langle G \rangle$. Let $k\colon D \to \langle G \rangle$ be the inclusion. We have the simplicial homomorphism $l\colon \langle G \rangle \to D$ such that $k \circ l + g \circ f = d$. We have $l \circ k = \mathrm{id}$.



The simplicial abelian group D is free. By the Freudenthal theorem, the map $i\colon E \to G$ is $(2n-1)$-connected. Since $m \leq 2n-1$, it is m-connected. Using the Dold–Thom theorem, we see that the simplicial homomorphism $\langle i \rangle\colon \langle E \rangle \to \langle G \rangle$ is m-connected. One easily sees that $(\langle i \rangle, k)\colon \langle E \rangle \oplus D \to \langle G \rangle$ is an isomorphism. Thus D is m-connected.

For $s \in \mathbf{N}$, let $D^{(s)} \subset D$ be the simplicial subgroup equal to: $\langle G \rangle^s_\vartriangle$ for $s \geq 2$, and D otherwise.

Decomposition of D. Let $r \in \mathbf{N}$, $r \geq 2$, be a number. By § 10, we have the decomposition $D = D^2 \oplus \ldots \oplus D^r$ where $\langle G \rangle^s_\vartriangle = D^s \oplus \ldots \oplus D^r$, $s = 2, \ldots, r$. (We have $D^s \cong \langle E^{\wedge s} \rangle_\vartriangle$ for $s < r$ and $D^r = \langle G \rangle^r_\vartriangle$.) Since D is free and m-connected, the groups D^s are free and m-connected.

The partition h. By (14.1), for each $s = 2, \ldots, r$, there is a partition $(h^s_z\colon D^s(\bar{\bar{z}}) \to D^s(L))_{z \in L}$. Combining them, we get the partition $(h_z\colon D(\bar{\bar{z}}) \to D(L))_{z \in L}$. We have $h_z(D^{(s)}(\bar{\bar{z}})) \subset D^{(s)}(L)$, $s \in \mathbf{N}$, $s \leq r$.

The simplicial homomorphism X. Let $\tilde{G}$ be the dummy of G, $p\colon \tilde{G} \to G$ be the projection. By (13.2), p is surjective. Thus, for the simplicial homomorphism $\langle p \rangle\colon \langle \tilde{G} \rangle \to \langle G \rangle$, we have $\langle p \rangle(\langle \tilde{G} \rangle^s_\vartriangle) = \langle G \rangle^s_\vartriangle$, $s \in \mathbf{N}$. Applying (11.1) to each component D^s of the decomposition of D, we get the simplicial homomorphism $X\colon D \to \langle \tilde{G} \rangle$ with the following properties:

(1) the diagram

$$\begin{array}{ccc} & & \langle \tilde{G} \rangle \\ & \overset{X|D_{(m)}}{\nearrow} & \downarrow \langle p \rangle \\ D_{(m)} & \xrightarrow{\text{inclusion}} & \langle G \rangle \end{array}$$

is commutative;

(2) $X(D^{(s)}) \subset \langle \tilde{G} \rangle^s_\vartriangle$, $s \in \mathbf{N}$, $s \leq r$.

We have $\operatorname{im} X \subset \langle \tilde{G} \rangle_\vartriangle$.

The homomorphism V. Let $J \colon \langle G(L) \rangle \to \langle G \rangle(L)$ be the fusion, $R \colon \langle \tilde{G} \rangle(L) \to \langle \tilde{G}(L) \rangle$ be the realization. We have the composition

$$V \colon D(L) \xrightarrow{X_\#} \langle \tilde{G} \rangle(L) \xrightarrow{R} \langle \tilde{G}(L) \rangle \xrightarrow{\langle p_\# \rangle} \langle G(L) \rangle.$$

We have $\operatorname{im} V \subset \langle G(L) \rangle_\vartriangle$.

(15.1) *The diagram*

$$\begin{array}{ccc} & & \langle G(L) \rangle \\ & \overset{V}{\nearrow} & \downarrow J \\ D(L) & \xrightarrow{k_\#} & \langle G \rangle(L) \end{array}$$

is commutative.

Proof. Let $\tilde{J} \colon \langle \tilde{G}(L) \rangle \to \langle \tilde{G} \rangle(L)$ be the fusion. The diagram

$$\begin{array}{ccccc} \langle \tilde{G} \rangle(L) & \xrightarrow{R} & \langle \tilde{G}(L) \rangle & \xrightarrow{\langle p_\# \rangle} & \langle G(L) \rangle \\ {\scriptstyle X_\#} \uparrow & & \downarrow {\scriptstyle \tilde{J}} & & \downarrow {\scriptstyle J} \\ D(L) & \xrightarrow{X_\#} & \langle \tilde{G} \rangle(L) & \xrightarrow{\langle p \rangle_\#} & \langle G \rangle(L) \end{array}$$

is commutative (we invoke (13.4) taking into account that $\operatorname{im} X \subset \langle \tilde{G} \rangle_\vartriangle$). We have $J \circ V = \langle p \rangle_\# \circ X_\# = k_\#$ by the property (1) of X. □

(15.2) *For $w \in D(L)$, we have $\Sigma(V(w)) \subset O_L(\sigma(w), 1)$.*

This follows from (13.5). □

(15.3) *We have $V(D^{(s)}(L)) \subset \langle G(L) \rangle^s_\vartriangle$, $s \in \mathbf{N}$, $s \leq r$.*

This follows from the property (2) of X and the claims (13.6) and (9.1). □

The maps P_z, P. For $z \in L$, we have the map $P_z \colon G(K) \to \langle G(L) \rangle$, $P_z(u) = (V \circ h_z)(l \circ j \circ u \circ e|_{\bar{\bar{z}}})$. We have $P_z(u) \in \langle G(L) \rangle_\Delta$ since $\operatorname{im} V \subset \langle \tilde{G}(L) \rangle_\Delta$. We have $\Sigma(P_z(u)) \subset O_L(z, b_1)$ (by the definition of a partition, the claim (15.2), and the inequality $b_1 \geq 2$).

We have the map $P \colon G(K) \to \langle G(L) \rangle$, $P(u) = V(l \circ j \circ u \circ e)$. We have

$$\sum_{z \in L} P_z(u) = P(u).$$

The homomorphism M. We have the additive homomorphism $M \colon \langle G(K) \rangle \to \langle G(L) \rangle$,

$$M(\text{`}u\text{'}) = \sum_{Z \subset L \,:\, \epsilon_L(Z) \geq b_3} (-1)^{\#Z} \text{`}u \circ e_Z\text{'} \prod_{z \in Z} P_z(u).$$

Here and in all our $\prod$'s, we mean that the order of factors is induced by some fixed order on L. (Moreover, one can see that the factors commute everywhere.)

(15.4) *For $U \in \langle G(K) \rangle$, we have $\theta(M(U)) \geq \min(\theta(U) + 1, \eta(U))$.*

Proof. Suppose that $\theta(U) \geq s - 1$ and $\eta(U) \geq s$ ($s \in \mathbf{N}_+$). We show that $\theta(M(U)) \geq s$. Take a set $T \subset L$ with $\#T < s$. We show that $M(U)\|_T = 0$.

The case $\epsilon_L(T) \geq b_4$. Put $I = \{Z \subset L : \epsilon_L(Z) \geq b_3\}$. For $u \in G(K)$, we have

$$M(\text{`}u\text{'})\|_T = \sum_{Z \in I} (-1)^{\#Z} \text{`}u \circ e_Z\text{'}\|_T \prod_{z \in Z} P_z(u)\|_T.$$

The sets $O_L(y, b_1)$, $y \in T$, (*balls*) do not intersect. Moreover, the distance (ρ_L) between simplices of distinct balls is at least b_3 (since $b_4 \geq 2b_1 + b_3$). The distance between simplices of a ball is smaller than b_3 (since $b_3 \geq 2b_1$). Let I_0 be the set of sets $Z \subset L$ that are contained in the union of the balls and have at most one simplex in each ball. Show that our sum over $Z \in I$ equals the same sum but over $Z \in I_0$. We have $I_0 \subset I$. If $Z \in I \setminus I_0$, there is a simplex $z \in Z \setminus O_L(T, b_1)$; then $P_z(u)\|_T = 0$ because: $P_z(u) \in \langle G(L) \rangle_\Delta$, $\Sigma(P_z(u)) \subset O_L(z, b_1)$, and $O_L(z, b_1) \cap T = \varnothing$. Thus the corresponding summand is zero.

Put

$$I_0' = \coprod_{S \subset T} W_S,$$

where W_S is the set of maps $w \colon S \to L$ such that $w(y) \in O_L(y, b_1)$, $y \in S$. We have the bijection $I_0' \to I_0$, $(S, w) \mapsto w(S)$. Thus

$$M(\text{`}u\text{'})\|_T = \sum_{(S,w) \in I_0'} (-1)^{\#S} \text{`}u \circ e_{w(S)}\text{'}\|_T \prod_{y \in S} P_{w(y)}(u)\|_T.$$

For $y \in T$, let $t_y \colon G(\bar{\bar{y}}) \to G_T$ be the canonical monomorphism of a factor to a product. Show that for $(S, w) \in I_0'$,

$$(u \circ e_{w(S)})\|_T = \prod_{y \in T \setminus S} t_y(u \circ e|_{\bar{\bar{y}}}).$$

If $y \in S$, we have $y \in O_L(w(y), b_1)$, and $e_{w(S)}$ sends the simplex y to a vertex of K; then $u \circ e_{w(S)}|_{\bar{\bar{y}}} = 1$ since $G_0 = 1$. If $y \in T \setminus S$, we have $y \notin O_L(w(S), b_2)$ (since $b_4 \geq b_1 + b_2$), and $e_{w(S)}|_{\bar{\bar{y}}} = e|_{\bar{\bar{y}}}$. Thus we have the desired equality.

For $(S, w) \in I'_0$ and $y \in S$, we have $P_{w(y)}(u)\|_T = \langle t_y \rangle (P_{w(y)}(u)|_{\bar{\bar{y}}})$. This is because $\Sigma(P_{w(y)}(u)) \subset O_L(w(y), b_1)$ and $O_L(w(y), b_1) \cap T = \{y\}$ (since $b_4 \geq 2b_1$).

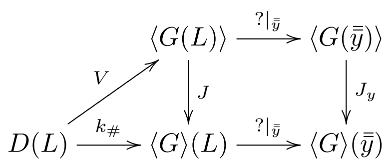
Thus

$$\begin{aligned} M(\text{`}u\text{'})\|_T &= \sum_{(S,w)\in I'_0} (-1)^{\#S} \Big( \prod_{y\in T\setminus S} \text{`}t_y(u \circ e|_{\bar{\bar{y}}})\text{'} \Big) \Big(\prod_{y \in S} \langle t_y \rangle (P_{w(y)}(u)|_{\bar{\bar{y}}}) \Big) = \\ &= \sum_{S\subset T} (-1)^{\#S} \Big(\prod_{y\in T\setminus S} \text{`}t_y(u \circ e|_{\bar{\bar{y}}})\text{'} \Big) \Big( \sum_{w\in W_S} \prod_{y\in S} \langle t_y \rangle (P_{w(y)}(u)|_{\bar{\bar{y}}}) \Big) = \\ &= \sum_{S\subset T} (-1)^{\#S} \Big( \prod_{y\in T\setminus S} \langle t_y \rangle (\text{`}u \circ e\text{'}|_{\bar{\bar{y}}}) \Big) \Big(\prod_{y\in S} \sum_{z\in O_L(y,b_1)} \langle t_y \rangle (P_z(u)|_{\bar{\bar{y}}}) \Big) = \\ &= \prod_{y\in T} \langle t_y \rangle \Big(\text{`}u \circ e\text{'}|_{\bar{\bar{y}}} - \sum_{z\in O_L(y,b_1)} P_z(u)|_{\bar{\bar{y}}} \Big). \end{aligned}$$

We may extend the domain of the last sum to $z \in L$ because for $z \in L \setminus O_L(y, b_1)$, we have $P_z(u)|_{\bar{\bar{y}}} = 0$ because: $P_z(u) \in \langle G(L) \rangle_\Delta$, $\Sigma(P_z(u)) \subset O_L(z, b_1)$, and $O_L(z, b_1) \cap \bar{\bar{y}} = \varnothing$ for such z. We have

$$M(\text{`}u\text{'})\|_T = \prod_{y\in T} \langle t_y \rangle (\text{`}u \circ e\text{'}|_{\bar{\bar{y}}} - P(u)|_{\bar{\bar{y}}}).$$

For $y \in T$, let $J_y \colon \langle G(\bar{\bar{y}}) \rangle \to \langle G \rangle(\bar{\bar{y}})$ be the fusion. Obviously, it is an isomorphism. We have the commutative diagram



(we invoke (15.1)). We have $J_y(\text{`}u \circ e\text{'}|_{\bar{\bar{y}}} - P(u)|_{\bar{\bar{y}}}) = J_y(\text{`}u \circ e\text{'}|_{\bar{\bar{y}}} - V(l \circ j \circ u \circ e)|_{\bar{\bar{y}}}) = j \circ u \circ e|_{\bar{\bar{y}}} - k \circ l \circ j \circ u \circ e|_{\bar{\bar{y}}} = 1 + g \circ f \circ j \circ u \circ e|_{\bar{\bar{y}}} = 1 + g \circ q \circ u \circ e|_{\bar{\bar{y}}}$. We have the homomorphism $a_y \colon G_K \to \langle G_T \rangle$ (in the additive group), $a_y(v) = (\langle t_y \rangle \circ J_y^{-1})((g \circ q \circ v \circ e)_y)$. We have

$$M(\text{`}u\text{'})\|_T = \prod_{y\in T} (1 + a_y(u\|_K)).$$

Since $\eta(U) > \#T$, by (8.1), $M(U)\|_T = 0$.

The converse case. There are distinct simplices $y_0, y_1 \in T$ with $\rho_L(y_0, y_1) < b_4$. For each $y \in T \setminus \{y_1\}$, consider the simplex $x \in K$ spanned by the set $e(O_L(y, b_5))$. Let $S \subset K$ be the set of these simplices. We have $\#S < s - 1$. For each $y \in T$, there exists a simplex $y' \in T \setminus \{y_1\}$ such that $O_L(y, 2b_2) \subset O_L(y', b_5)$:

we may let y' be equal to: y_0 if $y = y_1$, and y otherwise (we use the inequality $b_5 \geq 2b_2 + b_4$). Thus, for every $y \in T$, there exists a simplex $x \in S$ such that $e(O_L(y, 2b_2)) \subset \bar{\bar{x}}$. Let $e' \colon \overline{O_L(T, b_1)} \to \bar{S}$ be the abridgement of e (we use the inequality $b_1 \leq 2b_2$).

Take a set $Z \subset L$ such that $\epsilon_L(Z) \geq b_3$. Show that $e_Z(\bar{T}) \subset \bar{S}$. It suffices to check that $e_Z(y) \in \bar{S}$ for $y \in T$. If $y \notin O_L(Z, b_2)$, then $e_Z(y) = e(y) \in \bar{S}$. Otherwise, $y \in O_L(z, b_2)$ for some $z \in Z$. Then $e_Z(y) = e_z(y) \in \bar{\bar{x}}$, where $x \in K$ is the simplex spanned by $e(O_L(z, b_2))$. We have $e(O_L(z, b_2)) \subset e(O_L(y, 2b_2))$ Thus $e_Z(y) \in \bar{S}$. Let $\tilde{e}_Z \colon \bar{T} \to \bar{S}$ be the abridgement of e_Z.

We have the additive homomorphism $\tilde{M} \colon \langle G(\bar{S})\rangle \to \langle G(\bar{T})\rangle$,

$$\tilde{M}(\text{‘}\tilde{u}\text{’}) = \sum_{Z \subset O_L(T, b_1) \,:\, \epsilon_L(Z) \geq b_3} (-1)^{\#Z} \text{‘}\tilde{u} \circ \tilde{e}_Z\text{’} \prod_{z \in Z} (V \circ h_z)(l \circ j \circ \tilde{u} \circ e'|_{\bar{\bar{z}}})|_{\bar{T}}.$$

Show that the diagram

$$\begin{array}{ccc} \langle G(K)\rangle & \xrightarrow{M} & \langle G(L)\rangle \\ \Big\downarrow {\scriptstyle ?|_{\bar{S}}} & & \Big\downarrow {\scriptstyle ?|_{\bar{T}}} \\ \langle G(\bar{S})\rangle & \xrightarrow{\tilde{M}} & \langle G(\bar{T})\rangle \end{array}$$

is commutative. We have

$$M(\text{‘}u\text{’})|_{\bar{T}} = \sum_{Z \subset L \,:\, \epsilon_L(Z) \geq b_3} (-1)^{\#Z} \text{‘}u \circ e_Z\text{’}|_{\bar{T}} \prod_{z \in Z} P_z(u)|_{\bar{T}}.$$

The summands with $Z \not\subset O_L(T, b_1)$ equal zero (if $z \in Z \setminus O_L(T, b_1)$, then $P_z(u)|_{\bar{T}} = 0$ because: $P_z(u) \in \langle G(L)\rangle_\Delta$, $\Sigma(P_z(u)) \subset O_L(z, b_1)$, and $O_L(z, b_1) \cap \bar{T} = \varnothing$). We get

$$M(\text{‘}u\text{’})|_{\bar{T}} = \sum_{Z \subset O_L(T, b_1) \,:\, \epsilon_L(Z) \geq b_3} (-1)^{\#Z} \text{‘}u \circ e_Z\text{’}|_{\bar{T}} \prod_{z \in Z} (V \circ h_z)(l \circ j \circ u \circ e|_{\bar{\bar{z}}})|_{\bar{T}} =$$
$$= \tilde{M}(\text{‘}u\text{’}|_{\bar{S}}).$$

Since $\theta(U) > \#S$, $U\|_S = 0$. Thus $U|_{\bar{S}} = 0$. We get $M(U)|_{\bar{T}} = \tilde{M}(U|_{\bar{S}}) = 0$. Thus $M(U)\|_T = 0$. ☐

(15.5) *For $U \in \langle G(K)\rangle$, we have $\eta(M(U)) \geq \min(\eta(U), r)$.*

Proof. We have the additive homomorphism $N \colon \langle G_K\rangle \to \langle G_L\rangle$,

$$N(\text{‘}v\text{’}) = \sum_{Z \subset L \,:\, \epsilon_L(Z) \geq b_3} (-1)^{\#Z} \text{‘}v \circ e_Z\text{’} \prod_{z \in Z} (V \circ h_z)((l \circ j \circ v \circ e)_z)\|_L.$$

The diagram

$$\begin{array}{ccc} \langle G(K)\rangle & \xrightarrow{M} & \langle G(L)\rangle \\ \Big\downarrow {\scriptstyle ?\|_K} & & \Big\downarrow {\scriptstyle ?\|_L} \\ \langle G_K\rangle & \xrightarrow{N} & \langle G_L\rangle \end{array}$$

is commutative. It suffices to show that N is r-strict. For $z \in L$, we have the homomorphism $t_z\colon G_K \to G(\bar{\bar{z}})$, $t_z(v) = (v \circ e)_z$, and the additive homomorphism $B_z\colon \langle G(\bar{\bar{z}})\rangle \to \langle G(L)\rangle$, $B_z(\text{‘}v\text{’}) = (V \circ h_z)(l \circ j \circ v)$. We have the homomorphisms $e^{\#}_Z\colon G_K \to G_L$ and $?\|_L\colon G(L) \to G_L$. We have (for $v \in G_K$)

$$N(\text{‘}v\text{’}) = \sum_{Z \subset L\,:\, \epsilon_L(Z) \geq b_3} (-1)^{\#Z} \langle e^{\#}_Z\rangle(\text{‘}v\text{’}) \prod_{z \in Z} (B_z \circ \langle t_z\rangle)(\text{‘}v\text{’})\|_L.$$

By (9.1) and (9.2), it suffices to show that the homomorphisms B_z are r-strict. The homomorphism B_z equals the composition

$$\langle G(\bar{\bar{z}})\rangle \xrightarrow{J_z} \langle G\rangle(\bar{\bar{z}}) \xrightarrow{l_{\#}} D(\bar{\bar{z}}) \xrightarrow{h_z} D(L) \xrightarrow{V} \langle G(L)\rangle,$$

where J_z is the fusion. For $s \in \mathbf{N}$, we have: $J_z(\langle G(\bar{\bar{z}})\rangle^s_\triangle) = \langle G\rangle^s_\triangle(\bar{\bar{z}})$; $l_{\#}(\langle G\rangle^s_\triangle(\bar{\bar{z}})) = D^{(s)}(\bar{\bar{z}})$ (since l is identical on D); $h_z(D^{(s)}(\bar{\bar{z}})) \subset D^{(s)}(L)$ for $s \leq r$ (a property of the partition h); $V(D^{(s)}(L)) \subset \langle G(L)\rangle^s_\triangle$ for $s \leq r$ (by (15.3)). Thus $B_z(\langle G(\bar{\bar{z}})\rangle^s_\triangle) \subset \langle G(L)\rangle^s_\triangle$ for $s \leq r$, which is what we need. □

Put $Q = |K|$ $(= |L|)$.

(15.6) *For $U \in \langle G(K)\rangle$, we have $[|M(U)|] = [|U|]$ in the ring $\langle [Q, |G|]\rangle$.*

Proof. Take $u \in G(K)$ and $z \in L$. We have $P_z(u) \in \langle G(L)\rangle_\triangle$. By the construction of P_z, all the sections in the ensemble $P_z(u)$ lift to $\tilde{G}$. By (13.1), the space $|\tilde{G}|$ is contractible. Thus $[|P_z(u)|] = 0$. Applying the ring homomorphism $[|?|]\colon \langle G(L)\rangle \to \langle [Q, |G|]\rangle$ to the equality defining M, we get $[|M(\text{‘}u\text{’})|] = [|\text{‘}u \circ e\text{’}|] = [|\text{‘}u\text{’}|]$ since $|e|\colon Q \to Q$ is homotopic to the identity. □

16. Main procedure

Let K be a polyhedron with $\dim K \leq m$ $(m \in \mathbf{N})$ and E be an $(n-1)$-connected $(n \in \mathbf{N})$ simplicial set with a single vertex. Suppose that $m \leq 2n-1$. Put $Q = |K|$ and $G = FE$.

(16.1) *Let $U \in \langle G(K)\rangle$ be an ensemble with $\eta(U) \geq s$ ($s \in \mathbf{N}$). Then there exist a polyhedron L with the body Q and an ensemble $V \in \langle G(L)\rangle$ with $\theta(V) \geq s$ and $[|V|] = [|U|]$ in $\langle [Q, |G|]\rangle$.*

Proof. To get the desired pair (L, V), take the pair (K, U) and apply the pair (Δ^c, M) of operations of § 15 s times. We choose $r \geq s$. The desired properties follow from (15.4), (15.5), and (15.6). □

17. The function θ: the topological version

Let X and Y be spaces. For $A \in \langle C(X, Y)\rangle$, put $\theta(A) = \inf\{\#V : \text{finite } V \subset X,\ A|_V \neq 0\} \in \hat{\mathbf{N}}$.

Let X' and Y' be spaces, $g\colon X' \to X$ and $h\colon Y \to Y'$ be continuous maps. We have the map $t\colon C(X, Y) \to C(X', Y')$, $t(a) = h \circ a \circ g$. We have the homomorphism $\langle t\rangle\colon \langle C(X, Y)\rangle \to \langle C(X', Y')\rangle$.

(17.1) *For $A \in \langle C(X,Y)\rangle$, we have $\theta(\langle t\rangle(A)) \geq \theta(A)$.*

Proof. Take a finite $V' \subset X'$ with $\#V' < \theta(A)$. We show that $\langle t\rangle(A)|_{V'} = 0$. Put $V = g(V') \subset X$. We have $\#V < \theta(A)$. Thus $A|_V = 0$. Let $\tilde{g}\colon V' \to V$ be the abridgement of g. We have the map $\tilde{t}\colon C(V,Y) \to C(V',Y')$, $\tilde{t}(\tilde{a}) = h \circ \tilde{a} \circ \tilde{g}$. The diagram

$$\begin{array}{ccc} C(X,Y) & \xrightarrow{t} & C(X',Y') \\ {\scriptstyle ?|_V}\downarrow & & \downarrow{\scriptstyle ?|_{V'}} \\ C(V,Y) & \xrightarrow{\tilde{t}} & C(V',Y') \end{array}$$

is commutative. We have $\langle t\rangle(A)|_{V'} = \langle\tilde{t}\rangle(A|_V) = 0$. □

A characterization of the order. Let U be an abelian group and $f\colon [X,Y] \to U$ be a map. We have the homomorphism $\bar{f}\colon \langle[X,Y]\rangle \to U$, $\bar{f}(\text{`}w\text{'}) = f(w)$.

(17.2) *The condition* $\operatorname{ord} f \leq r$ *($r \in \mathbf{N}$) is equivalent to the condition that $\bar{f}([A]) = 0$ for every $A \in \langle C(X,Y)\rangle$ with $\theta(A) > r$.*

Proof. Let E_r, I_r, and D_r be as in § 1. We have the homomorphism $h\colon \langle C(X,Y)\rangle \to D_r$, $h(\text{`}a\text{'}) = I_r(a)$. It is surjective. One easily sees that for $A \in \langle C(X,Y)\rangle$, the conditions $h(A) = 0$ and $\theta(A) > r$ are equivalent. We have the homomorphism $\tilde{f}\colon \langle C(X,Y)\rangle \to U$, $\tilde{f}(A) = \bar{f}([A])$. The condition $\operatorname{ord} f \leq r$ is equivalent to the existence of a homomorphism $l\colon D_r \to U$ with $l \circ h = \tilde{f}$. The latter is equivalent to the condition $\tilde{f}|\ker h = 0$, that is, the condition that $\bar{f}([A]) = 0$ for every $A \in \langle C(X,Y)\rangle$ with $\theta(A) > r$. □

18. Geometric realization and simplicial approximation

Let K be a polyhedron and E be a simplicial set. Put $Q = |K|$.

(18.1) *For $U \in \langle E(K)\rangle$, we have $\theta(|U|) = \theta(U)$.* □

(18.2) *Let $B \in \langle C(Q,|E|)\rangle$ be an ensemble. Then there exist a polyhedron L with the body Q and an ensemble $V \in \langle E(L)\rangle$ with $\theta(V) \geq \theta(B)$ and $[|V|] = [B]$ in $\langle[Q,|E|]\rangle$.*

Proof. There are a finite set I, a map $k\colon I \to C(Q,|E|)$, and an element $g \in \langle I\rangle$ such that $\langle k\rangle(g) = B$. Put $b_i = k(i)$, $i \in I$. For $q \in Q$, we have the equivalence $R_q = \{(i,j) : b_i(q) = b_j(q)\}$ on I. For a finite set $W \subset Q$, put

$$R_W = \bigcap_{q \in W} R_q.$$

The map $i \mapsto b_i|_W$ is subordinate to the equivalence R_W (that is, constant on the classes of R_W). We have the commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{k} & C(Q,|E|) \\ {\scriptstyle p_W}\downarrow & & \downarrow{\scriptstyle ?|_W} \\ I/R_W & \xrightarrow{k_W} & C(W,|E|), \end{array}$$

where p_W is the projection. The map k_W is injective. We have $\langle k_W\rangle(\langle p_W\rangle(g)) = \langle k\rangle(g)|_W = B|_W$. If $\#W < \theta(B)$, then $B|_W = 0$, and thus $\langle p_W\rangle(g) = 0$.

We have the continuous map $b = (b_i)_{i\in I}\colon Q \to |E|^I$. Let $h\colon |E^I| \to |E|^I$ be the canonical continuous bijection. Since I is finite and Q is Haudorff and compact, the map $c = h^{-1} \circ b\colon Q \to |E^I|$ is continuous.

To each equivalence R on I assign the simplicial subset $D(R) \subset E^I$, $D(R)_n = \{(e_i)_{i\in I} \in E^I_n : (i,j) \in R \Rightarrow e_i = e_j\}$ (the diagonal). For $q \in Q$, we have $c(q) \in |D(R_q)| \subset |E^I|$. We have the simplicial subset $M \subset E^I$,

$$M = \bigcup_{q\in Q} D(R_q).$$

We have $c(Q) \subset |M| \subset |E^I|$. Let $c'\colon Q \to |M|$ be the abridgement of c. By the simplicial approximation theorem, there are a polyhedron L with the body Q and a section $u' \in M(L)$ such that the map $|u'|\colon Q \to |M|$ is homotopic to c'. Let $u \in E^I(L)$ be the composition of u' and the inclusion $M \to E^I$. We have $u = (u_i)_{i\in I}$, where $u_i \in E(L)$. The map $|u_i|\colon Q \to |E|$ is homotopic to b_i. We have the map $l\colon I \to E(L)$, $l(i) = u_i$. Put $V = \langle l\rangle(g)$. We have $[|V|] = [B]$.

For a simplex $y \in L$, $\dim y = s$, we have $u_s(y) \in M_s$, that is, there is a point $q = q_y \in Q$ such that $u_s(y) \in D(R_q)_s$, that is, $u_i|_{\bar{\bar{y}}} = u_j|_{\bar{\bar{y}}}$ for $(i,j) \in R_q$, that is, the map $i \mapsto u_i|_{\bar{\bar{y}}}$ is subordinate to R_q.

Take a set $T \subset L$. Put $W = \{q_y : y \in T\}$. We have $\#W \le \#T$. The map $i \mapsto u_i\|_T$ is subordinate to R_W. We have the commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{\;l\;} & E(L) \\ {\scriptstyle p_W}\downarrow & & \downarrow{\scriptstyle ?\|_T} \\ I/R_W & \xrightarrow{\;l_T\;} & E_T. \end{array}$$

We have $V\|_T = \langle l\rangle(g)\|_T = \langle l_T\rangle(\langle p_W\rangle(g))$. If $\#T < \theta(B)$, then: $\#W < \theta(B)$, $\langle p_W\rangle(g) = 0$, and $V\|_T = 0$. Thus $\theta(V) \ge \theta(B)$. □

19. Some subgroups of $\langle[Q, |G|]\rangle$.

Let Q be a polyhedral body, $\dim Q \le m$ ($m \in \mathbf{N}$), and E be a $(n-1)$-connected ($n \in \mathbf{N}$) simplicial set with a single vertex. Suppose that $m \le 2n - 1$. Put $G = FE$. Define the subgroups $P, M_s, J_s \subset \langle C(Q, |G|)\rangle$, $s \in \mathbf{N}$: put $P = \langle C(Q, |G_{(m)}|)\rangle$ (we have $C(Q, |G_{(m)}|) \subset C(Q, |G|)$), $M_s = \{B : \theta(B) \ge s\}$, and let J_s be generated by all elements of the form $(`b_1\text{'}-1)\ldots(`b_k\text{'}-1)$, where $k \in \mathbf{N}$, $b_l \in C(Q, |\gamma_{s_l} G|) \subset C(Q, |G|)$, and $s_1 + \ldots + s_k \ge s$. (M_s and J_s are ideals. Conjecture: $M_s \subset J_s$.) For a subgroup $S \subset \langle C(Q, |G|)\rangle$, let $[S] \subset \langle[Q, |G|]\rangle$ be its image under the homomorphism $[?]\colon \langle C(Q, |G|)\rangle \to \langle[Q, |G|]\rangle$.

(19.1) *For $s \in \mathbf{N}$, we have $[M_s] = [P \cap M_s] = [J_s]$.*

Proof. The inclusion $[M_s] \subset [J_s]$. Take an element $B \in M_s$. We have $\theta(B) \ge s$. By (18.2), there are a polyhedron L with the body Q and an ensemble $V \in$

$\langle G(L)\rangle$ with $\theta(V) \geq s$ and $[|V|] = [B]$. It suffices to show that $|V| \in J_s$. By (7.2), $\eta(V) \geq s$. Let $I_s \subset \langle G(L)\rangle$ be, as in § 7, the subgroup generated by all elements of the form $(`v_1\text{'}-1)\ldots(`v_k\text{'}-1)$, where $k \in \mathbf{N}$, $v_l \in (\gamma_{s_l}G)(L) \subset G(L)$, and $s_1 + \ldots + s_k \geq s$. By (7.1), $V \in I_s$. Obviously, $|V| \in J_s$.

The inclusion $[P \cap M_s] \supset [J_s]$. Take an element $B \in \langle C(Q, |G|)\rangle$, $B = (`b_1\text{'}-1)\ldots(`b_k\text{'}-1)$, where $k \in \mathbf{N}$, $b_l \in C(Q, |\gamma_{s_l}G|) \subset C(Q, |G|)$, and $s_1+\ldots+s_k \geq s$. Such elements generate J_s. Thus it suffices to show that $[B] \in [P \cap M_s]$. Choose a polyhedron K with the body Q. Since $\gamma_s G$ are Kan sets, there are sections $u_l \in (\gamma_{s_l}G)(K)$ with $[|u_l|] = [b_l]$ in $[Q, |G|]$. Put $U = (`u_1\text{'} - 1)\ldots(`u_k\text{'} - 1) \in \langle G(L)\rangle$. We have $[|U|] = [B]$ in $\langle [Q, |G|]\rangle$. By (7.1), $\eta(U) \geq s$. By (16.1), there are a polyhedron L with the body Q and an ensemble $V \in \langle G(L)\rangle$ with $\theta(V) \geq s$ and $[|V|] = [|U|]$ in $\langle [Q, |G|]\rangle$. Obviously, $|V| \in P$. By (18.1), $\theta(|V|) \geq s$. Thus, $[B] = [|V|]$ and $|V| \in P \cap M_s$. □

20. Step from $[Q, |G|]$ to $[X, Y]$

Let X be a finite CW-complex, $\dim X \leq m$ ($m \in \mathbf{N}$), and Y be an $(n-1)$-connected ($n \in \mathbf{N}$) CW-complex. Suppose that $m < 2n - 1$. We have the subgroups $L_s \subset \langle C(X, Y)\rangle$, $s \in \mathbf{N}$: $L_s = \{A : \theta(A) \geq s\}$. Let $B = (B_s)_{s=1}^{\infty}$ be the Curtis filtration of $[X, Y]$. For $s \in \mathbf{N}$, we have the subgroup $H_s \subset \langle [X, Y]\rangle$ generated by all elements of the form $(`w_1\text{'} - 1)\ldots(`w_k\text{'} - 1)$, where $k \in \mathbf{N}$, $w_l \in B_{s_l}$, and $s_1+\ldots+s_k \geq s$. (It is an ideal.) For a subgroup $R \subset \langle C(X, Y)\rangle$, let $[R] \subset \langle [X, Y]\rangle$ be its image under the homomorphism $[?] : \langle C(X, Y)\rangle \to \langle [X, Y]\rangle$.

(20.1) *We have* $[L_s] = H_s$, $s \in \mathbf{N}$.

Proof. There are a polyhedral body Q, $\dim Q \leq m$, and a homotopy equivalence $g : Q \to X$. Let $g' : X \to Q$ be a homotopy inverse map. There are a simplicial set E with a single vertex and a homotopy equivalence $k : Y \to |E|$. Put $G = FE$. Let $i : E \to G$ be the canonical simplicial map. By the Freudenthal theorem, it is $(2n-1)$-connected. The map $h = |i| \circ k : Y \to |G|$ is also $(2n-1)$-connected. Since $m \leq 2n-1$, there is a map $h' : |G_{(m)}| \to Y$ such that the map $h \circ h'$ is homotopic to the inclusion $|G_{(m)}| \to |G|$. We have the map $t : C(X, Y) \to C(Q, |G|)$, $t(a) = h \circ a \circ g$. Since $m < 2n - 1$, it induces an isomorphism $\bar{t} : [X, Y] \to [Q, |G|]$. We have the map $t' : C(Q, |G_{(m)}|) \to C(X, Y)$, $t'(b) = h' \circ b \circ g'$. For $b \in C(Q, |G_{(m)}|) \subset C(Q, |G|)$, we have $[t'(b)] = \bar{t}^{-1}([b])$. One can see that

$$\bar{t}(B_s) = \{[b] \in [Q, |G|] : b \in C(Q, |\gamma_s G|) \subset C(Q, |G|)\}, \qquad s \in \mathbf{N}. \qquad (*)$$

Let $P, M_s, J_s \subset \langle C(Q, |G|)\rangle$ be as in § 19. We have the homomorphisms $\langle t\rangle : \langle C(X, Y)\rangle \to \langle C(Q, |G|)\rangle$ and $\langle t'\rangle : P = \langle C(Q, |G_{(m)}|)\rangle \to \langle C(X, Y)\rangle$. By (17.1), $\langle t\rangle(L_s) \subset M_s$, and $\langle t'\rangle(P \cap M_s) \subset L_s$. We have the ring isomorphism $\langle \bar{t}\rangle : \langle [X, Y]\rangle \to \langle [Q, |G|]\rangle$. It follows from $(*)$ that $\langle \bar{t}\rangle(H_s) = [J_s]$. Using (19.1), we get $\langle \bar{t}\rangle([L_s]) = [\langle t\rangle(L_s)] \subset [M_s] = [J_s] = \langle \bar{t}\rangle(H_s)$. Hence $[L_s] \subset H_s$, and $[L_s] \supset [\langle t'\rangle(P \cap M_s)] = \langle \bar{t}^{-1}\rangle([P \cap M_s]) = \langle \bar{t}\rangle^{-1}([J_s]) = H_s$. □

Proof of Theorem (1.1). We have the homomorphism $\bar{f} : \langle [X, Y]\rangle \to U$, $\bar{f}(`w\text{'}) = f(w)$. By (17.2), the condition $\operatorname{ord} f < s$ ($s \in \mathbf{N}_+$) is equivalent to the condition

$\bar{f}|[L_s] = 0$. Obviously, the condition $\deg_B f < s$ is equivalent to the condition $\bar{f}|H_s = 0$. Now note that $[L_s] = H_s$ by (20.1). $\square$

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