An iterated sum formula for a spheroid's homotopy class modulo 2–torsion

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Abstract

Let X be a simply connected pointed space with finitely generated homotopy groups. Let $\Pi_n(X)$ denote the set of all continuous maps $a\colon I^n \to X$ taking ∂I^n to the basepoint. For $a \in \Pi_n(X)$, let $[a] \in \pi_n(X)$ be its homotopy class. For an open set $E \subset I^n$, let $\Pi(E, X)$ be the set of all continuous maps $a\colon E \to X$ taking $E \cap \partial I^n$ to the basepoint. For a cover Γ of I^n, let $\Gamma(r)$ be the set of all unions of at most r elements of Γ. Put $r = (n-1)!$. We prove that for any finite open cover Γ of I^n there exist maps $f_E\colon \Pi(E, X) \to \pi_n(X) \otimes \mathbf{Z}[1/2]$, $E \in \Gamma(r)$, such that

$$[a] \otimes 1 = \sum_{E \in \Gamma(r)} f_E(a|_E)$$

for all $a \in \Pi_n(X)$.

1. Introduction

Let X be a (pointed) space. An *n–spheroid* in X is a (continuous) map $a\colon I^n \to X$ taking ∂I^n to the basepoint. Let $\Pi_n(X)$ denote the set of all n–spheroids in X. For $a \in \Pi_n(X)$, let $[a] \in \pi_n(X)$ be its homotopy class. For an open set $E \subset I^n$, let $\Pi(E, X)$ be the set of all maps $a\colon E \to X$ taking $E \cap \partial I^n$ to the basepoint. For a cover Γ of I^n, let $\Gamma(r)$ be the set of all unions of at most r elements of Γ.

Let L be an abelian group. Consider a functional (i. e. a function) $f\colon \Pi_n(X) \to L$. We define the *degree* of f (denoted $\deg f$) to be the infimum of r such that for any finite open cover Γ of I^n there exist functionals $f_E\colon \Pi(E, X) \to L$, $E \in \Gamma(r)$, such that

$$f(a) = \sum_{E \in \Gamma(r)} f_E(a|_E)$$

for all $a \in \Pi_n(X)$. We are interested in functionals of finite degree.

This definition is motivated by the notion of order-restricted perceptron [2]. For relation to Vassiliev knot invariants, see [5]. An example of a functional $f\colon \Pi_n(X) \to \mathbf{R}$ with $\deg \leqslant r$ is given by

$$f(a) = \int_{u_1, \dots, u_r \in I^n} p(u_1, a(u_1), \dots, u_r, a(u_r))\, du_1 \dots du_r,$$

where $p\colon (I^n \times X)^r \to \mathbf{R}$ is any decent (say, bounded) measurable function.

We prove the following result.

1.1. Theorem. *Suppose X is simply connected and has finitely generated homotopy groups. Define $q\colon \Pi_n(X) \to \pi_n(X) \otimes \mathbf{Z}[1/2]$ by $q(a) = [a] \otimes 1$. Then $\deg q \leqslant (n-1)!$.*

This may be viewed as a finite sum version of Chen's iterated integrals. For a related result, see [3]. Possibly, the $\mathbf{Z}[1/2]$ factor and the factorial sign may be removed. Claim 1.2 implies that $\deg q \geqslant n-1$ for $X = S^2 \vee S^2$.

1.2. Claim. *Let $f\colon \Pi_n(X) \to L$ be a homotopy invariant functional with $\deg f \leqslant r$. Let $b_0, b \in \Pi_n(X)$ be spheroids with*

$$[b_0] = 0, \quad [b] = [\![\dots [\![[\![v_0, v_1]\!], v_2]\!], \dots, v_r]\!],$$

where $v_s \in \pi_{k_s}(X)$ ($k_0 + \ldots + k_r = n + r$; $[\![\cdot,\cdot]\!]$ denotes the Whitehead product). Then $f(b) = f(b_0)$.

(A functional $f\colon \Pi_n(X) \to L$ is *homotopy invariant* if $f(a)$ depends only on $[a]$.)

Proof (cf. [2, Theorem 3.2]). Put

$$T = S^{k_0} \vee \ldots \vee S^{k_r}.$$

Let $g\colon T \to X$ be a map whose restriction to the sth wedge summand represents v_s. Let $e_s \in \pi_{k_s}(T)$ be the elements represented by the canonical embeddings $S^{k_s} \to T$. Let $z \in \Pi_n(T)$ be a spheroid with $[z] = [\![\dots [\![[\![e_0, e_1]\!], e_2]\!], \dots, e_r]\!]$. We have $[g \circ z] = [b]$. For every k, choose maps $p_i^k\colon S^k \to S^k$, $i = 0, 1$, such that p_i^k has degree i and $p_0^k|_{U^k} = p_1^k|_{U^k}$ for some open neighbourhood U^k of the basepoint. Put

$$p_{i_0,\dots,i_r} = p_{i_0}^{k_0} \vee \ldots \vee p_{i_r}^{k_r} \colon T \to T, \qquad i_0, \dots, i_r = 0, 1.$$

We have

$$[p_{i_0,\dots,i_r} \circ z] = i_0 \dots i_r [z]$$

in $\pi_n(T)$ and thus

$$[g \circ p_{i_0,\dots,i_r} \circ z] = i_0 \dots i_r [b]$$

in $\pi_n(X)$. The sets

$$V_s = U^{k_0} \vee \ldots \vee S^{k_s} \vee \ldots \vee U^{k_r}, \qquad s = 0, \dots, r,$$

form an open cover of T. Put $\Gamma = \{\, z^{-1}(V_s) \mid s = 0, \dots, r \,\}$. Since $\deg f \leqslant r$, there are functionals $f_E\colon \Pi(E, X) \to L$, $E \in \Gamma(r)$, such that

$$f(a) = \sum_{E \in \Gamma(r)} f_E(a|_E)$$

for all $a \in \Pi_n(X)$. We have

$$(-1)^r(f(b_0) - f(b)) = \sum_{i_0,\dots,i_r=0,1} (-1)^{i_0+\dots+i_r} f(g \circ p_{i_0,\dots,i_r} \circ z) =$$
$$= \sum_{E\in\Gamma(r)} \sum_{i_0,\dots,i_r=0,1} (-1)^{i_0+\dots+i_r} f_E(g \circ p_{i_0,\dots,i_r} \circ z|_E).$$

Take $E \in \Gamma(r)$. We have

$$z(E) \subset S^{k_0} \vee \ldots \vee U^{k_s} \vee \ldots \vee S^{k_r}$$

for some s. Thus $p_{i_0,\dots,i_r} \circ z|_E$ does not depend on i_s. Thus the inner sum equals zero. $\square$

Conventions and notation. *Maps* are continuous, unlike *functions*. A *space* is a pointed space. A *subspace* contains the basepoint. Maps between spaces are basepoint preserving. This applies also to homotopies etc. A *cell space* is a pointed CW complex. $\mathrm{sk}_n X$ denotes the n-skeleton of a cell space X.

The homotopy relation is denoted $\sim$; $\sim_A$ is used for homotopy rel A. For a pair (X, A), $\mathrm{in}_{(X,A)} : A \to X$ and $\mathrm{pr}_{(X,A)} : X \to X/A$ are the inclusion and the projection. The subscript of in and pr is often omitted.

2. Making a loop space simply connected

The aim of this section is to prove Corollary 2.11. First fix some notation.

Homotopy fibres and cofibres. Let X and Y be spaces, $g : X \to Y$ be a map. Then we have the homotopy fibre sequence

$$F(g) \xrightarrow{p(g)} X \xrightarrow{g} Y,$$

where $F(g) = \{ (x, v) \in X \times PY \mid g(x) = v(1) \}$ and $p(g)$ is the fibraton defined by $p(g)(x, v) = x$. We have the homotopy cofibre sequence

$$X \xrightarrow{g} Y \xrightarrow{i(g)} C(g),$$

where $C(g)$ is the unreduced cone of g and $i(g)$ is the canonical embedding.

2.A. A Moore space

Fix $d > 0$. Let M_d be the space obtained from S^1 by attaching a 2-cell via a map $S^1 \to S^1$ of degree d. Our aim here is to prove Corollary 2.5.

2.1. Lemma. *Let*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a fibre sequence. Suppose B and E are path-connected, p induces an isomorphism on π_1, and the canonical action of $\pi_1(B)$ on $\pi_2(B)$ is trivial. Then the canonical action of $\pi_1(B)$ on $H_1(F)$ is trivial.

Proof. The group $\pi_1(E)$ acts canonically on $\pi_1(F)$ (see [1, 5.1.7.3]); it also acts on $\pi_2(B)$ and $H_1(F)$ through $p_*\colon \pi_1(E) \to \pi_1(B)$. The boundary homomorphism ∂ and the Hurewicz homomorphism h

$$\pi_2(B) \xrightarrow{\partial} \pi_1(F) \xrightarrow{h} H_1(F)$$

respect these actions (regarding ∂, see [1, 5.1.8.4]). Since ∂, h, and p_* are epimorphisms, $\pi_1(B)$ acts trivially on $H_1(F)$. □

2.2. Claim. $H_2(\Omega\Sigma M_d) \cong \mathbf{Z}_d$.

Easily seen from the homology spectral sequence of the path fibration of ΣM_d. □

2.3. Claim. *If d is odd, $\pi_3(\Sigma M_d) \cong \mathbf{Z}_d$.*

Proof. There is a fibration $U \to \Sigma M_d$ with fibre of homotopy type $K(\mathbf{Z}_d, 1)$ and U 2–connected. We have $\pi_3(\Sigma M_d) \cong H_3(U)$. Using the cohomology spectral sequence, we get[1] $H_3(U) \cong \mathbf{Z}_d$. □

2.4. Claim. *Suppose d is odd. Then the canonical embedding $j\colon M_d \to \Omega\Sigma M_d$ induces zero homomorphism on π_2.*

Proof. Consider the homotopy fibre sequence

$$F(j) \xrightarrow{p(j)} M_d \xrightarrow{j} \Omega\Sigma M_d.$$

By Lemma 2.1, the action of $\pi_1(\Omega\Sigma M_d)$ on $H_1(F(j))$ is trivial. Using the homology spectral sequence and Claim 2.2, we get $H_1(F(j)) \cong \mathbf{Z}_d$. The boundary homomorphism $\partial\colon \pi_2(\Omega\Sigma M_d) \to \pi_1(F(j))$ is an epimorhism. Thus $\pi_1(F(j))$ is abelian and thus isomorphic to $\mathbf{Z}_d$. By Claim 2.3, $\pi_2(\Omega\Sigma M_d) \cong \mathbf{Z}_d$. Thus ∂ is an isomorphism. Thus $j_*\colon \pi_2(M_d) \to \pi_2(\Omega\Sigma M_d)$ is zero. □

2.5. Corollary. *Suppose d is odd. Let X be a space, $f\colon M_d \to \Omega X$ be a map. Then f induces zero homomorphism on π_2.*

Proof. Let $g\colon \Sigma M_d \to X$ be the map adjoint to f. Then $f = \Omega g \circ j$, where j is as in Claim 2.4. □

2.B. Two technical lemmas

2.6. Lemma. *Let X be a space, $A \subset X$ be a closed subspace such that (X, A) is a Borsuk pair. Suppose X and A are homotopy equivalent to cell spaces of dimension at most n and $n-1$, respectively. Let Y be a cell space and $f\colon X \to Y$*

[1] This does not work for d even. This results in the $\mathbf{Z}[1/2]$ factor in Theorem 1.1.

be a map with $f(A) \subset \operatorname{sk}_n Y$*. Then* f *is rel* A *homotopic to a map* $h\colon X \to Y$ *with* $h(X) \subset \operatorname{sk}_n Y$*.*

This is used only for $n = 3$.

Proof. The map f is homotopic to a map $g\colon X \to Y$ with $g(X) \subset \operatorname{sk}_n Y$. Let $k\colon X \times [0,1] \to Y$ be the corresponding homotopy:

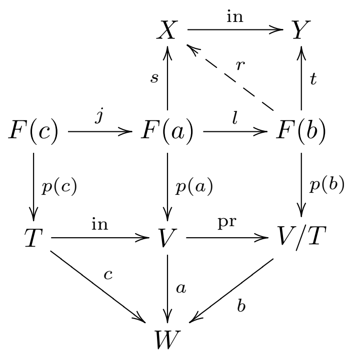
$$k(x,0) = f(x), \quad k(x,1) = g(x), \qquad x \in X.$$

The map $k|_{A\times[0,1]}$ is rel $A \times \{0,1\}$ homotopic to a map $q\colon A \times [0,1] \to Y$ with image in $\operatorname{sk}_n Y$. Since (X, A) is a Borsuk pair, there is a map $l\colon X \times [1,2] \to Y$ with image in $\operatorname{sk}_n Y$ such that $l(x,1) = g(x)$, $x \in X$, and $l(x, 1+t) = q(x, 1-t)$, $x \in A$, $t \in [0,1]$. Let $m\colon X \times [0,2] \to Y$ be the map with $m|_{X\times[0,1]} = k$, $m|_{X\times[1,2]} = l$. The map $m|_{A\times[0,2]}$ is rel $A \times \{0,2\}$ homotopic to the map $e\colon A \times [0,2] \to Y$, $e(x,t) = f(x)$. By the Strøm theorem [4, Lecture I, Proposition 2],

$$(X \times [0,2], X \times \{0,2\} \cup A \times [0,2])$$

is a Borsuk pair. Thus there is a map $\tilde{m}\colon X \times [0,2] \to Y$ with $\tilde{m}|_{X\times\{0,2\}} = m|_{X\times\{0,2\}}$, $\tilde{m}|_{A\times[0,2]} = e$. Put $h(x) = \tilde{m}(x,2)$. Then $\tilde{m}$ is the desired homotopy. □

2.7. Lemma. *Let* V *be a Hausdorff space,* $T \subset V$ *be a compact subspace. Let* W *be a space,* $a\colon V \to W$*,* $b\colon V/T \to W$*,* $c\colon T \to W$ *be maps such that* $b \circ \operatorname{pr}_{(V,T)} = a$*,* $c = a|_T$ *(so* c *is constant). Let* $j\colon F(c) \to F(a)$ *and* $l\colon F(a) \to F(b)$ *be the maps induced by* $\operatorname{in}_{(V,T)}$ *and* $\operatorname{pr}_{(V,T)}$*, respectively. Let* Y *be a space,* $X \subset Y$ *be a subspace,* $s\colon F(a) \to X$ *and* $t\colon F(b) \to Y$ *be maps such that* $\operatorname{in}_{(Y,X)} \circ s$ *and* $t \circ l$ *are rel* $j(F(c))$ *homotopic. Then there exists a unique map* $r\colon F(b) \to X$ *such that* $r \circ l = s$*. The maps* $\operatorname{in}_{(Y,X)} \circ r$ *and* t *are homotopic rel* $p(b)^{-1}(q_0)$*, where* $q_0 \in V/T$ *is the basepoint.*



This is because l is a quotient map. □

2.C. A $K(G,1)$ space and some its subquotients

Let G be a finitely generated abelian group and

$$G = G_1 \oplus \ldots \oplus G_r$$

be its decomposition into cyclic summands. For each $s = 1, \ldots, r$, take a cell space V_s of homotopy type $K(G_s, 1)$ such that $\mathrm{sk}_2\, V_s$ is either S^1 or M_d (see 2.A) for proper d. Put $T_s = \mathrm{sk}_1\, V_s$ $(= S^1)$, $U_s = \mathrm{sk}_3\, V_s$,

$$V = V_1 \times \ldots \times V_r, \quad T = T_1 \times \ldots \times T_r, \quad U = U_1 \times \ldots \times U_r.$$

We have $T \subset U \subset V$. V is a $K(G,1)$ space.

2.8. Claim. *Let X be a simply connected space with $\pi_2(X) \cong G$, Q be a space of weak homotopy type $K(G,2)$, $b\colon X \to Q$ be a map inducing an isomorphism on π_2, and $m\colon V \to \Omega Q$ be a weak homotopy equivalence. Suppose G has no 2–torsion. Then there exists a map $f\colon U \to \Omega X$ such that $\Omega b \circ f \sim m|_U$.*

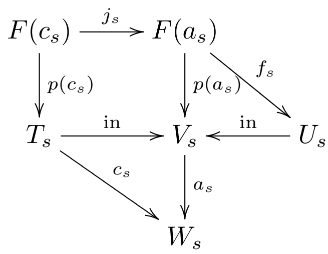
$$\begin{array}{ccc} U & \xrightarrow{\ \mathrm{in}\ } & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle m} \\ \Omega X & \xrightarrow{\ \Omega b\ } & \Omega Q \end{array}$$

Proof. We naturally have $U_s \subset U$, $s = 1, \ldots, r$. For every s, there is a map $f'_s\colon \mathrm{sk}_2\, U_s \to \Omega X$ such that $\Omega b \circ f'_s$ and $m|_{\mathrm{sk}_2\, U_s}$ induce the same homomorphism on π_1. By Corollary 2.5, f'_s induces zero homomorphism on π_2. Thus it extends to a map $f_s\colon U_s \to \Omega X$. Since ΩX is a loop space, there is a map $f\colon U \to \Omega X$ such that $f|_{U_s} = f_s$ for every s. The maps $\Omega b \circ f$ and $m|_U$ are homotopic since they induce the same homomorphism on π_1. $\square$

2.9. Lemma. *For every q, there exist a cell space Z and a map $k\colon Z \to U/T$ such that $\mathrm{in}_{(V/T, U/T)} \circ k$ induces an isomorphism $\pi_q(Z) \to \pi_q(V/T)$.*

Proof. For $q \leqslant 2$, put $Z = U/T$, $k = \mathrm{id}$. We shall construct a single k to serve all $q > 2$.

Take some s. If G_s if finite, let W_s be a space of homotopy type $K(\mathbf{Z}, 2)$; otherwise, let W_s be a point. There is a map $a_s\colon V_s \to W_s$ with $F(a_s)$ homotopy equivalent to S^1. The map $c_s = a_s|_{T_s}$ is null-homotopic. We choose a_s in such a way that c_s is constant. Let $j_s\colon F(c_s) \to F(a_s)$ be the map induced by $\mathrm{in}_{(V_s, T_s)}$. Since c_s is constant, $F(c_s) = T_s \times \Omega W_s$. Thus $F(c_s)$ is homotopy equivalent to a cell space of dimension at most 2. By [4, Lecture II, Proposition 5], $(F(a_s), j_s(F(c_s)))$ is a Borsuk pair. By Lemma 2.6, there is a map $f_s\colon F(a_s) \to U_s$ such that $\mathrm{in}_{(V_s, U_s)} \circ f_s$ is rel $j_s(F(c_s))$ homotopic to $p(a_s)$.



Put

$$W = W_1 \times \ldots \times W_r, \quad a = a_1 \times \ldots \times a_r \colon V \to W, \quad c = c_1 \times \ldots \times c_r \colon T \to W.$$

Since $a|_T$ is constant, there is a map $b\colon V/T \to W$ such that $b \circ \mathrm{pr}_{(V,T)} = a$. Let $j\colon F(c) \to F(a)$ and $l\colon F(a) \to F(b)$ be the maps induced by $\mathrm{in}_{(V,T)}$ and $\mathrm{pr}_{(V,T)}$, respectively.

We make natural identifications

$$F(a) = F(a_1) \times \ldots \times F(a_r), \quad F(c) = F(c_1) \times \ldots \times F(c_r).$$

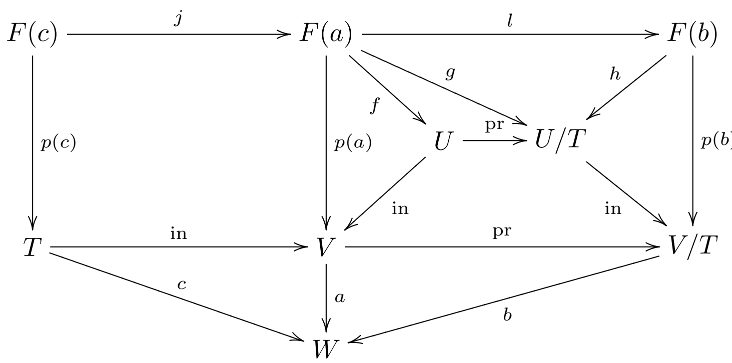
Then $p(a) = p(a_1) \times \ldots \times p(a_r)$ and $j = j_1 \times \ldots \times j_r$. Put

$$f = f_1 \times \ldots \times f_r \colon F(a) \to U.$$

The map $\mathrm{in}_{(V,U)} \circ f$ is rel $j(F(c))$ homotopic to $p(a)$. Put $g = \mathrm{pr}_{(U,T)} \circ f$. We have

$$\mathrm{in}_{(V/T,U/T)} \circ g = \mathrm{pr}_{(V,T)} \circ \mathrm{in}_{(V,U)} \circ f \sim_{j(F(c))} \mathrm{pr}_{(V,T)} \circ p(a) = p(b) \circ l.$$

By Lemma 2.7, there is a map $h\colon F(b) \to U/T$ such that $h \circ l = g$ and $\mathrm{in}_{(V/T,U/T)} \circ h \sim p(b)$. Since $\pi_q(W) = 0$ for $q \neq 2$, $p(b)$ induces isomorphisms on π_q, $q > 2$. Thus the map $\mathrm{in}_{(V/T,U/T)} \circ h$ does so. Thus for any $q > 2$ we may let $e\colon Z \to F(b)$ be a cell approximation and put $k = h \circ e$. $\square$



2.D. Treating a loop space

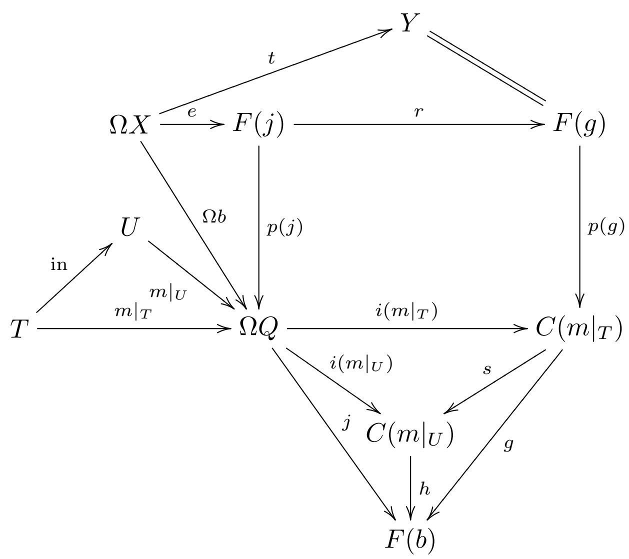
2.10. Lemma. *Let X be a simply connected space with finitely generated homotopy groups. Suppose $\pi_2(X)$ has no 2–torsion. Then there exist a simply connected space Y with finitely generated homotopy groups and a map $t\colon \Omega X \to Y$ inducing split monomorphisms on π_q, $q > 1$.*

(It follows easily that Y may be obtained by attaching 2–cells to ΩX.)

Proof. Put $G = \pi_2(X)$. There are a space Q of weak homotopy type $K(G, 2)$ and a map $b\colon X \to Q$ inducing an isomorphism on π_2. Consider the piece of the Puppe sequence of b:

$$\Omega X \xrightarrow{\Omega b} \Omega Q \xrightarrow{j} F(b).$$

There is a standard homotopy equivalence $e\colon \Omega X \to F(j)$ such that $p(j) \circ e = \Omega b$. Let V be the $K(G, 1)$ space considered in 2.C. Let $T \subset U \subset V$ be as there. There is a weak homotopy equivalence $m\colon V \to \Omega Q$. By Claim 2.8, $m|_U$ lifts (up to homotopy) along Ωb. Thus $j \circ m|_U$ is null-homotopic. Thus there is a map $h\colon C(m|_U) \to F(b)$ such that $h \circ i(m|_U) = j$. Let $s\colon C(m|_T) \to C(m|_U)$ be the map induced by $\mathrm{in}_{(U,T)}$. Put $g = h \circ s$. Let $r\colon F(j) \to F(g)$ be the map induced by $i(m|_T)$. Put $Y = F(g)$, $t = r \circ e$.



Let us check the desired properties. Since b is 3–connected, $F(b)$ is 2–connected. Since $m|_T$ is 1–connected, $C(m|_T)$ is 1–connected. Therefore g is 2–connected. Thus $F(g)$ is 1–connected, i. e. Y is simply connected. One checks similarly that $\pi_q(Y)$ are finitely generated.

We have the commutative diagram

$$\begin{array}{ccccc}
V/T & \xleftarrow{\ \rho\ } & C(\mathrm{in}_{(V,T)}) & \xrightarrow{\ m'\ } & C(m|_T) \\
{\scriptstyle \mathrm{pr}_{(V/T,U/T)}}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle s} \\
V/U & \xleftarrow{\ \sigma\ } & C(\mathrm{in}_{(V,U)}) & \xrightarrow{\ m''\ } & C(m|_U),
\end{array}$$

where ϕ is induced by $\mathrm{in}_{(U,T)}$, ρ and σ are the standard homotopy equivalences (contractions), m' and m'' are the weak equivalences induced by m. Take $q > 1$. Using Lemma 2.9, we get a cell space Z_q and a map $l_q\colon Z_q \to V/T$ inducing an isomorphism on π_q and such that $\mathrm{pr}_{(V/T,U/T)} \circ l_q$ is constant. Using the diagram, we get a map $v_q\colon Z_q \to C(m|_T)$ inducing an isomorphism on π_q and such that $s \circ v_q$ is null-homotopic. Since $g = h \circ s$, $g \circ v_q$ is null-homotopic. Thus v_q lifts along $p(g)$: there is a map $w_q\colon Z_q \to F(g)$ such that $p(g) \circ w_q = v_q$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & 0 \\
 & & \| & & & & & & \| \\
 & & \pi_{q+1}(\Omega Q) & \xrightarrow{j_*} & \pi_{q+1}(F(b)) & \xrightarrow{\partial_j} & \pi_q(F(j)) & \xrightarrow{p(j)_*} & \pi_q(\Omega Q) \\
 & & \downarrow{\scriptstyle i(m|_T)_*} & & \| & & \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle i(m|_T)_*} \\
\pi_{q+1}(F(g)) & \longrightarrow & \pi_{q+1}(C(m|_T)) & \xrightarrow{g_*} & \pi_{q+1}(F(b)) & \xrightarrow{\partial_g} & \pi_q(F(g)) & \longrightarrow & \pi_q(C(m|_T)) \\
 & {\scriptstyle w_{q+1\,*}}\nwarrow & \uparrow{\scriptstyle v_{q+1\,*}} & & & & & {\scriptstyle w_{q\,*}}\nwarrow & \uparrow{\scriptstyle v_{q\,*}} \\
 & & \pi_{q+1}(Z_{q+1}) & & & & & & \pi_q(Z_q),
\end{array}$$

where the rows are pieces of the exact sequences of the maps j and g. We see that ∂_j is an isomorphism and ∂_g is a split monomorphism. Thus r_* is a split monomorphism. Since $t = r \circ e$, where e is a homotopy equivalence, t induces a split monomorphism on π_q. $\square$

2.11. Corollary. *Let X be a simply connected space with finitely generated homotopy groups. Then there exist a simply connected space Y with finitely generated homotopy groups and a map $t\colon \Omega X \to Y$ such that for every $q > 1$ the homomorphism $t_* \otimes \mathrm{id}\colon \pi_q(\Omega X) \otimes \mathbf{Z}[1/2] \to \pi_q(Y) \otimes \mathbf{Z}[1/2]$ is a split monomorphism.*

Proof. By attaching 3– and 4–cells to X, we may quotient H_2 by 2–torsion without changing H_q for $q > 2$. By Serre, $\pi_q \otimes \mathbf{Z}[1/2]$ are not changed. Then apply Lemma 2.10. $\square$

3. Dimension descent

3.A. Concentrating spheroids near a codimension 2 net in I^n

3.1. Lemma. *Let X be a simply connected space, Γ be a finite open cover of I^n $(n > 1)$. Then there exist a finite open cover Δ of I^{n-1}, a function $\lambda\colon \Delta \to \Gamma(n-1)$, and a function $\Phi\colon \Pi_n(X) \to \Pi_n(X)$ such that (1) $[\Phi(a)] = [a]$ for all $a \in \Pi_n(X)$ and (2) for $F \in \Delta$, $a, a' \in \Pi_n(X)$, the implication holds*

$$a|_{\lambda(F)} = a'|_{\lambda(F)} \quad \Rightarrow \quad \Phi(a)|_{F\times I} = \Phi(a')|_{F\times I}.$$

Proof. Choose $\epsilon > 0$ such that every open ball $B(u, 3\epsilon)$ $(u \in I^n)$ is contained in some $E \in \Gamma$. Choose a rectilinear triangulation T of I^n with simplices of diameter less than ϵ. Let T' be the barycentric subdivision of T. By $\operatorname{cosk}^2 T$ we denote the union of all simplices of T' that do not intersect $\operatorname{sk}_1 T$. (So $\operatorname{cosk}^2 T$ is a polyhedron of dimension $n-2$; its complement collapses to $\operatorname{sk}_1 T$.)

There is a homeomorphism $l\colon I^n \to I^n$ preserving simplices of T and such that

$$|v \times I \cap l^{-1}(\operatorname{cosk}^2 T)| \leqslant n-1$$

for every $v \in I^{n-1}$. (Such l is obtained by generic simplex-wise perturbation of the identity map.) The map l preserves ∂I^n and has degree 1.

For $v \in I^{n-1}$, let $2\rho(v)$ be the least positive distance between $v \times I$ and $l^{-1}(t)$, where t runs over simplices of T'. Let Δ be a finite cover of I^{n-1} formed by some of the balls $B(v, \rho(v))$, $v \in I^{n-1}$.

Let us construct the function λ. Take $F \in \Delta$. We have $F = B(v, \rho(v))$ for some $v \in I^{n-1}$. For each

$$u \in v \times I \cap l^{-1}(\operatorname{cosk}^2 T)$$

(there are at most $n-1$ such u), choose $E \in \Gamma$ containing $B(u, 3\epsilon)$. Put $\lambda(F)$ be the union of these E.

Choose $\delta > 0$ such that $\delta \leqslant \epsilon$ and $\delta \leqslant \rho(v)$ for $B(v, \rho(v)) \in \Delta$. Let Q be the open δ–neighbourhood of $l^{-1}(\operatorname{cosk}^2 T)$. Put $P = l(I^n \setminus Q)$. The set P is closed and does not intersect $\operatorname{cosk}^2 T$. Thus there is a map $k\colon I^n \to I^n$ preserving the simplices of T with $k(P) \subset \operatorname{sk}_1 T$. The map k preserves ∂I^n and has degree 1.

Take $a \in \Pi_n(X)$. We homotop a to get a map $\Theta(a)$ taking $\operatorname{sk}_1 T$ to the basepoint. The homotopy is constructed by induction on the skeleta of T. There are no obstructions since X is simply connected. To extend the homotopy to some simplex s of T, we need to know only $a|_s$ and the homotopy constructed on ∂s (we need no information from the outside of s).

Therefore there is a function $\Theta\colon \Pi_n(X) \to \Pi_n(X)$ such that (a) $[\Theta(a)] = [a]$ for all $a \in \Pi_n(X)$, (b) for every simplex s of T and any $a, a' \in \Pi_n(X)$, the implication holds

$$a|_s = a'|_s \quad \Rightarrow \quad \Theta(a)|_s = \Theta(a')|_s,$$

and (c) for any $a \in \Pi_n(X)$, $\Theta(a)$ takes $\operatorname{sk}_1 T$ to the basepoint.

For $a \in \Pi_n(X)$, put $\Phi(a) = \Theta(a) \circ k \circ l$. The property (1) is obvious. Let us check the property (2). Take $F \in \Delta$ and $a, a' \in \Pi_n(X)$ such that $a|_{\lambda(F)} = a'|_{\lambda(F)}$. We should show that $\Phi(a)|_{F\times I} = \Phi(a')|_{F\times I}$. Take $u_0 \in F \times I$. Let us show that $\Phi(a)(u_0) = \Phi(a')(u_0)$.

If $l(u_0) \in P$, then $k(l(u_0)) \in \operatorname{sk}_1 T$ and thus $\Phi(a)(u_0) = \Phi(a')(u_0) = x_0$, where $x_0 \in X$ is the basepoint. Consider the converse case. Since $P = l(I^n \setminus Q)$, we have $u_0 \in Q$. Thus $\operatorname{dist}(u_0, l^{-1}(\operatorname{cosk}^2 T)) < \delta$. Thus there is a simplex t of $\operatorname{cosk}^2 T$ such that $\operatorname{dist}(u_0, l^{-1}(t)) < \delta$. We have $F = B(v, \rho(v))$ for some $v \in I^{n-1}$. We have

$$\operatorname{dist}(v \times I, l^{-1}(t)) \leqslant \operatorname{dist}(v \times I, u_0) + \operatorname{dist}(u_0, l^{-1}(t)) < \rho(v) + \delta \leqslant 2\rho(v).$$

By definition of $\rho(v)$, this means that $\operatorname{dist}(v \times I, l^{-1}(t)) = 0$. Thus there is a point $u \in v \times I \cap l^{-1}(t)$. Since l preserves simplices of T, $\operatorname{diam} l^{-1}(t) < \epsilon$. We have

$$\operatorname{dist}(u_0, u) \leqslant \operatorname{dist}(u_0, l^{-1}(t)) + \operatorname{diam} l^{-1}(t) < \delta + \epsilon \leqslant 2\epsilon.$$

The point u_0 belongs to some simplex s of T. We have

$$s \subset B(u_0, \epsilon) \subset B(u, 3\epsilon) \subset \lambda(F).$$

Thus $a|_s = a'|_s$. Thus $\Theta(a)|_s = \Theta(a')|_s$. Since k and l preserve s, $\Phi(a)|_s = \Phi(a')|_s$, which suffices. $\square$

3.B. Functionals of finite degree

For a space X, the formula $\Xi(a)(t_1, \ldots, t_{n-1})(t) = a(t_1, \ldots, t_{n-1}, t)$ $(t_1, \ldots, t_{n-1}, t \in I)$ defines a bijection $\Xi\colon \Pi_n(X) \to \Pi_{n-1}(\Omega X)$, which we call *standard*. The induced isomorphism $\xi\colon \pi_n(X) \to \pi_{n-1}(\Omega X)$ we also call standard. For an open set $F \subset I^{n-1}$, the bijection $\Xi_F\colon \Pi(F \times I, X) \to \Pi(F, \Omega X)$ defined by that formula is also called standard.

3.2. Corollary. *Let X be a simply connected space, L be an abelian group, $g\colon \Pi_{n-1}(\Omega X) \to L$ be a homotopy invariant functional ($n > 1$). Let $\Xi\colon \Pi_n(X) \to \Pi_{n-1}(\Omega X)$ be the standard bijection. Then* $\deg g \circ \Xi \leqslant (n-1) \deg g$.

Proof. Suppose $\deg g \leqslant r$. Let us show that $\deg g \circ \Xi \leqslant (n-1)r$. Let Γ be a finite open cover of I^n. By Lemma 3.1, there are a finite open cover Δ of I^{n-1}, a function $\lambda\colon \Delta \to \Gamma(n-1)$, and a function $\Phi\colon \Pi_n(X) \to \Pi_n(X)$ satisfying the conditions (1), (2) of the lemma.

For each $F \in \Delta(r)$, choose a decomposition $F = F_1 \cup \ldots \cup F_s$, $0 \leqslant s \leqslant r$, with $F_1, \ldots, F_s \in \Delta$ and put $\mu(F) = \lambda(F_1) \cup \ldots \cup \lambda(F_s)$. So we have a function $\mu\colon \Delta(r) \to \Gamma((n-1)r)$. It follows from the condition (2) that for $F \in \Delta(r)$, $a, a' \in \Pi_n(X)$ the implication holds

$$a|_{\mu(F)} = a'|_{\mu(F)} \quad \Rightarrow \quad \Phi(a)|_{F\times I} = \Phi(a')|_{F\times I}.$$

Thus for every $F \in \Delta(r)$ there is a function $\Phi_F\colon \Pi(\mu(F), X) \to \Pi(F \times I, X)$ such that $\Phi_F(a|_{\mu(F)}) = \Phi(a)|_{F\times I}$ for all $a \in \Pi_n(X)$. For every $F \in \Delta(r)$, we have the commutative diagram

$$\begin{array}{ccccc}
\Pi_n(X) & \xrightarrow{\Phi} & \Pi_n(X) & \xrightarrow{\Xi} & \Pi_{n-1}(\Omega X) \\
\downarrow & & \downarrow & & \downarrow \\
\Pi(\mu(F), X) & \xrightarrow{\Phi_F} & \Pi(F \times I, X) & \xrightarrow{\Xi_F} & \Pi(F, \Omega X),
\end{array}$$

where Ξ_F is the standard bijection and the vertical arrows are the restriction functions.

Since $\deg g \leqslant r$, there are functionals $g_F \colon \Pi(F, \Omega X) \to L$, $F \in \Delta(r)$, such that

$$g(b) = \sum_{F \in \Delta(r)} g_F(b|_F)$$

for all $b \in \Pi_{n-1}(\Omega X)$. For $E \in \Gamma((n-1)r)$, define a functional $f_E \colon \Pi(E, X) \to L$ by

$$f_E(a) = \sum_{F \in \mu^{-1}(E)} g_F(\Xi_F(\Phi_F(a))).$$

For $a \in \Pi_n(X)$, we have

$$g(\Xi(a)) = g(\Xi(\Phi(a))) = \sum_{F \in \Delta(r)} g_F(\Xi(\Phi(a))|_F) = \sum_{F \in \Delta(r)} g_F(\Xi_F(\Phi_F(a|_{\mu(F)}))) =$$

$$= \sum_{E \in \Gamma((n-1)r)} \sum_{F \in \mu^{-1}(E)} g_F(\Xi_F(\Phi_F(a|_E))) = \sum_{E \in \Gamma((n-1)r)} f_E(a|_E).$$

□

3.3. Lemma. *Let X and Y be spaces, $t \colon X \to Y$ be a map. Let L be an abelian group, $g \colon \Pi_n(Y) \to L$ be a functional. Then $\deg g \circ t_\# \leqslant \deg g$.*

(Obvious.) □

3.4. Lemma. *Let X be a space. Define $l \colon \Pi_n(X) \to H_n(X)$ by $l(a) = a_*(u)$, where $u \in H_n(I^n, \partial I^n)$ is the fundamental class. Then $\deg l \leqslant 1$.*

Proof. Let Γ be a finite open cover of I^n. Represent u by a (singular) cycle $U \in Z_n(I^n, \partial I^n)$ subordinate to Γ:

$$U = \sum_{E \in \Gamma} \mathrm{in}_{(I^n, E)\,\#}(U_E),$$

where $U_E \in C_n(E, E \cap \partial I^n)$ are some chains.

The subgroup $Z_n(X) \subset C_n(X)$ is a direct summand. Thus there is a homomorphism $k \colon C_n(X) \to H_n(X)$ such that $k(T) = [T]$ for all $T \in Z_n(X)$. For $E \in \Gamma$, define $l_E \colon \Pi(E, X) \to H_n(X)$ by $l_E(a) = k(a_\#(U_E))$. For $a \in \Pi_n(X)$, we have

$$l(a) = [a_\#(U)] = k(a_\#(U)) = \sum_{E \in \Gamma} k((a|_E)_\#(U_E)) = \sum_{E \in \Gamma} l_E(a|_E).$$

□

Proof of Theorem 1.1. Induction on n. For $n = 2$, consider the commutative diagram

$$\begin{array}{ccccc} & & \Pi_2(X) & & \\ & \swarrow^{l} & \downarrow^{p} & \searrow^{q} & \\ H_2(X) & \xleftarrow{h} & \pi_2(X) & \xrightarrow{m} & \pi_2(X) \otimes \mathbf{Z}[1/2], \end{array}$$

where p is the natural projection, m is defined by $m(v) = v \otimes 1$, h is the Hurewicz isomorphism, and l is as in Lemma 3.4, thus $\deg l \leqslant 1$. We see that $\deg p \leqslant 1$ and thus $\deg q \leqslant 1$.

Take $n > 2$. By Corollary 2.11, there are a simply connected space Y with finitely generated homotopy groups and a map $t\colon \Omega X \to Y$ such that for every $q > 1$ the homomorphism $t_* \otimes \mathrm{id}\colon \pi_q(\Omega X) \otimes \mathbf{Z}[1/2] \to \pi_q(Y) \otimes \mathbf{Z}[1/2]$ is a split monomorphism. Consider the commutative diagram

$$\begin{array}{ccccc} \Pi_n(X) & \xrightarrow{\Xi} & \Pi_{n-1}(\Omega X) & \xrightarrow{t_\#} & \Pi_{n-1}(Y) \\ \downarrow^{q} & & \downarrow^{q'} & & \downarrow^{q''} \\ \pi_n(X) \otimes \mathbf{Z}[1/2] & \xrightarrow{\xi \otimes \mathrm{id}} & \pi_{n-1}(\Omega X) \otimes \mathbf{Z}[1/2] & \xrightarrow{t_* \otimes \mathrm{id}} & \pi_{n-1}(Y) \otimes \mathbf{Z}[1/2], \end{array}$$

where Ξ is the standard bijection, ξ is the standard isomorphism, q' and q'' are defined similarly to q. Since $\xi \otimes \mathrm{id}$ is an isomorphism, $\deg q = \deg(\xi \otimes \mathrm{id}) \circ q = \deg q' \circ \Xi$. By Corollary 3.2, $\deg q' \circ \Xi \leqslant (n-1) \deg q'$. Since $t_* \otimes \mathrm{id}$ is a split monomorphism, $\deg q' = \deg(t_* \otimes \mathrm{id}) \circ q' = \deg q'' \circ t_\#$. By Lemma 3.3, $\deg q'' \circ t_\# \leqslant \deg q''$. By induction hypothesis, $\deg q'' \leqslant (n-2)!$. Therefore $\deg q \leqslant (n-1)!$. □

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