An iterated sum formula
for a spheroid’s homotopy class modulo 2–torsion

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Abstract
Let \( X \) be a simply connected pointed space with finitely generated homotopy groups. Let \( \Pi_n(X) \) denote the set of all continuous maps \( a: I^n \to X \) taking \( \partial I^n \) to the basepoint. For \( a \in \Pi_n(X) \), let \( [a] \in \pi_n(X) \) be its homotopy class. For an open set \( E \subset I^n \), let \( \Pi(E, X) \) be the set of all continuous maps \( a: E \to X \) taking \( E \cap \partial I^n \) to the basepoint. For a cover \( \Gamma \) of \( I^n \), let \( \Gamma(r) \) be the set of all unions of at most \( r \) elements of \( \Gamma \). Put \( r = (n-1)! \). We prove that for any finite open cover \( \Gamma \) of \( I^n \) there exist maps \( f_E: \Pi(E, X) \to \pi_n(X) \otimes \mathbb{Z}[1/2] \), \( E \in \Gamma(r) \), such that

\[
[a] \otimes 1 = \sum_{E \in \Gamma(r)} f_E(a|_E)
\]

for all \( a \in \Pi_n(X) \).

1. Introduction
Let \( X \) be a (pointed) space. An \( n \)-spheroid in \( X \) is a (continuous) map \( a: I^n \to X \) taking \( \partial I^n \) to the basepoint. Let \( \Pi_n(X) \) denote the set of all \( n \)-spheroids in \( X \). For \( a \in \Pi_n(X) \), let \( [a] \in \pi_n(X) \) be its homotopy class. For an open set \( E \subset I^n \), let \( \Pi(E, X) \) be the set of all maps \( a: E \to X \) taking \( E \cap \partial I^n \) to the basepoint. For a cover \( \Gamma \) of \( I^n \), let \( \Gamma(r) \) be the set of all unions of at most \( r \) elements of \( \Gamma \).

Let \( L \) be an abelian group. Consider a functional (i.e. a function) \( f: \Pi_n(X) \to L \). We define the degree of \( f \) (denoted \( \text{deg} f \)) to be the infimum of \( r \) such that for any finite open cover \( \Gamma \) of \( I^n \) there exist functionals \( f_E: \Pi(E, X) \to L, E \in \Gamma(r) \), such that

\[
f(a) = \sum_{E \in \Gamma(r)} f_E(a|_E)
\]

for all \( a \in \Pi_n(X) \). We are interested in functionals of finite degree.

This definition is motivated by the notion of order-restricted perceptron [2]. For relation to Vassiliev knot invariants, see [5]. An example of a functional \( f: \Pi_n(X) \to \mathbb{R} \) with \( \text{deg} \leq r \) is given by

\[
f(a) = \int_{u_1, \ldots, u_r \in I^n} p(u_1, a(u_1), \ldots, u_r, a(u_r)) \, du_1 \ldots du_r,
\]
there are functionals form an open cover of \( \pi \) in \( Z[1/2] \) by \( q(a) = [a] \otimes 1 \). Then \( \deg q \leq (n - 1)! \).

This may be viewed as a finite sum version of Chen’s iterated integrals. For a related result, see [3]. Possibly, the \( Z[1/2] \) factor and the factorial sign may be removed. Claim 1.2 implies that \( \deg a \leq n \) for \( X = S^2 \lor S^2 \).

1.2. Claim. Let \( f: \Pi_n(X) \to L \) be a homotopy invariant functional with \( \deg f \leq r \). Let \( b_0, b \in \Pi_n(X) \) be spheroids with

\[
[b_0] = 0, \quad [b] = [[[v_0, v_1], v_2], \ldots, v_r],
\]

where \( v_s \in \pi_{k_s}(X) \) (\( k_0 + \ldots + k_r = n + r ; [\cdot, \cdot] \) denotes the Whitehead product). Then \( f(b) = f(b_0) \).

(A functional \( f: \Pi_n(X) \to L \) is homotopy invariant if \( f(a) \) depends only on \([a]\).)

Proof (cf. [2, Theorem 3.2]). Put

\[
T = S^{k_0} \lor \ldots \lor S^{k_r}.
\]

Let \( g: T \to X \) be a map whose restriction to the \( s \)th wedge summand represents \( v_s \). Let \( e_s \in \pi_{k_s}(T) \) be the elements represented by the canonical embeddings \( S^{k_s} \to T \). Let \( z \in \Pi_n(T) \) be a spheroid with \( [z] = [[[e_0, e_1], e_2], \ldots, e_r] \). We have \( [g \circ z] = [b] \). For every \( k \), choose maps \( p_i^k: S^k \to S^k \), \( i = 0, 1 \), such that \( p_i^k \) has degree \( i \) and \( p_i^k|_{U^k} = p_i^k|_{U^k} \) for some open neighbourhood \( U^k \) of the basepoint. Put

\[
p_{i_0, \ldots, i_r} = p_{i_0}^{k_0} \lor \ldots \lor p_{i_r}^{k_r}: T \to T, \quad i_0, \ldots, i_r = 0, 1.
\]

We have

\[
[p_{i_0, \ldots, i_r} \circ z] = [i_0 \ldots i_r][z]
\]

in \( \pi_n(T) \) and thus

\[
[g \circ p_{i_0, \ldots, i_r} \circ z] = [i_0 \ldots i_r][b]
\]

in \( \pi_n(X) \). The sets

\[
V_s = U^{k_0} \lor \ldots \lor S^{k_r} \lor \ldots \lor U^{k_r}, \quad s = 0, \ldots, r,
\]

form an open cover of \( T \). Put \( \Gamma = \{ z^{-1}(V_s) \mid s = 0, \ldots, r \} \). Since \( \deg f \leq r \), there are functionals \( f_E: \Pi(E, X) \to L, E \in \Gamma(r) \), such that

\[
f(a) = \sum_{E \in \Gamma(r)} f_E(a|_E)
\]
for all \( a \in \Pi_n(X) \). We have
\[
(-1)^r (f(b_0) - f(b)) = \sum_{i_0, \ldots, i_r = 0, 1} (-1)^{i_0 + \cdots + i_r} f(g \circ p_{i_0, \ldots, i_r} \circ z) = \\
\sum_{E \in \Gamma(r)} \sum_{i_0, \ldots, i_r = 0, 1} (-1)^{i_0 + \cdots + i_r} f_E (g \circ p_{i_0, \ldots, i_r} \circ z|_E).
\]
Take \( E \in \Gamma(r) \). We have
\[
z(E) \subset S^{k_0} \lor \cdots \lor U^{k_s} \lor \cdots \lor S^{k_r}
\]
for some \( s \). Thus \( p_{i_0, \ldots, i_r} \circ z|_E \) does not depend on \( i_s \). Thus the inner sum equals zero. \( \square \)

**Conventions and notation.** Maps are continuous, unlike functions. A space is a pointed space. A subspace contains the basepoint. Maps between spaces are basepoint preserving. This applies also to homotopies etc. A cell space is a pointed CW complex. \( \text{sk}_n X \) denotes the \( n \)-skeleton of a cell space \( X \).

The homotopy relation is denoted \( \sim \); \( \sim_A \) is used for homotopy rel \( A \). For a pair \((X, A)\), \( \text{in}_{(X, A)} : A \to X \) and \( \text{pr}_{(X, A)} : X \to X/A \) are the inclusion and the projection. The subscript of \( \text{in} \) and \( \text{pr} \) is often omitted.

**2. Making a loop space simply connected**

The aim of this section is to prove Corollary 2.11. First fix some notation.

**Homotopy fibres and cofibres.** Let \( X \) and \( Y \) be spaces, \( g : X \to Y \) be a map. Then we have the homotopy fibre sequence
\[
F(g) \xrightarrow{p(g)} X \xrightarrow{g} Y,
\]
where \( F(g) = \{ (x, v) \in X \times PY \mid g(x) = v(1) \} \) and \( p(g) \) is the fibraton defined by \( p(g)(x, v) = x \). We have the homotopy cofibre sequence
\[
X \xrightarrow{g} Y \xrightarrow{i(g)} C(g),
\]
where \( C(g) \) is the unreduced cone of \( g \) and \( i(g) \) is the canonical embedding.

**2.A. A Moore space**

Fix \( d > 0 \). Let \( M_d \) be the space obtained from \( S^1 \) by attaching a 2-cell via a map \( S^1 \to S^1 \) of degree \( d \). Our aim here is to prove Corollary 2.5.

**2.1. Lemma.** Let
\[
F \xrightarrow{i} E \xrightarrow{p} B
\]
be a fibre sequence. Suppose $B$ and $E$ are path-connected, $p$ induces an isomorphism on $\pi_1$, and the canonical action of $\pi_1(B)$ on $\pi_2(B)$ is trivial. Then the canonical action of $\pi_1(B)$ on $H_1(F)$ is trivial.

**Proof.** The group $\pi_1(E)$ acts canonically on $\pi_1(F)$ (see [1, 5.1.7.3]); it also acts on $\pi_2(B)$ and $H_1(F)$ through $p_* : \pi_1(E) \to \pi_1(B)$. The boundary homomorphism $\partial$ and the Hurewicz homomorphism $h$

\[ \pi_2(B) \xrightarrow{\partial} \pi_1(F) \xrightarrow{h} H_1(F) \]

respect these actions (regarding $\partial$, see [1, 5.1.8.4]). Since $\partial$, $h$, and $p_*$ are epimorphisms, $\pi_1(B)$ acts trivially on $H_1(F)$. □

**2.2. Claim.** $H_2(\Omega \Sigma M_d) \cong \mathbb{Z}_d$.

Easily seen from the homology spectral sequence of the path fibration of $\Sigma M_d$. □

**2.3. Claim.** If $d$ is odd, $\pi_3(\Sigma M_d) \cong \mathbb{Z}_d$.

**Proof.** There is a fibration $U \to \Sigma M_d$ with fibre of homotopy type $K(\mathbb{Z}_d, 1)$ and $U$ 2-connected. We have $\pi_3(\Sigma M_d) \cong H_3(U)$. Using the cohomology spectral sequence, we get\(^1\) $H_3(U) \cong \mathbb{Z}_d$. □

**2.4. Claim.** Suppose $d$ is odd. Then the canonical embedding $j : M_d \to \Omega \Sigma M_d$ induces zero homomorphism on $\pi_2$.

**Proof.** Consider the homotopy fibre sequence

\[ F(j) \xrightarrow{p(j)} M_d \xrightarrow{j} \Omega \Sigma M_d. \]

By Lemma 2.1, the action of $\pi_1(\Omega \Sigma M_d)$ on $H_1(F(j))$ is trivial. Using the homology spectral sequence and Claim 2.2, we get $H_1(F(j)) \cong \mathbb{Z}_d$. The boundary homomorphism $\partial : \pi_2(\Omega \Sigma M_d) \to \pi_1(F(j))$ is an epimorphism. Thus $\pi_1(F(j))$ is abelian and thus isomorphic to $\mathbb{Z}_d$. By Claim 2.3, $\pi_2(\Omega \Sigma M_d) \cong \mathbb{Z}_d$. Thus $\partial$ is an isomorphism. Thus $j_* : \pi_2(M_d) \to \pi_2(\Omega \Sigma M_d)$ is zero. □

**2.5. Corollary.** Suppose $d$ is odd. Let $X$ be a space, $f : M_d \to \Omega X$ be a map. Then $f$ induces zero homomorphism on $\pi_2$.

**Proof.** Let $g : \Sigma M_d \to X$ be the map adjoint to $f$. Then $f = \Omega g \circ j$, where $j$ is as in Claim 2.4. □

**2.B. Two technical lemmas**

**2.6. Lemma.** Let $X$ be a space, $A \subset X$ be a closed subspace such that $(X, A)$ is a Borsuk pair. Suppose $X$ and $A$ are homotopy equivalent to cell spaces of dimension at most $n$ and $n-1$, respectively. Let $Y$ be a cell space and $f : X \to Y$

\(^1\)This does not work for $d$ even. This results in the $\mathbb{Z}[1/2]$ factor in Theorem 1.1.
be a map with \( f(A) \subset sk_n Y \). Then \( f \) is rel \( A \) homotopic to a map \( h: X \to Y \) with \( h(X) \subset sk_n Y \).

This is used only for \( n = 3 \).

**Proof.** The map \( f \) is homotopic to a map \( g: X \to Y \) with \( g(X) \subset sk_n Y \). Let \( k: X \times [0, 1] \to Y \) be the corresponding homotopy:

\[
k(x, 0) = f(x), \quad k(x, 1) = g(x), \quad x \in X.
\]

The map \( k|_{A \times [0,1]} \) is rel \( A \times \{0,1\} \) homotopic to a map \( q: A \times [0,1] \to Y \) with image in \( sk_n Y \). Since \((X, A)\) is a Borsuk pair, there is a map \( l: X \times [1,2] \to Y \) with image in \( sk_n Y \) such that \( l(x, 1) = q(x), x \in X \), and \( l(x, 1+t) = q(x, 1-t) \), \( x \in A \), \( t \in [0,1] \). Let \( m: X \times [0,2] \to Y \) be the map with \( m|_{X \times [0,1]} = k \), \( m|_{X \times [1,2]} = l \). The map \( m|_{A \times [0,2]} \) is rel \( A \times \{0,2\} \) homotopic to the map \( \tilde{c}: A \times [0,2] \to Y \), \( c(x,t) = f(x) \). By the Strom theorem [4, Lecture I, Proposition 2],

\[
(X \times [0,2], X \times \{0,2\} \cup A \times \{0,2\})
\]

is a Borsuk pair. Thus there is a map \( \tilde{m}: X \times [0,2] \to Y \) with \( \tilde{m}|_{A \times [0,2]} = m|_{A \times [0,2]} \), \( \tilde{m}|_{A \times [0,2]} = c \). Put \( h(x) = \tilde{m}(x, 2) \). Then \( h \) is the desired homotopy. \( \square \)

**2.7. Lemma.** Let \( V \) be a Hausdorff space, \( T \subset V \) be a compact subspace. Let \( W \) be a space, \( a: V \to W \), \( b: V/T \to W \), \( c: T \to W \) be maps such that \( b \circ \text{pr}_{(V,T)} = a \), \( c = a|_{T} \) (so \( c \) is constant). Let \( j: F(c) \to F(a) \) and \( l: F(a) \to F(b) \) be the maps induced by \( \text{in}_{(V,T)} \) and \( \text{pr}_{(V,T)} \), respectively. Let \( Y \) be a space, \( X \subset Y \) be a subspace, \( s: F(a) \to X \) and \( t: F(b) \to Y \) be maps such that \( \text{in}_{(Y,X)} \circ s \) and \( t \circ l \) are rel \( j(F(c)) \) homotopic. Then there exists a unique map \( r: F(b) \to X \) such that \( r \circ l = s \). The maps \( \text{in}_{(V,Y)} \circ r \) and \( t \) are homotopic rel \( p(b)^{-1}(q_0) \), where \( q_0 \in V/T \) is the basepoint.

This is because \( l \) is a quotient map. \( \square \)
2.C. A $K(G, 1)$ space and some its subquotients

Let $G$ be a finitely generated abelian group and

$$G = G_1 \oplus \ldots \oplus G_r$$

be its decomposition into cyclic summands. For each $s = 1, \ldots, r$, take a cell space $V_s$ of homotopy type $K(G_s, 1)$ such that $sk_2 V_s$ is either $S^1$ or $M_d$ (see 2.A) for proper $d$. Put $T_s = sk_1 V_s$ (=$S^1$), $U_s = sk_3 V_s$,

$$V = V_1 \times \ldots \times V_r, \quad T = T_1 \times \ldots \times T_r, \quad U = U_1 \times \ldots \times U_r.$$ We have $T \subset U \subset V$. $V$ is a $K(G, 1)$ space.

2.8. Claim. Let $X$ be a simply connected space with $\pi_2(X) \cong G$, $Q$ be a space of weak homotopy type $K(G, 2)$, $b: X \to Q$ be a map inducing an isomorphism on $\pi_2$, and $m: V \to \Omega Q$ be a weak homotopy equivalence. Suppose $G$ has no 2-torsion. Then there exists a map $f: U \to \Omega X$ such that $\Omega b \circ f \sim m|_U$.

\[ \begin{array}{ccc}
U & \xrightarrow{\text{in}} & V \\
\downarrow^{f} & & \downarrow^{m} \\
\Omega X & \xrightarrow{\Omega b} & \Omega Q
\end{array} \]

Proof. We naturally have $U_s \subset U$, $s = 1, \ldots, r$. For every $s$, there is a map $f'_s: sk_2 U_s \to \Omega X$ such that $\Omega b \circ f'_s$ and $m|_{sk_2 U_s}$ induce the same homomorphism on $\pi_1$. By Corollary 2.5, $f'_s$ induces zero homomorphism on $\pi_2$. Thus it extends to a map $f_s: U_s \to \Omega X$. Since $\Omega X$ is a loop space, there is a map $f: U \to \Omega X$ such that $f|_{U_s} = f_s$ for every $s$. The maps $\Omega b \circ f$ and $m|_U$ are homotopic since they induce the same homomorphism on $\pi_1$. \qed

2.9. Lemma. For every $q$, there exist a cell space $Z$ and a map $k: Z \to U/T$ such that $\text{in}_{(V/T, U/T)} \circ k$ induces an isomorphism $\pi_q(Z) \to \pi_q(V/T)$.

Proof. For $q \leq 2$, put $Z = U/T$, $k = \text{id}$. We shall construct a single $k$ to serve all $q > 2$.

Take some $s$. If $G_s$ is finite, let $W_s$ be a space of homotopy type $K(Z, 2)$; otherwise, let $W_s$ be a point. There is a map $a_s: V_s \to W_s$ with $F(a_s)$ homotopy equivalent to $S^1$. The map $c_s = a_s|_{T_s}$ is null-homotopic. We choose $a_s$ in such a way that $c_s$ is constant. Let $j_s: F(c_s) \to F(a_s)$ be the map induced by $\text{in}_{(V_s, T_s)}$. Since $c_s$ is constant, $F(c_s) = T_s \times \Omega W_s$. Thus $F(c_s)$ is homotopy equivalent to a cell space of dimension at most 2. By [4, Lecture II, Proposition 5], $(F(a_s), j_s(F(c_s)))$ is a Borsuk pair. By Lemma 2.6, there is a map $f_s: F(a_s) \to U_s$ such that $\text{in}_{(V_s, U_s)} \circ f_s$ is rel $j_s(F(c_s))$ homotopic to $p(a_s)$.  

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Put $W = W_1 \times \ldots \times W_r, \ a = a_1 \times \ldots \times a_r: V \to W, \ c = c_1 \times \ldots \times c_r: T \to W$.

Since $a|_T$ is constant, there is a map $b: V/T \to W$ such that $b \circ \text{pr}_{(V,T)} = a$. Let $j: F(c) \to F(a)$ and $l: F(a) \to F(b)$ be the maps induced by $\text{in}_{(V,T)}$ and $\text{pr}_{(V,T)}$, respectively.

We make natural identifications

$$F(a) = F(a_1) \times \ldots \times F(a_r), \quad F(c) = F(c_1) \times \ldots \times F(c_r).$$

Then $p(a) = p(a_1) \times \ldots \times p(a_r)$ and $j = j_1 \times \ldots \times j_r$. Put

$$f = f_1 \times \ldots \times f_r: F(a) \to U.$$

The map $\text{in}_{(V,U)} \circ f$ is rel $j(F(c))$ homotopic to $p(a)$. Put $g = \text{pr}_{(U,T)} \circ f$. We have

$$\text{in}_{(V/T,U/T)} \circ g = \text{pr}_{(V,T)} \circ \text{in}_{(V,U)} \circ f \sim_{j(F(c))} \text{pr}_{(V,T)} \circ p(a) = p(b) \circ l.$$

By Lemma 2.7, there is a map $h: F(b) \to U/T$ such that $h \circ l = g$ and $\text{in}_{(V/T,U/T)} \circ h \sim p(b)$. Since $\pi_q(W) = 0$ for $q \neq 2$, $p(b)$ induces isomorphisms on $\pi_q, q \geq 2$. Thus the map $\text{in}_{(V/T,U/T)} \circ h$ does so. Thus for any $q > 2$ we may let $e: Z \to F(b)$ be a cell approximation and put $k = h \circ e$. □

\begin{center}
\begin{tikzcd}
F(c) \arrow{r}{j} \arrow{d}{p(c)} & F(a) \arrow{r}{l} \arrow{d}{p(a)} & F(b) \\
V \arrow{r}{\text{in}} & U \arrow{r}{\text{in}} & V/T \\
T \arrow{u}{c} & \arrow{u}{a} & \arrow{u}{b}
\end{tikzcd}
\end{center}
2.D. Treating a loop space

2.10. Lemma. Let $X$ be a simply connected space with finitely generated homotopy groups. Suppose $\pi_2(X)$ has no 2–torsion. Then there exist a simply connected space $Y$ with finitely generated homotopy groups and a map $t: \Omega X \to Y$ inducing split monomorphisms on $\pi_q$, $q > 1$.

(It follows easily that $Y$ may be obtained by attaching 2–cells to $\Omega X$.)

Proof. Put $G = \pi_2(X)$. There are a space $Q$ of weak homotopy type $K(G, 2)$ and a map $b: X \to Q$ inducing an isomorphism on $\pi_2$. Consider the piece of the Puppe sequence of $b$:

$$\begin{array}{c}
(\Omega X \xrightarrow{\Omega b} \Omega Q \xrightarrow{j} F(b))
\end{array}$$

There is a standard homotopy equivalence $e: \Omega X \to F(b)$ such that $p(j) \circ e = \Omega b$. Let $V$ be the $K(G, 1)$ space considered in 2.C. Let $T \subset U \subset V$ be as there.

There is a weak homotopy equivalence $m: V \to \Omega Q$. By Claim 2.8, $m|_U$ lifts (up to homotopy) along $\Omega b$. Thus $j \circ m|_U$ is null-homotopic. Thus there is a map $h: C(m|_U) \to F(b)$ such that $h \circ m|_T = j$. Let $s: C(m|_T) \to C(m|_U)$ be the map induced by $m|_U$. Put $g = h \circ s$. Let $r: F(j) \to F(g)$ be the map induced by $i(m|_T)$. Put $Y = F(g)$, $t = r \circ e$.

Let us check the desired properties. Since $b$ is 3–connected, $F(b)$ is 2–connected. Since $m|_T$ is 1–connected, $C(m|_T)$ is 1–connected. Therefore $g$ is 2–connected. Thus $F(g)$ is 1–connected, i.e. $Y$ is simply connected. One checks similarly that $\pi_q(Y)$ are finitely generated.
We have the commutative diagram

\[
\begin{array}{ccc}
V/T & \xrightarrow{\rho} & C(in(V,T)) \\
\downarrow & & \downarrow m' \\
V/U & \xrightarrow{\sigma} & C(in(V,U))
\end{array}
\]

where \(\phi\) is induced by \(in(U,T)\), \(\rho\) and \(\sigma\) are the standard homotopy equivalences (contractions), \(m'\) and \(m''\) are the weak equivalences induced by \(m\). Take \(q > 1\).

Using Lemma 2.9, we get a cell space \(Z_q\) and a map \(l_q: Z_q \to V/T\) inducing an isomorphism on \(\pi_q\) and such that \(pr(V/T,U/T) \circ l_q\) is constant. Using the diagram, we get a map \(v_q: Z_q \to C(m|T)\) inducing an isomorphism on \(\pi_q\) and such that \(s \circ v_q\) is null-homotopic. Since \(g = h \circ s\), \(g \circ v_q\) is null-homotopic. Thus \(v_q\) lifts along \(p(g)\): there is a map \(w_q: Z_q \to F(g)\) such that \(p(g) \circ w_q = v_q\). Consider the commutative diagram

\[
\begin{array}{cccc}
\pi_{q+1}(\Omega Q) & \xrightarrow{j^*} & \pi_{q+1}(F(b)) & \xrightarrow{\partial_j} & \pi_q(F(j)) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow i(m|T)_* & & \downarrow r_* & & \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow i(m|T)_* \\
\pi_{q+1}(F(g)) & \xrightarrow{g^*} & \pi_{q+1}(C(m|T)) & \xrightarrow{\partial_q} & \pi_q(C(m|T)) \\
\downarrow w_{q+1} & & \downarrow v_{q+1} & & \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow v_q \\
\pi_{q+1}(Z_{q+1}) & \xrightarrow{w_q} & \pi_q(Z_q)
\end{array}
\]

where the rows are pieces of the exact sequences of the maps \(j\) and \(g\). We see that \(\partial_j\) is an isomorphism and \(\partial_q\) is a split monomorphism. Thus \(r_*\) is a split monomorphism. Since \(t = r \circ e\), where \(e\) is a homotopy equivalence, \(t\) induces a split monomorphism on \(\pi_q\). □

2.11. Corollary. Let \(X\) be a simply connected space with finitely generated homotopy groups. Then there exist a simply connected space \(Y\) with finitely generated homotopy groups and a map \(t: \Omega X \to Y\) such that for every \(q > 1\) the homomorphism \(t_* \otimes \text{id}: \pi_q(\Omega X) \otimes \mathbb{Z}[1/2] \to \pi_q(Y) \otimes \mathbb{Z}[1/2]\) is a split monomorphism.

Proof. By attaching 3- and 4-cells to \(X\), we may quotient \(H_2\) by 2-torsion without changing \(H_q\) for \(q > 2\). By Serre, \(\pi_q \otimes \mathbb{Z}[1/2]\) are not changed. Then apply Lemma 2.10. □
3. Dimension descent

3.1. Lemma. Let $X$ be a simply connected space, $\Gamma$ be a finite open cover of $I^n$ ($n > 1$). Then there exist a finite open cover $\Delta$ of $I^{n-1}$, a function $\lambda: \Delta \to \Gamma(n-1)$, and a function $\Phi: \Pi_n(X) \to \Pi_n(X)$ such that (1) $[\Phi(a)] = [a]$ for all $a \in \Pi_n(X)$ and (2) for $F \in \Delta$, $a, a' \in \Pi_n(X)$, the implication holds

$$a|_{\lambda(F)} = a'|_{\lambda(F)} \Rightarrow \Phi(a)|_{F \times I} = \Phi(a')|_{F \times I}.$$ 

Proof. Choose $\epsilon > 0$ such that every open ball $B(u, 3\epsilon)$ ($u \in I^n$) is contained in some $E \in \Gamma$. Choose a rectilinear triangulation $T$ of $I^n$ with simplices of diameter less than $\epsilon$. Let $T'$ be the barycentric subdivision of $T$. By cosk$^2 T$ we denote the union of all simplices of $T'$ that do not intersect $sk_1 T$. (So cosk$^2 T$ is a polyhedron of dimension $n - 2$; its complement collapses to $sk_1 T$.)

There is a homeomorphism $l: I^n \to I^n$ preserving simplices of $T$ and such that

$$|v \times I \cap l^{-1}(\cosk^2 T)| \leq n - 1$$

for every $v \in I^{n-1}$. (Such $l$ is obtained by generic simplex-wise perturbation of the identity map.) The map $l$ preserves $\partial I^n$ and has degree 1.

For $v \in I^{n-1}$, let $2\rho(v)$ be the least positive distance between $v \times I$ and $l^{-1}(t)$, where $t$ runs over simplices of $T'$. Let $\Delta$ be a finite cover of $I^{n-1}$ formed by some of the balls $B(v, \rho(v))$, $v \in I^{n-1}$.

Let us construct the function $\lambda$. Take $F \in \Delta$. We have $F = B(v, \rho(v))$ for some $v \in I^{n-1}$. For each

$$u \in v \times I \cap l^{-1}(\cosk^2 T)$$

there are at most $n - 1$ such $u$), choose $E \in \Gamma$ containing $B(u, 3\epsilon)$. Put $\lambda(F)$ be the union of these $E$.

Choose $\delta > 0$ such that $\delta \leq \epsilon$ and $\delta \leq \rho(v)$ for $B(v, \rho(v)) \in \Delta$. Let $Q$ be the open $\delta$-neighbourhood of $l^{-1}(\cosk^2 T)$. Put $P = l(I^n \setminus Q)$. The set $P$ is closed and does not intersect $\cosk^2 T$. Thus there is a map $k: I^n \to I^n$ preserving the simplices of $T$ with $k(P) \subset sk_1 T$. The map $k$ preserves $\partial I^n$ and has degree 1.

Take $a \in \Pi_n(X)$. We homotop $a$ to get a map $\Theta(a)$ taking $sk_1 T$ to the basepoint. The homotopy is constructed by induction on the skeleta of $T$. There are no obstructions since $X$ is simply connected. To extend the homotopy to some simplex $s$ of $T$, we need to know only $a|_s$ and the homotopy constructed on $\partial s$ (we need no information from the outside of $s$).

Therefore there is a function $\Theta: \Pi_n(X) \to \Pi_n(X)$ such that (a) $[\Theta(a)] = [a]$ for all $a \in \Pi_n(X)$, (b) for every simplex $s$ of $T$ and any $a, a' \in \Pi_n(X)$, the implication holds

$$a|_s = a'|_s \Rightarrow \Theta(a)|_s = \Theta(a')|_s,$$

and (c) for any $a \in \Pi_n(X)$, $\Theta(a)$ takes $sk_1 T$ to the basepoint.
For $a \in \Pi_n(X)$, put $\Phi(a) = \Theta(a) \circ k \circ l$. The property (1) is obvious. Let us check the property (2). Take $F \in \Delta$ and $a, a' \in \Pi_n(X)$ such that $a|_{\lambda(F)} = a'|_{\lambda(F)}$. We should show that $\Phi(a)|_{F \times I} = \Phi(a')|_{F \times I}$. Take $u_0 \in F \times I$. Let us show that $\Phi(a)(u_0) = \Phi(a')(u_0)$.

If $l(u_0) \in P$, then $k(l(u_0)) \in sk_3 T$ and thus $\Phi(a)(u_0) = \Phi(a')(u_0) = x_0$, where $x_0 \in X$ is the basepoint. Consider the converse case. Since $P = l(F \setminus Q)$, we have $u_0 \in Q$. Thus $\dist(u_0, l^{-1}(\cos^2 T)) < \delta$. Thus there is a simplex $t$ of $\cos^2 T$ such that $\dist(u_0, l^{-1}(t)) < \delta$. We have $F = B(v, \rho(v))$ for some $v \in I^{n-1}$. We have

$$\dist(v \times I, l^{-1}(t)) \leq \dist(v \times I, u_0) + \dist(u_0, l^{-1}(t)) < \rho(v) + \delta \leq 2\rho(v).$$

By definition of $\rho(v)$, this means that $\dist(v \times I, l^{-1}(t)) = 0$. Thus there is a point $u \in v \times I \cap l^{-1}(t)$. Since $l$ preserves simplices of $T$, $\diam l^{-1}(t) < \epsilon$. We have

$$\dist(u_0, u) \leq \dist(u_0, l^{-1}(t)) + \diam l^{-1}(t) < \delta + \epsilon \leq 2\epsilon.$$

The point $u_0$ belongs to some simplex $s$ of $T$. We have

$$s \subset B(u_0, \epsilon) \subset B(u, 3\epsilon) \subset \lambda(F).$$

Thus $a|_s = a'|_s$. Thus $\Theta(a)|_s = \Theta(a')|_s$. Since $k$ and $l$ preserve $s$, $\Phi(a)|_s = \Phi(a')|_s$, which suffices. □

3.B. Functionals of finite degree

For a space $X$, the formula $\Xi(a)(t_1, \ldots, t_{n-1})(t) = a(t_1, \ldots, t_{n-1}, t) \ (t_1, \ldots, t_{n-1}, t \in I)$ defines a bijection $\Xi : \Pi_n(X) \to \Pi_{n-1}(\Omega X)$, which we call standard. The induced isomorphism $\xi : \pi_n(X) \to \pi_{n-1}(\Omega X)$ we also call standard. For an open set $F \subset I^{n-1}$, the bijection $\Xi_F : \Pi(F \times I, X) \to \Pi(F, \Omega X)$ defined by that formula is also called standard.

3.2. Corollary. Let $X$ be a simply connected space, $L$ be an abelian group, $g : \Pi_{n-1}(\Omega X) \to L$ be a homotopy invariant functional ($n > 1$). Let $\Xi : \Pi_n(X) \to \Pi_{n-1}(\Omega X)$ be the standard bijection. Then $\deg g \circ \Xi \leq (n-1) \deg g$.

Proof. Suppose $\deg g \leq r$. Let us show that $\deg g \circ \Xi \leq (n-1) \deg g$. Let $\Gamma$ be a finite open cover of $I^n$. By Lemma 3.1, there are a finite open cover $\Delta$ of $I^{n-1}$, a function $\lambda : \Delta \to \Gamma((n-1) r)$, and a function $\Phi : \Pi_n(X) \to \Pi_n(X)$ satisfying the conditions (1), (2) of the lemma.

For each $F \in \Delta(r)$, choose a decomposition $F = F_1 \cup \ldots \cup F_s$, $0 \leq s \leq r$, with $F_1, \ldots, F_s \in \Delta$ and put $\mu(F) = \lambda(F_1) \cup \ldots \cup \lambda(F_s)$. So we have a function $\mu : \Delta(r) \to \Gamma((n-1)r)$. It follows from the condition (2) that for $F \in \Delta(r)$, $a, a' \in \Pi_n(X)$ the implication holds

$$a|_{\mu(F)} = a'|_{\mu(F)} \Rightarrow \Phi(a)|_{F \times I} = \Phi(a')|_{F \times I}.$$

Thus for every $F \in \Delta(r)$ there is a function $\Phi_F : \Pi(\mu(F), X) \to \Pi(F \times I, X)$ such that $\Phi_F(a|_{\mu(F)}) = \Phi(a)|_{F \times I}$ for all $a \in \Pi_n(X)$. For every $F \in \Delta(r)$, we
have the commutative diagram

\[
\begin{array}{ccc}
\Pi_n(X) & \xrightarrow{\Phi} & \Pi_n(X) \\
\downarrow & & \downarrow \\
\Pi(\mu(F), X) & \xrightarrow{\Phi_F} & \Pi(F \times I, X)
\end{array}
\xrightarrow{\Xi_F} \begin{array}{c}
\Pi_n(X) \\
\downarrow \\
\Pi(F, \Omega X),
\end{array}
\]

where $\Xi_F$ is the standard bijection and the vertical arrows are the restriction functions.

Since $\deg g \leq r$, there are functionals $g_F : \Pi(F, \Omega X) \to L$, $F \in \Delta(r)$, such that

\[g(b) = \sum_{F \in \Delta(r)} g_F(b|_F)\]

for all $b \in \Pi_{n-1}(\Omega X)$. For $E \in \Gamma((n-1)r)$, define a functional $f_E : \Pi(E, X) \to L$ by

\[f_E(a) = \sum_{F \in \mu^{-1}(E)} g_F(\Xi_F(\Phi_F(a))).\]

For $a \in \Pi_n(X)$, we have

\[g(\Xi(a)) = g(\Xi(\Phi(a))) = \sum_{F \in \Delta(r)} g_F(\Xi(\Phi(a)|_F)) = \sum_{F \in \Delta(r)} g_F(\Xi_F(\Phi_F(a|_{\mu(F)}))) = \sum_{E \in \Gamma((n-1)r)} \sum_{F \in \mu^{-1}(E)} g_F(\Xi_F(\Phi_F(a|_E))) = \sum_{E \in \Gamma((n-1)r)} f_E(a|_E).\]

\[\square\]

**3.3. Lemma.** Let $X$ and $Y$ be spaces, $t : X \to Y$ be a map. Let $L$ be an abelian group, $g : \Pi_n(Y) \to L$ be a functional. Then $\deg g \circ t_\# \leq \deg g$.

(Obvious.) \[\square\]

**3.4. Lemma.** Let $X$ be a space. Define $l : \Pi_n(X) \to H_n(X)$ by $l(a) = a_\#(u)$, where $u \in H_n(I^n, \partial I^n)$ is the fundamental class. Then $\deg l \leq 1$.

**Proof.** Let $\Gamma$ be a finite open cover of $I^n$. Represent $u$ by a (singular) cycle $U \in Z_n(I^n, \partial I^n)$ subordinate to $\Gamma$:

\[U = \sum_{E \in \Gamma} \text{in}(I^n, E)_\#(U_E),\]

where $U_E \in C_n(E, E \cap \partial I^n)$ are some chains.

The subgroup $Z_n(X) \subset C_n(X)$ is a direct summand. Thus there is a homomorphism $k : C_n(X) \to H_n(X)$ such that $k(T) = [T]$ for all $T \in Z_n(X)$. For $E \in \Gamma$, define $l_E : \Pi(E, X) \to H_n(X)$ by $l_E(a) = k(a_\#(U_E))$. For $a \in \Pi_n(X)$, we have

\[l(a) = [a_\#(U)] = k(a_\#(U)) = \sum_{E \in \Gamma} k((a|_E)_\#(U_E)) = \sum_{E \in \Gamma} l_E(a|_E).\]

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Proof of Theorem 1.1. Induction on \( n \). For \( n = 2 \), consider the commutative diagram

\[
\begin{array}{ccc}
\Pi_2(X) & \xrightarrow{t} & H_2(X) \\
\downarrow{l} & & \downarrow{h} \\
\pi_2(X) & \xrightarrow{p} & \pi_2(X) \otimes \mathbb{Z}[1/2],
\end{array}
\]

where \( p \) is the natural projection, \( m \) is defined by \( m(v) = v \oplus 1 \), \( h \) is the Hurewicz isomorphism, and \( l \) is as in Lemma 3.4, thus \( \deg l \leq 1 \). We see that \( \deg p \leq 1 \) and thus \( \deg q \leq 1 \).

Take \( n > 2 \). By Corollary 2.11, there are a simply connected space \( Y \) with finitely generated homotopy groups and a map \( t: \Omega X \to Y \) such that for every \( q > 1 \) the homomorphism \( t_* \otimes \text{id}: \pi_q(\Omega X) \otimes \mathbb{Z}[1/2] \to \pi_q(Y) \otimes \mathbb{Z}[1/2] \) is a split monomorphism. Consider the commutative diagram

\[
\begin{array}{ccc}
\Pi_n(X) & \xrightarrow{\Xi} & \Pi_{n-1}(\Omega X) \\
\downarrow{q} & & \downarrow{q'} \\
\pi_n(X) \otimes \mathbb{Z}[1/2] & \xrightarrow{\xi \otimes \text{id}} & \pi_{n-1}(\Omega X) \otimes \mathbb{Z}[1/2] \\
\downarrow{t_* \otimes \text{id}} & & \downarrow{t_* \otimes \text{id}} \\
\pi_n(Y) \otimes \mathbb{Z}[1/2] & \xrightarrow{\xi \otimes \text{id}} & \pi_{n-1}(Y) \otimes \mathbb{Z}[1/2],
\end{array}
\]

where \( \Xi \) is the standard bijection, \( \xi \) is the standard isomorphism, \( q' \) and \( q'' \) are defined similarly to \( q \). Since \( \xi \otimes \text{id} \) is an isomorphism, \( \deg q = \deg(\xi \otimes \text{id}) \circ q = \deg q' \circ \Xi \). By Corollary 3.2, \( \deg q' \circ \Xi \leq (n-1) \deg q' \). Since \( t_* \otimes \text{id} \) is a split monomorphism, \( \deg q' = \deg(t_* \otimes \text{id}) \circ q' = \deg q'' \circ t_* \). By Lemma 3.3, \( \deg q'' \circ t_* \leq \deg q'' \). By induction hypothesis, \( \deg q'' \leq (n-2)! \). Therefore \( \deg q \leq (n-1)! \). □

References


