On homotopy invariants of finite degree

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Abstract

We prove that homotopy invariants of finite degree distinguish homotopy classes of maps of a connected compact CW-complex to a nilpotent connected CW-complex with finitely generated homotopy groups.

§ 1. Introduction

$\mathbf{N} = \{0, 1, \dots\}$. *Space* means "pointed topological space". *CW-complexes* are also pointed (the basepoint being a vertex). *Map* means "basepoint preserving continuous map". Homotopies, the notation $[X, Y]$, etc. are to be understood in the pointed sense.

Invariants of finite degree. Let X and Y be spaces, V be an abelian group, and $f\colon [X, Y] \to V$ be a function (a homotopy invariant). Let us define a number $\operatorname{Deg} f \in \mathbf{N} \cup \{\infty\}$, the *degree* of f. Given a map $a\colon X \to Y$ and a number $r \in \mathbf{N}$, we have the map $a^r\colon X^r \to Y^r$ (the Cartesian power), which induces the homomorphism $C_0(a^r)\colon C_0(X^r) \to C_0(Y^r)$ between the groups of (unreduced) zero-dimensional chains with the coefficients in $\mathbf{Z}$. Let the inequality $\operatorname{Deg} f \leqslant r$ be equivalent to the existence of a homomorphism $l\colon \operatorname{Hom}(C_0(X^r), C_0(Y^r)) \to V$ such that $f([a]) = l(C_0(a^r))$ for all maps $a\colon X \to Y$. As one easily sees, $\operatorname{Deg} f$ is well defined by this condition. *Finite-degree* invariants are those of finite degree.

Main results.

1.1. Theorem. *Let X be a connected compact CW-complex, Y be a nilpotent connected CW-complex with finitely generated homotopy groups, and $u_1, u_2 \in [X, Y]$ be distinct classes. Then, for some prime p, there exists a finite-degree invariant $f\colon [X, Y] \to \mathbf{Z}_p$ such that $f(u_1) \neq f(u_2)$.*

Related facts were known for certain cases where $[X, Y]$ is an abelian group [10, 11]. Theorem 1.1 follows (see § 11) from a result of Bousfield–Kan and Theorem 1.2.

We call a group *p-finite* (for a prime p) if it is finite and its order is a power of p.

1.2. Theorem. *Let p be a prime, X be a compact CW-complex, and Y be a connected CW-complex with p-finite homotopy groups. Then every invariant $f\colon [X, Y] \to \mathbf{Z}_p$ has finite degree.*

Probably, Theorem 1.2 can be deduced from Shipley's convergence theorem [12], which we do not use. We use an (approximate) simplicial model of Y that admits a harmonic (see § 6) embedding in a simplicial $\mathbf{Z}_p$-module.

Non-nilpotent examples. The following examples show the importance of the nilpotency assumption in Theorem 1.1. We consider finite-degree invariants on $\pi_n(Y) = [S^n, Y]$.

1.3. *Let Y be a space with $\pi_1(Y)$ perfect. Then, for any abelian group V, any finite-degree invariant $f: \pi_1(Y) \to V$ is constant.*

This follows from Lemmas 12.2 and 3.6. □

1.4. *Take $n > 1$. Let Y be a space such that $\pi_n(Y) \cong \mathbf{Z}^2$ and an element $g \in \pi_1(Y)$ induces an order 6 automorphism on $\pi_n(Y)$. Then, for any abelian group V, any finite-degree invariant $f: \pi_n(Y) \to V$ is constant.*

This follows from Lemmas 12.2 and 12.3 and claim 3.7. □

An example: maps $S^{n-1} \times S^n \to S^n_{(\mathbf{Q})}$ (cf. [1, Example 4.6]). Take an even $n > 0$. Let $c: S^{n-1} \times S^n \to S^{2n-1}$ be a map of degree 1. Put $i = [\mathrm{id}] \in \pi_n(S^n)$, $j = i * i \in \pi_{2n-1}(S^n)$ (the Whitehead square), and $u(q) = (qj) \circ [c] \in [S^{n-1} \times S^n, S^n]$, $q \in \mathbf{Z}$. Let $l: S^n \to S^n_{\mathbf{Q}}$ be the rationalization. Put $\bar{u}(q) = [l] \circ u(q) \in [S^{n-1} \times S^n, S^n_{\mathbf{Q}}]$. The classes $u(q)$, $q \in \mathbf{Z}$, are pairwise distinct; moreover, the classes $\bar{u}(q)$, $q \in \mathbf{Z}$, are pairwise distinct (the proof is omitted).

Is it true that, under the assumptions of Theorem 1.1, there must exist an $r \in \mathbf{N}$ such that the elements of $[X, Y]$ are distinguished by invariants of degree at most r? No, as the following claim shows.

1.5. *Let V be an abelian group and $f: [S^{n-1} \times S^n, S^n] \to V$ be an invariant of degree at most $r \in \mathbf{N}$. Then $f(u(q)) = f(u(0))$ whenever $r! \mid q$.*

The following claim shows the importance of the assumption of Theorem 1.1 that Y has finitely generated homotopy groups.

1.6. *Let V be an abelian group and $f: [S^{n-1} \times S^n, S^n_{\mathbf{Q}}] \to V$ be an invariant of finite degree. Then $f(\bar{u}(q)) = f(\bar{u}(0))$, $q \in \mathbf{Z}$.*

The following claim shows that, under the assumptions of Theorem 1.1, finite-degree invariants taking values in $\mathbf{Q}$ may not distinguish rationally distinct homotopy classes.

1.7. *Let $f: [S^{n-1} \times S^n, S^n] \to \mathbf{Q}$ be an invariant of finite degree. Then $f(u(q)) = f(u(0))$, $q \in \mathbf{Z}$.*

Elusive elements of $H_0(Y^X)$. The space of maps $X \to Y$ is denoted Y^X. An invariant $f: [X, Y] \to V$ gives rise to the homomorphism ${}^+f: H_0(Y^X) \to V$, $\lfloor u \rfloor \mapsto f(u)$ (here $\lfloor u \rfloor$ denotes the basic element corresponding to u). Is it true that, under the assumptions of Theorem 1.1, for any non-zero element $w \in H_0(Y^X)$ there exist an abelian group V and a finite-degree invariant $f: [X, Y] \to V$ such that ${}^+f(w) \neq 0$? No, as the following claim shows.

1.8. *Take $n > 1$. Let Y be a space and $u_1, u_2 \in \pi_n(Y)$ be elements of coprime finite orders. Put $w = \lfloor u_1 + u_2 \rfloor - \lfloor u_1 \rfloor - \lfloor u_2 \rfloor + \lfloor 0 \rfloor$. Let V be an abelian group and $f\colon \pi_n(Y) \to V$ be an invariant of finite degree. Then ${}^{+}f(w) = 0$.*

This follows from Lemmas 12.2 and 3.8. □

If the group $\pi_n(Y)$ is torsion and divisible, then the same is true for any elements $u_1, u_2 \in \pi_n(Y)$ (this follows from Lemmas 12.2 and 3.9). In this case, $\pi_n(Y)$ cannot be finitely generated (without being zero). In return, Y can be p-local, e. g. $Y = \mathcal{K}(P, n)$ (the Eilenberg–MacLane space) for $P = \mathbf{Z}[1/p]/\mathbf{Z}$.

§ 2. Preliminaries

We say *crew* for "pointed set" and *archism* for "basepoint preserving function". We use the standard model structure on the category of simplicial crews (and archisms) [7, Corollary 3.6.6]. The words *fibration*, *cofibration*, etc. refer to it. A *fibring* simplicial archism is a fibration. An *isotypical* simplicial archism, or an *isotypy*, is a weak equivalence. *Isotypic* simplicial crews are weakly equivalent ones.

An abelian group is a crew (the basepoint being 0); a simplicial abelian group is a simplicial crew.

We call a simplicial crew T *compact* if it is generated by a finite number of simplices, and *gradual* if the crews T_q, $q \in \mathbf{N}$, are finite.

For simplicial crews K and T, we have a simplicial crew T^K, the function object (denoted $\hom_*(K, T)$ in [2, Ch. VIII, 4.8]). A simplicial archism $f\colon K \to L$ induces a simplicial archism $T^f\colon T^L \to T^K$, etc. We use this notation in the topological case as well.

The sign $\sim$ denotes the homotopy relation; the sign $\simeq$ denotes the homotopy equivalence of spaces.

Main homomorphisms. By default, chains and homology have coefficients in a commutative ring $\mathcal{R}$; $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{R}}$. (In § 1, we had $\mathcal{R} = \mathbf{Z}$ implicitly.)

For spaces X and Y, define $\mathcal{R}$-homomorphisms

$$ {}^{X}_{Y}\mu_r\colon C_0(Y^X) \to \mathrm{Hom}(C_0(X^r), C_0(Y^r)), \qquad \lfloor a \rfloor \mapsto C_0(a^r), $$

$r \in \mathbf{N}$. We have the projection $\mathcal{R}$-homomorphism

$$ {}^{X}_{Y}\nu\colon C_0(Y^X) \to H_0(Y^X). $$

For simplicial crews K and T, define $\mathcal{R}$-homomorphisms

$$ {}^{K}_{T}\mu_r\colon C_0(T^K) \to \mathrm{Hom}_0(C_*(K^r), C_*(T^r)), \qquad \lfloor b \rfloor \mapsto C_*(b^r), $$

$r \in \mathbf{N}$. Here $\lfloor b \rfloor$ is the basic chain corresponding to a simplex $b \in (T^K)_0$, i. e. a simplicial archism $b\colon K \to T$; $b^r\colon K^r \to T^r$ is the Cartesian power; $C_*(b^r)\colon C_*(K^r) \to C_*(T^r)$ is the induced $\mathcal{R}$-homomorphism of graded $\mathcal{R}$-modules of chains; Hom_0 denotes the $\mathcal{R}$-module of grading-preserving $\mathcal{R}$-homomorphisms. We have the projection $\mathcal{R}$-homomorphism

$$ {}^{K}_{T}\nu\colon C_0(T^K) \to H_0(T^K). $$

§ 3. Group algebras and gentle functions

Let $\mathcal{R}\lfloor G\rfloor$ denote the group $\mathcal{R}$-algebra of a group G. An element $g \in G$ has the corresponding basic element $\lfloor g\rfloor \in \mathcal{R}\lfloor G\rfloor$. The augmentation ideal $\rceil\mathcal{R}\lfloor G\rfloor \subseteq \mathcal{R}\lfloor G\rfloor$ is the kernel of the $\mathcal{R}$-homomorphism $\mathcal{R}\lfloor G\rfloor \to \mathcal{R}$, $\lfloor g\rfloor \mapsto 1$. The ideal $\rceil\mathcal{R}\lfloor G\rfloor^s$ $(s > 0)$ is $\mathcal{R}$-generated by elements of the form $(1 - \lfloor g_1\rfloor)\ldots(1 - \lfloor g_s\rfloor)$.

Let V be an abelian group. A function $f\colon G \to V$ gives rise to the homomorphism ${}^+f\colon \mathbf{Z}\lfloor G\rfloor \to V$, $\lfloor g\rfloor \mapsto f(g)$. We call f *r-gentle* if ${}^+f \mid \rceil\mathbf{Z}\lfloor G\rfloor^{r+1} = 0$, and *gentle* (or *polynomial*) if it is r-gentle for some $r \in \mathbf{N}$ [9, Ch. V].

Let p be a prime.

3.1. Lemma. *Let U be a finite $\mathbf{Z}_p$-module of dimension m. Then $\rceil\mathbf{Z}_p\lfloor U\rfloor^{(p-1)m+1} = 0$.* □

3.2. Corollary. *Let U and V be $\mathbf{Z}_p$-modules. If U is finite, then every function $f\colon U \to V$ is gentle.* □

3.3. Lemma [4, Proposition 1.2]. *Let U, V, and W be abelian groups, $f\colon U \to V$ be an r-gentle function, and $g\colon V \to W$ be an s-gentle one ($r, s \in \mathbf{N}$). Then the function $g \circ f\colon U \to W$ is rs-gentle.*

This follows from [9, Ch. V, Theorem 2.1]. □

A function $f\colon U \to V$ between abelian groups induces the $\mathcal{R}$-homomorphism $f_{\mathcal{R}}\colon \mathcal{R}\lfloor U\rfloor \to \mathcal{R}\lfloor V\rfloor$, $\lfloor u\rfloor \mapsto \lfloor f(u)\rfloor$.

3.4. Corollary. *Let U and V be abelian groups and $f\colon U \to V$ be an r-gentle ($r \in \mathbf{N}$) function. Then, for any $s \in \mathbf{N}$, the $\mathcal{R}$-homomorphism $f_{\mathcal{R}}$ maps the ideal $\rceil\mathcal{R}\lfloor U\rfloor^{rs+1}$ to the ideal $\rceil\mathcal{R}\lfloor V\rfloor^{s+1}$.* □

3.5. Lemma. *Let I be a set. For each $i \in I$, let U_i and V_i be abelian groups and $f_i\colon U_i \to V_i$ be an r-gentle ($r \in \mathbf{N}$) function. The the function*

$$\prod_{i\in I} f_i\colon \prod_{i\in I} U_i \to \prod_{i\in I} V_i$$

is r-gentle. □

The following claims are used only in discussion of the examples of § 1, not in the proof of the main results.

3.6. Lemma. *Let G be a perfect group and V be an abelian group. Then any gentle function $f\colon G \to V$ is constant.*

This follows from [9, Ch. III, Corollary 1.3]. □

3.7. *Let U be an abelian group isomorphic to $\mathbf{Z}^2$, $J\colon U \to U$ be an automorphism of order 6, V be an abelian group, and $f\colon U \to V$ be a gentle function. Suppose that the function $\mathbf{Z} \times U \to V$, $(t, u) \mapsto f(J^t u - u)$, is gentle. Then f is constant.*

The proof is omitted. $\square$

3.8. Lemma. *Let U and V be abelian groups, $f\colon U \to V$ be a gentle function, and $u_1, u_2 \in U$ be elements of coprime finite orders. Then $f(u_1+u_2) - f(u_1) - f(u_2) + f(0) = 0$.* $\square$

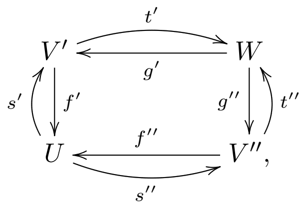
3.9. Lemma. *Let U be a divisible torsion abelian group, and V be an abelian group. Then every gentle function $f\colon U \to V$ is 1-gentle.* $\square$

3.10. Lemma. *Let G and H be groups. Then the ideal $\rceil\mathcal{R}\lfloor G \times H\rfloor^s$ ($s > 1$) is $\mathcal{R}$-generated by elements of the form $(1-\lfloor a_1\rfloor)\dots(1-\lfloor a_{s-q}\rfloor)(1-\lfloor b_1\rfloor)\dots(1-\lfloor b_q\rfloor)$, where $0 \leqslant q \leqslant s$, $a_t \in G \times 1 \subseteq G \times H$, and $b_t \in 1 \times H \subseteq G \times H$.* $\square$

3.11. Lemma. *A function $F\colon \mathbf{Z} \to \mathbf{Q}$ is r-gentle ($r \in \mathbf{N}$) if and only if it is given by a polynomial of degree at most r.* $\square$

§ 4. Keys of a commutative square

Let E be a commutative ring. Consider the diagram of simplicial E-modules and E-homomorphisms

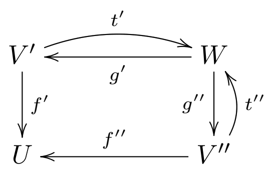


where the square is commutative: $f' \circ g' = f'' \circ g''$. We call the quadruple (s', s'', t', t'') a *key* of this square if we have $(-s', s'') \circ (-f', f'') + (g', g'') \circ (t', t'') = \mathrm{id}$ in the diagram

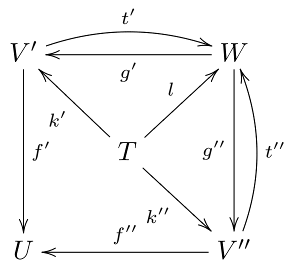
$$U \underset{(-s',s'')}{\overset{(-f',f'')}{\rightleftarrows}} V' \oplus V'' \underset{(t',t'')}{\overset{(g',g'')}{\rightleftarrows}} W.$$

The pair (t', t'') is called a *half-key* in this case.

4.1. Lemma. *Let*



be a commutative square of simplicial E-modules and E-homomorphisms with a half-key, T be a simplicial crew, and $k'\colon T \to V'$ and $k''\colon T \to V''$ be simplicial *archisms such that $f' \circ k' = f'' \circ k''$. Consider the simplicial archism $l = t' \circ k' + t'' \circ k'' \colon T \to W$. Then $g' \circ l = k'$ and $g'' \circ l = k''$.*



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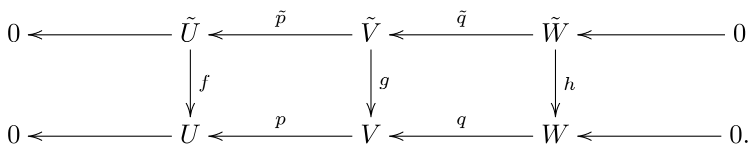
By a *sector* of a simplicial E-homomorphism $h \colon \tilde{W} \to W$ we mean a simplicial E-homomorphism $s \colon W \to \tilde{W}$ such that $h \circ s = \mathrm{id}$.

4.2. Lemma. *Consider a commutative diagram of simplicial E-modules and E-homomorphisms*

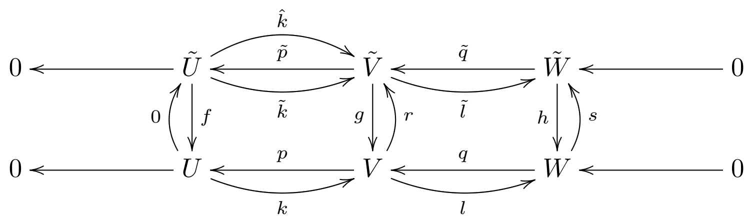


Suppose that its rows are split exact and h has a sector. Then the left-hand square has a key.

Proof. Let (k, l) and $(\tilde{k}, \tilde{l})$ (see the diagram below) be splittings:

$$p \circ k = \mathrm{id}, \qquad l \circ q = \mathrm{id}, \qquad k \circ p + q \circ l = \mathrm{id},$$
$$\tilde{p} \circ \tilde{k} = \mathrm{id}, \qquad \tilde{l} \circ \tilde{q} = \mathrm{id}, \qquad \tilde{k} \circ \tilde{p} + \tilde{q} \circ \tilde{l} = \mathrm{id},$$

and s be a sector: $h \circ s = \mathrm{id}$. Put $r = \tilde{q} \circ s \circ l$ and $\hat{k} = \tilde{k} + r \circ (k \circ f - g \circ \tilde{k})$. Then $(0, k, \hat{k}, r)$ is a key.



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4.3. Lemma. *Let L and M be simplicial crews, $j \colon L \to M$ be an isotypical cofibration, and Q be a fibrant simplicial crew. Then $Q^j \colon Q^M \to Q^L$ is an isotypical fibration.* ∎

4.4. Lemma. *Let Q and R be simplicial crews, $c\colon Q \to R$ be a fibration, and N be a simplicial crew isotypic to a point. Then $c^N : Q^N \to R^N$ is an isotypical fibration.* □

4.5. Lemma. *Suppose that E is a field. Let V and W be simplicial E-modules and $f\colon W \to V$ be an isotypical fibring simplicial E-homomorphism. Then f has a sector.* □

4.6. Lemma. *Suppose that E is a field. Let L and M be simplicial crews, $j\colon L \to M$ be an isotypical cofibration, Q and R be simplicial E-modules, and $c\colon Q \to R$ be a fibring simplicial E-homomorphism. Then the commutative square*

$$\begin{array}{ccc} Q^L & \xleftarrow{Q^j} & Q^M \\ \downarrow{\scriptstyle c^L} & & \downarrow{\scriptstyle c^M} \\ R^L & \xleftarrow{R^j} & R^M \end{array}$$

has a key.

Proof. Consider the (strictly) cofibration sequence

$$L \xrightarrow{j} M \xrightarrow{k} N.$$

Since j is isotypical, the simplicial crew N is isotypic to a point. We have the following diagram of simplicial E-modules and E-homomorphisms:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & Q^L & \xleftarrow{Q^j} & Q^M & \xleftarrow{Q^k} & Q^N & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle c^L} & & \downarrow{\scriptstyle c^M} & & \downarrow{\scriptstyle c^N} & & \\ 0 & \longleftarrow & R^L & \xleftarrow{R^j} & R^M & \xleftarrow{R^k} & R^N & \longleftarrow & 0. \end{array}$$

We show that the rows are split exact. Consider the upper row. Obviously, it is exact in the middle and the right-hand terms. Q is fibrant since it is a simplicial abelian group. By Lemma 4.3, Q^j is an isotypical fibration. By Lemma 4.5, Q^j has a sector. Therefore, the upper row is split exact. The same is true for the lower row. By Lemma 4.4, c^N is an isotypical fibration. By Lemma 4.5, c^N has a sector. By Lemma 4.2, the desired key exists. □

§ 5. Quasi-simplicial archisms

A *quasi-simplicial archism* $f\colon K \dashrightarrow L$ between simplicial crews K and L is a sequence of archisms $f_q\colon K_q \to L_q$, $q \in \mathbf{N}$. Let $\tilde{\mathrm{s}}\mathrm{Ar}(K, L)$ denote the crew of quasi-simplicial archisms and $\mathrm{sAr}(K, L)$ denote the subcrew of simplicial ones.

A quasi-simplicial archism $f\colon U \dashrightarrow V$ between simplicial abelian groups is *r-gentle* if the archisms $f_q\colon U_q \to V_q$ are r-gentle.

Let T be a simplicial crew. For $m, q \in \mathbf{N}$, let $[m|q]$ be the set of non-strictly increasing functions $[m] \to [q]$ (where $[q] = \{0, \dots, q\}$) and consider the archism

$$T(m,q) = (T(h))_{h \in [m|q]} \colon T_q \to T_m^{[m|q]}.$$

We call T *m-soluble* if, for any q, the archism $T(m,q)$ is injective.

Let p be a prime.

5.1. Lemma. *Let T be a gradual simplicial crew, U be a gradual simplicial $\mathbf{Z}_p$-module, R be an m-soluble ($m \in \mathbf{N}$) simplicial $\mathbf{Z}_p$-module, $d \colon T \to U$ be a cofibration, and $k \colon T \to R$ be a simplicial archism. Then, for some $r \in \mathbf{N}$, there exists an r-gentle quasi-simplicial archism $w \colon U \dashrightarrow R$ such that $w \circ d = k$.*

$$U \xleftarrow{\ d\ } T \xrightarrow{\ k\ } R, \qquad w \colon U \dashrightarrow R$$

Proof. Since $d_m \colon T_m \to U_m$ is injective, there exists an archism $v \colon U_m \to R_m$ such that $v \circ d_m = k_m$. By Corollary 3.2, v is r-gentle for some $r \in \mathbf{N}$. Take $q \in \mathbf{N}$. We have the commutative diagram

$$\begin{array}{ccccc}
U_q & \xleftarrow{\ d_q\ } & T_q & \xrightarrow{\ k_q\ } & R_q \\
\downarrow{\scriptstyle U(m,q)} & & \downarrow{\scriptstyle T(m,q)} & & \downarrow{\scriptstyle R(m,q)} \\
U_m^{[m|q]} & \xleftarrow{\ d_m^{[m|q]}\ } & T_m^{[m|q]} & \xrightarrow{\ k_m^{[m|q]}\ } & R_m^{[m|q]}.
\end{array}
\qquad v^{[m|q]} \colon U_m^{[m|q]} \to R_m^{[m|q]}$$

By Lemma 3.5, the archism $v^{[m|q]}$ is r-gentle. Since the $\mathbf{Z}_p$-homomorphism $R(m,q)$ is injective, there exists a $\mathbf{Z}_p$-homomorphism $f \colon R_m^{[m|q]} \to R_q$ such that $f \circ R(m,q) = \mathrm{id}$. Consider the r-gentle archism

$$w_q \colon U_q \xrightarrow{\ U(m,q)\ } U_m^{[m|q]} \xrightarrow{\ v^{[m|q]}\ } R_m^{[m|q]} \xrightarrow{\ f\ } R_q.$$

Using the diagram, we get $w_q \circ d_q = k_q$. □

5.2. Lemma. *Let M be a simplicial crew, U and V be simplicial abeliam groups, and $t \colon U \dashrightarrow V$ be an r-gentle ($r \in \mathbf{N}$) quasi-simplicial archism. Then the archism $t_\# \colon \tilde{\mathrm{s}}\mathrm{Ar}(M,U) \to \tilde{\mathrm{s}}\mathrm{Ar}(M,V)$, $f \mapsto t \circ f$, is r-gentle.*

Proof. This follows from Lemma 3.5 because of the commutative diagram

$$\begin{array}{ccc}
\tilde{\mathrm{s}}\mathrm{Ar}(M,U) & \xrightarrow{\ t_\#\ } & \tilde{\mathrm{s}}\mathrm{Ar}(M,V) \\
\Vert & & \Vert \\
\prod\limits_{q \in \mathbf{N}, k \in M_q^\times} U_q & \xrightarrow{\ \prod_{q \in \mathbf{N}, k \in M_q^\times} t_q\ } & \prod\limits_{q \in \mathbf{N}, k \in M_q^\times} V_q,
\end{array}$$

where $M_q^\times = M_q \setminus \{\text{basepoint}\}$. □

5.3. Lemma. *Let M and T be simplicial crews, U and R be simplicial $\mathbf{Z}_p$-modules, $d\colon T \to U$ and $k\colon T \to R$ be simplicial archisms, and $w\colon U \dashrightarrow R$ be an r-gentle ($r \in \mathbf{N}$) quasi-simplicial archism such that $w \circ d = k$. Then there exists an r-gentle quasi-simplicial archism $z\colon U^M \dashrightarrow R^M$ such that $z \circ d^M = k^M$.*

$$U^M \xleftarrow{\ d^M\ } T^M \xrightarrow{\ k^M\ } R^M, \qquad z\colon U^M \dashrightarrow R^M$$

Proof. Take $q \in \mathbf{N}$. We have the commutative diagram

$$\begin{array}{ccccc}
(U^M)_q & \xleftarrow{(d^M)_q} & (T^M)_q & \xrightarrow{(k^M)_q} & (R^M)_q \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle j} \\
\tilde{\mathrm{s}}\mathrm{Ar}(\Delta^q_+ \wedge M, U) & & \xrightarrow{\quad w_\# \quad} & & \tilde{\mathrm{s}}\mathrm{Ar}(\Delta^q_+ \wedge M, R),
\end{array}$$

where the $\mathbf{Z}_p$-homomorphism $i\colon (U^M)_q = \mathrm{sAr}(\Delta^q_+ \wedge M, U) \to \tilde{\mathrm{s}}\mathrm{Ar}(\Delta^q_+ \wedge M, U)$ is the inclusion and j is analogous. By Lemma 5.2, the archism $w_\#$ is r-gentle. There is a $\mathbf{Z}_p$-homomorphism $f\colon \tilde{\mathrm{s}}\mathrm{Ar}(\Delta^q_+ \wedge M, R) \to (R^M)_q$ such that $f \circ j = \mathrm{id}$. Consider the r-gentle archism

$$z_q\colon (U^M)_q \xrightarrow{\ i\ } \tilde{\mathrm{s}}\mathrm{Ar}(\Delta^q_+ \wedge M, U) \xrightarrow{\ w_\#\ } \tilde{\mathrm{s}}\mathrm{Ar}(\Delta^q_+ \wedge M, R) \xrightarrow{\ f\ } (R^M)_q.$$

Using the diagram, we get $z_q \circ (d^M)_q = (k^M)_q$. □

§ 6. Harmonic cofibrations

Let T be a simplicial crew and U be a simplicial abelian group. A cofibration $d\colon T \to U$ is called *r-harmonic* ($r \in \mathbf{N}$) if, for any compact simplicial crews L and M and any isotypical cofibration $j\colon L \to M$, there exist a simplicial archism $x\colon T^L \to T^M$ and an r-gentle quasi-simplicial archism $y\colon U^L \dashrightarrow U^M$ such that $d^M \circ x = y \circ d^L$ and $T^j \circ x = \mathrm{id}$.

$$\begin{array}{ccc}
T^L & \underset{T^j}{\overset{x}{\rightleftarrows}} & T^M \\
\downarrow{\scriptstyle d^L} & & \downarrow{\scriptstyle d^M} \\
U^L & \underset{y\ (\dashrightarrow)}{\overset{U^j}{\longleftarrow}} & U^M
\end{array}$$

A cofibration is *harmonic* if it is r-harmonic for some $r \in \mathbf{N}$.

By the *height* of a 0-connected space Y we mean the supremum of those $q \in \mathbf{N}$ for which $\pi_q(Y) \neq 1$ (the supremum of the empty set is 0).

6.1. Lemma. *Let p be a prime and Y be a connected CW-complex of finite height with p-finite homotopy groups. Then there exist a gradual simplicial crew T with $|T| \simeq Y$, a gradual simplicial $\mathbf{Z}_p$-module U, and a harmonic cofibration $d\colon T \to U$.*

Proof. (Induction along the Postnikov decomposition of Y with fibres of the form $\mathcal{K}(\mathbf{Z}_p, q)$.) Let n be the height of Y. If $n = 0$, then Y is contractible, we put $T = U = 0$ and that is all. Otherwise, choose an order p element $e \in \pi_n(Y)$ fixed by the canonical action of $\pi_1(Y)$. Its existence follows from the well-known congruence $|\mathrm{Fix}_G X| \equiv |X| \pmod p$ for an action of a p-finite group G on a finite set X (cf. the remark in [2, Ch. II, Example 5.2(iv)]). We attach cells to Y to get a map $Y \to \bar{Y}$ inducing isomorphisms on π_q, $q \neq n$, and an epimorphism with the kernel generated by e on π_n. The space Y is homotopy equivalent to the homotopy fibre of some map $\bar{Y} \to \mathcal{K}(\mathbf{Z}_p, n+1)$ [6, Lemma 4.70].

We assume (as an induction hypothesis) that there are gradual simplicial crew $\bar{T}$ with $|\bar{T}| \simeq \bar{Y}$, gradual simplicial $\mathbf{Z}_p$-module $\bar{U}$, and r-harmonic ($r \geqslant 1$) cofibration $\bar{d}\colon \bar{T} \to \bar{U}$.

Let R be a gradual $(n+1)$-soluble simplicial $\mathbf{Z}_p$-module with $|R| \simeq \mathcal{K}(\mathbf{Z}_p, n+1)$, Q be a gradual simplicial $\mathbf{Z}_p$-module isotypic to a point, and $c\colon Q \to R$ be a fibring simplicial $\mathbf{Z}_p$-homomorphism (see [3]). There is a Cartesian square of simplicial crews and archisms

$$\begin{array}{ccc} T & \xrightarrow{\ h\ } & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle c} \\ \bar{T} & \xrightarrow{\ k\ } & R, \end{array}$$

where $|T| \simeq Y$. Put $U = \bar{U} \times Q$. Let $\mathbf{Z}_p$-homomorphisms $a\colon U \to \bar{U}$ and $b\colon U \to Q$ be the projections. Let $d\colon T \to U$ be the simplicial archism given by the conditions $a \circ d = \bar{d} \circ f$ and $b \circ d = h$. Obviously, d is a cofibration.

By Lemma 5.1, for some $s \geqslant 1$ there is an s-gentle quasi-simplicial archism $w\colon \bar{U} \dashrightarrow R$ such that $w \circ \bar{d} = k$.

We show that d is rs-harmonic. Take compact simplicial crews L and M and an isotypical cofibration $j\colon L \to M$. We need a simplicial archism $x\colon T^L \to T^M$ and an rs-gentle quasi-simplicial archism $y\colon U^L \dashrightarrow U^M$ such that $d^M \circ x = y \circ d^L$ and $T^j \circ x = \mathrm{id}$. Since $\bar{d}$ is r-harmonic, there are a simplicial archism $\bar{x}\colon \bar{T}^L \to \bar{T}^M$ and an r-gentle quasi-simplicial archism $\bar{y}\colon \bar{U}^L \dashrightarrow \bar{U}^M$ such that $\bar{d}^M \circ \bar{x} = \bar{y} \circ \bar{d}^L$ and $\bar{T}^j \circ \bar{x} = \mathrm{id}$.

We have the commutative square of simplicial $\mathbf{Z}_p$-modules and $\mathbf{Z}_p$-homomorphisms with a half-key

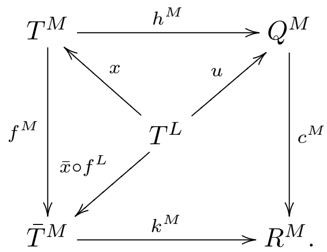
$$\begin{array}{ccc} Q^L & \underset{Q^j}{\overset{t'}{\rightleftarrows}} & Q^M \\ {\scriptstyle c^L}\downarrow & & \downarrow{\scriptstyle c^M}\ \uparrow{\scriptstyle t''} \\ R^L & \xleftarrow{\ R^j\ } & R^M \end{array}$$

(the half-key exists by Lemma 4.6). We have the simplicial archism

$$u = t' \circ h^L + t'' \circ k^M \circ \bar{x} \circ f^L\colon T^L \to Q^M.$$

We have $c^L \circ h^L = k^L \circ f^L = k^L \circ \bar{T}^j \circ \bar{x} \circ f^L = R^j \circ k^M \circ \bar{x} \circ f^L$. Therefore, by Lemma 4.1, $Q^j \circ u = h^L$ and $c^M \circ u = k^M \circ \bar{x} \circ f^L$.

Define the desired x by the conditions $f^M \circ x = \bar{x} \circ f^L$ and $h^M \circ x = u$:



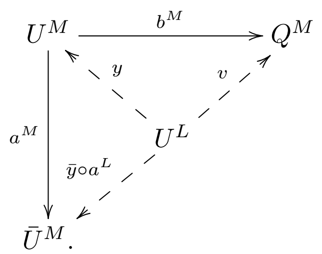
This is possible because the square is Cartesian and the conditions are compatible: $k^M \circ \bar{x} \circ f^L = c^M \circ u$. We have $T^j \circ x = \mathrm{id}$ because $f^L \circ T^j \circ x = \bar{T}^j \circ f^M \circ x = \bar{T}^j \circ \bar{x} \circ f^L = f^L$ and $h^L \circ T^j \circ x = Q^j \circ h^M \circ x = Q^j \circ u = h^L$.

By Lemma 5.3, there is an s-gentle quasi-simplicial archism $z \colon \bar{U}^M \dashrightarrow R^M$ such that $z \circ \bar{d}^M = k^M$. We have the quasi-simplicial archism

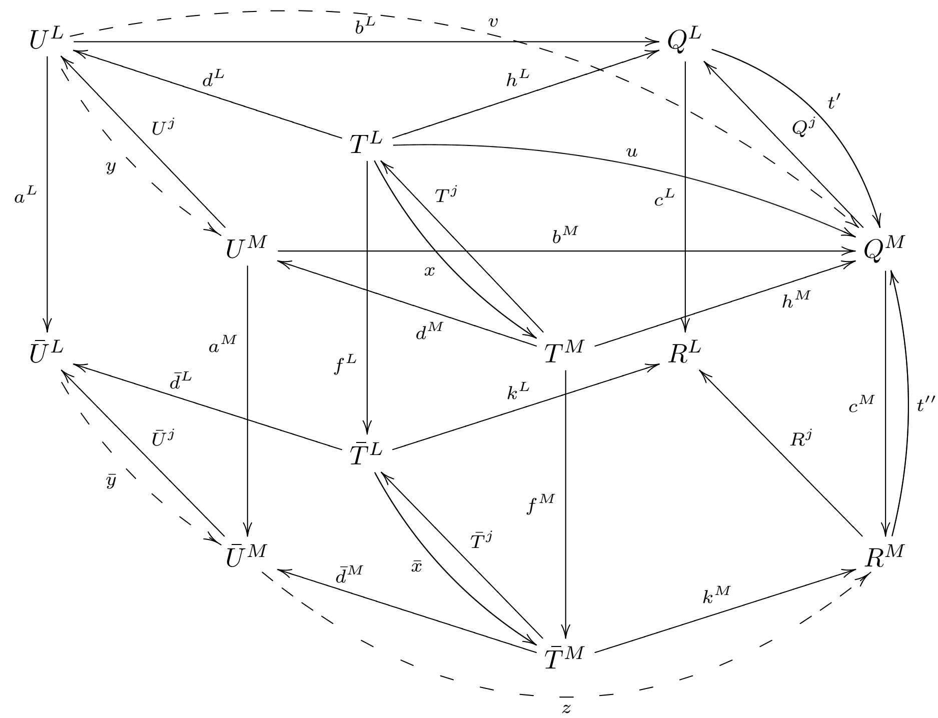
$$v = t' \circ b^L + t'' \circ z \circ \bar{y} \circ a^L \colon U^L \dashrightarrow Q^M.$$

By Lemma 3.3, it is rs-gentle.

Define the desired y by the conditions $a^M \circ y = \bar{y} \circ a^L$ and $b^M \circ y = v$:



This is possible because $(a^M, b^M) \colon U^M \to \bar{U}^M \times Q^M$ is an isomorphism. Obviously, y is rs-gentle. We have $d^M \circ x = y \circ d^L$ because $a^M \circ d^M \circ x = \bar{d}^M \circ f^M \circ x = \bar{d}^M \circ \bar{x} \circ f^L = \bar{y} \circ \bar{d}^L \circ f^L = \bar{y} \circ a^L \circ d^L = a^M \circ y \circ d^L$ and $b^M \circ d^M \circ x = h^M \circ x = u = t' \circ h^L + t'' \circ k^M \circ \bar{x} \circ f^L = t' \circ h^L + t'' \circ z \circ \bar{d}^M \circ \bar{x} \circ f^L = t' \circ h^L + t'' \circ z \circ \bar{y} \circ \bar{d}^L \circ f^L = t' \circ b^L \circ d^L + t'' \circ z \circ \bar{y} \circ a^L \circ d^L = v \circ d^L = b^M \circ y \circ d^L$.



(The straight arrows of this diagram form a commutative subdiagram.) □

§ 7. Two filtrations of the module $C_0(U^K)$

7.1. Lemma. *Let U_i, $i \in I$, be a finite collection of abelian groups. Put*

$$U_J = \bigoplus_{i \in J} U_i, \qquad J \subseteq I,$$

and $U = U_I$. Let $p_J \colon U \to U_J$ be the projections. Then for any $r \in \mathbf{N}$

$$\bigcap_{J \subseteq I \,:\, |J| \leqslant r} \ker(p_J)_{\mathcal{R}} \subseteq \rceil \mathcal{R} \lfloor U \rfloor^{r+1}.$$

in the $\mathcal{R}$-algebra $\mathcal{R}\lfloor U \rfloor$.

Proof. Let $s_J \colon U_J \to U$ be the canonical embeddings. Put $q_J = s_J \circ p_J \colon U \to U$. We assume $|I| > r$ (otherwise, the assertion is trivial). For $u \in U$, we have (cf.

[5, Lemma 5.5])

$$\begin{aligned}\lfloor u\rfloor - \sum_{J\subseteq I\,:\,|J|\leqslant r} (-1)^{r-|J|}\binom{|I|-|J|-1}{r-|J|}\lfloor q_J(u)\rfloor = \\ = \sum_{J\subseteq I}\Big(\sum_{M\subseteq I\,:\,M\supseteq J,\,|M|>r}(-1)^{|M|-|J|}\Big)\lfloor q_J(u)\rfloor = \\ = \sum_{M\subseteq I\,:\,|M|>r}\Big(\sum_{J\subseteq M}(-1)^{|M|-|J|}\lfloor q_J(u)\rfloor\Big) = \\ = \sum_{M\subseteq I\,:\,|M|>r}\prod_{i\in M}(\lfloor q_{\{i\}}(u)\rfloor - 1) \in \rceil\mathcal{R}\lfloor U\rfloor^{r+1}.\end{aligned}$$

It follows that for $w\in\mathcal{R}\lfloor U\rfloor$ we have

$$w - \sum_{J\subseteq I\,:\,|J|\leqslant r}(-1)^{r-|J|}\binom{|I|-|J|-1}{r-|J|}(q_J)_{\mathcal{R}}(w)\in\rceil\mathcal{R}\lfloor U\rfloor^{r+1}.$$

If

$$w\in\bigcap_{J\subseteq I\,:\,|J|\leqslant r}\ker(p_J)_{\mathcal{R}},$$

then, using that $\ker(p_J)_{\mathcal{R}} = \ker(q_J)_{\mathcal{R}}$, we get $w\in\rceil\mathcal{R}\lfloor U\rfloor^{r+1}$. □

For a simplicial abelian group V, the module $C_0(V) = \mathcal{R}\lfloor V_0\rfloor$ has the filtration $C_0^{\rceil s}(V) = \rceil\mathcal{R}\lfloor V_0\rfloor^s$, $s\in\mathbf{N}$.

7.2. Corollary. *Let K be a compact simplicial crew, E be a field, U be a simplicial E-module, and $r\in\mathbf{N}$ be a number. Consider the $\mathcal{R}$-homomorphism*

$$C_0(U^K)\xrightarrow{{}^K_U\mu_r}\operatorname{Hom}_0(C_*(K^r), C_*(U^r)).$$

Then $\ker{}^K_U\mu_r\subseteq C_0^{\rceil r+1}(U^K)$.

Proof. Take an element $B\in\ker{}^K_U\mu_r$. We show that $B\in C_0^{\rceil r+1}(U^K)$.

There is $n\in\mathbf{N}$ such that the simplicial crew K is generated by a finite collection of n-simplices: $g_i\in K_n$, $i\in I$. We have the E-homomorphism $h\colon (U^K)_0\to U_n^I$, $b\mapsto(b(g_i))_{i\in I}$. It is injective. Therefore, there is an E-homomorphism $f\colon U_n^I\to(U^K)_0$ such that $f\circ h = \mathrm{id}$. It suffices to show that $h_{\mathcal{R}}(B)\in\rceil\mathcal{R}\lfloor U_n^I\rfloor^{r+1}$. Indeed, then $B = f_{\mathcal{R}}(h_{\mathcal{R}}(B))\in\rceil\mathcal{R}\lfloor(U^K)_0\rfloor^{r+1} = C_0^{\rceil r+1}(U^K)$.

For $J\subseteq I$, let $p_J\colon U_n^I\to U_n^J$ be the projection. Take $J\subseteq I$ with $|J|\leqslant r$. By Lemma 7.1, it suffices to verify that $(p_J)_{\mathcal{R}}(h_{\mathcal{R}}(B)) = 0$.

Choose a function $t\colon J\to\{1,\dots,r\}$ and a simplex $k = (k_1,\dots,k_r)\in K_n^r$ such that $k_{t(i)} = g_i$, $i\in J$. We have the E-homomorphism $U_n^t\colon U_n^r\to U_n^J$, the

$\mathcal{R}$-homomorphism $(U^t_n)_{\mathcal{R}}\colon C_n(U^r) = \mathcal{R}\lfloor U^r_n\rfloor \to \mathcal{R}\lfloor U^J_n\rfloor$, and the commutative diagram

$$\begin{array}{ccc} \mathcal{R}\lfloor (U^K)_0\rfloor & \xrightarrow{\quad h_{\mathcal{R}}\quad} & \mathcal{R}\lfloor U^I_n\rfloor \\ {\scriptstyle {}^K_U\mu_r}\big\downarrow & & \big\downarrow{\scriptstyle (p_J)_{\mathcal{R}}} \\ \mathrm{Hom}_0(C_*(K^r),C_*(U^r)) & \xrightarrow{v\mapsto (U^t_n)_{\mathcal{R}}(v(\lfloor k\rfloor))} & \mathcal{R}\lfloor U^J_n\rfloor. \end{array}$$

Since ${}^K_U\mu_r(B) = 0$, we get $(p_J)_{\mathcal{R}}(h_{\mathcal{R}}(B)) = 0$. □

§ 8. Simplicial approximation

8.1. Lemma. *Let K be a compact simplicial crew, W be a simplicial crew, and $f\colon |K| \to |W|$ be a map. Then there exist a compact simplicial crew L, an isotypy $e\colon L \to K$, and a simplicial archism $g\colon L \to W$ such that $f \circ |e| \sim |g|$.*

See [8, Corollary 4.8]. □

For simplicial crews L and T, the geometrical realization $|?|\colon (T^L)_0 \to |T|^{|L|}$ induces an $\mathcal{R}$-homomorphism $\|?\|\colon H_0(T^L) \to H_0(|T|^{|L|})$.

8.2. Lemma. *Let K be a compact simplicial crew, T be a simplicial crew, and $r \in \mathbf{N}$ be a number. Then, for any $A \in \ker {}^{|K|}_{|T|}\mu_r$, there exist a compact simplicial crew L, an isotypy $e\colon L \to K$, and an element $B \in \ker {}^L_T\mu_r$ such that $H_0(|T|^{|e|})({}^{|K|}_{|T|}\nu(A)) = \|{}^L_T\nu(B)\|$:*

$$\begin{array}{ccccc} \mathrm{Hom}_0(C_*(L^r),C_*(T^r)) & \xleftarrow{\quad {}^L_T\mu_r \quad} & C_0(T^L) & \xrightarrow{\quad {}^L_T\nu \quad} & H_0(T^L) \\ & & B & & \big\downarrow{\scriptstyle \|?\|} \\ & & & & H_0(|T|^{|L|}) \\ & & A & & \big\uparrow{\scriptstyle H_0(|T|^{|e|})} \\ \mathrm{Hom}(C_0(|K|^r),C_0(|T|^r)) & \xleftarrow{\quad {}^{|K|}_{|T|}\mu_r \quad} & C_0(|T|^{|K|}) & \xrightarrow{\quad {}^{|K|}_{|T|}\nu \quad} & H_0(|T|^{|K|}). \end{array}$$

Proof. We have

$$A = \sum_{i=1}^{m} v_i \lfloor a_i\rfloor,$$

where $m \in \mathbf{N}$, $v_i \in \mathcal{R}$, and $a_i \in |T|^{|K|}$. For $x \in |K|$, define an equivalence (relation) $c(x)$ on the set $I = \{1, \dots, m\}$: $c(x) = \{\, (i,j) : a_i(x) = a_j(x) \,\}$. Put $E = \{\, c(x) : x \in |K| \,\}$.

We call an equivalence on I *neutral* if

$$\sum_{i \in J} v_i = 0$$

for all its classes $J \subseteq I$. We show that for any $h_1, \dots, h_r \in E$ the equivalence $h = h_1 \cap \dots \cap h_r$ is neutral. For each $s = 1, \dots, r$, there is a point $x_s \in |K|$ such that $h_s = c(x_s)$. Put $x = (x_1, \dots, x_r) \in |K|^r$. In $C_0(|T|^r)$, we have

$$\sum_{i \in I} v_i \lfloor a_i^r(x) \rfloor = {}^{|K|}_{|T|}\mu_r(A) = 0.$$

It follows that h is neutral because

$$a_i^r(x) = a_j^r(x) \iff (i, j) \in h$$

for $i, j \in I$.

For each equivalence h on I, there is the corresponding simplicial subcrew $V(h) \subseteq T^m$ (the diagonal):

$$V(h)_q = \{ (t_1, \dots, t_m) \in T_q^m : t_i = t_j \text{ for all } (i, j) \in h \}.$$

Put

$$W = \bigcup_{h \in E} V(h) \subseteq T^m.$$

We have the maps $a = (a_1, \dots, a_m)\colon |K| \to |T|^m$ and $\tilde{a} = d^{-1} \circ a\colon |K| \to |T^m|$, where $d\colon |T^m| \to |T|^m$ is the canonical bijective map. For $x \in |K|$, we have $\tilde{a}(x) \in |V(c(x))|$. Therefore $\operatorname{im} \tilde{a} \subseteq |W|$. Using Lemma 8.1, we find a compact simplicial crew L, an isotypy $e\colon L \to K$, and a simplicial archism $b = (b_1, \dots, b_m)\colon L \to T^m$ such that $\operatorname{im} b \subseteq W$ and $\tilde{a} \circ |e| \sim |b|$. Put

$$B = \sum_{i=1}^m v_i \lfloor b_i \rfloor.$$

We have $a_i \circ |e| \sim |b_i|$. Therefore $H_0(|T|^{|e|})({}^{|K|}_{|T|}\nu(A)) = \|{}^L_T\nu(B)\|$. We show that ${}^K_T\mu_r(B) = 0$. For $k = (k_1, \dots, k_r) \in K_q^r$ ($q \in \mathbf{N}$), we have

$${}^K_T\mu_r(B)(\lfloor k \rfloor) = \sum_{i=1}^m v_i \lfloor b_i^r(k) \rfloor.$$

Take $s = 1, \dots, r$. Since $\operatorname{im} b \subseteq W$, there is $h_s \in E$ such that $b(k_s) \in V(h_s)$. Therefore, the function $i \mapsto b_i(k_s)$ is subordinate to (i. e. constant on the classes of) the equivalence h_s. Since $b_i^r(k) = (b_i(k_1), \dots, b_i(k_r))$, the function $i \mapsto b_i^r(k)$ is subordinate to the equivalence $h = h_1 \cap \dots \cap h_r$. Since h is neutral, we get ${}^K_T\mu_r(B)(\lfloor k \rfloor) = 0$. □

§ 9. The inclusion $\ker {}^X_Y\mu_r \subseteq \ker {}^X_Y\nu$ for large r

9.1. Lemma. *Let X, Y, $\tilde{X}$, and $\tilde{Y}$ be spaces. Suppose that $X \simeq \tilde{X}$ and $Y \simeq \tilde{Y}$. Then, for any $r \in \mathbf{N}$, we have*

$$\ker {}^X_Y\mu_r \subseteq \ker {}^X_Y\nu \iff \ker {}^{\tilde{X}}_{\tilde{Y}}\mu_r \subseteq \ker {}^{\tilde{X}}_{\tilde{Y}}\nu.$$

Proof. There are homotopy euivalences $k\colon X \to \tilde{X}$ and $h\colon \tilde{Y} \to Y$. We have the commutative diagram of $\mathcal{R}$-modules and $\mathcal{R}$-homomorphisms:

$$\begin{array}{ccccc}
\mathrm{Hom}(C_0(\tilde{X}^r), C_0(\tilde{Y}^r)) & \xleftarrow{\;{}^{\tilde{X}}_{\tilde{Y}}\mu_r\;} & C_0(\tilde{Y}^{\tilde{X}}) & \xrightarrow{\;{}^{\tilde{X}}_{\tilde{Y}}\nu\;} & H_0(\tilde{Y}^{\tilde{X}}) \\
\big\downarrow & & \big\downarrow{\scriptstyle C_0(h^k)} & & \big\downarrow{\scriptstyle H_0(h^k)} \\
\mathrm{Hom}(C_0(X^r), C_0(Y^r)) & \xleftarrow{\;{}^{X}_{Y}\mu_r\;} & C_0(Y^X) & \xrightarrow{\;{}^{X}_{Y}\nu\;} & H_0(Y^X),
\end{array}$$

where the vertical arrows are induced by k and h. Since $H_0(h^k)$ is an isomorphism, we get the implication $\Rightarrow$. The implication $\Leftarrow$ is analogous. $\square$

Let p be a prime. Assume $\mathcal{R} = \mathbf{Z}_p$.

9.2. *Let X be a compact CW-complex and Y be a connected CW-complex of finite height with p-finite homotopy groups. Then, for any sufficiently large $r \in \mathbf{N}$, we have $\ker {}^{X}_{Y}\mu_r \subseteq \ker {}^{X}_{Y}\nu$ in the diagram*

$$\mathrm{Hom}(C_0(X^r), C_0(Y^r)) \xleftarrow{\;{}^{X}_{Y}\mu_r\;} C_0(Y^X) \xrightarrow{\;{}^{X}_{Y}\nu\;} H_0(Y^X).$$

Proof. By Lemma 6.1, for some $s \in \mathbf{N}$, there are a gradual simplicial crew T with $|T| \simeq Y$, a gradual simplicial $\mathbf{Z}_p$-module U, and an s-harmonic cofibration $d\colon T \to U$. We have $X \simeq |K|$ for some compact simplicial crew K. Obviously, $(U^K)_0$ is a finite $\mathbf{Z}_p$-module. By Lemma 3.1, $C_0^{\rceil t+1}(U^K) = 0$ for some $t \in \mathbf{N}$. Take $r \geqslant st$. We show that $\ker {}^{|K|}_{|T|}\mu_r \subseteq \ker {}^{|K|}_{|T|}\nu$ in the diagram

$$\mathrm{Hom}(C_0(|K|^r), C_0(|T|^r)) \xleftarrow{\;{}^{|K|}_{|T|}\mu_r\;} C_0(|T|^{|K|}) \xrightarrow{\;{}^{|K|}_{|T|}\nu\;} H_0(|T|^{|K|}).$$

This will suffice by Lemma 9.1.

Take an element $A \in \ker {}^{|K|}_{|T|}\mu_r$. We show that $A \in \ker {}^{|K|}_{|T|}\nu$. By Lemma 8.2, there are a compact simplicial crew L, an isotypy $e\colon L \to K$, and an element $B \in \ker {}^{L}_{T}\mu_r$ such that $H_0(|T|^{|e|})({}^{|K|}_{|T|}\nu(A)) = \|{}^{L}_{T}\nu(B)\|$. Since $|e|$ is a homotopy equivalence, $H_0(|T|^{|e|})$ is an isomorphism. Therefore it suffices to show that ${}^{L}_{T}\nu(B) = 0$.

Let a simplicial crew M be the (reduced) cylinder of e. We have the homotopy commutative diagram

$$\begin{array}{ccc}
K & \xleftarrow{\;e\;} & L \\
{\scriptstyle i}\searrow & & \swarrow{\scriptstyle j} \\
& M, &
\end{array}$$

where i and j are the canonical cofibrations. By the definition of a cylinder, i is an isotypy. Since e is an isotypy, j is an isotypy too. Since d is s-harmonic, there is the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{id} & & \\ T^L & \xrightarrow{\quad x \quad} & T^M & \xrightarrow{\quad T^j \quad} & T^L \\ {\scriptstyle d^L}\big\downarrow & & \big\downarrow{\scriptstyle d^M} & & \\ U^L & \overset{y}{\dashrightarrow} & U^M, & & \end{array}$$

where x is a simplicial archism and y is an s-gentle quasi-simplicial archism. We have the commutative diagram of $\mathbf{Z}_p$-homomorphisms:

$$\begin{array}{ccccccc} & & B & & B_1 & & B_2 \\ \mathrm{Hom}_0(C_*(L^r), C_*(T^r)) & \xleftarrow{{}^L_T\mu_r} & C_0(T^L) & \xrightarrow{C_0(x)} & C_0(T^M) & \xrightarrow{C_0(T^i)} & C_0(T^K) \\ \big\downarrow & & \big\downarrow{\scriptstyle C_0(d^L)} & & \big\downarrow{\scriptstyle C_0(d^M)} & & \big\downarrow{\scriptstyle C_0(d^K)} \\ \mathrm{Hom}_0(C_*(L^r), C_*(U^r)) & \xleftarrow{{}^L_U\mu_r} & C_0(U^L) & \xrightarrow{C_0(y)} & C_0(U^M) & \xrightarrow{C_0(U^i)} & C_0(U^K), \\ & & B' & & B'_1 & & B'_2 \end{array}$$

where the vertical arrows are induced by the cofibration d; $B_1, \ldots, B'_2$ are the images of B in the corresponding modules. Since ${}^L_T\mu_r(B) = 0$, we have ${}^L_U\mu_r(B') = 0$. By Corollary 7.2, $B' \in C_0^{\rceil r+1}(U^L)$. Since $r \geqslant st$ and the archism y_0 is s-gentle, we have, by Corollary 3.4, $B'_1 \in C_0^{\rceil t+1}(U^M)$. Since $(U^i)_0$ is a homomorphism, $B'_2 \in C_0^{\rceil t+1}(U^K)$. We have $C_0^{\rceil t+1}(U^K) = 0$. It follows that $B'_2 = 0$. Since d is a cofibration, $C_0(d^K)$ is injective. Therefore $B_2 = 0$.

We have the commutative diagram of $\mathbf{Z}_p$-homomorphisms

$$\begin{array}{ccccccc} \underset{B}{C_0(T^L)} & \xrightarrow{\mathrm{id}} & & \underset{}{\overset{B}{C_0(T^L)}} & \xrightarrow{{}^L_T\nu} & & H_0(T^L) \\ & {\scriptstyle C_0(x)}\searrow & & \nearrow{\scriptstyle C_0(T^j)} & & \nearrow{\scriptstyle H_0(T^j)} & \big\uparrow{\scriptstyle H_0(T^e)} \\ & & \underset{B_1}{C_0(T^M)} & \xrightarrow{{}^M_T\nu} & H_0(T^M) & & \\ & & {\scriptstyle C_0(T^i)}\searrow & & & \searrow{\scriptstyle H_0(T^i)} & \\ & & & \underset{B_2}{C_0(T^K)} & \xrightarrow{{}^K_T\nu} & & H_0(T^K). \end{array}$$

Since $B_2 = 0$, we get ${}^L_T\nu(B) = 0$. □

Consider the filtration of the complex $C_*(Y^X)$ formed by the kernels of the $\mathbf{Z}_p$-homomorphisms

$$C_q(Y^X) \xrightarrow{i_q} C_0(Y^{\Delta^q_+ \wedge X}) \xrightarrow{{}^{\Delta^q_+ \wedge X}_{\quad Y}\mu_r} \mathrm{Hom}(C_0((\Delta^q_+ \wedge X)^r), C_0(Y^r)),$$

where i_q are the obvious isomorphisms. Does this filtration converge?

§ 10. Deducing Theorem 1.2 from claim 9.2

10.1. Lemma. *Let X, Y, $\tilde{X}$, and $\tilde{Y}$ be spaces, $k\colon X \to \tilde{X}$ and $h\colon \tilde{Y} \to Y$ be maps, V be an abelian group, and $f\colon [X,Y] \to V$ be an invariant. Consider the invariant $\tilde{f}\colon [\tilde{X},\tilde{Y}] \to V$, $\tilde{u} \mapsto f([h] \circ \tilde{u} \circ [k])$. Then* $\operatorname{Deg} \tilde{f} \leqslant \operatorname{Deg} f$.

Proof. Take $r \in \mathbf{N}$. The maps k and h induce a homomorphism

$$t\colon\ \operatorname{Hom}(C_0(\tilde{X}^r), C_0(\tilde{Y}^r)) \to \operatorname{Hom}(C_0(X^r), C_0(Y^r)).$$

We have $t(C_0(\tilde{a}^r)) = C_0((h \circ \tilde{a} \circ k)^r)$, $\tilde{a} \in \tilde{Y}^{\tilde{X}}$. Assume that $\operatorname{Deg} f \leqslant r$. There is a homomorphism $l\colon\ \operatorname{Hom}(C_0(X^r), C_0(Y^r)) \to V$ such that $f([a]) = l(C_0(a^r))$ for all $a \in Y^X$. Consider the homomorphism $\tilde{l} = l \circ t\colon\ \operatorname{Hom}(C_0(\tilde{X}^r), C_0(\tilde{Y}^r)) \to V$. For $\tilde{a} \in \tilde{Y}^{\tilde{X}}$ we have $\tilde{f}([\tilde{a}]) = f([h \circ \tilde{a} \circ k]) = l(C_0((h \circ \tilde{a} \circ k)^r)) = l(t(C_0(\tilde{a}^r))) = \tilde{l}(C_0(\tilde{a}^r))$. Therefore $\operatorname{Deg} \tilde{f} \leqslant r$. □

Proof of Theorem 1.2. (1) *Case of Y of finite height.* It suffices to show that the "universal" invariant $F\colon [X,Y] \to H_0(Y^X; \mathbf{Z}_p)$, $u \mapsto \lfloor u \rfloor$, has finite degree. For $r \in \mathbf{N}$ we have the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Hom}_{\mathbf{Z}}(C_0(X^r;\mathbf{Z}), C_0(Y^r;\mathbf{Z})) & \xleftarrow{{}^X_Y\tilde{\mu}_r} & C_0(Y^X;\mathbf{Z}) & & \\
\Big\downarrow{\scriptstyle m'} & & \Big\downarrow{\scriptstyle m} & & \\
\operatorname{Hom}_{\mathbf{Z}_p}(C_0(X^r;\mathbf{Z}_p), C_0(Y^r;\mathbf{Z}_p)) & \xleftarrow{{}^X_Y\mu_r} & C_0(Y^X;\mathbf{Z}_p) & \xrightarrow{{}^X_Y\nu} & H_0(Y^X;\mathbf{Z}_p),
\end{array}$$

where m and m' are the homomorphisms of reduction modulo p; the tilde over μ in the upper row means "over $\mathbf{Z}$". By claim 9.2, we have $\ker {}^X_Y\mu_r \subseteq \ker {}^X_Y\nu$ for sufficiently large r. Then there is a $\mathbf{Z}_p$-homomorphism $t\colon\ \operatorname{Hom}_{\mathbf{Z}_p}(C_0(X^r;\mathbf{Z}_p), C_0(Y^r;\mathbf{Z}_p)) \to H_0(Y^X;\mathbf{Z}_p)$ such that $t \circ {}^X_Y\mu_r = {}^X_Y\nu$. For $a \in Y^X$, we have $F([a]) = ({}^X_Y\nu \circ m)(\lfloor a \rfloor) = (t \circ m' \circ {}^X_Y\tilde{\mu}_r)(\lfloor a \rfloor) = (t \circ m')(C_0(a^r;\mathbf{Z}))$. Therefore $\operatorname{Deg} F \leqslant r$.

(2) *General case.* There are a connected CW-complex $\bar{Y}$ of finite height with p-finite homotopy groups and a $(\dim X + 1)$-connected map $h\colon Y \to \bar{Y}$ ($\bar{Y}$ is obtained from Y by attaching cells of high dimensions). The induced function $h_\#\colon [X,Y] \to [X,\bar{Y}]$ is bijective. Consider the invariant $\bar{f} = f \circ h_\#^{-1}\colon [X,\bar{Y}] \to \mathbf{Z}_p$. By Lemma 10.1, $\operatorname{Deg} f \leqslant \operatorname{Deg} \bar{f}$. By (1), $\operatorname{Deg} \bar{f} < \infty$. □

§ 11. Deducing Theorem 1.1 from Theorem 1.2

11.1. Lemma [2, Ch. VI, Proposition 8.6]. *Let X be a connected compact CW-complex, Y be a nilpotent connected CW-complex with finitely generated homotopy groups, and $u_1, u_2 \in [X,Y]$ be distinct classes. Then, for some prime p, there exist a connected CW-complex $\bar{Y}$ with p-finite homotopy groups and a map $h\colon Y \to \bar{Y}$ such that $[h] \circ u_1 \neq [h] \circ u_2$ in $[X,\bar{Y}]$.* □

Proof of Theorem 1.1. By Lemma 11.1, for some prime p there are a connected CW-complex $\bar{Y}$ with p-finite homotopy groups, and a map $h\colon Y \to \bar{Y}$

such that the classes $\bar{u}_i = [h] \circ u_i$, $i = 1, 2$, are distinct. There is an invariant $\bar{f}\colon [X, \bar{Y}] \to \mathbf{Z}_p$ such that $\bar{f}(\bar{u}_1) \neq \bar{f}(\bar{u}_2)$. By Theorem 1.2, $\operatorname{Deg} \bar{f} < \infty$. Consider the invariant $f = \bar{f} \circ h_\#\colon [X, Y] \to \mathbf{Z}_p$. By Lemma 10.1, $\operatorname{Deg} f < \infty$. We have $f(u_1) = \bar{f}(\bar{u}_1) \neq \bar{f}(\bar{u}_2) = f(u_2)$. □

§ 12. Properties of finite-degree invariants

Put $\mathcal{E} = \{0, 1\} \subseteq \mathbf{Z}$. For $e = (e_1, \ldots, e_n) \in \mathcal{E}^n$, put $|e| = e_1 + \ldots + e_n$.

Cosider a wedge of spaces $W = T_1 \vee \ldots \vee T_n$. Let $\mathrm{in}_k^W\colon T_k \to W$ be the inclusions. For $e \in \mathcal{E}^n$, put $M_e^W = m_1 \vee \ldots \vee m_n\colon W \to W$, where $m_k\colon T_k \to T_k$ is: the identity if $e_k = 1$, and the constant map otherwise.

12.1. Lemma. *Let X and Y be spaces, V be an abelian group, $f\colon [X, Y] \to V$ be an invariant of degree at most $r \in \mathbf{N}$, $W = T_1 \vee \ldots \vee T_{r+1}$ be a wedge of spaces, and $k\colon X \to W$ and $h\colon W \to Y$ be maps. Then*

$$\sum_{e \in \mathcal{E}^{r+1}} (-1)^{|e|} f([h \circ M_e^W \circ k]) = 0.$$

Proof. Consider the invariant $\tilde{f}\colon [W, W] \to V$, $\tilde{u} \mapsto f([h] \circ \tilde{u} \circ [k])$. We show that

$$\sum_{e \in \mathcal{E}^{r+1}} (-1)^{|e|} \tilde{f}([M_e^W]) = 0.$$

By Lemma 10.1, $\operatorname{Deg} \tilde{f} \leqslant r$, i. e. there is a homomorphism $l\colon \operatorname{Hom}(C_0(W^r), C_0(W^r)) \to V$ such that $\tilde{f}([\tilde{a}]) = l(C_0(\tilde{a}^r))$ for all $\tilde{a} \in W^W$ (hereafter, $\mathcal{R} = \mathbf{Z}$). Therefore it suffices to show that

$$\sum_{e \in \mathcal{E}^{r+1}} (-1)^{|e|} C_0((M_e^W)^r) = 0.$$

Take a point $w = (w_1, \ldots, w_r) \in W^r$. There is $s \in \{1, \ldots, r+1\}$ such that $\{w_1, \ldots, w_r\} \cap T_s \subseteq \{\text{basepoint}\}$. The point $(M_e^W)^r(w) \in W^r$ does not depend on the sth component of e. Since $C_0((M_e^W)^r)(\lfloor w \rfloor) = \lfloor (M_e^W)^r(w) \rfloor$, it follows that

$$\sum_{e \in \mathcal{E}^{r+1}} (-1)^{|e|} C_0((M_e^W)^r)(\lfloor w \rfloor) = 0.$$

□

Maps $S^n \to Y$. In this subsection, we use multiplicative notation for homotopy groups.

12.2. Lemma. *Let $n \geqslant 1$ be a number, Y be a space, V be an abelian group, and $f\colon \pi_n(Y) \to V$ be an invariant of degree at most $r \in \mathbf{N}$. Then f is r-gentle.*

Proof. Take elements $u_1, \ldots, u_{r+1} \in \pi_n(Y)$. We show that ${}^+f((1 - \lfloor u_1 \rfloor) \ldots (1 - \lfloor u_{r+1} \rfloor)) = 0$. Put $W = S^n \vee \ldots \vee S^n$ ($r+1$ summands). Let $k\colon S^n \to W$ be a map with $[k] = [\mathrm{in}_1^W] \dots [\mathrm{in}_{r+1}^W]$ in $\pi_n(W)$, and $h\colon W \to Y$ be a map with $[h \circ \mathrm{in}_s^W] = u_s$ in $\pi_n(Y)$. By Lemma 12.1,

$$\sum_{e \in \mathcal{E}^{r+1}} (-1)^{|e|} f([h \circ M_e^W \circ k]) = 0.$$

This is what we need because $[h \circ M_e^W \circ k] = u_1^{e_1} \dots u_{r+1}^{e_{r+1}}$ in $\pi_n(Y)$. □

We denote the Whitehead product by the sign $*$.

12.3. Lemma. *Let $m, n \geqslant 1$ be numbers, Y be a space, and $f\colon \pi_{m+n-1}(Y) \to V$ be an invariant of degree at most $r \in \mathbf{N}$. Then the function $b\colon \pi_m(Y) \times \pi_n(Y) \to V$, $(u, v) \mapsto f(u * v)$, is r-gentle.*

Proof. Assume $r > 0$ (otherwise, the claim is trivial). Take elements $u_1, \dots, u_p \in \pi_m(Y)$ and $v_1, \dots, v_q \in \pi_n(Y)$, where $p, q \geqslant 0$ and $p+q = r+1$. By Lemma 3.10, it suffices to show that ${}^+b((1 - \lfloor \hat{u}_1 \rfloor) \dots (1 - \lfloor \hat{u}_p \rfloor)(1 - \lfloor \hat{v}_1 \rfloor) \dots (1 - \lfloor \hat{v}_q \rfloor)) = 0$, where $\hat{u}_s = (u_s, 1) \in \pi_m(Y) \times \pi_n(Y)$ and $\hat{v}_s = (1, v_s) \in \pi_m(Y) \times \pi_n(Y)$. Put $W = S^m \vee \dots \vee S^m \vee S^n \vee \dots \vee S^n$ (p times S^m and q times S^n). Let $k\colon S^{m+n-1} \to W$ be a map with $[k] = ([\mathrm{in}_1^W] \dots [\mathrm{in}_p^W]) * ([\mathrm{in}_{p+1}^W] \dots [\mathrm{in}_{r+1}^W])$ in $\pi_{m+n-1}(W)$ and $h\colon W \to Y$ be a map with $[h \circ \mathrm{in}_s^W] = u_s$ in $\pi_m(Y)$ for $s = 1, \dots, p$ and $[h \circ \mathrm{in}_{p+t}^W] = v_t$ in $\pi_n(Y)$ for $t = 1, \dots, q$. By Lemma 12.1,

$$\sum_{e \in \mathcal{E}^{r+1}} (-1)^{|e|} f([h \circ M_e^W \circ k]) = 0.$$

This is what we need because $[h \circ M_e^W \circ k] = (u_1^{e_1} \dots u_p^{e_p}) * (v_1^{e_{p+1}} \dots v_q^{e_{r+1}})$ in $\pi_{m+n-1}(Y)$ and, consequently, $f([h \circ M_e^W \circ k]) = b(u_1^{e_1} \dots u_p^{e_p}, v_1^{e_{p+1}} \dots v_q^{e_{r+1}}) = b(\hat{u}_1^{e_1} \dots \hat{u}_p^{e_p} \hat{v}_1^{e_{p+1}} \dots \hat{v}_q^{e_{r+1}})$. □

Maps $S^{n-1} \times S^n \to S^n_{(\mathbf{Q})}$. In this subsection, we prove claims 1.5–1.7 and use the objects defined in the corresponding subsection of § 1. For $u \in \pi_p(Y)$ and $v \in \pi_q(Y)$, the class $(u, v) \in [S^p \vee S^q, Y]$ is defined in the obvious way.

Let $x\colon S^n \vee S^{2n-1} \to S^n \times S^{2n-1}$ be the canonical embedding of a wedge in the product. Consider the map $(\mathrm{pr}_2, c)\colon S^{n-1} \times S^n \to S^n \times S^{2n-1}$, where $\mathrm{pr}_2\colon S^{n-1} \times S^n \to S^n$ is the projection and $c\colon S^{n-1} \times S^n \to S^{2n-1}$ is the map defined in § 1. There exists a (unique up to homotopy) map $b\colon S^{n-1} \times S^n \to S^n \vee S^{2n-1}$ such that $x \circ b \sim (\mathrm{pr}_2, c)$. For $p, q \in \mathbf{Z}$, we have the homotopy classes

$$v(p,q)\colon S^{n-1} \times S^n \overset{[b]}{\rightsquigarrow} S^n \vee S^{2n-1} \overset{(pi, qj)}{\rightsquigarrow} S^n$$

(wavy arrows present homotopy classes) and $\bar{v}(p,q) = [l] \circ v(p,q) \in [S^{n-1} \times S^n, S^n_{\mathbf{Q}}]$. Obviously, $v(0,q) = u(q)$ and $\bar{v}(0,q) = \bar{u}(q)$. We have $v(p,q) = v(p,0)$ if $p \mid q$ (the proof is omitted) and $\bar{v}(p,q) = \bar{v}(p,0)$ if $p \neq 0$ [1, Example 4.6].

Proof of 1.5. Take $q \in \mathbf{Z}$. Put $W = S^n \vee \dots \vee S^n \vee S^{2n-1}$ (r times S^n). Let $d\colon S^n \vee S^{2n-1} \to W$ be a map with $[d] = ([\mathrm{in}_1^W] + \dots + [\mathrm{in}_r^W], [\mathrm{in}_{r+1}^W])$. Put

$k = d \circ b\colon S^{n-1} \times S^n \to W$. Let $h\colon W \to S^n$ be a map with $[h] = (i, \dots, i, qj)$. By Lemma 12.1,

$$\sum_{e \in \mathcal{E}^{r+1}} (-1)^{|e|} f([h \circ M_e^W \circ k]) = 0.$$

Since $[h \circ M_e^W \circ k] = v(e_1 + \ldots + e_r, e_{r+1} q)$, we have

$$\sum_{e' \in \mathcal{E}^r} (-1)^{|e'|} \sum_{e'' \in \mathcal{E}} (-1)^{e''} f(v(|e'|, e'' q)) = 0.$$

Assume $r! \mid q$. If $e' \neq (0, \dots, 0)$, the inner sum vanishes because then $|e'| \mid q$ and, consequently, the class $v(|e'|, e''q)$ does not depend on e''. We get $f(v(0,0)) - f(v(0,q)) = 0$, i. e. $f(u(q)) = f(u(0))$. □

Proof of 1.6. Assume $\operatorname{Deg} f \leqslant r \in \mathbf{N}$. Take $q \in \mathbf{Z}$. As in the proof of 1.5, we get

$$\sum_{e' \in \mathcal{E}^r} (-1)^{|e'|} \sum_{e'' \in \mathcal{E}} (-1)^{e''} f(\bar{v}(|e'|, e'' q)) = 0.$$

If $e' \neq (0, \dots, 0)$, the class $\bar{v}(|e'|, e''q)$ does not depend on e''. As in the proof of 1.5, we get $f(\bar{u}(q)) = f(\bar{u}(0))$. □

Proof of 1.7. Assume $\operatorname{Deg} f \leqslant r \in \mathbf{N}$. Consider the invariant $\tilde{f}\colon \pi_{2n-1}(S^n) \to \mathbf{Q}$, $\tilde{u} \mapsto f(\tilde{u} \circ [c])$. By Lemma 10.1, $\operatorname{Deg} \tilde{f} \leqslant r$. By Lemma 12.2, $\tilde{f}$ is gentle. Consider the function $F\colon \mathbf{Z} \to \mathbf{Q}$, $q \mapsto f(u(q))$. We have $F(q) = \tilde{f}(qj)$. Therefore F is gentle, i. e., by Lemma 3.11, is given by a polynomial. By 1.5, $F(q) = F(0)$ if $r! \mid q$. It follows that F is constant. □

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