On homology of map spaces

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Abstract
Following an idea of Bendersky–Gitler, we construct an isomorphism between Anderson’s and Arone’s complexes modelling the chain complex of a map space. This allows us to apply Shipley’s convergence theorem to Arone’s model. As a corollary, we reduce the problem of homotopy equivalence for certain “toy” spaces to a problem in homological algebra.

A space is a pointed simplicial set. A map is a basepoint-preserving simplicial map. Chains, homology etc. are reduced with coefficients in a commutative ring $R$.

Fix spaces $X$ and $Y$. We are interested in the homology of $Y^X$, the space of maps $X \to Y$.

0.1. Lemma. The functors $M_n(X)$ are projective objects of the abelian category of functors $\Omega^* \to R$-Mod.

Proof is given in 1.B.

The boundary operators $\partial: C_n(X^s) \to C_{n-1}(X^s)$ form a functor morphism $\partial : M_n(X) \to M_{n-1}(X)$. Thus $M_*(X)$ is a chain complex of functors.

0.2. Corollary. If a map $e: X \to Y$ is a weak equivalence, then the induced chain homomorphism $M_*(e): M_*(X) \to M_*(Y)$ is a chain homotopy equivalence.

We have the (unbounded) chain complex of $R$-modules

$$G_*(X,Y) = \text{Hom}_R(M_*(X), M_*(Y))$$
and a chain homomorphism
\[ \lambda_*(X, Y): C_*(Y^X) \to G_*(X, Y), \]
see 2.C, 2.D. A natural filtration of \( G_*(X, Y) \) yields the Arone spectral sequence
\[ H_{t-s}(\text{Hom}_{\Sigma_n}(C_*(X^{(s)}), C_*(Y^{\wedge s}))) = 1E_t^s \Rightarrow H_{t-s}(G_*(X, Y)), \]
where \( X^{(s)} = X^{\wedge s}/(\text{fat diagonal}) \) [4], [1]. [6, Theorem 9.2] ensures conditional convergence. If \( Y \) is (dim \( X \))-connected, then the convergence is strong and \( \lambda_*(X, Y) \) is a quasi-isomorphism, see [4] for the precise statement. (A similar result was obtained in [11, Ch. III, § 5].) We wish to get free of the connectivity assumption.

0.B. Main results. Here we suppose \( R = \mathbb{Z}/\ell, \ell \) a prime. We call \( Y \) \( \ell \)-toy if \( \pi_0(Y) \) is finite and \( \pi_n(Y, y) \) is a finite \( \ell \)-group for all \( y \in Y_0 \) and \( n > 0 \).

0.3. Theorem. Suppose that \( X \) is essentially compact\(^1\) and \( Y \) is fibrant and \( \ell \)-toy. Then \( \lambda_*(X, Y) \) is a quasi-isomorphism.

This follows from Theorems 0.5 and 0.6 below, see § 4 for details. Under the assumptions of the theorem, the convergence of \((*)\) is strong by [6, Theorem 7.1].

0.4. Corollary. Suppose that \( X \) and \( Y \) are essentially compact and \( \ell \)-toy. Suppose that the complexes \( M_*(X) \) and \( M_*(Y) \) are chain homotopy equivalent. Then \( X \) and \( Y \) are weakly equivalent.

The proof is given in § 5. There seems to be no easy/functorial way to extract \( \pi_1(X) \) or the ring structure of \( H^*(X) \) from \( M_*(X) \). The corollary has an algebraic analogue [9].

0.C. Anderson’s approach. For a pointed set \( S \), the space \( Y^S \) is defined to be the fibre of the projection
\[ \prod_{s \in S} Y \to Y \]
corresponding to \( s = * \) (this agrees with our convention that maps preserve basepoints).

We have an (unbounded) chain complex \( D_*(X, Y) \) with
\[ D_n(X, Y) = \prod_{q-p=n} C_q(Y^{X_p}) \]
and a chain homomorphism
\[ \mu_*(X, Y): C_*(Y^X) \to D_*(X, Y), \]
\(^1\)A space is compact (or finite) if it is generated by a finite number of simplices. Essentially compact means weakly equivalent to a compact space.
see 2.F, 2.G for details. A natural filtration of $D_\ast(X,Y)$ yields the Anderson spectral sequence

$$H_q(Y^X) = E_1^{p,q} \Rightarrow H_{q-p}(D_\ast(X,Y)).$$

If $Y$ is $(\dim X)$-connected, then $\mu_\ast(X,Y)$ is a quasi-isomorphism, see [2] and [7, 4.2] for precise statements. Shipley got rid of the connectivity assumption [10].

**0.5. Theorem.** Suppose that $R = \mathbb{Z}/\ell$, $\ell$ a prime. Suppose that $X$ is compact and $Y$ is fibrant and $\ell$-toy. Then $\mu_\ast(X,Y)$ is a quasi-isomorphism.

This is a special case of Shipley’s strong convergence theorem, see § 3 for details.

**0.D. Comparing $G_\ast(X,Y)$ and $D_\ast(X,Y)$.** We construct a chain homomorphism

$$\epsilon_\ast(X,Y): D_\ast(X,Y) \rightarrow G_\ast(X,Y)$$

such that the diagram

$$\begin{array}{ccc}
D_\ast(X,Y) & \xrightarrow{\mu_\ast(X,Y)} & C_\ast(Y^X) \\
\downarrow{\epsilon_\ast(X,Y)} & & \downarrow{\lambda_\ast(X,Y)} \\
G_\ast(X,Y) & \rightarrow & G_\ast(X,Y)
\end{array}$$

is commutative, see 2.H.

**0.6. Theorem.** Suppose that $X$ is gradua\textsuperscript{P}. Then $\epsilon_\ast(X,Y)$ is an isomorphism.

Proof is given in 2.I.

**Remark.** In some cases, the $^2E$ term of the Anderson spectral sequence [5, Theorem 7.1 (2)] and the $^1E$ term of the Arone spectral sequence differ in the grading only. This suggested relation of the two approaches [1, footnote 1] and motivated this work. Our construction of $\epsilon_\ast(X,Y)$ follows the line of [5, § 6].

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\footnote{A space $X$ is gradua (or finite type) if the sets $X_n$, $n \geq 0$, are finite.}
1. Preliminaries

1.A. Notation. For a pointed set $S$, we put $S^\times = S \setminus \{\ast\}$. $\Delta^p_\ast$ is the standard $p$-simplex with an added basepoint. Let $\iota_p \in (\Delta^p_\ast)_p$ be the fundamental simplex.

For $x \in X_n$, $[x] \in C_n(X)$ is the chain consisting of the single simplex $x$ with the coefficient 1.

Given functors $F, F': \Omega^p \to R$-Mod, a functor morphism $T: F \to F'$ consists of homomorphisms $T_s(x) = F_s(x) \to F'_s(x)$.

1.B. Proof of Lemma 0.1 (cf. [3, § 2]). Fix a linear order on $X_n^\times$. Introduce the set

$$I = \prod_{s>0} \{(x_1, \ldots, x_s) \mid x_1, \ldots, x_s \in X_n^\times, x_1 < \ldots < x_s\}.$$

For $i = (x_1, \ldots, x_s) \in I$, put $|i| = s$ and $e_i = [x_1 \ldots x_s] \in C_n(X^\times) = M_n(X)(\langle s \rangle)$. The elements $e_i$ form a basis of $M_n(X)$ in the following sense.

For any functor $F: \Omega^p \to R$-Mod and elements $a_i \in F(\langle |i| \rangle)$, $i \in I$, there exists a unique functor morphism $T: M_n(X) \to F$ such that $|i|T(e_i) = a_i$ for all $i \in I$. Therefore, for a functor epimorphism $\tilde{F} \to F$, any functor morphism $M_n(X) \to F$ lifts to $\tilde{F}$. □

2. Main constructions

2.A. Diagonal complexes. A bicomplex $W^\ast_\ast$ (of $R$-modules) has differentials $d': W^{p-1}_q \to W^p_q$ and $d''': W^p_q \to W^p_{q-1}$, which commute: $d''d' = d' d''$. The diagonal (or complete total) chain complex $\text{diag} W^\ast_\ast = W_\ast$ of $W^\ast_\ast$ has

$$W_n = \prod_{q-p=n} W^p_q.$$

For $w \in W_n$, we have $w = (w^p_q)_{q-p=n}$, where $w^p_q \in W^p_q$. The differential $\partial: W_n \to W_{n-1}$ is defined by

$$(\partial w)^p_q = d''(w^p_{q+1}) - (-1)^n d'(w^p_{q-1}), \quad q - p = n - 1.$$

2.B. The complex $\text{Hom}_\ast(U_\ast, V_\ast)$. Given chain complexes $U_\ast$ and $V_\ast$ in some $R$-linear category, we define the bicomplex $\text{Hom}_\ast(U_\ast, V_\ast)$ with $\text{Hom}_\ast(U_\ast, V_\ast) = \text{Hom}(U_p, V_q)$ and the differentials induced by those of $U_\ast$ and $V_\ast$. We have

$$\text{Hom}_\ast(U_\ast, V_\ast) = \text{diag} \text{Hom}_\ast(U_\ast, V_\ast).$$

2.C. The complex $G_\ast(X, Y)$. We put

$$G^\ast_\ast(X, Y) = \text{Hom}_\ast(M_\ast(X), M_\ast(Y)), \quad G_\ast(X, Y) = \text{Hom}_\ast(M_\ast(X), M_\ast(Y)).$$
2.D. Construction of $\lambda_*(X,Y)$. For $s > 0$, let $^s\eta: Y^X \land X^{\land s} \to Y^{\land s}$ be the evaluation map. For $s > 0$ and $p, q \in \mathbb{Z}$, we have the homomorphism $C_q(^s\eta): C_q(Y^X \land X^{\land s}) \to C_q(Y^{\land s})$ and define the homomorphism

$$^s\lambda^p_q: C_{q-p}(Y^X) \to \text{Hom}(C_p(X^{\land s}), C_q(Y^{\land s}))$$

by

$$^s\lambda^p_q(z) = C_q(^s\eta)(z \times u), \quad u \in C_p(X^{\land s}), \quad z \in C_{q-p}(Y^X).$$

The homomorphisms $^s\lambda^p_q$ form the promised chain homomorphism $\lambda_*(X,Y)$.

2.E. The complex $D_*(V)$. For a cosimplicial space $V$, we have the bicomplex $D^*_p(V)$ with $D^*_p(V) = C^*_p(V^p)$ and the following differentials. The differential $d': C_q(V^{p-1}) \to C_q(V^p)$ is defined by

$$d' = \sum_{i=0}^p (-1)^i C_q(\delta^i),$$

where $\delta^i: V^{p-1} \to V^p$ are the coface maps. The differential $d'': C_q(Y^{X^p}) \to C_{q-1}(Y^{X^p})$ is the ordinary boundary operator. We put $D_*(V) = \text{diag}_p D^*_p(V)$.

2.F. The complex $D_*(X,Y)$. Consider the cosimplicial space $V = \text{hom}(X,Y)$ with $V^p = Y^{X^p}$ [8, Ch. X, 2.2 (ii)]. We put

$$D^*_p(X,Y) = D^*_p(V), \quad D_*(X,Y) = D_*(V).$$

2.G. Construction of $\mu_*(X,Y)$. For $x \in X_p$, we have the composite map

$$\theta^x: Y^X \land \Delta^p_+ \xrightarrow{\text{id} \land \varphi} Y^X \land X \xrightarrow{\eta} Y,$$

where $\varphi: \Delta^p_+ \to X$ is the characteristic map of the simplex $x$ and $\eta$ is the evaluation map. Combining $\theta^x$ over all $x \in X_p$, we get a map

$$\theta^p: Y^X \land \Delta^p_+ \to Y^{X^p}.$$

For $p \geq 0$ and $q \in \mathbb{Z}$, we have the homomorphism $C_q(\theta^p): C_q(Y^X \land \Delta^p_+) \to C_q(Y^{X^p})$ and introduce the homomorphism

$$\mu^p_q: C_{q-p}(Y^X) \to C_q(Y^{X^p}), \quad \mu^p_q(z) = C_q(\theta^p)(z \times \{x\}).$$

The homomorphisms $\mu^p_q$ form the promised chain homomorphism $\mu_*(X,Y)$.

2.H. Construction of $\epsilon_*(X,Y)$. A simplex $v \in (Y^{X^p})_q$ is a basepoint-preserving function $v: X_p \to Y_q$. For $s > 0$ and $p, q \geq 0$, we define the homomorphism

$$^s\epsilon^p_q: C_q(Y^{X^p}) \to \text{Hom}(C_p(X^{\land s}), C_q(Y^{\land s}))$$

by

$$^s\epsilon^p_q([v])([x_1 \ldots x_s]) = [v(x_1) \ldots v(x_s)], \quad x_1, \ldots, x_s \in X_p, \quad v \in (Y^{X^p})_q.$$
The homomorphisms \( s^p \) form a homomorphism of bicomplexes
\[
\varphi^p_s(X, Y) : D^p_s(X, Y) \to G^p_s(X, Y)
\]
and thus the promised chain homomorphism \( \epsilon_* (X, Y) \).

Remark. The bicomplexes \( D^p_s(X, Y) \) and \( G^p_s(X, Y) \) are in fact cosimplicial simplicial \( R \)-modules. (To see this, recall that, for every space \( Z \), \( C_\ast(Z) \) is in fact a simplicial \( R \)-module and thus \( M_\ast(Z) \) is a simplicial functor.) The homomorphism \( \varphi^p_s(X, Y) \) preserves this structure.

One easily verifies that \( \epsilon_* (X, Y) \circ \mu_* (X, Y) = \lambda_* (X, Y) \).

2.1. Proof of Theorem 0.6. Take \( p, q \geq 0 \). It suffices to prove that the homomorphism
\[
epsilon^p_q (X, Y) : C_q (Y^X_p) \to \text{Hom}(M_p(X), M_q(Y))
\]
is an isomorphism. We construct a homomorphism
\[
\epsilon^p_q : \text{Hom}(M_p(X), M_q(Y)) \to C_q (Y^X_p)
\]
and leave to the reader to verify that \( \epsilon^p_q \circ \epsilon^p_q \) and \( \epsilon^p_q \circ \epsilon^p_q \) are the identities.

Fix a linear order on \( X_p^\times \). Suppose we are given sets \( E, F \subseteq X_p^\times \) such that \( E \supseteq F \neq \emptyset \). We have \( E = \{x_1, \ldots, x_s\} \) for some \( x_1 < \ldots < x_s \). Put \( \kappa_E = x_1 \ldots x_s \in (X^\times)^p \). For \( y_1, \ldots, y_s \in Y_q \), define the function \( \phi^F_E (y_1, \ldots, y_s) : X_p \to Y_q \) by the rules
\[
x_t \mapsto y_t \text{ for } t = 1, \ldots, s \text{ such that } x_t \in F;
\]
x \mapsto * \text{ for all other } x \in X_p.

We have the homomorphism \( \Phi^F_E : C_q (Y^{\times s}) \to C_q (Y^X_p) \) with \( \Phi^F_E ([y_1, \ldots, y_s]) = [\phi^F_E (y_1, \ldots, y_s)] \) for \( y_1, \ldots, y_s \in Y_q^s \). Define the homomorphism
\[
\psi^F_E : \text{Hom}_{\Sigma^s} (C_p (X^\times^s), C_q (Y^{\times s})) \to C_q (Y^X_p)
\]
by \( \psi^F_E (t) = \Phi^F_E (t([\kappa_E])) \). (One may note that \( \psi^F_E \) does not depend on the order on \( X^X_p \).) For a functor morphism \( T : M_p(X) \to M_q(Y) \), we set
\[
\epsilon^p_q (T) = \sum_{E, F \subseteq X^X_p : E \supseteq F \neq \emptyset} (-1)^{|E||F|} \psi^F_E (|E|T).
\]

\( \square \)

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3. Anderson’s model

3.A. General cosimplicial case. We follow [7, § 2]. Let $V$ be a cosimplicial space. We have the (unbounded) chain complex $D_*(V)$ (see 2.E). There is the chain homomorphism $\mu_*(V) : C_*(\text{Tot} V) \to D_*(V)$ formed by the homomorphisms

$$\mu^p_q : C_q(\text{Tot} V) \to C_q(V^p)$$

that are defined in the following way. A simplex $w \in (\text{Tot} V)_n$ is a sequence $(w^p)_{p \geq 0}$ of maps $w^p : \Delta^n_+ \wedge \Delta^p_+ \to V^p$. For $w \in (\text{Tot} V)_{q-p}$, we have the homomorphism $C_q(w^p) : C_q(\Delta^{q-p}_+ \wedge \Delta^p_+) \to C_q(V^p)$ and set

$$\mu^p_q([w]) = C_q(w^p)([\iota_{q-p}] \times [\iota_p]).$$

3.1. Theorem. Suppose that $R = \mathbb{Z}/\ell$, $\ell$ a prime, $V$ is fibrant and the spaces $V^p$, $p \geq 0$, and $\text{Tot} V$ are $\ell$-toy. Then $\mu_*(V)$ is a quasi-isomorphism.

Proof. Apply Shipley’s strong convergence theorem [10, Theorem 6.1] and [7, Lemma 2.3].

3.B. Proof of Theorem 0.5. We have the cosimplicial space $V = \text{hom}(X, Y)$ and the canonical isomorphism $Y^X = \text{Tot} V$ [8, Ch. X, 3.3 (i)]. The diagram

$$\begin{array}{ccc}
C_*(Y^X) & \xrightarrow{\mu_*(X,Y)} & D_*(X,Y) \\
\| & & \| \\
C_*(\text{Tot} V) & \xrightarrow{\mu_*(V)} & D_*(V)
\end{array}$$

is commutative.

The cosimplicial space $V$ is fibrant by [8, Ch. X, 4.7 (ii)]. The spaces $V^p$ are $\ell$-toy since $X$ is gradual and $Y$ is $\ell$-toy. The spaces $Y^X$ and thus $\text{Tot} V$ are $\ell$-toy since $X$ is compact and $Y$ is fibrant and $\ell$-toy. By Theorem 3.1, $\mu_*(V)$ is a quasi-isomorphism.

4. Arone’s model

4.A. Homotopy invariance.

4.1. Lemma. Let $e : X' \to X$ and $f : Y \to Y'$ be weak equivalences of spaces. Suppose that $Y$ and $Y'$ are fibrant. Then $\lambda_*(X,Y)$ is a quasi-isomorphism if and only if $\lambda_*(X',Y')$ is.
Proof. The maps $e$ and $f$ induce a map $g: Y^X \to Y'^{X'}$. We have the commutative diagram

$$
\begin{array}{ccc}
C_*(Y^X) & \xrightarrow{\lambda_*(X,Y)} & G_*(X,Y) \\
C_*(g) & & G_*(e,f) \\
C_*(Y'^{X'}) & \xrightarrow{\lambda_*(X',Y')} & G_*(X',Y') \\
\end{array}
$$

$C_*(g)$ is a quasi-isomorphism since $g$ is a weak equivalence. It follows from Corollary 0.2 that $G_*(e,f)$ is a quasi-isomorphism. The desired equivalence is clear now. 

4.B. Proof of Theorem 0.3. If $X$ is compact, the assertion follows immediately from Theorems 0.5 and 0.6. In general, $X$ is weakly equivalent to a compact space $X^\circ$. Using Lemma 4.1, we pass from $\lambda_*(X^\circ, Y)$ to $\lambda_*(X, Y)$. 

5. Reconstructing $X$ from $M_*(X)$

5.A. Composition of maps and homomorphisms.

5.1. Lemma. Let $X$, $Y$ and $Z$ be spaces and $\gamma: Z^Y \land Y^X \to Z^X$ be the composition map. Then the diagram of chain complexes and chain homomorphisms

$$
\begin{array}{ccc}
C_*(Z^Y) \otimes C_*(Y^X) & \xrightarrow{\text{cross product}} & C_*(Z^Y \land Y^X) \\
& \xrightarrow{\lambda_*(Y,Z) \otimes \lambda_*(X,Y)} & C_*(Z^X) \\
G_*(Y,Z) \otimes G_*(X,Y) & \xrightarrow{\text{composition}} & G_*(X,Z) \\
\end{array}
$$

is commutative.

This follows from the associativity of the cross product. 

5.B. Proof of Corollary 0.4. Corollary 0.2 allows us to assume $X$ and $Y$ fibrant. Note that $H_0(G_*(X,Y)) = [M_*(X), M_*(Y)]$, the $R$-module of chain homotopy classes. By Lemma 5.1, we have the commutative diagram

$$
\begin{array}{ccc}
H_0(X^Y) \otimes H_0(Y^X) & \xrightarrow{\text{cross product}} & H_0(X^Y \land Y^X) \\
& \xrightarrow{H_0(\lambda_*(Y,X)) \otimes H_0(\lambda_*(X,Y))} & H_0(X^X) \\
[M_*(Y), M_*(X)] \otimes [M_*(X), M_*(Y)] & \xrightarrow{\text{composition}} & [M_*(X), M_*(X)], \\
\end{array}
$$

where $\gamma: X^Y \land Y^X \to X^X$ is the composition map. We use the notation $B \otimes A \mapsto B \circ A$ for the upper line homomorphism $H_0(X^Y) \otimes H_0(Y^X) \to H_0(X^X)$. By Theorem 0.3, $H_0(\lambda_*(X,Y))$, $H_0(\lambda_*(Y,X))$ and $H_0(\lambda_*(X,X))$ are isomorphisms.
Let $f: M_*(X) \to M_*(Y)$ and $g: M_*(Y) \to M_*(X)$ be mutually inverse chain homotopy equivalences. We have $[f] = H_0(\lambda_*(X,Y))(A)$ for some $A \in H_0(Y^X)$ and $[g] = H_0(\lambda_*(Y,X))(B)$ for some $B \in H_0(X^Y)$. By the diagram, $B \circ A = 1$ in $H_0(X^X)$. Thus there are maps $a: X \to Y$ and $b: Y \to X$ such that $b \circ a \sim \text{id}_X$. Interchanging $X$ and $Y$ in this reasoning, we get maps $a': X \to Y$ and $b': Y \to X$ such that $a' \circ b' \sim \text{id}_Y$. Since $X$ and $Y$ are $\ell$-toy, these four maps are weak equivalences.

References


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