On homology of map spaces

S. S. Podkorytov

Abstract

Following an idea of Bendersky–Gitler, we construct an isomorphism between Anderson's and Arone's complexes modelling the chain complex of a map space. This allows us to apply Shipley's convergence theorem to Arone's model. As a corollary, we reduce the problem of homotopy equivalence for certain "toy" spaces to a problem in homological algebra.

A *space* is a pointed simplicial set. A *map* is a basepoint-preserving simplicial map. Chains, homology etc. are reduced with coefficients in a commutative ring R.

Fix spaces X and Y. We are interested in the homology of Y^X, the space of maps $X \to Y$.

0.A. Arone's approach. Let Ω be the category whose objects are the sets $\langle s \rangle = \{1, \ldots, s\}$, $s > 0$, and whose morphisms are surjective functions. Let Ω° denote the dual category. For $n \in \mathbf{Z}$, let us define a functor $M_n(X)\colon \Omega^\circ \to R$-Mod. Set $M_n(X)(\langle s \rangle) = C_n(X^{\wedge s})$, where $X^{\wedge s}$ is the sth smash power. For a morphism $h\colon \langle t \rangle \to \langle s \rangle$, set $M_n(X)(h) = C_n(h^\sharp)\colon C_n(X^{\wedge s}) \to C_n(X^{\wedge t})$, where the map $h^\sharp\colon X^{\wedge s} \to X^{\wedge t}$ is given by $h^\sharp(x_1 \ldots x_s) = x_{h(1)} \ldots x_{h(t)}$ for $x_1, \ldots, x_s \in X_n$, $n \geqslant 0$. Here the simplex $x_1 \ldots x_s \in (X^{\wedge s})_n$ is the image of the simplex $(x_1, \ldots, x_s) \in (X^s)_n$ under the projection.

0.1. Lemma. *The functors $M_n(X)$ are projective objects of the abelian category of functors $\Omega^\circ \to R$-Mod.*

Proof is given in 1.B.

The boundary operators $\partial\colon C_n(X^{\wedge s}) \to C_{n-1}(X^{\wedge s})$ form a functor morphism $\partial : M_n(X) \to M_{n-1}(X)$. Thus $M_*(X)$ is a chain complex of functors.

0.2. Corollary. *If a map $e\colon X \to Y$ is a weak equivalence, then the induced chain homomorphism $M_*(e)\colon M_*(X) \to M_*(Y)$ is a chain homotopy equivalence.* □

We have the (unbounded) chain complex of R-modules

$$G_*(X, Y) = \mathrm{Hom}_*(M_*(X), M_*(Y))$$

and a chain homomorphism

$$\lambda_*(X,Y)\colon C_*(Y^X) \to G_*(X,Y),$$

see 2.C, 2.D. A natural filtration of $G_*(X,Y)$ yields the Arone spectral sequence

$$H_{t-s}(\mathrm{Hom}_{\Sigma_s *}(C_*(X^{(s)}), C_*(Y^{\wedge s}))) = {}^1E^s_t \Rightarrow H_{t-s}(G_*(X,Y)), \qquad (*)$$

where $X^{(s)} = X^{\wedge s}/(\text{fat diagonal})$ [4], [1]. [6, Theorem 9.2] ensures conditional convergence. If Y is $(\dim X)$-connected, then the convergence is strong and $\lambda_*(X,Y)$ is a quasi-isomorphism, see [4] for the precise statement. (A similar result was obtained in [11, Ch. III, § 5].) We wish to get free of the connectivity assumption.

0.B. Main results. Here we suppose $R = \mathbf{Z}/\ell$, ℓ a prime. We call Y *ℓ-toy* if $\pi_0(Y)$ is finite and $\pi_n(Y,y)$ is a finite ℓ-group for all $y \in Y_0$ and $n > 0$.

0.3. Theorem. *Suppose that X is essentially compact*[1] *and Y is fibrant and ℓ-toy. Then $\lambda_*(X,Y)$ is a quasi-isomorphism.*

This follows from Theorems 0.5 and 0.6 below, see § 4 for details. Under the assumptions of the theorem, the convergence of $(*)$ is strong by [6, Theorem 7.1].

0.4. Corollary. *Suppose that X and Y are essentially compact and ℓ-toy. Suppose that the complexes $M_*(X)$ and $M_*(Y)$ are chain homotopy equivalent. Then X and Y are weakly equivalent.*

The proof is given in § 5. There seems to be no easy/functorial way to extract $\pi_1(X)$ or the ring structure of $H^*(X)$ from $M_*(X)$. The corollary has an algebraic analogue [9].

0.C. Anderson's approach. For a pointed set S, the space Y^S is defined to be the fibre of the projection

$$\prod_{s\in S} Y \to Y$$

corresponding to $s = *$ (this agrees with our convention that maps preserve basepoints).

We have an (unbounded) chain complex $D_*(X,Y)$ with

$$D_n(X,Y) = \prod_{q-p=n} C_q(Y^{X_p})$$

and a chain homomorphism

$$\mu_*(X,Y)\colon C_*(Y^X) \to D_*(X,Y),$$

[1] A space is *compact* (or *finite*) if it is generated by a finite number of simplices. *Essentially compact* means weakly equivalent to a compact space.

see 2.F, 2.G for details. A natural filtration of $D_*(X,Y)$ yields the Anderson spectral sequence

$$H_q(Y^{X_p}) = {}^1E^p_q \Rightarrow H_{q-p}(D_*(X,Y)).$$

If Y is $(\dim X)$-connected, then $\mu_*(X,Y)$ is a quasi-isomorphism, see [2] and [7, 4.2] for precise statements. Shipley got rid of the connectivity assumption [10].

0.5. Theorem. *Suppose that $R = \mathbf{Z}/\ell$, ℓ a prime. Suppose that X is compact and Y is fibrant and ℓ-toy. Then $\mu_*(X,Y)$ is a quasi-isomorphism.*

This is a special case of Shipley's strong convergence theorem, see § 3 for details.

0.D. Comparing $G_*(X,Y)$ and $D_*(X,Y)$. We construct a chain homomorphism

$$\epsilon_*(X,Y)\colon D_*(X,Y) \to G_*(X,Y)$$

such that the diagram

$$\begin{tikzcd}
 & D_*(X,Y) \arrow[dd, "\epsilon_*(X,Y)"] \\
C_*(Y^X) \arrow[ur, "\mu_*(X,Y)"] \arrow[dr, "\lambda_*(X,Y)"'] & \\
 & G_*(X,Y)
\end{tikzcd}$$

is commutative, see 2.H.

0.6. Theorem. *Suppose that X is gradual*[2]*. Then $\epsilon_*(X,Y)$ is an isomorphism.*

Proof is given in 2.I.

Remark. In some cases, the ${}^2E$ term of the Anderson spectral sequence [5, Theorem 7.1 (2)] and the ${}^1E$ term of the Arone spectral sequence differ in the grading only. This suggested relation of the two approaches [1, footnote 1] and motivated this work. Our construction of $\epsilon_*(X,Y)$ follows the line of [5, § 6].

Acknoledgement. I am grateful to S. Betley, V. A. Vassiliev and M. Yu. Zvagel′skiĭ for useful discussions.

[2]A space X is *gradual* (or *finite type*) if the sets X_n, $n \geqslant 0$, are finite.

1. Preliminaries

1.A. Notation. For a pointed set S, we put $S^{\times} = S \setminus \{*\}$.

Δ^p_+ is the standard p-simplex with an added basepoint. Let $\iota_p \in (\Delta^p_+)_p$ be the fundamental simplex.

For $x \in X_n$, $[x] \in C_n(X)$ is the chain consisting of the single simplex x with the coefficient 1.

Given functors $F, F'\colon \Omega^\circ \to R$-Mod, a functor morphism $T\colon F \to F'$ consists of homomorphisms ${}^sT\colon F(\langle s\rangle) \to F'(\langle s\rangle)$.

1.B. Proof of Lemma 0.1 (cf. [3, § 2]). Fix a linear order on $X_n^{\times}$. Introduce the set

$$I = \coprod_{s>0} \{(x_1 \dots, x_s) \mid x_1 \dots, x_s \in X_n^{\times},\ x_1 < \dots < x_s\}.$$

For $i = (x_1 \dots, x_s) \in I$, put $|i| = s$ and $e_i = [x_1 \dots x_s] \in C_n(X^{\wedge s}) = M_n(X)(\langle s\rangle)$. The elements e_i form a basis of $M_n(X)$ in the following sense. For any functor $F\colon \Omega^\circ \to R$-Mod and elements $a_i \in F(\langle |i| \rangle)$, $i \in I$, there exists a unique functor morphism $T\colon M_n(X) \to F$ such that ${}^{|i|}T(e_i) = a_i$ for all $i \in I$. Therefore, for a functor epimorphism $\tilde{F} \to F$, any functor morphism $M_n(X) \to F$ lifts to $\tilde{F}$. □

2. Main constructions

2.A. Diagonal complexes. A *bicomplex* $\underline{W}^*_*$ (of R-modules) has differentials $d'\colon \underline{W}^{p-1}_q \to \underline{W}^p_q$ and $d''\colon \underline{W}^p_q \to \underline{W}^p_{q-1}$, which commute: $d''d' = d'd''$. The *diagonal* (or *complete total*) chain complex $\operatorname{diag}_* \underline{W}^*_* = W_*$ of $\underline{W}^*_*$ has

$$W_n = \prod_{q-p=n} \underline{W}^p_q.$$

For $w \in W_n$, we have $w = (w^p_q)_{q-p=n}$, where $w^p_q \in \underline{W}^p_q$. The differential $\partial\colon W_n \to W_{n-1}$ is defined by

$$(\partial w)^p_q = d''(w^p_{q+1}) - (-1)^n d'(w^{p-1}_q), \quad q-p = n-1.$$

2.B. The complex $\operatorname{Hom}_*(U_*, V_*)$***.*** Given chain complexes U_* and V_* in some R-linear category, we define the bicomplex $\underline{\operatorname{Hom}}^*_*(U_*, V_*)$ with $\underline{\operatorname{Hom}}^p_q(U_*, V_*) = \operatorname{Hom}(U_p, V_q)$ and the differentials induced by those of U_* and V_*. We have

$$\operatorname{Hom}_*(U_*, V_*) = \operatorname{diag}_* \underline{\operatorname{Hom}}^*_*(U_*, V_*).$$

2.C. The complex $G_*(X, Y)$***.*** We put

$$\underline{G}^*_*(X, Y) = \underline{\operatorname{Hom}}^*_*(M_*(X), M_*(Y)), \quad G_*(X, Y) = \operatorname{Hom}_*(M_*(X), M_*(Y)).$$

2.D. Construction of $\lambda_*(X,Y)$***.*** For $s>0$, let ${}^s\eta\colon Y^X\wedge X^{\wedge s}\to Y^{\wedge s}$ be the evaluation map. For $s>0$ and $p,q\in\mathbf{Z}$, we have the homomorphism $C_q({}^s\eta)\colon C_q(Y^X\wedge X^{\wedge s})\to C_q(Y^{\wedge s})$ and define the homomorphism

$$ {}^s\lambda^p_q\colon C_{q-p}(Y^X)\to \mathrm{Hom}(C_p(X^{\wedge s}),C_q(Y^{\wedge s})) $$

by

$$ {}^s\lambda^p_q(z)(u)=C_q({}^s\eta)(z\times u),\quad u\in C_p(X^{\wedge s}),\quad z\in C_{q-p}(Y^X). $$

The homomorphisms ${}^s\lambda^p_q$ form the promised chain homomorphism $\lambda_*(X,Y)$.

2.E. The complex $D_*(V)$***.*** For a cosimplicial space V, we have the bicomplex $\underline{D}^*_*(V)$ with $\underline{D}^p_q(V)=C_q(V^p)$ and the following differentials. The differential $d'\colon C_q(V^{p-1})\to C_q(V^p)$ is defined by

$$ d'=\sum_{i=0}^{p}(-1)^iC_q(\delta^i), $$

where $\delta^i\colon V^{p-1}\to V^p$ are the coface maps. The differential $d''\colon C_q(Y^{X_p})\to C_{q-1}(Y^{X_p})$ is the ordinary boundary operator. We put $D_*(V)=\mathrm{diag}_*\,\underline{D}^*_*(V)$.

2.F. The complex $D_*(X,Y)$***.*** Consider the cosimplicial space $V=\mathrm{hom}(X,Y)$ with $V^p=Y^{X_p}$ [8, Ch. X, 2.2 (ii)]. We put

$$ \underline{D}^*_*(X,Y)=\underline{D}^*_*(V),\quad D_*(X,Y)=D_*(V). $$

2.G. Construction of $\mu_*(X,Y)$***.*** For $x\in X_p$, we have the composite map

$$ \theta^x\colon Y^X\wedge\Delta^p_+\xrightarrow{\mathrm{id}\wedge\overline{x}}Y^X\wedge X\xrightarrow{\ \eta\ }Y, $$

where $\overline{x}\colon\Delta^p_+\to X$ is the characteristic map of the simplex x and η is the evaluation map. Combining θ^x over all $x\in X_p$, we get a map

$$ \theta^p\colon Y^X\wedge\Delta^p_+\to Y^{X_p}. $$

For $p\geqslant 0$ and $q\in\mathbf{Z}$, we have the homomorphism $C_q(\theta^p)\colon C_q(Y^X\wedge\Delta^p_+)\to C_q(Y^{X_p})$ and introduce the homomorphism

$$ \mu^p_q\colon C_{q-p}(Y^X)\to C_q(Y^{X_p}),\quad \mu^p_q(z)=C_q(\theta^p)(z\times[\iota_p]). $$

The homomorphisms μ^p_q form the promised chain homomorphism $\mu_*(X,Y)$.

2.H. Construction of $\epsilon_*(X,Y)$***.*** A simplex $v\in(Y^{X_p})_q$ is a basepoint-preserving function $v\colon X_p\to Y_q$. For $s>0$ and $p,q\geqslant 0$, we define the homomorphism

$$ {}^s\epsilon^p_q\colon C_q(Y^{X_p})\to \mathrm{Hom}(C_p(X^{\wedge s}),C_q(Y^{\wedge s})) $$

by

$$ {}^s\epsilon^p_q([v])([x_1\dots x_s])=[v(x_1)\dots v(x_s)],\quad x_1,\dots,x_s\in X_p,\quad v\in(Y^{X_p})_q. $$

The homomorphisms ${}^s\epsilon^p_q$ form a homomorphism of bicomplexes

$$\underline{\epsilon}^*_*(X,Y)\colon \underline{D}^*_*(X,Y) \to \underline{G}^*_*(X,Y)$$

and thus the promised chain homomorphism $\epsilon_*(X,Y)$.

Remark. The bicomplexes $\underline{D}^*_*(X,Y)$ and $\underline{G}^*_*(X,Y)$ are in fact cosimplicial simplicial R-modules. (To see this, recall that, for every space Z, $C_*(Z)$ is in fact a simplicial R-module and thus $M_*(Z)$ is a simplicial functor.) The homomorphism $\underline{\epsilon}^*_*(X,Y)$ preserves this structure.

One easily verifies that $\epsilon_*(X,Y) \circ \mu_*(X,Y) = \lambda_*(X,Y)$.

2.I. Proof of Theorem 0.6. Take $p, q \geqslant 0$. It suffices to prove that the homomorphism

$$\epsilon^p_q = ({}^s\epsilon^p_q)_{s>0}\colon C_q(Y^{X_p}) \to \operatorname{Hom}(M_p(X), M_q(Y))$$

is an isomorphism. We construct a homomorphism

$$\xi^p_q\colon \operatorname{Hom}(M_p(X), M_q(Y)) \to C_q(Y^{X_p})$$

and leave to the reader to verify that $\xi^p_q \circ \epsilon^p_q$ and $\epsilon^p_q \circ \xi^p_q$ are the identities.

Fix a linear order on $X^\times_p$. Suppose we are given sets $E, F \subseteq X^\times_p$ such that $E \supseteq F \neq \varnothing$. We have $E = \{x_1, \ldots, x_s\}$ for some $x_1 < \ldots < x_s$. Put $\kappa_E = x_1 \ldots x_s \in (X^{\wedge s})_p$. For $y_1, \ldots, y_s \in Y_q$, define the function $\phi^F_E(y_1, \ldots, y_s)\colon X_p \to Y_q$ by the rules

$x_t \mapsto y_t$ for $t = 1, \ldots, s$ such that $x_t \in F$;

$x \mapsto *$ for all other $x \in X_p$.

We have the homomorphism $\Phi^F_E\colon C_q(Y^{\wedge s}) \to C_q(Y^{X_p})$ with $\Phi^F_E([y_1 \ldots y_s]) = [\phi^F_E(y_1, \ldots, y_s)]$ for $y_1, \ldots, y_s \in Y^\times_q$. Define the homomorphism

$$\psi^F_E\colon \operatorname{Hom}_{\Sigma_s}(C_p(X^{\wedge s}), C_q(Y^{\wedge s})) \to C_q(Y^{X_p})$$

by $\psi^F_E(t) = \Phi^F_E(t([\kappa_E]))$. (One may note that ψ^F_E does not depend on the order on $X^\times_p$.) For a functor morphism $T\colon M_p(X) \to M_q(Y)$, we set

$$\xi^p_q(T) = \sum_{E,F \subseteq X^\times_p : E \supseteq F \neq \varnothing} (-1)^{|E|-|F|} \psi^F_E({}^{|E|}T).$$

□

3. Anderson's model

3.A. General cosimplicial case. We follow [7, § 2]. Let V be a cosimplicial space. We have the (unbounded) chain complex $D_*(V)$ (see 2.E). There is the chain homomorphism

$$\mu_*(V)\colon C_*(\operatorname{Tot} V) \to D_*(V)$$

formed by the homomorphisms

$$\mu^p_q\colon C_{q-p}(\operatorname{Tot} V) \to C_q(V^p)$$

that are defined in the following way. A simplex $w \in (\operatorname{Tot} V)_n$ is a sequence $(w^p)_{p\geqslant 0}$ of maps $w^p\colon \Delta^n_+ \wedge \Delta^p_+ \to V^p$. For $w \in (\operatorname{Tot} V)_{q-p}$, we have the homomorphism $C_q(w^p)\colon C_q(\Delta^{q-p}_+ \wedge \Delta^p_+) \to C_q(V^p)$ and set

$$\mu^p_q([w]) = C_q(w^p)([\iota_{q-p}] \times [\iota_p]).$$

3.1. Theorem. *Suppose that $R = \mathbf{Z}/\ell$, ℓ a prime, V is fibrant and the spaces V^p, $p \geqslant 0$, and $\operatorname{Tot} V$ are ℓ-toy. Then $\mu_*(V)$ is a quasi-isomorphism.*

Proof. Apply Shipley's strong convergence theorem [10, Theorem 6.1] and [7, Lemma 2.3]. □

3.B. Proof of Theorem 0.5. We have the cosimplicial space $V = \hom(X, Y)$ and the canonical isomorphism $Y^X = \operatorname{Tot} V$ [8, Ch. X, 3.3 (i)]. The diagram

$$\begin{tikzcd} C_*(Y^X) \arrow[r, "\mu_*(X,Y)"] \arrow[d, equal] & D_*(X,Y) \arrow[d, equal] \\ C_*(\operatorname{Tot} V) \arrow[r, "\mu_*(V)"] & D_*(V) \end{tikzcd}$$

is commutative.

The cosimplicial space V is fibrant by [8, Ch. X, 4.7 (ii)]. The spaces V^p are ℓ-toy since X is gradual and Y is ℓ-toy. The spaces Y^X and thus $\operatorname{Tot} V$ are ℓ-toy since X is compact and Y is fibrant and ℓ-toy. By Theorem 3.1, $\mu_*(V)$ is a quasi-isomorphism. □

4. Arone's model

4.A. Homotopy invariance.

4.1. Lemma. *Let $e\colon X' \to X$ and $f\colon Y \to Y'$ be weak equivalences of spaces. Suppose that Y and Y' are fibrant. Then $\lambda_*(X, Y)$ is a quasi-isomorphism if and only if $\lambda_*(X', Y')$ is.*

Proof. The maps e and f induce a map $g\colon Y^X \to {Y'}^{X'}$. We have the commutative diagram

$$\begin{tikzcd} C_*(Y^X) \arrow[r, "\lambda_*(X,Y)"] \arrow[d, "C_*(g)"'] & G_*(X,Y) \arrow[d, "G_*(e,f)"] \\ C_*({Y'}^{X'}) \arrow[r, "\lambda_*(X',Y')"] & G_*(X',Y'). \end{tikzcd}$$

$C_*(g)$ is a quasi-isomorphism since g is a weak equivalence. It follows from Corollary 0.2 that $G_*(e,f)$ is a quasi-isomorphism. The desired equivalence is clear now. □

4.B. Proof of Theorem 0.3. If X is compact, the assertion follows immediately from Theorems 0.5 and 0.6. In general, X is weakly equivalent to a compact space X°. Using Lemma 4.1, we pass from $\lambda_*(X^\circ, Y)$ to $\lambda_*(X,Y)$. □

5. Reconstructing X from $M_*(X)$

5.A. Composition of maps and homomorphisms.

5.1. Lemma. *Let X, Y and Z be spaces and $\gamma\colon Z^Y \wedge Y^X \to Z^X$ be the composition map. Then the diagram of chain complexes and chain homomorphisms*

$$\begin{tikzcd} C_*(Z^Y) \otimes C_*(Y^X) \arrow[r, "\text{cross product}"] \arrow[d, "\lambda_*(Y,Z)\otimes\lambda_*(X,Y)"] & C_*(Z^Y \wedge Y^X) \arrow[r, "C_*(\gamma)"] & C_*(Z^X) \arrow[d, "\lambda_*(X,Z)"'] \\ G_*(Y,Z) \otimes G_*(X,Y) \arrow[rr, "\text{composition}"] & & G_*(X,Z) \end{tikzcd}$$

is commutative.

This follows from the associativity of the cross product. □

5.B. Proof of Corollary 0.4. Corollary 0.2 allows us to assume X and Y fibrant. Note that $H_0(G_*(X,Y)) = [M_*(X), M_*(Y)]$, the R-module of chain homotopy classes. By Lemma 5.1, we have the commutative diagram

$$\begin{tikzcd} H_0(X^Y) \otimes H_0(Y^X) \arrow[r, "\text{cross product}"] \arrow[d, "H_0(\lambda_*(Y,X))\otimes H_0(\lambda_*(X,Y))"] & H_0(X^Y \wedge Y^X) \arrow[r, "H_0(\gamma)"] & H_0(X^X) \arrow[d, "H_0(\lambda_*(X,X))"'] \\ {[M_*(Y), M_*(X)]} \otimes [M_*(X), M_*(Y)] \arrow[rr, "\text{composition}"] & & {[M_*(X), M_*(X)]}, \end{tikzcd}$$

where $\gamma\colon X^Y \wedge Y^X \to X^X$ is the composition map. We use the notation $B \otimes A \mapsto B \circ A$ for the upper line homomorphism $H_0(X^Y) \otimes H_0(Y^X) \to H_0(X^X)$. By Theorem 0.3, $H_0(\lambda_*(X,Y))$, $H_0(\lambda_*(Y,X))$ and $H_0(\lambda_*(X,X))$ are isomorphisms.

Let $f\colon M_*(X) \to M_*(Y)$ and $g\colon M_*(Y) \to M_*(X)$ be mutually inverse chain homotopy equivalences. We have $[f] = H_0(\lambda_*(X,Y))(A)$ for some $A \in H_0(Y^X)$ and $[g] = H_0(\lambda_*(Y,X))(B)$ for some $B \in H_0(X^Y)$. By the diagram, $B \circ A = 1$ in $H_0(X^X)$. Thus there are maps $a\colon X \to Y$ and $b\colon Y \to X$ such that $b \circ a \sim \mathrm{id}_X$. Interchanging X and Y in this reasoning, we get maps $a'\colon X \to Y$ and $b'\colon Y \to X$ such that $a' \circ b' \sim \mathrm{id}_Y$. Since X and Y are ℓ-toy, these four maps are weak equivalences. $\square$

References

[1] S. T. Ahearn, N. J. Kuhn, Product and other fine structure in polynomial resolutions of mapping spaces, Alg. Geom. Topology **2** (2002), 591–647.

[2] D. W. Anderson, A generalization of the Eilenberg–Moore spectral sequence, Bull. AMS, **78** (1972), No. 5, 784–786.

[3] J. Antosz, S. Betley, Homological algebra in the category of Γ-modules, Commun. Algebra **33** (2005), No. 6, 1913–1936.

[4] G. Arone, A generalization of Snaith-type filtration, Trans. AMS **351** (1999), No. 3, 1123–1150.

[5] M. Bendersky, S. Gitler, The cohomology of certain function spaces, Trans. AMS **326** (1991), No. 1, 423–440.

[6] J. M. Boardman, Conditionally convergent spectral sequences, Contemp. Math. **239** (1999), 49–84.

[7] A. K. Bousfield, On the homology spectral sequence of a cosimplicial space, Amer. J. Math. **109** (1987), No. 2, 361–394.

[8] A. K. Bousfield, D. M. Kan, Homotopy limits, completions and localizations, Lect. Notes Math. 304, Springer, 1972.

[9] S. S. Podkorytov, Commutative algebras and representations of the category of finite sets, arxiv:1011.6192.

[10] B. E. Shipley, Convergence of the homology spectral sequence of a cosimplicial space, Amer. J. Math. **118** (1996), No. 1, 179–207.

[11] V. A. Vassiliev, Complements of discriminants of smooth maps: topology and applications, AMS, 1992.

ssp@pdmi.ras.ru
http://www.pdmi.ras.ru/~ssp