# On the algebra of the Möbius crown

S. S. Podkorytov

**Abstract**

A commutative algebra over a field gives rise to a representation of the category of finite sets and surjective maps. We consider the restriction of this representation to the subcategory of sets of cardinality at most $r$. For each $r$, we present two non-isomorphic algebras that give rise to isomorphic representations of this subcategory.

Let $\Omega_r$ ($r = 0, 1, \ldots, \infty$) be the category whose objects are the sets $\langle p \rangle = \{1, \ldots, p\}$, $p = 1, 2, \ldots$, $p \leqslant r$, and whose morphisms are surjective maps. Let $\boldsymbol{k}$ be a field. We imply it when saying about vector spaces, tensor products, etc. By an *algebra* we mean a commutative non-unital $\boldsymbol{k}$-algebra. An algebra $A$ gives rise to the functor $L^r(A)\colon \Omega_r \to \boldsymbol{k}\textbf{-Mod}$ (a *representation* of $\Omega_r$) that takes an object $\langle p \rangle$ to the vector space $A^{\otimes p}$ and takes a morphism $s\colon \langle p \rangle \to \langle q \rangle$ to the linear map

$$A^{\otimes p} \to A^{\otimes q}, \qquad a_1 \otimes \ldots \otimes a_p \mapsto m_1 \otimes \ldots \otimes m_q,$$

where

$$m_j = \prod_{i \in s^{-1}(j)} a_i$$

(a variant of the Loday functor [3, Proposition 6.4.4]).

Must algebras $A$ and $B$ be isomorphic if the representations $L^r(A)$ and $L^r(B)$ are isomorphic? Yes if $r = \infty$, the field $\boldsymbol{k}$ is algebraically closed and the algebras have finite (vector-space) dimension ([4], cf. [1]). Our aim here is to show that this is false for arbitrarily large finite $r$. For each $r = 1, 2, \ldots$ and arbitrary $\boldsymbol{k}$, we present two non-isomorphic finite-dimensional algebras $A$ and $B$ with isomorphic representations $L^r(A)$ and $L^r(B)$. These algebras are obtained from the Stanley–Reisner algebras of certain graphs ("crowns") by taking the homogeneous components of degrees 1 and 2.

***The functor $L^r$.*** The correspondence $A \mapsto L^r(A)$ is covariant in an obvious way. So we have the functor $L^r\colon \boldsymbol{k}\textbf{-Alc} \to \textbf{Fun}(\Omega_r, \boldsymbol{k}\textbf{-Mod})$, where $\boldsymbol{k}\textbf{-Alc}$ is the category of algebras and $\textbf{Fun}(\Omega_r, \boldsymbol{k}\textbf{-Mod})$ is that of functors $\Omega_r \to \boldsymbol{k}\textbf{-Mod}$ (representations).

***The action category $M \mathbin{\backslash\backslash} S$.*** Let a monoid $M$ act on a set $S$ from the left. For $s, t \in S$, put $M(s,t) = \{m : m \cdot s = t\} \subseteq M$. We have the category $M \mathbin{\backslash\backslash} S$, where $\operatorname{Ob} M \mathbin{\backslash\backslash} S = S$, a bijection

$$M(s,t) \to \operatorname{Mor}_{M \backslash\backslash S}(s,t), \qquad m \mapsto m|_{s \to t},$$

is given for each $s, t \in S$, $1_s = 1|_{s\to s}$, and the composition of morphisms is given by the multiplication in $M$.

We have the not necessarily commutative unital $\boldsymbol{k}$-algebra $\boldsymbol{k}[M]$. For $s, t \in S$, we have the subspace $\boldsymbol{k}[M(s,t)] \subseteq \boldsymbol{k}[M]$.

Consider the linear category $\boldsymbol{k}[M \mathbin{\backslash\backslash} S]$. For $s, t \in S$, we have the linear map

$$\boldsymbol{k}[M(s,t)] \to \mathrm{Mor}_{\boldsymbol{k}[M\backslash\backslash S]}(s,t), \qquad X \mapsto X\|_{s\to t},$$

given by the rule $[m] \mapsto [m|_{s\to t}]$. Clearly, $1\|_{s\to s} = 1_s$ $(s \in S)$. If $X \in \boldsymbol{k}[M(s,t)]$, $Y \in \boldsymbol{k}[M(t,u)]$ $(s,t,u \in S)$, then $YX \in \boldsymbol{k}[M(s,u)]$ and

$$(YX)\|_{s\to u} = Y\|_{t\to u} \circ X\|_{s\to t}.$$

***The monoid $\boldsymbol{W_n}$ and the elements $\boldsymbol{T_n}$ and $\boldsymbol{Z_n}$.*** Introduce the multiplicative submonoid $\boldsymbol{V} = \{1, -1, 0\} \subseteq \mathbb{Z}$ and its submonoids $\boldsymbol{U} = \{1, -1\}$ and $\boldsymbol{E} = \{1, 0\}$. We denote the elements 1 and $-1$ also by $+$ and $-$ (respectively).

Let $W_n \subseteq \boldsymbol{V}^{2n+1}$ be the submonoid formed by the collections

$$w = (w_1, w_2, \dots, w_{2n+1})$$

in which $w_{2i+1} \in \boldsymbol{U}$ $(i = 0, \dots, n)$ and $w_j w_{j+1} \in \boldsymbol{E}$ $(j = 1, \dots, 2n)$.

Introduce the elements $g_i, h_i \in W_n$ $(i = 1, \dots, n)$:

$$g_i = (+, \dots, +, \underset{2i}{0}, +, \dots, +), \qquad h_i = (-, \dots, -, \underset{2i}{0}, +, \dots, +)$$

and $T_n, Z_n \in \boldsymbol{k}[W_n]$:

$$T_n = \sum_{i=1}^{n} (1 - [g_1]) \dots (1 - [g_{i-1}])[h_i], \qquad Z_n = (1 - [g_1]) \dots (1 - [g_n]).$$

Using commutativity of $W_n$ and the relations $g_i^2 = h_i^2 = g_i$ and $g_i h_i = h_i$, we get

$$T_n^2 = 1 - Z_n.$$

***Two actions of $\boldsymbol{W_n}$ and their categories.*** The monoid $W_n$ acts on the set $\boldsymbol{U}$ from the left by the rule $w \cdot s = w_1 w_{2n+1} s$. Since $T_n \in \boldsymbol{k}[W_n(s, -s)]$ and $Z_n \in \boldsymbol{k}[W_n(s,s)]$ for each $s \in \boldsymbol{U}$, we have

$$T_n\|_{-s\to s} \circ T_n\|_{s\to -s} = 1_s - Z_n\|_{s\to s} \tag{1}$$

in $\boldsymbol{k}[W_n \mathbin{\backslash\backslash} \boldsymbol{U}]$.

Consider the one-element set $\{\star\}$ with the left action of $W_n$. The map $\boldsymbol{U} \to \{\star\}$ induces the functors $\omega_n \colon W_n \mathbin{\backslash\backslash} \boldsymbol{U} \to W_n \mathbin{\backslash\backslash} \{\star\}$ and $\boldsymbol{k}[\omega_n] \colon \boldsymbol{k}[W_n \mathbin{\backslash\backslash} \boldsymbol{U}] \to \boldsymbol{k}[W_n \mathbin{\backslash\backslash} \{\star\}]$. For any $s, t \in \boldsymbol{U}$ and $X \in \boldsymbol{k}[W_n(s,t)]$, we have

$$\boldsymbol{k}[\omega_n] \colon X\|_{s\to t} \mapsto X\|_{\star\to\star}. \tag{2}$$

***Graphs.*** By a *graph* we mean a pair $G = (G_1, G_2)$, where $G_1$ is a set and $G_2 \subseteq G_1 \times G_1$ is a reflexive symmetric relation. The *vertices* of $G$ are the elements of $G_1$; its *edges* are the sets $\{x, y\}$, where $(x, y) \in G_2$, $x \neq y$.

A morphism $f\colon G \to H$ of graphs is a pair $f = (f_1, f_2)$, where $f_p\colon G_p \to H_p$, $p = 1, 2$, are maps such that $f_2(x, y) = (f_1(x), f_1(y))$, $(x, y) \in G_2$. Graphs and their morphisms form a category **Graph**.

***The cofunctor Q: the algebra of a graph.*** Let $G$ be a graph. The symmetric group $\Sigma_2$ acts on $G_2 \subseteq G_1 \times G_1$ by permuting the cooordinates. We have the projection

$$\boldsymbol{k}^{G_2} \to (\boldsymbol{k}^{G_2})_{\Sigma_2}, \qquad u \mapsto \bar{u}.$$

Let $A^\bullet$ be the graded algebra concentrated in degrees 1 and 2:

$$A^1 = \boldsymbol{k}^{G_1}, \qquad A^2 = (\boldsymbol{k}^{G_2})_{\Sigma_2},$$

where, if $a, b \in A^1$, then $ab = \bar{u} \in A^2$, where $u \in \boldsymbol{k}^{G_2}$, $u(x, y) = a(x)b(y)$.

Put $Q^\bullet(G) = A^\bullet$. Let $Q(G)$ be the same algebra considered without the grading. The correspondence $G \mapsto Q(G)$ is contravariant in an obvious way. So we have the cofunctor $Q\colon$ **Graph** $\to$ $\boldsymbol{k}$-**Alc**. We need the following properties of $Q$.

1°. If $G$ is finite, then $Q(G)$ has finite dimension.

2°. If graph morphisms $f_i\colon G_i \to H$, $i \in I$, form a *cover*, i. e.,

$$\bigcup_{i \in I} \operatorname{Im} f_{i\,p} = H_p, \qquad p = 1, 2,$$

then the linear map

$$(Q(f_i))_{i \in I}\colon Q(H) \to \prod_{i \in I} Q(G_i)$$

is injective.

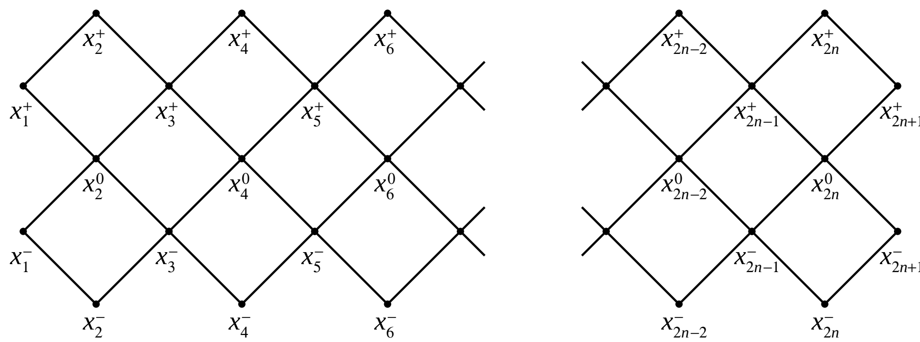
3°. If finite graphs $G$ and $H$ are non-isomorphic, then the algebras $Q(G)$ and $Q(H)$ are non-isomorphic too. This follows from the Gubeladze theorem [2, Theorem 3.1]. We give simpler arguments that suffice in the special case that we will need.

Call a graph $G$ *admissible* if, for any distinct $x, y \in G_1$, there exists $z \in G_1$ such that $(x, z) \notin G_2$ and $(y, z) \in G_2$. (For example, any graph without triangles and pendant vertices is admissible.) We show that an admissible graph $G$ can be reconstructed from $Q(G)$.

Let $A^\bullet$ be a graded algebra concentrated in degrees 1 and 2. Consider the projective space $P(A^1)$. Let $[\ ]\colon A^1 \setminus \{0\} \to P(A^1)$ be the projection. Define on $P(A^1)$ a symmetric relation $\#$ (*dependence*): $[a] \# [b] \Leftrightarrow ab \neq 0$, and a preorder $\lesssim$: $p \lesssim q \Leftrightarrow p^\# \subseteq q^\#$, where $r^\# = \{s : r \# s\}$. Let $R \subseteq P(A^1)$ be the set of *minimal* points, i. e. those points $p$ for which $\{s : s \lesssim p\} = \{p\}$. If $A^\bullet = Q^\bullet(G)$ for some graph $G$, then there is the injective map $e\colon G_1 \to P(A^1)$, $x \mapsto [\delta_x]$, where $\delta_x \in A^1 = \boldsymbol{k}^{G_1}$, $\delta_x(y)$ equals 1 if $y = x$ and 0 otherwise.

The inverse image of # under $e$ equals $G_2$. It is not hard to check that, if $G$ is admissible, then $\operatorname{Im} e = R$. It remains to add that the graded algebra $A^\bullet$ can be reconstructed from the ungraded algebra $A = Q(G)$: $A^\bullet$ is canonically isomorphic to the graded algebra $B^\bullet$ with the components $B^1$ and $B^2$, where $B^2 = \{b : bA = 0\} \subseteq A$ and $B^1 = A/B^2$ (so $B^2 = A^2$ and $B^1 \cong A^1$), and the multiplication induced by that in $A$.

***The graph $B_n$.*** Let $B_n$ be the graph shown on the figure. Its vertices are $x_j^v$, where $j = 1, \dots, 2n+1$, $v \in \boldsymbol{V}$, and $v \in \boldsymbol{U}$ if $j$ is odd.




The monoid $W_n$ acts on $B_n$ from the left by the rule $w \cdot x_j^v = x_j^{w_j v}$. Let $w_* \colon B_n \to B_n$ be the action of $w \in W_n$. The graph $B_n$ with the action of $W_n$ gives rise to the functor

$$\underline{B}_n \colon W_n \mathbin{\backslash\!\backslash} \{\star\} \to \mathbf{Graph}, \qquad \star \mapsto B_n, \qquad w|_{\star\to\star} \mapsto w_*.$$

Since $\mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod})$ is a linear category, the cofunctor

$$W_n \mathbin{\backslash\!\backslash} \{\star\} \xrightarrow{\underline{B}_n} \mathbf{Graph} \xrightarrow{Q} \boldsymbol{k}\text{-}\mathbf{Alc} \xrightarrow{L^r} \mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod})$$

extends to a linear cofunctor

$$b_n^r \colon \boldsymbol{k}[W_n \mathbin{\backslash\!\backslash} \{\star\}] \to \mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod}).$$

**Lemma.** *We have* $b_n^{n-1}(Z_n\|_{\star\to\star}) = 0$.

*Proof.* Take $p = 1, \dots, n-1$. The monoid $W_n$ acts on $B_n$ from the left. The induced right action on the vector space $Q(B_n)^{\otimes p}$ makes it a right $\boldsymbol{k}[W_n]$-module. We should show that $Q(B_n)^{\otimes p} Z_n = 0$.

For $i = 1, \dots, n$, let $F_i$ be the subgraph of $B_n$ spanned by the vertices $x_j^v$ with $|j - 2i| \leqslant 1$ and let $e_i \colon F_i \to B_n$ be the inclusion morphism. Since the subgraphs $F_i$ cover $B_n$, the linear map

$$(Q(e_i))_{i=1}^n \colon Q(B_n) \to \bigoplus_{i=1}^n Q(F_i)$$

is injective (by the property 2°). Raising it to the tensor power $p$, we get an injective linear map

$$E_p\colon Q(B_n)^{\otimes p} \to \bigoplus_{i_1,\dots,i_p} S_{i_1\dots i_p}, \qquad S_{i_1\dots i_p} = Q(F_{i_1}) \otimes \ldots \otimes Q(F_{i_p}).$$

The subgraphs $F_i$ are invariant under the action of $W_n$. The induced right action on the vector spaces $S_{i_1\dots i_p}$ makes them right $\boldsymbol{k}[W_n]$-modules. The map $E_p$ is a homomorphism of $\boldsymbol{k}[W_n]$-modules. Since it is injective, it suffices to show that $S_{i_1\dots i_p} Z_n = 0$.

Each element $g_i$ acts trivially on the subgraphs $F_{i'}$, $i' \neq i$. Thus, if $i$ is distinct from $i_1, \dots, i_p$, the element $g_i$ acts trivially on $S_{i_1\dots i_p}$ and thus $S_{i_1\dots i_p} Z_n = 0$. Since $p < n$, such an $i$ exists for any $i_1, \dots, i_p$. □

***The graphs $C_n^s$ (crowns).*** Take $n \geqslant 2$. For $s \in \boldsymbol{U}$, let $C_n^s$ be the graph obtained from $B_n$ by identifying $x_{2n+1}^v$ with $x_1^{sv}$ for each $v \in \boldsymbol{U}$. Let $f_n^s\colon B_n \to C_n^s$ be the projection morphism. We call $C_n^+$ the *simple crown* and $C_n^-$ the *Möbius* one.

The graphs $C_n^s$, $s \in \boldsymbol{U}$, are non-isomorphic (the edges containing vertices of valency 2 form two cycles in $C_n^+$ and one cycle in $C_n^-$). They are finite and admissible, and thus (see the properties 1° and 3°) their algebras $Q(C_n^s)$ are finite-dimensional and non-isomorphic. We show that the representations $L^{n-1}(Q(C_n^s))$, $s \in \boldsymbol{U}$, are isomorphic.

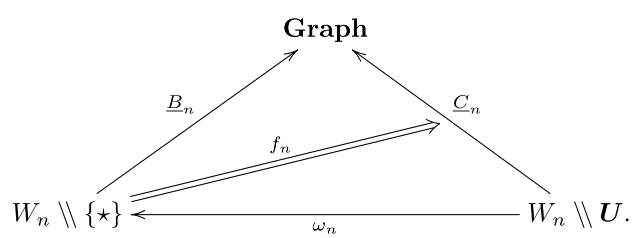
For $s, t \in \boldsymbol{U}$ and $w \in W_n(s,t)$, let $w_*\colon C_n^s \to C_n^t$ be the morphism such that the following diagram is commutative:

$$\begin{array}{ccc} B_n & \xrightarrow{\;f_n^s\;} & C_n^s \\ {\scriptstyle w_*}\downarrow & & \downarrow{\scriptstyle w_*} \\ B_n & \xrightarrow{\;f_n^t\;} & C_n^t. \end{array}$$

So we have the functor

$$\underline{C}_n\colon W_n \mathbin{\backslash\!\backslash} \boldsymbol{U} \to \mathbf{Graph}, \qquad s \mapsto C_n^s, \qquad w|_{s\to t} \mapsto w_*.$$

The morphisms $f_n^s$, $s \in \boldsymbol{U}$, form a morphism of functors $f_n\colon \underline{B}_n \circ \omega_n \to \underline{C}_n$:




Since $\mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod})$ is a linear category, the cofunctor

$$W_n \mathbin{\backslash\!\backslash} \boldsymbol{U} \xrightarrow{\underline{C}_n} \mathbf{Graph} \xrightarrow{Q} \boldsymbol{k}\text{-}\mathbf{Alc} \xrightarrow{L^r} \mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod})$$

extends to a linear cofunctor

$$c_n^r\colon \boldsymbol{k}[W_n \mathbin{\backslash\!\backslash} \boldsymbol{U}] \to \mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod}).$$

The morphism $f_n$ induces a morphism of cofunctors

$$\begin{array}{ccc} & \mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod}) & \\ {\scriptstyle b_n^r}\nearrow & \Leftarrow & \nwarrow{\scriptstyle c_n^r} \\ \boldsymbol{k}[W_n \mathbin{\backslash\!\backslash} \{\star\}] & \xleftarrow{\ \boldsymbol{k}[\omega_n]\ } & \boldsymbol{k}[W_n \mathbin{\backslash\!\backslash} \boldsymbol{U}], \end{array}$$

i. e., for any $s, t \in \boldsymbol{U}$ and $X \in \boldsymbol{k}[W_n(s,t)]$, we have the commutative diagram

$$\begin{array}{ccc} L^r(Q(B_n)) & \xleftarrow{\ L^r(Q(f_n^s))\ } & L^r(Q(C_n^s)) \\ {\scriptstyle b_n^r(X\|_{\star\to\star})}\uparrow & & \uparrow{\scriptstyle c_n^r(X\|_{s\to t})} \\ L^r(Q(B_n)) & \xleftarrow{\ L^r(Q(f_n^t))\ } & L^r(Q(C_n^t)) \end{array}$$

(we used the rule (2)). Since $f_n^s$ is a cover, the homomorphism $Q(f_n^s)\colon Q(C_n^s) \to Q(B_n)$ is injective (by the property 2°), and thus the morphism $L^r(Q(f_n^s))$ is objectwise injective.

Now assume $r = n-1$, $s = t$ and $X = Z_n$. By Lemma, $b_n^{n-1}(Z_n\|_{\star\to\star}) = 0$. Thus $c_n^{n-1}(Z_n\|_{s\to s}) = 0$ (by commutativity of the diagram and the mentioned objectwise injectivity). We show that the arrows of the diagram

$$L^{n-1}(Q(C_n^+)) \underset{c_n^{n-1}(T_n\|_{+\to-})}{\overset{c_n^{n-1}(T_n\|_{-\to+})}{\rightleftarrows}} L^{n-1}(Q(C_n^-))$$

are mutually inverse. For each $s \in \boldsymbol{U}$, we have

$$\begin{aligned} c_n^{n-1}(T_n\|_{s\to -s}) \circ c_n^{n-1}(T_n\|_{-s\to s}) &= c_n^{n-1}(T_n\|_{-s\to s} \circ T_n\|_{s\to -s}) = \\ &= c_n^{n-1}(1_s - Z_n\|_{s\to s}) = 1_{L^{n-1}(Q(C_n^s))} - c_n^{n-1}(Z_n\|_{s\to s}) = 1_{L^{n-1}(Q(C_n^s))} \end{aligned}$$

(we used the equality (1)).

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## References

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St. Petersburg Department of Steklov Mathematical Institute
of Russian Academy of Sciences

ssp@pdmi.ras.ru
http://www.pdmi.ras.ru/~ssp