## On the algebra of the Möbius crown

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## Abstract

A commutative algebra over a field gives rise to a representation of the category of finite sets and surjective maps. We consider the restriction of this representation to the subcategory of sets of cardinality at most r. For each r, we present two non-isomorphic algebras that give rise to isomorphic representations of this subcategory.

Let  $\Omega_r$   $(r = 0, 1, ..., \infty)$  be the category whose objects are the sets  $\langle p \rangle = \{1, ..., p\}, p = 1, 2, ..., p \leqslant r$ , and whose morphisms are surjective maps. Let  $\mathbf{k}$  be a field. We imply it when saying about vector spaces, tensor products, etc. By an *algebra* we mean a commutative non-unital  $\mathbf{k}$ -algebra. An algebra A gives rise to the functor  $L^r(A): \Omega_r \to \mathbf{k}$ -Mod (a *representation* of  $\Omega_r$ ) that takes an object  $\langle p \rangle$  to the vector space  $A^{\otimes p}$  and takes a morphism  $s: \langle p \rangle \to \langle q \rangle$  to the linear map

$$A^{\otimes p} \to A^{\otimes q}, \qquad a_1 \otimes \ldots \otimes a_p \mapsto m_1 \otimes \ldots \otimes m_q,$$

where

$$m_j = \prod_{i \in s^{-1}(j)} a_i$$

(a variant of the Loday functor [3, Proposition 6.4.4]).

Must algebras A and B be isomorphic if the representations  $L^{r}(A)$  and  $L^{r}(B)$  are isomorphic? Yes if  $r = \infty$ , the field  $\mathbf{k}$  is algebraically closed and the algebras have finite (vector-space) dimension ([4], cf. [1]). Our aim here is to show that this is false for arbitrarily large finite r. For each r = 1, 2, ... and arbitrary  $\mathbf{k}$ , we present two non-isomorphic finite-dimensional algebras A and B with isomorphic representations  $L^{r}(A)$  and  $L^{r}(B)$ . These algebras are obtained from the Stanley–Reisner algebras of certain graphs ("crowns") by taking the homogeneous components of degrees 1 and 2.

The functor  $L^r$ . The correspondence  $A \mapsto L^r(A)$  is covariant in an obvious way. So we have the functor  $L^r: \mathbf{k}\text{-}\mathbf{Alc} \to \mathbf{Fun}(\Omega_r, \mathbf{k}\text{-}\mathbf{Mod})$ , where  $\mathbf{k}\text{-}\mathbf{Alc}$  is the category of algebras and  $\mathbf{Fun}(\Omega_r, \mathbf{k}\text{-}\mathbf{Mod})$  is that of functors  $\Omega_r \to \mathbf{k}\text{-}\mathbf{Mod}$ (representations).

The action category  $M \setminus S$ . Let a monoid M act on a set S from the left. For  $s, t \in S$ , put  $M(s,t) = \{m : m \cdot s = t\} \subseteq M$ . We have the category  $M \setminus S$ , where  $Ob M \setminus S = S$ , a bijection

$$M(s,t) \to \operatorname{Mor}_{M \setminus S}(s,t), \qquad m \mapsto m|_{s \to t},$$

is given for each  $s, t \in S$ ,  $1_s = 1|_{s \to s}$ , and the composition of morphisms is given by the multiplication in M.

We have the not necessarily commutative unital k-algebra k[M]. For  $s, t \in S$ , we have the subspace  $k[M(s,t)] \subseteq k[M]$ .

Consider the linear category  $k[M \setminus S]$ . For  $s, t \in S$ , we have the linear map

$$\boldsymbol{k}[M(s,t)] \to \operatorname{Mor}_{\boldsymbol{k}[M \setminus \! \backslash S]}(s,t), \qquad X \mapsto X \|_{s \to t},$$

given by the rule  $[m] \mapsto [m|_{s \to t}]$ . Clearly,  $1||_{s \to s} = 1_s$   $(s \in S)$ . If  $X \in \mathbf{k}[M(s,t)]$ ,  $Y \in \mathbf{k}[M(t,u)]$   $(s,t,u \in S)$ , then  $YX \in \mathbf{k}[M(s,u)]$  and

$$(YX)\|_{s\to u} = Y\|_{t\to u} \circ X\|_{s\to t}.$$

The monoid  $W_n$  and the elements  $T_n$  and  $Z_n$ . Introduce the multiplicative submonoid  $V = \{1, -1, 0\} \subseteq \mathbb{Z}$  and its submonoids  $U = \{1, -1\}$  and  $E = \{1, 0\}$ . We denote the elements 1 and -1 also by + and - (respectively).

Let  $W_n \subseteq V^{2n+1}$  be the submonoid formed by the collections

$$w = (w_1, w_2, \dots, w_{2n+1})$$

in which  $w_{2i+1} \in U$  (i = 0, ..., n) and  $w_j w_{j+1} \in E$  (j = 1, ..., 2n). Introduce the elements  $g_i, h_i \in W_n$  (i = 1, ..., n):

$$g_i = (+, \dots, +, 0, +, \dots, +), \qquad h_i = (-, \dots, -, 0, +, \dots, +)$$

and  $T_n, Z_n \in \mathbf{k}[W_n]$ :

$$T_n = \sum_{i=1}^n (1 - [g_1]) \dots (1 - [g_{i-1}])[h_i], \qquad Z_n = (1 - [g_1]) \dots (1 - [g_n]).$$

Using commutativity of  $W_n$  and the relations  $g_i^2 = h_i^2 = g_i$  and  $g_i h_i = h_i$ , we get

$$T_n^2 = 1 - Z_n.$$

Two actions of  $W_n$  and their categories. The monoid  $W_n$  acts on the set U from the left by the rule  $w \cdot s = w_1 w_{2n+1} s$ . Since  $T_n \in \mathbf{k}[W_n(s, -s)]$  and  $Z_n \in \mathbf{k}[W_n(s, s)]$  for each  $s \in U$ , we have

$$T_n\|_{-s \to s} \circ T_n\|_{s \to -s} = 1_s - Z_n\|_{s \to s} \tag{1}$$

in  $\boldsymbol{k}[W_n \setminus \boldsymbol{U}]$ .

Consider the one-element set  $\{\star\}$  with the left action of  $W_n$ . The map  $U \to \{\star\}$  induces the functors  $\omega_n \colon W_n \setminus U \to W_n \setminus \{\star\}$  and  $k[\omega_n] \colon k[W_n \setminus U] \to k[W_n \setminus \{\star\}]$ . For any  $s, t \in U$  and  $X \in k[W_n(s, t)]$ , we have

$$\boldsymbol{k}[\omega_n] \colon X \|_{s \to t} \mapsto X \|_{\star \to \star}.$$
<sup>(2)</sup>

**Graphs.** By a graph we mean a pair  $G = (G_1, G_2)$ , where  $G_1$  is a set and  $G_2 \subseteq G_1 \times G_1$  is a reflexive symmetric relation. The vertices of G are the elements of  $G_1$ ; its edges are the sets  $\{x, y\}$ , where  $(x, y) \in G_2, x \neq y$ .

A morphism  $f: G \to H$  of graphs is a pair  $f = (f_1, f_2)$ , where  $f_p: G_p \to H_p$ , p = 1, 2, are maps such that  $f_2(x, y) = (f_1(x), f_1(y)), (x, y) \in G_2$ . Graphs and their morphisms form a category **Graph**.

The cofunctor Q: the algebra of a graph. Let G be a graph. The symmetric group  $\Sigma_2$  acts on  $G_2 \subseteq G_1 \times G_1$  by permuting the coordinates. We have the projection

$$\mathbf{k}^{G_2} \to (\mathbf{k}^{G_2})_{\Sigma_2}, \qquad u \mapsto \bar{u}.$$

Let  $A^{\bullet}$  be the graded algebra concentrated in degrees 1 and 2:

$$A^1 = \boldsymbol{k}^{G_1}, \qquad A^2 = (\boldsymbol{k}^{G_2})_{\Sigma_2}$$

where, if  $a, b \in A^1$ , then  $ab = \overline{u} \in A^2$ , where  $u \in \mathbf{k}^{G_2}$ , u(x, y) = a(x)b(y).

Put  $Q^{\bullet}(G) = A^{\bullet}$ . Let Q(G) be the same algebra considered without the grading. The correspondence  $G \mapsto Q(G)$  is contravariant in an obvious way. So we have the cofunctor Q:**Graph**  $\to k$ -**Alc**. We need the following properties of Q.

1°. If G is finite, then Q(G) has finite dimension.

2°. If graph morphisms  $f_i: G_i \to H, i \in I$ , form a *cover*, i. e.,

$$\bigcup_{i \in I} \operatorname{Im} f_{i p} = H_p, \qquad p = 1, 2,$$

then the linear map

$$(Q(f_i))_{i \in I} \colon Q(H) \to \prod_{i \in I} Q(G_i)$$

is injective.

3°. If finite graphs G and H are non-isomorphic, then the algebras Q(G) and Q(H) are non-isomorphic too. This follows from the Gubeladze theorem [2, Theorem 3.1]. We give simpler arguments that suffice in the special case that we will need.

Call a graph G admissible if, for any distinct  $x, y \in G_1$ , there exists  $z \in G_1$ such that  $(x, z) \notin G_2$  and  $(y, z) \in G_2$ . (For example, any graph without triangles and pendant vertices is admissible.) We show that an admissible graph G can be reconstructed from Q(G).

Let  $A^{\bullet}$  be a graded algebra concentrated in degrees 1 and 2. Consider the projective space  $P(A^1)$ . Let  $[]: A^1 \setminus \{0\} \to P(A^1)$  be the projection. Define on  $P(A^1)$  a symmetric relation # (dependence):  $[a] \# [b] \Leftrightarrow ab \neq 0$ , and a preorder  $\leq: p \leq q \Leftrightarrow p^{\#} \subseteq q^{\#}$ , where  $r^{\#} = \{s: r \ \# \ s\}$ . Let  $R \subseteq P(A^1)$  be the set of minimal points, i. e. those points p for which  $\{s: s \leq p\} = \{p\}$ . If  $A^{\bullet} = Q^{\bullet}(G)$  for some graph G, then there is the injective map  $e: G_1 \to P(A^1)$ ,  $x \mapsto [\delta_x]$ , where  $\delta_x \in A^1 = \mathbf{k}^{G_1}, \ \delta_x(y)$  equals 1 if y = x and 0 otherwise.

The inverse image of # under e equals  $G_2$ . It is not hard to check that, if G is admissible, then  $\operatorname{Im} e = R$ . It remains to add that the graded algebra  $A^{\bullet}$  can be reconstructed from the ungraded algebra A = Q(G):  $A^{\bullet}$  is canonically isomorphic to the graded algebra  $B^{\bullet}$  with the components  $B^1$  and  $B^2$ , where  $B^2 = \{b : bA = 0\} \subseteq A$  and  $B^1 = A/B^2$  (so  $B^2 = A^2$  and  $B^1 \cong A^1$ ), and the multiplication induced by that in A.

**The graph**  $B_n$ . Let  $B_n$  be the graph shown on the figure. Its vertices are  $x_j^v$ , where  $j = 1, ..., 2n + 1, v \in V$ , and  $v \in U$  if j is odd.



The monoid  $W_n$  acts on  $B_n$  from the left by the rule  $w \cdot x_j^v = x_j^{w_j v}$ . Let  $w_* \colon B_n \to B_n$  be the action of  $w \in W_n$ . The graph  $B_n$  with the action of  $W_n$  gives rise to the functor

$$\underline{B}_n \colon W_n \setminus \{\star\} \to \mathbf{Graph}, \qquad \star \mapsto B_n, \qquad w|_{\star \to \star} \mapsto w_*$$

Since  $\mathbf{Fun}(\Omega_r, \mathbf{k}\text{-}\mathbf{Mod})$  is a linear category, the cofunctor

$$W_n \setminus \{\star\} \xrightarrow{\underline{B}_n} \mathbf{Graph} \xrightarrow{Q} k\text{-Alc} \xrightarrow{L^r} \mathbf{Fun}(\Omega_r, k\text{-Mod})$$

extends to a linear cofunctor

$$b_n^r \colon \mathbf{k}[W_n \setminus \{\star\}] \to \mathbf{Fun}(\Omega_r, \mathbf{k}\text{-}\mathbf{Mod}).$$

**Lemma.** We have  $b_n^{n-1}(Z_n \|_{\star \to \star}) = 0.$ 

*Proof.* Take p = 1, ..., n-1. The monoid  $W_n$  acts on  $B_n$  from the left. The induced right action on the vector space  $Q(B_n)^{\otimes p}$  makes it a right  $\mathbf{k}[W_n]$ -module. We should show that  $Q(B_n)^{\otimes p}Z_n = 0$ .

For i = 1, ..., n, let  $F_i$  be the subgraph of  $B_n$  spanned by the vertices  $x_j^v$  with  $|j - 2i| \leq 1$  and let  $e_i \colon F_i \to B_n$  be the inclusion morphism. Since the subgraphs  $F_i$  cover  $B_n$ , the linear map

$$(Q(e_i))_{i=1}^n \colon Q(B_n) \to \bigoplus_{i=1}^n Q(F_i)$$

is injective (by the property 2°). Raising it to the tensor power p, we get an injective linear map

$$E_p: Q(B_n)^{\otimes p} \to \bigoplus_{i_1, \dots, i_p} S_{i_1 \dots i_p}, \qquad S_{i_1 \dots i_p} = Q(F_{i_1}) \otimes \dots \otimes Q(F_{i_p}).$$

The subgraphs  $F_i$  are invariant under the action of  $W_n$ . The induced right action on the vector spaces  $S_{i_1...i_p}$  makes them right  $\mathbf{k}[W_n]$ -modules. The map  $E_p$  is a homomorphism of  $\mathbf{k}[W_n]$ -modules. Since it is injective, it suffices to show that  $S_{i_1...i_p}Z_n = 0$ .

Each element  $g_i$  acts trivially on the subgraphs  $F_{i'}$ ,  $i' \neq i$ . Thus, if i is distinct from  $i_1, \ldots, i_p$ , the element  $g_i$  acts trivially on  $S_{i_1 \ldots i_p}$  and thus  $S_{i_1 \ldots i_p} Z_n = 0$ . Since p < n, such an i exists for any  $i_1, \ldots, i_p$ .

The graphs  $C_n^s$  (crowns). Take  $n \ge 2$ . For  $s \in U$ , let  $C_n^s$  be the graph obtained from  $B_n$  by identifying  $x_{2n+1}^v$  with  $x_1^{sv}$  for each  $v \in U$ . Let  $f_n^s \colon B_n \to C_n^s$  be the projection morphism. We call  $C_n^+$  the simple crown and  $C_n^-$  the Möbius one.

The graphs  $C_n^s$ ,  $s \in U$ , are non-isomorphic (the edges containing vertices of valency 2 form two cycles in  $C_n^+$  and one cycle in  $C_n^-$ ). They are finite and admissible, and thus (see the properties 1° and 3°) their algebras  $Q(C_n^s)$ are finite-dimensional and non-isomorphic. We show that the representations  $L^{n-1}(Q(C_n^s))$ ,  $s \in U$ , are isomorphic.

For  $s, t \in U$  and  $w \in W_n(s, t)$ , let  $w_* \colon C_n^s \to C_n^t$  be the morphism such that the following diagram is commutative:



So we have the functor

$$\underline{C}_n: W_n \setminus U \to \mathbf{Graph}, \qquad s \mapsto C_n^s, \qquad w|_{s \to t} \mapsto w_*.$$

The morphisms  $f_n^s$ ,  $s \in U$ , form a morphism of functors  $f_n : \underline{B}_n \circ \omega_n \to \underline{C}_n$ :



Since  $\mathbf{Fun}(\Omega_r, \mathbf{k}\text{-}\mathbf{Mod})$  is a linear category, the cofunctor

$$W_n \setminus U \xrightarrow{\underline{C}_n} \mathbf{Graph} \xrightarrow{Q} k\text{-}\mathbf{Alc} \xrightarrow{L^r} \mathbf{Fun}(\Omega_r, k\text{-}\mathbf{Mod})$$

extends to a linear cofunctor

 $c_n^r \colon \boldsymbol{k}[W_n \setminus \boldsymbol{U}] \to \mathbf{Fun}(\Omega_r, \boldsymbol{k}\text{-}\mathbf{Mod}).$ 

The morphism  $f_n$  induces a morphism of cofunctors



i. e., for any  $s, t \in U$  and  $X \in k[W_n(s,t)]$ , we have the commutative diagram

$$L^{r}(Q(B_{n})) \stackrel{L^{r}(Q(f_{n}^{s}))}{\longleftarrow} L^{r}(Q(C_{n}^{s}))$$

$$b_{n}^{r}(X||_{\star \to \star}) \stackrel{\uparrow}{\longrightarrow} L^{r}(Q(f_{n}^{t})) \stackrel{fc_{n}^{r}(X||_{s \to t})}{\longleftarrow} L^{r}(Q(C_{n}^{t}))$$

(we used the rule (2)). Since  $f_n^s$  is a cover, the homomorphism  $Q(f_n^s): Q(C_n^s) \to Q(B_n)$  is injective (by the property 2°), and thus the morphism  $L^r(Q(f_n^s))$  is objectwise injective.

Now assume r = n - 1, s = t and  $X = Z_n$ . By Lemma,  $b_n^{n-1}(Z_n \|_{\star \to \star}) = 0$ . Thus  $c_n^{n-1}(Z_n \|_{s \to s}) = 0$  (by commutativity of the diagram and the mentioned objectwise injectivity). We show that the arrows of the diagram

$$L^{n-1}(Q(C_n^+)) \underbrace{\sum_{c_n^{n-1}(T_n \parallel_{-\to +})}^{c_n^{n-1}(T_n \parallel_{-\to +})}}_{C_n^{n-1}(T_n \parallel_{+\to -})} L^{n-1}(Q(C_n^-))$$

are mutually inverse. For each  $s \in U$ , we have

$$c_n^{n-1}(T_n \|_{s \to -s}) \circ c_n^{n-1}(T_n \|_{-s \to s}) = c_n^{n-1}(T_n \|_{-s \to s} \circ T_n \|_{s \to -s}) =$$
$$= c_n^{n-1}(1_s - Z_n \|_{s \to s}) = 1_{L^{n-1}(Q(C_n^s))} - c_n^{n-1}(Z_n \|_{s \to s}) = 1_{L^{n-1}(Q(C_n^s))}$$

(we used the equality (1)).

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