Straight homotopy invariants

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Abstract

Let X and Y be spaces and M be an abelian group. A homotopy invariant $f\colon [X,Y] \to M$ is called straight if there exists a homomorphism $F\colon L(X,Y) \to M$ such that $f([a]) = F(\langle a\rangle)$ for all $a \in C(X,Y)$. Here $\langle a\rangle\colon \langle X\rangle \to \langle Y\rangle$ is the homomorphism induced by a between the abelian groups freely generated by X and Y and $L(X,Y)$ is a certain group of "admissible" homomorphisms. We show that all straight invariants can be expressed through a "universal" straight invariant of homological nature.

§ 1. Introduction

We define straight homotopy invariants of maps and give their characterization, which reduces them to the classical homology theory. Straight invariants are a variant of the notion of homotopy invariants of degree at most 1 [10, 8, 9, 11]. This variant has especially simple homological characterization. Homotopy invariants of finite degree are a homotopy analogue of Vassiliev invariants [8].

The group $L(X,Y)$. For a set X, let $\langle X\rangle$ be the (free) abelian group with the basis $X^\sharp \subseteq \langle X\rangle$ endowed with the bijection $X \to X^\sharp$, $x \mapsto {<}x{>}$. For sets X and Y, let $L(X,Y) \subseteq \operatorname{Hom}(\langle X\rangle, \langle Y\rangle)$ be the subgroup generated by the homomorphisms u such that $u(X^\sharp) \subseteq Y^\sharp \cup \{0\}$. (Elements of $L(X,Y)$ are the homomorphisms bounded with respect to the ℓ_1-norm.) A map $a\colon X \to Y$ induces the homomorphism $\langle a\rangle \in L(X,Y)$, $\langle a\rangle({<}x{>}) = {<}a(x){>}$.

Straight homotopy invariants. Let X and Y be spaces, M be an abelian group, and $f\colon [X,Y] \to M$ be a map (a homotopy invariant). The invariant f is called *straight* if there exists a homomorphism $F\colon L(X,Y) \to M$ such that $f([a]) = F(\langle a\rangle)$ for all $a \in C(X,Y)$.

(If M is divisible, the group $L(X,Y)$ can be replaced here by $\operatorname{Hom}(\langle X\rangle, \langle Y\rangle)$ because any homomorphism $L(X,Y) \to M$ extends to $\operatorname{Hom}(\langle X\rangle, \langle Y\rangle)$ in this case. In general, this replacement is inadequate. For example, let X and Y be circles. Then the invariant "degree" $[X,Y] \to \mathbb{Z}$ is straight by Theorem 1.1 (or Corollary 6.8). At the same time, every homomorphism $F\colon \operatorname{Hom}(\langle X\rangle, \langle Y\rangle) \to \mathbb{Z}$ factors through the restriction homomorphism $\operatorname{Hom}(\langle X\rangle, \langle Y\rangle) \to \operatorname{Hom}(\langle T\rangle, \langle Y\rangle)$ for some finite set $T \subseteq X$ [2, § 94]. Thus F cannot give rise to a non-constant homotopy invariant.)

The main invariant $h\colon [X,Y] \to [SX,SY]$. For a space X, let SX be its singular chain complex. Let X and Y be spaces. Let $[SX,SY]$ be the group of homotopy classes of morphisms $SX \to SY$. There is a (non-naturally) split exact natural sequence

$$0 \longrightarrow \prod_{i\in\mathbb{Z}} \operatorname{Ext}(H_{i-1}X, H_iY) \longrightarrow [SX, SY] \longrightarrow \prod_{i\in\mathbb{Z}} \operatorname{Hom}(H_iX, H_iY) \longrightarrow 0$$

("the universal coefficient theorem", cf. [12, Theorem 5.5.3]). For $a \in C(X, Y)$, let $Sa\colon SX \to SY$ be the induced morphism and $[Sa] \in [SX, SY]$ be its homotopy class. The invariant $h\colon [X, Y] \to [SX, SY]$, $[a] \mapsto [Sa]$, is called *main*.

The main result. We call a space *valid* if it is homotopy equivalent to a CW-complex; we call it *finitary* if it is weakly homotopy equivalent to a compact CW-complex.

1.1. Theorem. *Let X be a finitary valid space, Y be a valid space, $h\colon [X, Y] \to [SX, SY]$ be the main invariant, M be an abelian group, and $f\colon [X, Y] \to M$ be an invariant. The invariant f is straight if and only if there exists a homomorphism $d\colon [SX, SY] \to M$ such that $f = d \circ h$.*

The theorem follows from Propositions 7.3 and 12.2. □

The theorem says that the main invariant is a "universal" straight invariant. A weaker and slightly complicated result is [7, Theorem II]. If M is divisible, then the sufficiency ("if") follows easily from an appropriate form of the Dold–Thom theorem (see § 7), and the necessity ("only if") follows from [7, Theorem II] (but any abelian group is a subgroup of a divisible one). The validity and finitarity hypotheses are essential, see §§ 13, 14.

In § 15, we consider K-straight invariants taking values in modules over a commutative ring K (by definitions, straight = $\mathbb{Z}$-straight).

§ 2. Notation

The question mark. The expression [?] denotes the map $a \mapsto [a]$ between sets indicated in the context. We similarly use $\langle ? \rangle$, etc. This notation is also used for functors.

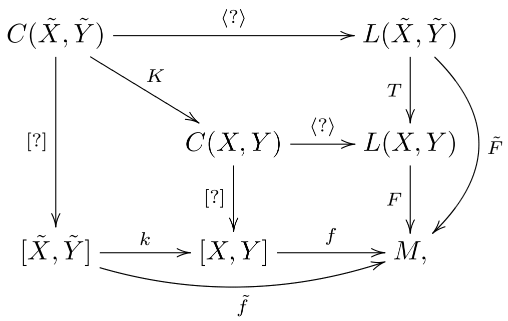
Sets and abelian groups. For a set X, let $c_X\colon X \to \langle X \rangle$ be the canonical map $x \mapsto {<}x{>}$. For $v \in \langle X \rangle$ and $x \in X$, let $v/x \in \mathbb{Z}$ be the coefficient of ${<}x{>}$ in v. For an abelian group T, a map $a\colon X \to T$ gives rise to the homomorphism $a^+\colon \langle X \rangle \to T$, ${<}x{>} \mapsto a(x)$. T^X is the group of maps $X \to T$.

Simplicial sets. For simplicial sets U and V, let $\operatorname{Si}(U, V)$ be the set of simplicial maps and $[U, V]$ be the set of their homotopy classes (two simplicial maps are homotopic if they are connected by a sequence of homotopies). The functor $\langle ? \rangle$ takes simplicial sets to simplicial abelian groups degreewise. There is the canonical simplicial map $c_U\colon U \to \langle U \rangle$. For a simplicial abelian group Z, a simplicial map $s\colon U \to Z$ gives rise to the simplicial homomorphism $s^+\colon \langle U \rangle \to Z$. For a simplicial set T, a simplicial map $s\colon U \to V$ induces the maps $s^T_\#\colon \operatorname{Si}(T, U) \to \operatorname{Si}(T, V)$, $s^\#_T\colon \operatorname{Si}(V, T) \to \operatorname{Si}(U, T)$, $s^T_*\colon [T, U] \to [T, V]$, and $s^*_T\colon [V, T] \to [U, T]$. This notation is also used in the topological case.

§ 3. Induced straight invariants

3.1. Lemma. *Let X, $\tilde{X}$, $\tilde{Y}$, and Y be spaces, $r\colon X \to \tilde{X}$ and $s\colon \tilde{Y} \to Y$ be continuous maps, M be an abelian group and $f\colon [X,Y] \to M$ be a straight invariant. Then the invariant $\tilde{f}\colon [\tilde{X},\tilde{Y}] \to M$, $\tilde{f}([\tilde{a}]) = f([s \circ \tilde{a} \circ r])$, $\tilde{a} \in C(\tilde{X},\tilde{Y})$, is straight.*

Proof. There is a homomorphism $F\colon L(X,Y) \to M$ such that $f([a]) = F(\langle a \rangle)$, $a \in C(X,Y)$. We have the commutative diagram

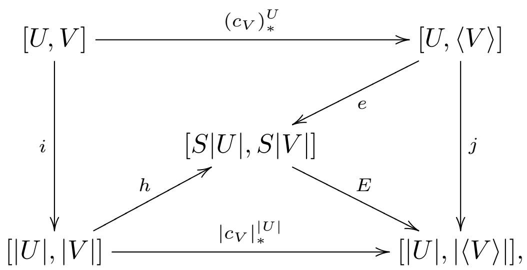


where the maps K and k and the homomorphism T are induced by the pair (r,s) (that is, $K(\tilde{a}) = s \circ \tilde{a} \circ r$, $k([\tilde{a}]) = [s \circ \tilde{a} \circ r]$, $T(\tilde{u}) = \langle s \rangle \circ \tilde{u} \circ \langle r \rangle$), and $\tilde{F} = F \circ T$. Thus $\tilde{f}$ is straight. $\square$

§ 4. The main invariant $h\colon [|U|,|V|] \to [S|U|, S|V|]$

The geometric realization $|Z|$ of a simplicial abelian group Z has a structure of an abelian group. $|Z|$ is a topological abelian group if Z is countable; in general, it is a group of the category of compactly generated Hausdorff spaces. For a simplicial set T, $C(|T|,|Z|)$ and $[|T|,|Z|]$ are abelian groups with respect to pointwise addition. Clearly, $\mathrm{Si}(T,Z)$ and $[T,Z]$ are also abelian groups.

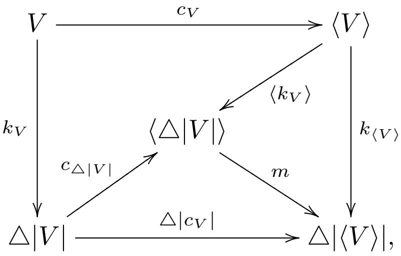
4.1 Lemma. *Let U and V be simplicial sets. Then there exists a commutative diagram*



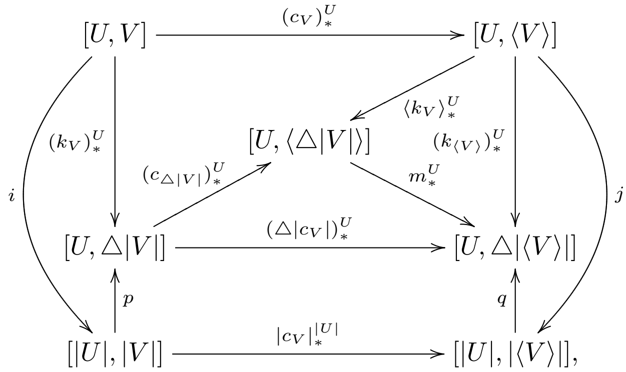
where $i\colon [s] \mapsto [|s|]$ (the map induced by the geometric realization map), j is similar, h is the main invariant, and e, E are some isomorphisms.

This is a version of the Dold–Thom theorem [3, § 4.K].

Proof. Let $\triangle$ be the singular functor. For a simplicial set T, let $k_T\colon T \to \triangle|T|$ be the canonical weak equivalence. If T is a simplicial abelian group, k_T is a simplicial homomorphism. We have the commutative diagram



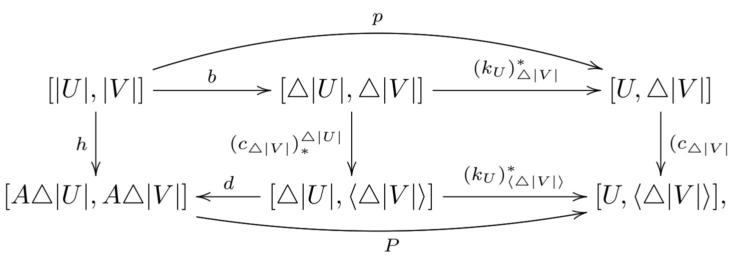
where $m = (\triangle|c_V|)^+$. $k_{\langle V\rangle}$, $\langle k_V\rangle$, and thus m are weak equivalences. Consider the commutative diagram



where the upper part is the result of applying the functor $[U, ?]$ to the previous diagram and p and q are the standard adjunction bijections for the functors $|?|$ and $\triangle$. $\langle k_V\rangle^U_*$, m^U_*, and q are isomorphisms.

We will find an isomorphism $P\colon [S|U|, S|V|] \to [U, \langle\triangle|V|\rangle]$ such that $P\circ h = (c_{\triangle|V|})^U_* \circ p$. Then it will be enough to set $e = P^{-1}\circ\langle k_V\rangle^U_*$ and $E = q^{-1}\circ m^U_*\circ P$.

For a simplicial set T, let AT be its chain complex, so that $(AT)_n = \langle T_n\rangle$, $n \geqslant 0$. Then $SX = A\triangle X$ for any space X. A simplicial map $s\colon T \to \langle W\rangle$ gives rise to the morphism $v\colon AT \to AW$, $v_n = s_n^+$, $n \geqslant 0$. This rule yields an isomorphism $d\colon [T, \langle W\rangle] \to [AT, AW]$ (the Dold–Kan correspondence). We set $T = \triangle|U|$ and $W = \triangle|V|$. Consider the commutative diagram



where the map b is given by the functor $\triangle$ and $P = (k_U)^*_{\langle\triangle|V|\rangle} \circ d^{-1}$. Since $(k_U)^*_{\langle\triangle|V|\rangle}$ is an isomorphism, P is an isomorphism too. $\square$

§ 5. Nöbeling–Bergman theory

By a *ring* we mean a (non-unital) commutative ring; *subring* is understood accordingly. The following facts follow from [5, Theorem 2 and its proof], cf. [2, § 97].

5.1. Lemma. *Let E be a torsion-free ring generated by idempotents. Then E is a free abelian group.* $\square$

An example: the ring $B(X)$ of bounded functions $X \to \mathbb{Z}$, where X is an arbitrary set.

5.2. Lemma. *Let E be a torsion-free ring and $F \subseteq E$ be a subring, both generated by idempotents. Then the abelian group E/F is free.* $\square$

For $F = 0$, this is Lemma 5.1.

§ 6. Maps to a space with addition

Let X be a space and T be a Hausdorff space.

For a set $V \subseteq T$, we introduce the homomorphism $s_V : L(X,T) \to \mathbb{Z}^X$, $s_V(u)(x) = I_V^+(u(\text{<}x\text{>}))$, $x \in X$, where $I_V : T \to \mathbb{Z}$ is the indicator function of the set V.

The subgroup $R \subseteq L(X,T)$. For $p \in X$, $q \in T$, let $R(p,q) \subseteq L(X,T)$ be the subgroup of homomorphisms u such that, for any sufficiently small (open) neighbourhood V of q, the function $s_V(u)$ is constant in some neighbourhood of p. Let $R \subseteq L(X,T)$ be the intersection of the subgroups $R(p,q)$, $p \in X$, $q \in T$.

6.1. Lemma. *For $a \in C(X,T)$, we have $\langle a\rangle \in R$.*

Proof. Take $p \in X$, $q \in T$. We show that $\langle a\rangle \in R(p,q)$. If $a(p) = q$, then, for any neighbourhood V of q, we take the neighbourhood $U = a^{-1}(V)$ of p and get $s_V(\langle a\rangle)|_U = 1$. Otherwise, choose disjoint neighbourhoods W of q and W_1 of $a(p)$. Consider the neighbourhood $U = a^{-1}(W_1)$ of p. For any $V \subseteq W$, we have $s_V(\langle a\rangle)|_U = 0$. $\square$

6.2. Lemma. *The abelian group $L(X,T)/R$ is free.*

Proof. Let O_T be the set of open sets in T. Consider the ring $E = B(X \times X \times O_T)$. For $p \in X$, $q \in T$, let $I(p,q) \subseteq E$ be the ideal of functions f such that, for any sufficiently small neighbourhood V of q, the function $X \to \mathbb{Z}$, $x \mapsto f(p,x,V)$, vanishes in some neighbourhood of p. Let $I \subseteq E$ be the intersection of the ideals $I(p,q)$, $p \in X$, $q \in T$. The ring E/I is torsion-free and generated by idempotents. By Lemma 5.1, E/I is a free abelian group. Consider the homomorphism $k\colon L(X,T) \to E$, $k(u)(p,x,V) = s_V(u)(x) - s_V(u)(p)$, $p, x \in X$, $V \in O_T$, $u \in L(X,T)$. We have $k^{-1}(I(p,q)) = R(p,q)$ and thus $k^{-1}(I) = R$. Therefore, k induces a monomorphism $L(X,T)/R \to E/I$. It follows that the abelian group $L(X,T)/R$ is free. □

The set Q and the homomorphisms $e(D,a)$. Let Q be the set of pairs (D,a), where $D \subseteq X$ is a closed set and $a \in C(D,T)$. For $(D,a) \in Q$, introduce the homomorphism $e(D,a) \in L(X,T)$,

$$e(D,a)(<x>) = \begin{cases} <a(x)> & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases}$$

$x \in X$.

6.3. Lemma. *Let $(D,a) \in Q$, $p \in X$, and $q \in T$. If $e(D,a) \notin R(p,q)$, then $p \in D$ and $a(p) = q$.*

Proof. Put $u = e(D,a)$. *The case $p \notin D$.* Consider the neighbourhood $U = X \setminus D$ of p. We have $s_V(u)|_U = 0$ for any $V \subseteq T$. Thus $u \in R(p,q)$. *The case $p \in D$, $a(p) \neq q$.* Choose disjoint neighbourhoods W of q and W_1 of $a(p)$. There is a neighbourhood U of p such that $a(D \cap U) \subseteq W_1$. We have $s_V(u)|_U = 0$ for any $V \subseteq W$. Thus $u \in R(p,q)$. □

The subgroup $K \subseteq L(X,T)$. Let $K \subseteq L(X,T)$ be the subgroup generated by $e(D,a)$, $(D,a) \in Q$.

6.4. Lemma. *The abelian group $L(X,T)/K$ is free.*

Proof. Consider the monomorphism $j\colon L(X,T) \to B(X \times T)$, $j(u)(x,t) = u(<x>)/t$. For $(D_i, a_i) \in Q$, $i = 1, 2$, we have $j(e(D_1,a_1))j(e(D_2,a_2)) = j(e(D,a))$, where $D = \{x \in D_1 \cap D_2 : a_1(x) = a_2(x)\}$ and $a = a_1|_D = a_2|_D$. In particular, $j(e(D,a))$, $(D,a) \in Q$, are idempotents. Therefore, $j(K)$ is a subring generated by idempotents. By Lemma 5.2, the abelian group $B(X \times T)/j(K)$ is free. Since j induces a monomorphism $L(X,T)/K \to B(X \times T)/j(K)$, the abelian group $L(X,T)/K$ is free. □

6.5. Lemma. *The abelian group $L(X,T)/(K \cap R)$ is free.*

Proof. The quotients in the chain $L(X,T) \supseteq K \supseteq K \cap R$ are free: $L(X,T)/K$ by Lemma 6.4, and $K/(K \cap R)$ as a subgroup of $L(X,T)/R$, which is free by Lemma 6.2. □

The homomorphism $G\colon L(X,T) \to T^X$. Let T have a structure of an abelian group such that, $(*)$ for any closed set $D \subseteq X$, the set $C(D,T)$ becomes an abelian group with respect to pointwise addition[1]. Consider the homomorphism $G\colon L(X,T) \to T^X$, $G(u)(x) = r(u(<x>))$, $x \in X$, $u \in L(X,T)$, where $r = \mathrm{id}^+\colon \langle T\rangle \to T$.

[1] The condition $(*)$ is satisfied if T is a topological abelian group or if $X = |U|$ and $T = |Z|$, where U is a simplicial set and Z is a simplicial abelian group.

6.6. Lemma. $G(K \cap R) \subseteq C(X,T)$.

Proof. Take $u \in K \cap R$. We show that $G(u) \in C(X,T)$. Since $u \in K$, we have

$$u = \sum_{i \in I} u_i, \qquad\qquad u_i = k_i e(D_i, a_i),$$

where I is a finite set, $k_i \in \mathbb{Z}$, and $(D_i, a_i) \in Q$. For $J \subseteq I$, put

$$u_J = \sum_{i \in J} u_i, \qquad\qquad D_J = \bigcap_{i \in J} D_i \subseteq X$$

(so $D_\varnothing = X$) and

$$b_J = \sum_{i \in J} k_i a_i|_{D_J} \in C(D_J, T), \qquad\qquad k_J = \sum_{i \in J} k_i.$$

Take $p \in X$. We verify that $G(u)$ is continuous at p. Put $N = \{i \in I : p \notin D_i\}$. For $q \in T$, put $I(q) = \{i \in I : p \in D_i,\ a_i(p) = q\}$. We have

$$u = u_N + \sum_{q \in T} u_{I(q)}$$

(almost all summands are zero). Clearly, $G(u_N)$ vanishes in some neighbourhood of p. Take $q \in T$. It suffices to show that $G(u_{I(q)})$ is continuous at p. Put $t_0 = G(u_{I(q)}) \in T$. We have $t_0 = k_{I(q)} q$. Let W be a neighbourhood of t_0. We seek a neighbourhood U of p such that $G(u_{I(q)})(U) \subseteq W$.

Put $E = \{J \subseteq I(q) : k_J = k_{I(q)}\}$. For $J \in E$, we have $p \in D_J$ and $b_J(p) = t_0$. There is a neighbourhood U_1 of p such that $b_J(D_J \cap U_1) \subseteq W$ for all $J \in E$.

By Lemma 6.3, $u_i \in R(p,q)$ for $i \in I \setminus I(q)$. Since $u \in R(p,q)$, we have $u_{I(q)} \in R(p,q)$. Therefore, there is a neighbourhood $V \subseteq T$ of q such that the function $s_V(u_{I(q)})$ is constant in some neighbourhood U_2 of p.

There is a neighbourhood U_3 of p such that $a_i(D_i \cap U_3) \subseteq V$ for all $i \in I(q)$. For $x \in X$, put $J(x) = \{i \in I(q) : x \in D_i\}$. For $x \in U_2 \cap U_3$, we have $k_{J(x)} = s_V(u_{I(q)})(x) = s_V(u_{I(q)})(p) = k_{I(q)}$, i. e. $J(x) \in E$.

Set $U = U_1 \cap U_2 \cap U_3$. Take $x \in U$. We have $G(u_{I(q)})(x) = b_{J(x)}(x) \in W$ because $J(x) \in E$. □

6.7. Lemma. *There exists a homomorphism $g\colon L(X,T) \to C(X,T)$ such that $g(\langle a \rangle) = a$ for all $a \in C(X,T)$.*

Proof. We have $G(\langle a \rangle) = a$ for all $a \in T^X$. Since $G(K \cap R) \subseteq C(X,T)$ (by Lemma 6.6) and the abelian group $L(X,T)/(K \cap R)$ is free (by Lemma 6.5), there is a homomorphism $g\colon L(X,T) \to C(X,T)$ such that $g(u) = G(u)$ for $u \in K \cap R$. For $a \in C(X,T)$, we have $\langle a \rangle \in K$ (because $\langle a \rangle = e(X,a)$) and $\langle a \rangle \in R$ (by Lemma 6.1). We get $g(\langle a \rangle) = G(\langle a \rangle) = a$. □

6.8. Corollary. *Suppose that* ($*$) $[X,T]$ *is an abelian group with respect to pointwise addition*[2]. *Then the invariant* $\mathrm{id}\colon [X,T] \to [X,T]$ *is straight.*

Proof. By Lemma 6.7, there is a homomorphism $g\colon L(X,T) \to C(X,T)$ such that $g(\langle a\rangle) = a$ for all $a \in C(X,T)$. Consider the homomorphism $F\colon L(X,T) \to [X,T]$, $u \mapsto [g(u)]$. For $a \in C(X,T)$, we have $[a] = [g(\langle a\rangle)] = F(\langle a\rangle)$. □

§ 7. Sufficiency in Theorem 1.1

The proof of sufficiency in Theorem 1.1 relies on Corollary 6.8. If the group M is divisible, it is easy to use Lemma 7.1 instead (then the stuff of §§ 5, 6 is needless).

7.1. Lemma (cf. [10, Lemma 1.2]). *Let* X *and* T *be spaces and* T *have a structure of an abelian group such that* ($*$) *the sets* $C(X,T)$ *and* $[X,T]$ *become abelian groups with respect to pointwise addition*[3]. *Let* M *be a divisible abelian group and* $f\colon [X,T] \to M$ *be a homomorphism. Then* f *is a straight invariant.*

Proof. Consider the homomorphism $G\colon L(X,T) \to T^X$, $G(u)(x) = r(u(<x>))$, $x \in X$, $u \in L(X,T)$, where $r = \mathrm{id}^+\colon \langle T\rangle \to T$. Let $D \subseteq L(X,T)$ be the subgroup generated by the homomorphisms $\langle a\rangle$, $a \in C(X,T)$. Clearly, $G(\langle a\rangle) = a$ for $a \in C(X,T)$. Therefore, $G(D) \subseteq C(X,T)$. Consider the homomorphism $F_0\colon D \to M$, $u \mapsto f([G(u)])$. Since M is divisible, there is a homomorphism $F\colon L(X,T) \to M$ such that $F|_D = F_0$. For $a \in C(X,T)$, we have $f([a]) = f([G(\langle a\rangle)]) = F_0(\langle a\rangle) = F(\langle a\rangle)$. □

7.2. Claim. *Let* U *and* V *be simplicial sets. Then the main invariant* $h\colon [|U|,|V|] \to [S|U|,S|V|]$ *is straight.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} [|U|,|V|] & \xrightarrow{h} & [S|U|,S|V|] \\ & \searrow^{|c_V|_*^{|U|}} & \downarrow E \\ & & [|U|,|\langle V\rangle|], \end{array}$$

where E is the isomorphism from Lemma 4.1. By Corollary 6.8, the invariant $\mathrm{id}\colon [|U|,|\langle V\rangle|] \to [|U|,|\langle V\rangle|]$ is straight. Therefore, by Lemma 3.1, the invariant $|c_V|_*^{|U|}$ is straight. Since E is an isomorphism, h is also straight. □

7.3. Proposition. *Let* X *be a space and* Y *be a valid space. Then the main invariant* $h\colon [X,Y] \to [SX,SY]$ *is straight.*

[2] See footnote 1.
[3] See footnote 1.

Proof. There are homology equivalences $r\colon |U| \to X$ and $s\colon Y \to |V|$, where U and V are simplicial sets. Consider the commutative diagram

$$\begin{array}{ccc} [X,Y] & \xrightarrow{\ h\ } & [SX,SY] \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle l} \\ [|U|,|V|] & \xrightarrow{\ \tilde{h}\ } & [S|U|,S|V|], \end{array}$$

where $\tilde{h}$ is the main invariant and the map k and the isomorphism l are induced by the pair (r,s). By Claim 7.2, $\tilde{h}$ is straight. By Lemma 3.1, the invariant $\tilde{h}\circ k$ is straight. Since $h = l^{-1}\circ\tilde{h}\circ k$, h is also straight. $\square$

§ 8. The superposition $Z\colon \langle \mathrm{Si}(U,V)\rangle_0 \to \mathrm{Si}(U,\langle V\rangle_0)$

For a set X, let $\langle X\rangle_0 \subseteq \langle X\rangle$ be the kernel of the homomorphism $\langle X\rangle \to \mathbb{Z}$, $<x> \mapsto 1$. We apply the functor $\langle ?\rangle_0$ to simplicial sets degreewise.

Let U and V be simplicial sets. The canonical simplicial map $c = c_V\colon V \to \langle V\rangle$ gives rise to the map $c^U_\#\colon \mathrm{Si}(U,V) \to \mathrm{Si}(U,\langle V\rangle)$ and the homomorphism $(c^U_\#)^+\colon \langle \mathrm{Si}(U,V)\rangle \to \mathrm{Si}(U,\langle V\rangle)$. We have the commutative diagram

$$\begin{array}{ccc} \langle \mathrm{Si}(U,V)\rangle_0 & \xrightarrow{\ Z\ } & \mathrm{Si}(U,\langle V\rangle_0) \\ \downarrow & & \downarrow \\ \langle \mathrm{Si}(U,V)\rangle & \xrightarrow{\ (c^U_\#)^+\ } & \mathrm{Si}(U,\langle V\rangle), \end{array}$$

where the vertical arrows are induced by the canonical inclusion $\langle ?\rangle_0 \to \langle ?\rangle$ and Z is a new homomorphism called the *superposition.*

§ 9. Surjectivity of the superposition

Our aim here is Lemma 9.1. We follow [10, §§ 12, 13].

Extension of simplicial maps. For $n \geqslant 0$, let Δ^n be the combinatorial standard n-simplex (a simplicial set) and $\partial\Delta^n$ be its boundary.

Let W be a contractible fibrant simplicial set. For each $n \geqslant 0$, choose a map $e_n\colon \mathrm{Si}(\partial\Delta^n,W) \to \mathrm{Si}(\Delta^n,W)$ such that $e_n(q)|_{\partial\Delta^n} = q$ for any $q \in \mathrm{Si}(\partial\Delta^n,W)$.

Let U be a simplicial set. For each simplicial subset $A \subseteq U$, we introduce the map $E_A\colon \mathrm{Si}(A,W) \to \mathrm{Si}(U,W)$, $x \mapsto t$, where $t|_A = x$ and $t\circ p = e_n(t\circ p|_{\partial\Delta^n})$ for the characteristic map $p\colon \Delta^n \to U$ of each non-degenerate simplex outside A. Clearly,

(1) $E_A(x)|_A = x$;

(2) $E_A(x)|_B = E_{A\cap B}(x|_{A\cap B})|_B$,

where $A, B \subseteq U$ are simplicial subsets and $x \in \mathrm{Si}(A, W)$.

The ring $\langle Q\rangle$ and its identity I. Let Q be the system of simplicial subsets of U consisting of all subsets isomorphic to Δ^n, $n \geqslant 0$, and the empty subset. Suppose that the simplicial set U is *polyhedral*, i. e. Q is its cover closed under intersection, and *compact*, i. e. generated by a finite number of simplices. Q is finite.

We introduce multiplication in $\langle Q\rangle$ by putting $<A><B> = <A\cap B>$ for $A, B \in Q$. The ring $\langle Q\rangle$ has an identity I. Indeed, the homomorphism $e\colon \langle Q\rangle \to \mathbb{Z}^Q$,

$$e(<A>)(B) = \begin{cases} 1 & \text{if } A \supseteq B, \\ 0 & \text{otherwise,} \end{cases}$$

$A, B \in Q$, is an isomorphism ("an upper unitriangular matrix") preserving multiplication. Therefore, $I = e^{-1}(1)$ is an identity.

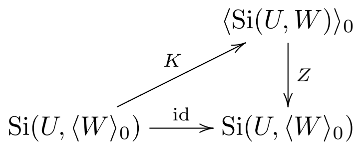
The homomorphism K: $\mathbf{Si}(U, \langle W\rangle_0) \to \langle \mathbf{Si}(U, W)\rangle_0$. For a simplicial set T, let $Z_T\colon \langle \mathrm{Si}(T, W)\rangle_0 \to \mathrm{Si}(T, \langle W\rangle_0)$ be the superposition. For simplicial sets $T \supseteq A$, let $r^T_A\colon \mathrm{Si}(T, W) \to \mathrm{Si}(A, W)$ and $s^T_A\colon \mathrm{Si}(T, \langle W\rangle_0) \to \mathrm{Si}(A, \langle W\rangle_0)$ be the restriction maps. s^T_A is a homomorphism. If $T = U$, we omit the corresponding sub/superscript in this notation.

Note that Z_A is an isomorphism for $A \in Q$. Consider the map $k\colon Q \to \mathrm{Hom}(\mathrm{Si}(U, \langle W\rangle_0), \langle \mathrm{Si}(U, W)\rangle_0)$, $A \mapsto \langle E_A\rangle_0 \circ Z_A^{-1} \circ s_A$:

$$k(A)\colon \mathrm{Si}(U, \langle W\rangle_0) \xrightarrow{s_A} \mathrm{Si}(A, \langle W\rangle_0) \xrightarrow{Z_A^{-1}} \langle \mathrm{Si}(A, W)\rangle_0 \xrightarrow{\langle E_A\rangle_0} \langle \mathrm{Si}(U, W)\rangle_0.$$

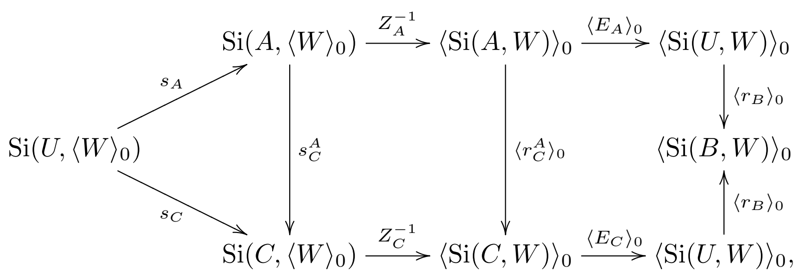
Put $K = k^+(I)$.

9.1. Lemma. *The diagram*



is commutative.

Proof. Take $A, B \in Q$. We have the commutative diagram



where $C = A \cap B$ (commutativity of the "pentagon" follows from the property (2) of the family E). Therefore, $\langle r_B\rangle_0 \circ k(A) = \langle r_B\rangle_0 \circ k(A \cap B)$. Therefore, $\langle r_B\rangle_0 \circ k^+(X) = \langle r_B\rangle_0 \circ k^+(X{<}B{>})$ for $X \in \langle Q\rangle$. We have $\langle r_B\rangle_0 \circ K = \langle r_B\rangle_0 \circ k^+(I) = \langle r_B\rangle_0 \circ k^+(I{<}B{>}) = \langle r_B\rangle_0 \circ k^+({<}B{>}) = \langle r_B\rangle_0 \circ k(B) = \langle r_B\rangle_0 \circ \langle E_B\rangle_0 \circ Z_B^{-1} \circ s_B = Z_B^{-1} \circ s_B$, because $r_B \circ E_B = \mathrm{id}$ by property (1) of the family E. We get $s_B \circ Z \circ K = Z_B \circ \langle r_B\rangle_0 \circ K = s_B$. Since B is arbitrary, $Z \circ K = \mathrm{id}$. □

§ 10. A cocartesian square

Let U be a compact polyhedral simplicial set and V be a fibrant simplicial set. The canonical simplicial map $c = c_V \colon V \to \langle V\rangle$ induces the maps $c^U_\# \colon \mathrm{Si}(U,V) \to \mathrm{Si}(U,\langle V\rangle)$ and $c^U_* \colon [U,V] \to [U,\langle V\rangle]$. Consider the commutative square of abelian groups and homomorphisms

$$\begin{array}{ccc}
\langle \mathrm{Si}(U,V)\rangle & \xrightarrow{(c^U_\#)^+} & \mathrm{Si}(U,\langle V\rangle) \\
{\scriptstyle \langle p\rangle}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
\langle [U,V]\rangle & \xrightarrow{(c^U_*)^+} & [U,\langle V\rangle],
\end{array}$$

where $p = [?] \colon \mathrm{Si}(U,V) \to [U,V]$ and $q = [?]$ (the projections).

10.1. Lemma. *This square is cocartesian.*

Proof. Since $\langle p\rangle$ and q are epimorphisms, it suffices to show that $\operatorname{Ker} q = (c^U_\#)^+(\operatorname{Ker}\langle p\rangle)$.

Suppose we have a decomposition

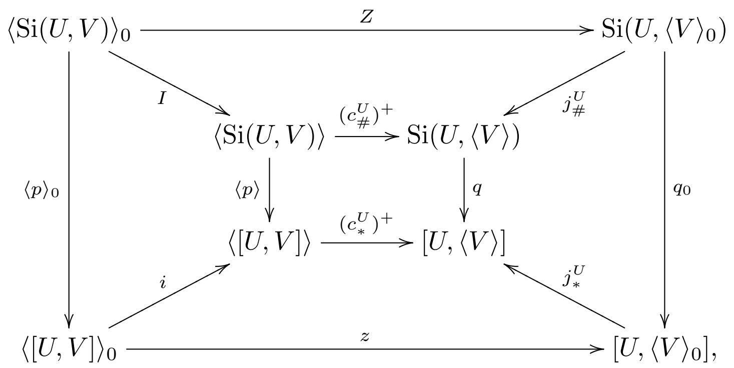
$$V = \coprod_{i \in I} V_i.$$

Consider the commutative diagram

$$\begin{array}{ccccc}
\bigoplus_{i\in I}\langle \mathrm{Si}(U,V_i)\rangle & & \xrightarrow{\quad\bigoplus_{i\in I}((c_i)^U_\#)^+\quad} & & \bigoplus_{i\in I}\mathrm{Si}(U,\langle V_i\rangle) \\
\big\downarrow & \searrow & & \swarrow{\scriptstyle E} & \big\downarrow \\
\big\downarrow & \langle \mathrm{Si}(U,V)\rangle & \xrightarrow{(c^U_\#)^+} & \mathrm{Si}(U,\langle V\rangle) & \big\downarrow \\
{\scriptstyle \bigoplus_{i\in I}\langle p_i\rangle}\big\downarrow & {\scriptstyle \langle p\rangle}\big\downarrow & & \big\downarrow{\scriptstyle q} & \big\downarrow{\scriptstyle \bigoplus_{i\in I} q_i} \\
\big\downarrow & \langle [U,V]\rangle & \xrightarrow{(c^U_*)^+} & [U,\langle V\rangle] & \big\downarrow \\
\big\downarrow & \nearrow & & \nwarrow{\scriptstyle e} & \big\downarrow \\
\bigoplus_{i\in I}\langle [U,V_i]\rangle & & \xrightarrow{\quad\bigoplus_{i\in I}((c_i)^U_*)^+\quad} & & \bigoplus_{i\in I}[U,\langle V_i\rangle],
\end{array}$$

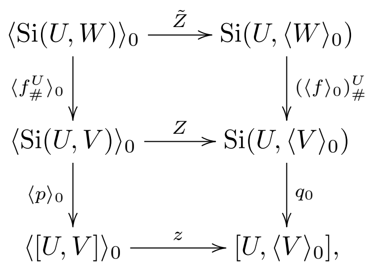
where c_i, p_i, and q_i are similar to c, p, and q (respectively) and the slanting arrows are induced by the inclusions $V_i \to V$. Since U is compact, E and e are isomorphisms. Therefore, is suffices to show that $\operatorname{Ker} q_i = ((c_i)^U_\#)^+(\operatorname{Ker}\langle p_i\rangle)$ for each $i \in I$. This reduction allows us to assume that V is 0-connected.

Consider the commutative diagram



where $q_0 = [?]$ (the projection), Z is the superposition, z is the homomorphism such that the outer square is commutative, I and i are the inclusion homomorphisms, and $j\colon \langle V\rangle_0 \to \langle V\rangle$ is the inclusion simplicial homomorphism. Clearly, $\operatorname{Ker} q = j^U_\#(\operatorname{Ker} q_0)$. Therefore, it suffices to show that $\operatorname{Ker} q_0 = Z(\operatorname{Ker}\langle p\rangle_0)$.

Since V is fibrant and 0-connected, there is a surjective simplicial map $f\colon W \to V$, where W is a contractible fibrant simplicial set. Consider the commutative diagram



where the map $f^U_\#\colon \operatorname{Si}(U, W) \to \operatorname{Si}(U, V)$ and the simplicial homomorphism $\langle f\rangle_0\colon \langle W\rangle_0 \to \langle V\rangle_0$ are induced by f and $\tilde{Z}$ is the superposition. Since $\langle f\rangle_0$ is surjective, it is a fibration. Therefore, $\operatorname{Ker} q_0 \subseteq \operatorname{Im}(\langle f\rangle_0)^U_\#$. By Lemma 9.1, $\tilde{Z}$ is surjective. Since W is contractible, $\operatorname{Im}\langle f^U_\#\rangle_0 \subseteq \operatorname{Ker}\langle p\rangle_0$. Therefore, $\operatorname{Ker} q_0 \subseteq Z(\operatorname{Ker}\langle p\rangle_0)$. The reverse inclusion is obvious. $\square$

§ 11. The homomorphism $P : \operatorname{Si}(U, \langle V\rangle) \to L(|U|, |V|)$

For $n \geqslant 0$, let $\mathbf{\Delta}^n$ be the geometric standard n-simplex and $\mathring{\mathbf{\Delta}}^n$ be its interior. For a simplicial set U and a point $z \in \mathbf{\Delta}^n$, there is a canonical map $z_U\colon U_n \to |U|$. The map $\mathbf{\Delta}^n \times U_n \to |U|$, $(z, u) \mapsto z_U(u)$, is the canonical pairing of geometric realization.

Let U and V be simplicial sets. We define a homomorphism $\tilde{P} : \mathrm{Si}(U, \langle V\rangle) \to \mathrm{Hom}(\langle|U|\rangle, \langle|V|\rangle)$. For $t \in \mathrm{Si}(U, \langle V\rangle)$ and $x \in |U|$, $x = z_U(u)$, where $z \in \mathbf{\Delta}^n$ and $u \in U_n$ $(n \geqslant 0)$, put $\tilde{P}(t)(<\!x\!>) = \langle z_V\rangle(t_n(u))$:

$$u \in U_n \xrightarrow{t_n} \langle V\rangle_n = \langle V_n\rangle \xrightarrow{\langle z_V\rangle} \langle|V|\rangle.$$

$\tilde{P}$ is well-defined.

Suppose that U is compact.

11.1. Lemma. $\operatorname{Im} \tilde{P} \subseteq L(|U|, |V|)$.

Proof. Let $U_n^\times \subseteq U_n$ $(n \geqslant 0)$ be the set of non-degenerate simplices. For $u \in U_n^\times$ $(n \geqslant 0)$, we define a homomorphism $I_u \colon \langle V_n\rangle \to L(|U|, |V|)$. For $v \in V_n$, $x \in |U|$, put

$$I_u(<\!v\!>)(<\!x\!>) = \begin{cases} <\!z_V(v)\!> & \text{if } x = z_U(u) \text{ for } z \in \mathring{\mathbf{\Delta}}^n, \\ 0 & \text{otherwise.} \end{cases}$$

This equality is preserved if we replace $<\!v\!>$ by $w \in \langle V_n\rangle$ and $<\!z_V(v)\!>$ by $\langle z_V\rangle(w)$. It suffices to show that

$$\tilde{P}(t) = \sum_{n \geqslant 0,\, u \in U_n^\times} I_u(t_n(u)), \qquad t \in \mathrm{Si}(U, \langle V\rangle).$$

Evaluating each side at $<\!x\!>$, $x = z_U(u)$, where $z \in \mathring{\mathbf{\Delta}}^n$ and $u \in U_n^\times$ $(n \geqslant 0)$, we get $\langle z_V\rangle(t_n(u))$. □

Lemma 11.1 allows us to introduce the homomorphism $P : \mathrm{Si}(U, \langle V\rangle) \to L(|U|, [V|)$, $P(t) = \tilde{P}(t)$.

11.2. Lemma. *The diagram*

$$\begin{array}{ccc} \mathrm{Si}(U, V) & \xrightarrow{c^U_\#} & \mathrm{Si}(U, \langle V\rangle) \\ \big\downarrow{\scriptstyle |?|} & & \big\downarrow{\scriptstyle P} \\ C(|U|, |V|) & \xrightarrow{\langle ?\rangle} & L(|U|, |V|), \end{array}$$

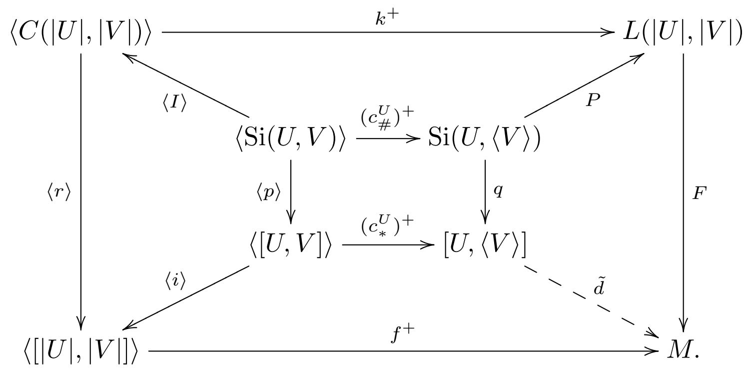
where $c = c_V \colon V \to \langle V\rangle$ is the canonical simplicial map, is commutative.

Proof. For $s \in \mathrm{Si}(U, V)$ and $x \in |U|$, $x = z_U(u)$, where $z \in \mathbf{\Delta}^n$ and $u \in U_n$ $(n \geqslant 0)$, we have $(P \circ c^U_\#)(s)(<\!x\!>) = P(c \circ s)(<\!x\!>) = \langle z_V\rangle((c \circ s)_n(u)) = <\!z_V(s_n(u))\!> = <\!|s|(z_U(u))\!> = <\!|s|(x)\!> = \langle|s|\rangle(<\!x\!>)$. □

§ 12. Necessity in Theorem 1.1

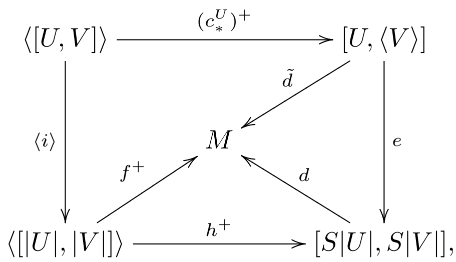
12.1. Claim. *Let U be a compact polyhedral simplicial set, V be a fibrant simplicial set, $h \colon [|U|, |V|] \to [S|U|, S|V|]$ be the main invariant, M be an abelian group, and $f \colon [|U|, |V|] \to M$ be a straight invariant. Then there exists a homomorphism $d \colon [S|U|, S|V|] \to M$ such that $f = d \circ h$.*

Proof. Since f is straight, there is a homomorphism $F\colon L(|U|,|V|) \to M$ such that $f([a]) = F(\langle a\rangle)$ for $a \in C(|U|,|V|)$. Consider the diagram of abelian groups and homomorphisms



Here the inner square is as in § 10, $r = [?]\colon C(|U|,|V|) \to [|U|,|V|]$ (the projection), $k = \langle ?\rangle\colon C(|U|,|V|) \to L(|U|,|V|)$, $I = |?|\colon \mathrm{Si}(U,V) \to C(|U|,|V|)$ (the geometric realization map), $i\colon [U,V] \to [|U|,|V|]$, $[s] \mapsto [|s|]$, and P is as in § 11. By Lemma 11.2, the upper trapezium is commutative. The solid arrows are defined and form a commutative subdiagram. Since the inner square is cocartesian by Lemma 10.1, the dashed arrow $\tilde d$ is well-defined by the condition of commutativity of the diagram.

Consider the diagram

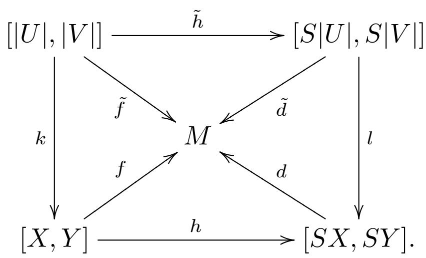


where e is the isomorphism from Lemma 4.1 and $d = \tilde d \circ e^{-1}$. The square is commutative by Lemma 4.1. We have $\tilde d \circ (c^U_*)^+ = f^+ \circ \langle i\rangle$. Since V is fibrant, i is a bijection, and thus $\langle i\rangle$ is an isomorphism. We get $f^+ = d \circ h^+$ (so the diagram is commutative). Therefore, $f = d \circ h$. $\square$

12.2. Proposition. *Let X be finitary valid space, Y be a space, $h\colon [X,Y] \to [SX,SY]$ be the main invariant, M be an abelian group, and $f\colon [X,Y] \to M$ be a straight invariant. Then there exists a homomorphism $d\colon [SX,SY] \to M$ such that $f = d \circ h$.*

Proof. There are a homotopy equivalence $r\colon X \to |U|$ and a weak homotopy equivalence $s\colon |V| \to Y$, where U is a compact polyhedral simplicial set and V

is a fibrant simplicial set. We construct the commutative diagram



Here the bijection k and the isomorphism l are induced by the pair (r,s) and $\tilde{h}$ is the main invariant. The square is commutative. By Lemma 3.1, the invariant $\tilde{f}=f\circ k$ is straight. By Claim 12.1, there is a homomorphism $\tilde{d}$ such that $\tilde{f}=\tilde{d}\circ\tilde{h}$. Set $d=\tilde{d}\circ l^{-1}$. Since k is a bijection, we get $f=d\circ h$ (so the diagram is commutative). □

§ 13. Three counterexamples

The Hawaiian ear-ring. Let us show that the hypothesis of validity of Y in Theorem 1.1 and Proposition 7.3 is essential. Let X be the one-point compactification of the ray $\mathbb{R}_+=(0,\infty)$ (a circle) and Y be that of the space $\mathbb{R}_+\setminus\mathbb{N}$ (the Hawaiian ear-ring [3, Example 1.25]). We define a map $m\in C(X,Y)$ by putting

$$m(x)=\left[\frac{x+1}{2}\right]+(-1)^{[x/2]}\{-x\}$$

for $x\in\mathbb{R}_+\setminus\mathbb{N}$. Here $[t]$ and $\{t\}$ are the integral and the fractional (respectively) parts of a number $t\in\mathbb{R}$. The element of $\pi_1(Y,\infty)$ represented by the loop m is the (reasonably understood) infinite product of commutators

$$\prod_{p=0}^{\infty}[u_{2p},u_{2p+1}], \qquad (*)$$

where u_q is the element realized by the closure of the interval $(q,q+1)$. Let $e\in H_1(X)$ be the standard generator. As in [4, p. 76], we get that the element $m_*(e)\in H_1(Y)$ has infinite order. Therefore, there is a homomorphism $k\colon H_1(Y)\to\mathbb{Q}$ such that $k(m_*(e))=1$. We define a homomorphism $d\colon [SX,SY]\to\mathbb{Q}$ by putting $d([v])=k(v_*(e))$ for a morphism $v\colon SX\to SY$. Let $h\colon [X,Y]\to[SX,SY]$ be the main invariant. We show that *the invariants* $d\circ h$ *and thus* h *are not straight.*

For $y\in Y$ and $i=0,1$, put $y_{(i)}\in Y$ equal to ∞ if $i=1$ and to y otherwise. For $i,j=0,1$, we define a map $r_{ij}\in C(Y,Y)$. For $y\in\mathbb{R}_+\setminus\mathbb{N}$, we put $r_{ij}(y)$ equal to $y_{(j)}$ if $[y]$ is odd and to $y_{(i)}$ otherwise. For elements z_{ij}, $i,j=0,1$, of an abelian group, put $\vee_{ij}z_{ij}=z_{00}-z_{10}-z_{01}+z_{11}$. Clearly, $\vee_{ij}\langle r_{ij}\rangle=0$ in $L(Y,Y)$. Put $a_{ij}=r_{ij}\circ m\in C(X,Y)$. We get $\vee_{ij}\langle a_{ij}\rangle=0$ in $L(X,Y)$. Therefore, $\vee_{ij}f([a_{ij}])=0$ for any straight invariant f. We show that this is false for the invariant $d \circ h$. We have $a_{00} = m$; the map a_{11} is constant. It is easy to see that the maps a_{10} and a_{01} are null-homotopic (this "follows formally" from the presentation $(*)$ and the equalities $r_{10\,*}(u_{2p}) = r_{01\,*}(u_{2p+1}) = 1$). We get $\vee_{ij}(d \circ h)([a_{ij}]) = (d \circ h)([m]) = k(m_*(e)) = 1$. $\square$

Using [1, Theorem 2], one can make the spaces X and Y simply-connected in this example.

The Warsaw circle. Let us show that the hypothesis of validity of X in Theorem 1.1 and Proposition 12.2 is essential. Let X be the Warsaw circle [3, Exercise 7 in § 1.3] and Y be the unit circle in $\mathbb{C}$. Y is a topological abelian group. The group $[X, Y]$ is non-zero by [3, Exercise 7 in § 1.3, Proposition 1.30] and torsion-free by [6, Theorem 1 in § 56-III]. Therefore, there is a non-zero homomorphism $f\colon [X, Y] \to \mathbb{Q}$. By Lemma 7.1, f is a straight invariant. Since X is weakly homotopy equivalent to a point [3, Exercise 10 in § 4.1] and Y is 0-connected, the main invariant $h\colon [X, Y] \to [SX, SY]$ is constant. Therefore *there exists no homomorphism* $d\colon [SX, SY] \to \mathbb{Q}$ *such that* $f = d \circ h$. $\square$

An infinite discrete space. Let us show that the hypothesis of finitarity of X in Theorem 1.1 and Proposition 12.2 is essential (see also § 14).

Note that, for an infinite set X, the subgroup $B(X) \subseteq \mathbb{Z}^X$ is not a direct summand because the group $\mathbb{Z}^X$ is reduced and the group $\mathbb{Z}^X/B(X)$ is divisible and non-zero.

Let X and Y be discrete spaces, X infinite and $Y = \{y_0, y_1\}$. Introduce the function $k\colon Y \to \mathbb{Z}$, $y_i \mapsto i$, $i = 0, 1$. Consider the invariant $f\colon [X, Y] \to B(X)$, $[a] \mapsto k \circ a$, $a \in C(X, Y)$.

The invariant f is straight because, for the homomorphism $F\colon L(X, Y) \to B(X)$, $F(u)(x) = k^+(u({<}x{>}))$, $x \in X$, $u \in L(X, Y)$, we have $f([a]) = F(\langle a \rangle)$, $a \in C(X, Y)$.

Let $h\colon [X, Y] \to [SX, SY]$ be the main invariant. We show that *there exists no homomorphism* $d\colon [SX, SY] \to B(X)$ *such that* $f = d \circ h$. Assume that there is such a d.

Consider the homomorphism $l\colon \mathbb{Z}^X \to \mathrm{Hom}(\langle X \rangle, \langle Y \rangle)$, $l(v)({<}x{>}) = v(x)({<}y_1{>} - {<}y_0{>})$, $x \in X$, $v \in \mathbb{Z}^X$. We have $l(f([a])) = \langle a \rangle - \langle a_0 \rangle$, $a \in C(X, Y)$, where $a_0\colon X \to Y$, $x \mapsto y_0$. Clearly, there is an isomorphism $e\colon \mathrm{Hom}(\langle X \rangle, \langle Y \rangle) \to [SX, SY]$ such that $e(\langle a \rangle) = h([a])$, $a \in C(X, Y)$. Consider the composition

$$r\colon \mathbb{Z}^X \xrightarrow{l} \mathrm{Hom}(\langle X \rangle, \langle Y \rangle) \xrightarrow{e} [SX, SY] \xrightarrow{d} B(X).$$

For $a \in C(X, Y)$, we have $r(f([a])) = (d \circ e \circ l \circ f)([a]) = d(e(\langle a \rangle - \langle a_0 \rangle)) = d(h([a]) - h([a_0])) = f([a]) - f([a_0]) = f([a])$. Since the elements $f([a])$, $a \in C(X, Y)$, generate $B(X)$, we get $r|_{B(X)} = \mathrm{id}$, which is impossible. $\square$

§ 14. Invariants of maps $\mathbb{R}P^\infty \to \mathbb{R}P^\infty$

Here we show that the hypothesis of finitarity of X in Theorem 1.1 and Proposition 12.2 is essential even if M is divisible. (Possibly, if M is divisible and/or Y is (simply-)connected, the hypothesis of finitarity of X can be replaced by the weaker one that X is weakly homotopy equivalent to a finite-dimensional CW-complex.)

Let X and Y be spaces. A set $E \subseteq X$ is called Y*-representative* if any maps $a, b \in C(X, Y)$ equal on E are homotopic. X is called Y*-unitary* if any finite cover of X contains a Y-representative set.

14.1. Lemma. *Let M be a divisible group. If X is Y-unitary, then any invariant $f\colon [X, Y] \to M$ is straight.*

Proof. Introduce the maps $r = [?]\colon C(X, Y) \to [X, Y]$ (the projection) and $k = \langle ? \rangle\colon C(X, Y) \to L(X, Y)$. We seek a homomorphism F giving the commutative diagram

$$\begin{array}{ccc} \langle C(X,Y)\rangle & \xrightarrow{k^+} & L(X,Y) \\ {\scriptstyle\langle r\rangle}\downarrow & & \downarrow{\scriptstyle F} \\ \langle [X,Y]\rangle & \xrightarrow{f^+} & M. \end{array}$$

Since M is divisible, it suffices to show that $\operatorname{Ker} k^+ \subseteq \operatorname{Ker}\langle r\rangle$. Take an element $w \in \operatorname{Ker} k^+$. We show that $w \in \operatorname{Ker}\langle r\rangle$. There are a finite set I, a map $l\colon I \to C(X, Y)$, and an element $v \in \langle I\rangle$ such that $\langle l\rangle(v) = w$. Put $a_i = l(i)$, $i \in I$. For an equivalence d on I, let $p_d\colon I \to I/d$ be the projection. Let N be the set of equivalences d on I such that $\langle p_d\rangle(v) = 0$ in $\langle I/d\rangle$.

Take $x \in X$. Consider the equivalence $d(x) = \{(i, j) : a_i(x) = a_j(x)\}$ on I. We show that $d(x) \in N$. We have the commutative diagrams

$$\begin{array}{ccc} I & \xrightarrow{l} & C(X,Y) \\ {\scriptstyle p_{d(x)}}\downarrow & & \downarrow{\scriptstyle e_x} \\ I/d(x) & \xrightarrow{l_x} & Y, \end{array} \qquad \begin{array}{ccc} \langle C(X,Y)\rangle & \xrightarrow{k^+} & L(X,Y) \\ {\scriptstyle\langle e_x\rangle}\downarrow & \swarrow{\scriptstyle h_x} & \\ \langle Y\rangle, & & \end{array}$$

where the map l_x is defined by the condition of commutativity of the diagram, e_x is the map of evaluation at x, and h_x is the homomorphism of evaluation at <x>. We get $\langle l_x\rangle(\langle p_{d(x)}\rangle(v)) = \langle e_x\rangle(\langle l\rangle(v)) = \langle e_x\rangle(w) = h_x(k^+(w)) = 0$. Since l_x is injective, we get $\langle p_{d(x)}\rangle(v) = 0$, which is what we promised.

For an equivalence d on I, put $E_d = \{x \in X : (i, j) \in d \Rightarrow a_i(x) = a_j(x)\}$. Since $x \in E_{d(x)}$ for any $x \in X$, the family E_d, $d \in N$, is a cover of X. Since X is Y-unitary, E_d is Y-representative for some $d \in N$. For $(i, j) \in d$, the maps a_i and a_j are equal on E_d and thus homotopic. Therefore, there is a map m giving the commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{l} & C(X,Y) \\ {\scriptstyle p_d}\downarrow & & \downarrow{\scriptstyle r} \\ I/d & \xrightarrow{m} & [X,Y]. \end{array}$$

We get $\langle r\rangle(w) = \langle r\rangle(\langle l\rangle(v)) = \langle m\rangle(\langle p_d\rangle(v)) = 0$ because $d \in N$. □

Hereafter, let X and Y be homeomorphic to $\mathbb{R}P^\infty$.

14.2. Lemma. *X is Y-unitary.*

Proof. Let $H^\bullet$ be the $\mathbb{Z}_2$-cohomology. Let $g \in H^1X$ and $h \in H^1Y$ be the non-zero classes.

We show that $(*)$ a set $E \subseteq X$ is Y-representative if $g|_U \neq 0$ for any neighbourhood U of E. If maps $a, b \in C(X,Y)$ are equal on E, they are homotopic on some neighbourhood U of E. Then $a^*(h)|_U = b^*(h)|_U$. Since $g|_U \neq 0$, the homomorphism $?|_U \colon H^1X \to H^1U$ is injective. Therefore, $a^*(h) = b^*(h)$. Since Y is a $\mathcal{K}(\mathbb{Z}_2, 1)$ space, a and b are homotopic, as needed.

We show that X is Y-unitary. Assume that $X = E_1 \cup \ldots \cup E_n$, where the sets E_i are not Y-representative. By $(*)$, each E_i has a neighbourhood U_i with $g|_{U_i} = 0$. Since $U_1 \cup \ldots \cup U_n = X$, we get $g^n = 0$, which is false. $\square$

We have $[X,Y] = \{u_0, u_1\}$, where u_0 is the class of a constant map and u_1 is that of a homeomorphism. Consider the invariant $f\colon [X,Y] \to \mathbb{Q}$, $u_i \mapsto i$, $i = 0, 1$. By Lemmas 14.2 and 14.1, f is straight. Let $h\colon [X,Y] \to [SX, SY]$ be the main invariant. Using the isomorphism

$$[SX, SY] \longrightarrow \prod_{i \in \mathbb{Z}} \operatorname{Hom}(H_iX, H_iY), \qquad [v] \mapsto v_*,$$

we get $2h(u_0) = 2h(u_1)$. Therefore, *there exists no homomorphism $d\colon [SX, SY] \to \mathbb{Q}$ such that $f = d \circ h$.* $\square$

§ 15. *K*-straight invariants

Let K be a unital ring. *K-modules* are unital.

***K*-module $L_K(X,Y)$.** For a set X, let $\langle X \rangle_K$ be the (free) K-module with the basis $X^\sharp_K \subseteq \langle X \rangle_K$ endowed with the bijection $X \to X^\sharp_K$, $x \mapsto {<}x{>}_K$. For sets X and Y, let $L_K(X,Y) \subseteq \operatorname{Hom}_K(\langle X \rangle_K, \langle Y \rangle_K)$ be the K-submodule generated by the K-homomorphisms u such that $u(X^\sharp_K) \subseteq Y^\sharp_K \cup \{0\}$. A map $a\colon X \to Y$ induces a K-homomorphism $\langle a \rangle_K \in L_K(X,Y)$, $\langle a \rangle_K({<}x{>}_K) = {<}a(x){>}_K$.

***K*-straight invariants.** Let X and Y be spaces and M be a K-module. An invariant $f\colon [X,Y] \to M$ is called *K-straight* if there exists a K-homomorphism $\tilde{F}\colon L_K(X,Y) \to M$ such that $f([a]) = \tilde{F}(\langle a \rangle_K)$ for all $a \in C(X,Y)$.

15.1. Proposition. *An invariant $f\colon [X,Y] \to M$ is K-straight if and only if it is straight.*

Proof is given in § 16.

***The K-main invariant* $\tilde{h}\colon [X,Y] \to [S_KX, S_KY]_K$.** Let S_KX be the K-complex of singular chains of X with coefficients in K and $[S_KX, S_KY]_K$ be the K-module of K-homotopy classes of K-morphisms $S_KX \to S_KY$. For $a \in C(X,Y)$, let $S_Ka\colon S_KX \to S_KY$ be the induced K-morphism and $[S_Ka]_K \in [S_KX, S_KY]_K$ be its K-homotopy class. The invariant $\tilde{h}\colon [X,Y] \to [S_KX, S_KY]_K$, $[a] \mapsto [S_Ka]_K$, is called *K-main.*

15.2. Theorem. *Suppose that X is valid and finitary and Y is valid. An invariant $f\colon [X,Y] \to M$ is K-straight if and only if there exists a K-homomorphism $\tilde d\colon [S_K X, S_K Y]_K \to M$ such that $f = \tilde d \circ \tilde h$.*

Proof is given in § 16. For $K = \mathbb{Z}$, this is Theorem 1.1.

§ 16. *K*-straight invariants: proofs

Let X and Y be sets. We define a homomorphism $e\colon L(X,Y) \to L_K(X,Y)$. For $u \in L(X,Y)$, let $e(u)$ be the K-homomorphism giving the commutative diagram

$$\begin{array}{ccc} \langle X\rangle & \xrightarrow{\;u\;} & \langle Y\rangle \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ \langle X\rangle_K & \xrightarrow{\;e(u)\;} & \langle Y\rangle_K, \end{array}$$

where i_X is the homomorphism $<x> \mapsto <x>_K$ and i_Y is similar.

For an abelian group A, a K-module M, and a homomorphism $t\colon A \to M$, we introduce the K-homomorphism $t^{(K)}\colon K \otimes A \to M$, $1 \otimes a \mapsto t(a)$.

16.1. Lemma. $e^{(K)}\colon K \otimes L(X,Y) \to L_K(X,Y)$ *is a K-isomorphism.*

Proof. For $w \in \langle Y\rangle_K$ and $y \in Y$, let $w/y \in K$ be the coefficient of $<y>_K$ in w. For $v \in L_K(X,Y)$ and $k \in K \setminus \{0\}$, we introduce the homomorphism $v_k \in L(X,Y)$,

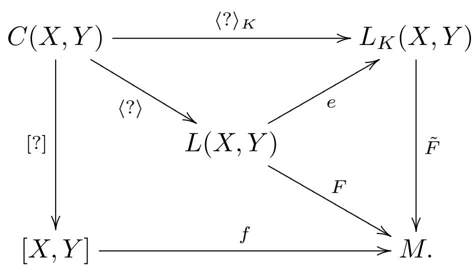
$$v_k(<x>) = \sum_{y \in Y : v(<x>_K)/y = k} <y>, \qquad x \in X.$$

It is not difficult to verify that the map $d\colon L_K(X,Y) \to K \otimes L(X,Y)$,

$$d(v) = \sum_{k \in K \setminus \{0\}} k \otimes v_k,$$

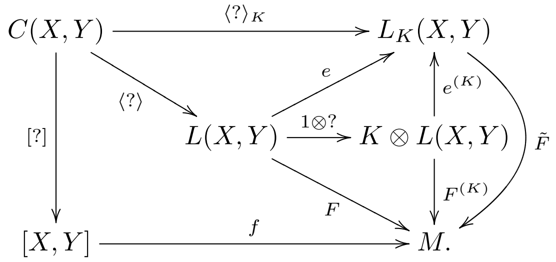
is a K-homomorphism. Using this, we get $e^{(K)} \circ d = \mathrm{id}$ and $d \circ e^{(K)} = \mathrm{id}$. □

Proof of Proposition 15.1. Necessity. Let f be K-straight. There is a K-homomorphism $\tilde F\colon L_K(X,Y) \to M$ such that $f([a]) = \tilde F(\langle a\rangle_K)$, $a \in C(X,Y)$. Consider the homomorphism $F = \tilde F \circ e$:



The diagram is commutative. We get $f([a]) = F(\langle a \rangle)$, $a \in C(X,Y)$. Therefore, f is straight.

Sufficiency. Let f be straight. There is a homomorphism $F\colon L(X,Y) \to M$ such that $f([a]) = F(\langle a \rangle)$, $a \in C(X,Y)$. By Lemma 16.1, $e^{(K)}$ is a K-isomorphism. Consider the homomorphism $\tilde{F} = F^{(K)} \circ (e^{(K)})^{-1}$:



The diagram is commutative. We get $f([a]) = \tilde{F}(\langle a \rangle_K)$, $a \in C(X,Y)$. Therefore, f is K-straight. □

The homomorphism $I\colon [SX, SY] \to [S_K X, S_K Y]_K$. Let X and Y be spaces. A morphism $v\colon SX \to SY$ induces a K-morphism

$$S_K X = K \otimes SX \xrightarrow{\mathrm{id} \otimes v} K \otimes SY = S_K Y.$$

Consider the homomorphism $I\colon [SX, SY] \to [S_K X, S_K Y]_K$, $[v] \mapsto [\mathrm{id} \otimes v]_K$.

16.2. Lemma. *If the group $H_\bullet(X)$ is finitely generated, then the K-homomorphism*

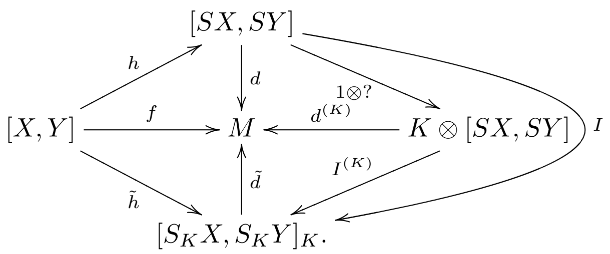
$$I^{(K)}\colon K \otimes [SX, SY] \to [S_K X, S_K Y]_K$$

is a K-split K-monomorphism, i. e. there exists a K-homomorphism $R\colon [S_K X, S_K Y]_K \to K \otimes [SX, SY]$ such that $R \circ I^{(K)} = \mathrm{id}$.

This is a variant of the universal coefficient theorem, cf. [12, Theorems 5.2.8 and 5.5.10]. □

Proof of Theorem 15.2. We have $\tilde{h} = I \circ h$, where $h\colon [X,Y] \to [SX, SY]$ is the main invariant. By Proposition 7.3, h is straight. Therefore, $\tilde{h}$ is straight. By Proposition 15.1, $\tilde{h}$ is K-straight.

This gives the sufficiency. Necessity. Let f be K-straight. By Proposition 15.1, f is straight. By Proposition 12.2, there is a homomorphism $d\colon [SX, SY] \to M$ such that $f = d \circ h$. By Lemma 16.2, there is a K-homomorphism $\tilde{d}$ such that $\tilde{d} \circ I^{(K)} = d^{(K)}$:



The diagram is commutative. In particular, $f = \tilde{d} \circ \tilde{h}$. □

References

[1] M. G. Barratt, J. Milnor, An example of anomalous singular homology, Proc. Amer. Math. Soc. **13** (1962), 293–297.

[2] L. Fuchs, Infinite abelian groups, vol. 2, Academic Press, 1973.

[3] A. Hatcher, Algebraic topology, Cambridge University Press, 2002.

[4] G. Higman, Unrestricted free products, and varieties of topological groups, J. Lond. Math. Soc. **27** (1952), 73–81.

[5] P. Hill, The additive group of commutative rings generated by idempotents, Proc. Amer. Math. Soc. **38** (1973), 499–502.

[6] K. Kuratowski, Topology, vol. 2, Academic Press, PWN, 1968.

[7] S. S. Podkorytov, An alternative proof of a weak form of Serre's theorem (Russian), Zap. Nauchn. Sem. POMI **261** (1999), 210–221. English translation: J. Math. Sci. (N. Y.) **110** (2002), no. 4, 2875–2881.

[8] S. S. Podkorytov, Mappings of the sphere to a simply connected space (Russian), Zap. Nauchn. Sem. POMI **329** (2005), 159–194. English translation: J. Math. Sci. (N. Y.) **140** (2007), no. 4, 589–610.

[9] S. S. Podkorytov, An iterated sum formula for a spheroid's homotopy class modulo 2-torsion, arXiv:math/0606528 (2006).

[10] S. S. Podkorytov, The order of a homotopy invariant in the stable case (Russian), Mat. Sb. **202** (2011), no. 8, 95–116. English translation: Sb. Math. **202** (2011), no. 8, 1183—1206.

[11] S. S. Podkorytov, On homotopy invariants of finite degree (Russian), Zap. Nauchn. Sem. POMI **415** (2013), 109–136. English preprint: arXiv:1209.1952 (2012).

[12] E. H. Spanier, Algebraic topology, McGraw-Hill, 1966.

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