Straight homotopy invariants

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Abstract

Let $X$ and $Y$ be spaces and $M$ be an abelian group. A homotopy invariant $f: [X, Y] \rightarrow M$ is called straight if there exists a homomorphism $F: L(X, Y) \rightarrow M$ such that $f([a]) = F(⟨a⟩)$ for all $a ∈ C(X, Y)$. Here $⟨a⟩: (X) \rightarrow (Y)$ is the homomorphism induced by $a$ between the abelian groups freely generated by $X$ and $Y$ and $L(X, Y)$ is a certain group of “admissible” homomorphisms. We show that all straight invariants can be expressed through a “universal” straight invariant of homological nature.

§ 1. Introduction

We define straight homotopy invariants of maps and give their characterization, which reduces them to the classical homology theory. Straight invariants are a variant of the notion of homotopy invariants of degree at most 1 [10, 8, 9, 11]. This variant has especially simple homological characterization. Homotopy invariants of finite degree are a homotopy analogue of Vassiliev invariants [8].

The group $L(X, Y)$. For a set $X$, let $⟨X⟩$ be the (free) abelian group with the basis $X^♯ ⊆ ⟨X⟩$ endowed with the bijection $X → X^♯$, $x ↦ ⟨x⟩$. For sets $X$ and $Y$, let $L(X, Y) ⊆ \text{Hom}(⟨X⟩, ⟨Y⟩)$ be the subgroup generated by the homomorphisms $u$ such that $u(X^♯) ⊆ Y^♯ ∪ \{0\}$. (Elements of $L(X, Y)$ are the homomorphisms bounded with respect to the $ℓ_1$-norm.) A map $a: X → Y$ induces the homomorphism $⟨a⟩ ∈ L(X, Y)$, $⟨a⟩⟨x⟩ = ⟨a(x)⟩$.

Straight homotopy invariants. Let $X$ and $Y$ be spaces, $M$ be an abelian group, and $f: [X, Y] → M$ be a map (a homotopy invariant). The invariant $f$ is called straight if there exists a homomorphism $F: L(X, Y) → M$ such that $f([a]) = F(⟨a⟩)$ for all $a ∈ C(X, Y)$.

(If $M$ is divisible, the group $L(X, Y)$ can be replaced here by $\text{Hom}(⟨X⟩, ⟨Y⟩)$ because any homomorphism $L(X, Y) → M$ extends to $\text{Hom}(⟨X⟩, ⟨Y⟩)$ in this case. In general, this replacement is inadequate. For example, let $X$ and $Y$ be circles. Then the invariant “degree” $[X, Y] → \mathbb{Z}$ is straight by Theorem 1.1 (or Corollary 6.8). At the same time, every homomorphism $F: \text{Hom}(⟨X⟩, ⟨Y⟩) → \mathbb{Z}$ factors through the restriction homomorphism $\text{Hom}(⟨T⟩, ⟨Y⟩)$ for some finite set $T ⊆ X$ [2, § 94]. Thus $F$ cannot give rise to a non-constant homotopy invariant.)

The main invariant $h: [X, Y] → [SX, SY]$. For a space $X$, let $SX$ be its singular chain complex. Let $X$ and $Y$ be spaces. Let $[SX, SY]$ be the group
of homotopy classes of morphisms \(SX \to SY\). There is a (non-naturally) split
exact natural sequence

\[
0 \to \prod_{i \in \mathbb{Z}} \text{Ext}(H_{i-1}X, H_1Y) \to [SX, SY] \to \prod_{i \in \mathbb{Z}} \text{Hom}(H_iX, H_iY) \to 0
\]

(“the universal coefficient theorem”, cf. [12, Theorem 5.5.3]). For \(a \in C(X, Y)\),
let \(Su: SX \to SY\) be the induced morphism and \([Su] \in [SX, SY]\) be its
homotopy class. The invariant \(h: [X, Y] \to [SX, SY], [a] \mapsto [Su]\), is called main.

**The main result.** We call a space valid if it is homotopy equivalent to a CW-complex; we call it finitary if it is weakly homotopy equivalent to a compact CW-complex.

**1.1. Theorem.** Let \(X\) be a finitary valid space, \(Y\) be a valid space,
\(h: [X, Y] \to [SX, SY]\) be the main invariant, \(M\) be an abelian group, and \(f: [X, Y] \to M\) be
an invariant. The invariant \(f\) is straight if and only if there exists a homomorphism \(d: [SX, SY] \to M\) such that \(f = d \circ h\).

The theorem follows from Propositions 7.3 and 12.2.

The theorem says that the main invariant is a “universal” straight invariant.
A weaker and slightly complicated result is [7, Theorem II]. If \(M\) is divisible, then
the sufficiency (“\(\text{if}\)”) follows easily from an appropriate form of the Dold-Thom
theorem (see §7), and the necessity (“\(\text{only if}\)”) follows from [7, Theorem II] (but
any abelian group is a subgroup of a divisible one). The validity and finitarity
hypotheses are essential, see §§13, 14.

In §15, we consider \(K\)-straight invariants taking values in modules over a
commutative ring \(K\) (by definitions, straight = \(\mathbb{Z}\)-straight).

§ 2. Notation

**The question mark.** The expression \([?]\) denotes the map \(a \mapsto [a]\) between sets
indicated in the context. We similarly use \((?)\), etc. This notation is also used for
functors.

**Sets and abelian groups.** For a set \(X\), let \(c_X: X \to \langle X \rangle\) be the canonical
map \(x \mapsto <x>\). For \(v \in \langle X \rangle\) and \(x \in X\), let \(v/x \in \mathbb{Z}\) be the coefficient of \(<x>\) in \(v\).
For an abelian group \(T\), a map \(a: X \to T\) gives rise to the homomorphism \(a^+: \langle X \rangle \to T, <x> \mapsto a(x)\).
\(T_X\) is the group of maps \(X \to T\).

**Simplicial sets.** For simplicial sets \(U\) and \(V\), let \(\text{Si}(U, V)\) be the set of simplicial
maps and \([U, V]\) be the set of their homotopy classes (two simplicial maps
are homotopic if they are connected by a sequence of homotopies). The functor \((?)\)
takes simplicial sets to simplicial abelian groups degreewise. There is
the canonical simplicial map \(c_U: U \to \langle U \rangle\).
For a simplicial abelian group \(Z\), a simplicial map \(s: U \to Z\) gives rise to the simplicial homomorphism \(s^+: \langle U \rangle \to Z\).
For a simplicial set \(T\), a simplicial map \(s: U \to V\) induces the
maps \(s^+_T: \text{Si}(T, U) \to \text{Si}(T, V), s^+_U: \text{Si}(V, T) \to \text{Si}(U, T), s^+_T: [T, U] \to [T, V]\),
and \(s^+_T: [V, T] \to [U, T]\). This notation is also used in the topological case.
§ 3. Induced straight invariants

3.1. Lemma. Let $X, \tilde{X}, \tilde{Y},$ and $Y$ be spaces, $r: X \to \tilde{X}$ and $s: \tilde{Y} \to Y$ be continuous maps, $M$ be an abelian group and $f: [X, Y] \to M$ be a straight invariant. Then the invariant $\sim f: [\tilde{X}, \tilde{Y}] \to M, \sim f([\tilde{a}]) = f([s \circ \tilde{a} \circ r]), \tilde{a} \in C(\tilde{X}, \tilde{Y}),$ is straight.

Proof. There is a homomorphism $F: L(X, Y) \to M$ such that $f([a]) = F([a]), a \in C(X, Y).$ We have the commutative diagram

\[
\begin{array}{ccc}
C(\tilde{X}, \tilde{Y}) & \xrightarrow{\sim f} & L(\tilde{X}, \tilde{Y}) \\
\downarrow K & & \downarrow T \\
C(X, Y) & \xrightarrow{f} & L(X, Y) \\
\downarrow k & & \downarrow F \\
[X, \tilde{Y}] & \xrightarrow{\sim} & M,
\end{array}
\]

where the maps $K$ and $k$ and the homomorphism $T$ are induced by the pair $(r, s)$ (that is, $K(\tilde{a}) = s \circ \tilde{a} \circ r, k([\tilde{a}]) = [s \circ \tilde{a} \circ r], T(\tilde{a}) = \langle s \circ \tilde{a} \circ \rangle,$) and $F = F \circ T.$ Thus $\sim f$ is straight.

§ 4. The main invariant $h: [U], [V] \to [S[U], S[V]]$

The geometric realization $|Z|$ of a simplicial abelian group $Z$ has a structure of an abelian group. $|Z|$ is a topological abelian group if $Z$ is countable; in general, it is a group of the category of compactly generated Hausdorff spaces. For a simplicial set $T, C([T], |Z|)$ and $|[T], |Z||$ are abelian groups with respect to pointwise addition. Clearly, $Si(T, Z)$ and $[T, Z]$ are also abelian groups.

4.1 Lemma. Let $U$ and $V$ be simplicial sets. Then there exists a commutative diagram

\[
\begin{array}{ccc}
[U, V] & \xrightarrow{(cV)_U} & [U, (V)] \\
\downarrow i \downarrow c & & \downarrow j \\
[S[U], S[V]] & \xrightarrow{h} & [U, [V]] \\
\downarrow E & & \downarrow \text{[cV]U} \downarrow E \\
|[U], [V]| & & |[U], [V]|,
\end{array}
\]

where $i: [s] \to [\langle s \rangle]$ (the map induced by the geometric realization map), $j$ is similar, $h$ is the main invariant, and $c, E$ are some isomorphisms.

This is a version of the Dold–Thom theorem [3, § 4.K].
Proof. Let $\triangle$ be the singular functor. For a simplicial set $T$, let $k_T: T \to \triangle|T|_*$ be the canonical weak equivalence. If $T$ is a simplicial abelian group, $k_T$ is a simplicial homomorphism. We have the commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{c_V} & \langle V \rangle \\
\downarrow{k_V} & & \downarrow{(k_V)_V^{\circ}} \\
\triangle|V| & \xrightarrow{\triangle|c_V|} & \triangle|\langle V \rangle|,
\end{array}
\]

where $m = (\triangle|c_V|)^+$. $k(V)$, $(k_V)_V$, and thus $m$ are weak equivalences. Consider the commutative diagram

\[
\begin{array}{ccc}
[U, V] & \xrightarrow{\langle c_V \rangle_U^V} & \langle U, (V) \rangle \\
\downarrow{(k_V)_V^U} & & \downarrow{(k_V)_V^U} \\
[U, \triangle|V|] & \xrightarrow{\langle \triangle|c_V| \rangle_U^V} & \langle U, \triangle|\langle V \rangle| \rangle \\
\downarrow{\langle p \rangle_U^V} & & \downarrow{\langle q \rangle_U^V} \\
[[U, V]] & \xrightarrow{\langle c_V \rangle_U^V} & [[U, \langle V \rangle]],
\end{array}
\]

where the upper part is the result of applying the functor $[U, ?]$ to the previous diagram and $p$ and $q$ are the standard adjunction bijections for the functors $[?]$ and $\triangle$. $(k_V)_V^U$, $m^U$, and $q$ are isomorphisms.

We will find an isomorphism $P: [S[U], S[V]] \to [U, \langle \triangle|V| \rangle]$ such that $P \circ h = (\triangle|c_V|)^+ \circ p$. Then it will be enough to set $e = P^{-1} \circ (k_V)_V^U$ and $E = q^{-1} \circ m^U \circ p$.

For a simplicial set $T$, let $AT$ be its chain complex, so that $(AT)_n = (T_n)$, $n \geq 0$. Then $SX = A\triangle X$ for any space $X$. A simplicial map $s: T \to (W)$ gives rise to the morphism $v: AT \to AW$, $v_n = s^+_{n+}$, $n \geq 0$. This rule yields an isomorphism $d: [T, (W)] \to [AT, AW]$ (the Dold–Kan correspondence). We set $T = \triangle|U|$ and $W = \triangle|V|$. Consider the commutative diagram

\[
\begin{array}{ccc}
[[U], [V]] & \xrightarrow{b} & [\triangle|U|, \triangle|V|] \\
\downarrow{h} & & \downarrow{h} \\
[A\triangle|U|, A\triangle|V|] & \xrightarrow{d} & [\triangle|U|, \langle \triangle|V| \rangle].
\end{array}
\]
where the map \( b \) is given by the functor \( \Delta \) and \( P = (k_U)_{\Delta(V)} \circ d^{-1} \). Since 
\((k_U)_{\Delta(V)} \) is an isomorphism, \( P \) is an isomorphism too.

§ 5. Nöbeling–Bergman theory

By a ring we mean a (non-unital) commutative ring; subring is understood accordingly. The following facts follow from [5, Theorem 2 and its proof], cf. [2, § 97].

5.1. Lemma. Let \( E \) be a torsion-free ring generated by idempotents. Then \( E \) is a free abelian group.

An example: the ring \( B(X) \) of bounded functions \( X \to \mathbb{Z} \), where \( X \) is an arbitrary set.

5.2. Lemma. Let \( E \) be a torsion-free ring and \( F \subseteq E \) be a subring, both generated by idempotents. Then the abelian group \( E/F \) is free.

For \( F = 0 \), this is Lemma 5.1.

§ 6. Maps to a space with addition

Let \( X \) be a space and \( T \) be a Hausdorff space.

For a set \( V \subseteq T \), we introduce the homomorphism \( s_V : L(X, T) \to \mathbb{Z}^X \), 
\( s_V(u)(x) = I_V^T(u(x)) \), \( x \in X \), where \( I_V : T \to \mathbb{Z} \) is the indicator function of the set \( V \).

The subgroup \( R \subseteq L(X, T) \). For \( p \in X, q \in T \), let \( R(p, q) \subseteq L(X, T) \) be the subgroup of homomorphisms \( u \) such that, for any sufficiently small (open) neighbourhood \( V \) of \( q \), the function \( s_V(u) \) is constant in some neighbourhood of \( p \). Let \( R \subseteq L(X, T) \) be the intersection of the subgroups \( R(p, q) \), \( p \in X, q \in T \).

6.1. Lemma. For \( a \in C(X, T) \), we have \( \langle a \rangle \in R \).

Proof. Take \( p \in X, q \in T \). We show that \( \langle a \rangle \in R(p, q) \). If \( a(p) = q \), then, for any neighbourhood \( V \) of \( q \), we take the neighbourhood \( U = a^{-1}(V) \) of \( p \) and get \( s_V((a))|U = 1 \). Otherwise, choose disjoint neighbourhoods \( W \) of \( q \) and \( W_1 \) of \( a(p) \). Consider the neighbourhood \( U = a^{-1}(W_1) \) of \( p \). For any \( V \subseteq W \), we have \( s_V((a))|V = 0 \).

6.2. Lemma. The abelian group \( L(X, T)/R \) is free.

Proof. Let \( O_T \) be the set of open sets in \( T \). Consider the ring \( E = B(X \times X \times O_T) \). For \( p \in X, q \in T \), let \( I(p, q) \subseteq E \) be the ideal of functions \( f \) such that, for any sufficiently small neighbourhood \( V \) of \( q \), the function \( X \to \mathbb{Z}, x \mapsto f(p, x, V) \), vanishes in some neighbourhood of \( p \). Let \( I \subseteq E \) be the intersection of the ideals \( I(p, q) \), \( p \in X, q \in T \). The ring \( E/I \) is torsion-free and generated by idempotents. By Lemma 5.1, \( E/I \) is a free abelian group. Consider the
homomorphism \( k: \text{L}(X,T) \to E \), \( k(u)(p,x,V) = s_V(u)(x) - s_V(u)(p) \), \( p,x \in X \), \( V \subseteq O_T \), \( u \in \text{L}(X,T) \). We have \( k^{-1}(I(p,q)) = R(p,q) \) and thus \( k^{-1}(I) = R \). Therefore, \( k \) induces a monomorphism \( \text{L}(X,T)/R \to E/I \). It follows that the abelian group \( \text{L}(X,T)/R \) is free.

\[ \tag{\text{6.4. Lemma}} \]

\textbf{The set Q and the homomorphisms e(D,a).} Let \( Q \) be the set of pairs \((D,a)\), where \( D \subseteq X \) is a closed set and \( a \in \text{C}(D,T) \). For \((D,a) \in Q\), introduce the homomorphism \( e(D,a) \in \text{L}(X,T) \),

\[ e(D,a)(<x>) = \begin{cases} <a(x)> & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases} \]

\( x \in X \).

\[ \tag{\text{6.3. Lemma}} \]

\textbf{The subgroup K \subseteq L(X,T).} Let \( K \subseteq \text{L}(X,T) \) be the subgroup generated by \( e(D,a) \), \((D,a) \in Q\).

\[ \tag{\text{6.4. Lemma}} \]

\textbf{The abelian group L(X,T)/K is free.} \( j: \text{L}(X,T) \to B(X \times T) \), \( j(u)(x,t) = u(<x>)/t \). For \((D_i,a_i) \in Q\), \( i = 1,2 \), we have \( j(e(D_1,a_1))j(e(D_2,a_2)) = j(e(D,a)) \), where \( D = \{ x \in D_1 \cap D_2 : a_1(x) = a_2(x) \} \) and \( a = a_1|_D = a_2|_D \). In particular, \( j(e(D,a)) \), \((D,a) \in Q\), are idempotents. Therefore, \( j(K) \) is a subring generated by idempotents. By Lemma 5.2, the abelian group \( B(X \times T)/j(K) \) is free. Since \( j \) induces a monomorphism \( \text{L}(X,T)/K \to B(X \times T)/j(K) \), the abelian group \( \text{L}(X,T)/K \) is free.

\[ \tag{\text{6.5. Lemma}} \]

\textbf{The abelian group L(X,T)/(K \cap R) is free.} \( j: \text{L}(X,T) \to B(X \times T) \), \( j(u)(x,t) = u(<x>)/t \). For \((D_i,a_i) \in Q\), \( i = 1,2 \), we have \( j(e(D_1,a_1))j(e(D_2,a_2)) = j(e(D,a)) \), where \( D = \{ x \in D_1 \cap D_2 : a_1(x) = a_2(x) \} \) and \( a = a_1|_D = a_2|_D \). In particular, \( j(e(D,a)) \), \((D,a) \in Q\), are idempotents. Therefore, \( j(K) \) is a subring generated by idempotents. By Lemma 5.2, the abelian group \( B(X \times T)/j(K) \) is free. Since \( j \) induces a monomorphism \( \text{L}(X,T)/K \to B(X \times T)/j(K) \), the abelian group \( \text{L}(X,T)/K \) is free.

\[ \tag{\text{6.5. Lemma}} \]

\textbf{The homomorphism G: L(X,T) \to T}\textsuperscript{X}.} Let \( T \) have a structure of an abelian group such that, \((*)\) for any closed set \( D \subseteq X \), the set \( \text{C}(D,T) \) becomes an abelian group with respect to pointwise addition\(^1\). Consider the homomorphism \( G: \text{L}(X,T) \to T^X \), \( G(u)(x) = r(u(<x>)) \), \( x \in X \), \( u \in \text{L}(X,T) \), where \( r = \text{id}^+: (T) \to T \).

\[ ^1\text{The condition } (*) \text{ is satisfied if } T \text{ is a topological abelian group or if } X = |U| \text{ and } T = |Z|, \text{ where } U \text{ is a simplicial set and } Z \text{ is a simplicial abelian group.} \]
6.6. Lemma. $G(K \cap R) \subseteq C(X, T)$.

Proof. Take $u \in K \cap R$. We show that $G(u) \in C(X, T)$. Since $u \in K$, we have

$$u = \sum_{i \in I} u_i,$$

where $I$ is a finite set, $k_i \in \mathbb{Z}$, and $(D_i, a_i) \in Q$. For $J \subseteq I$, put

$$u_J = \sum_{i \in J} u_i, \quad D_J = \bigcap_{i \in J} D_i \subseteq X$$

(so $D_{\emptyset} = X$) and

$$b_J = \sum_{i \in J} k_i a_i|_{D_J} \in C(D_J, T), \quad k_J = \sum_{i \in J} k_i.$$

Take $p \in X$. We verify that $G(u)$ is continuous at $p$. Put $N = \{i \in I : p \notin D_i\}$. For $q \in T$, put $I(q) = \{i \in I : p \in D_i, a_i(p) = q\}$. We have

$$u = u_N + \sum_{q \in T} u_{I(q)}$$

(almost all summands are zero). Clearly, $G(u_N)$ vanishes in some neighbourhood of $p$. Take $q \in T$. It suffices to show that $G(u_{I(q)})$ is continuous at $p$. Put $t_0 = G(u_{I(q)}) \in T$. We have $t_0 = k_{I(q)}q$. Let $W$ be a neighbourhood of $t_0$. We seek a neighbourhood $U$ of $p$ such that $G(u_{I(q)})(U) \subseteq W$.

Put $E = \{J \subseteq I(q) : k_J = k_{I(q)}\}$. For $J \in E$, we have $p \in D_J$ and $b_J(p) = t_0$. There is a neighbourhood $U_1$ of $p$ such that $b_J(D_J \cap U_1) \subseteq W$ for all $J \in E$.

By Lemma 6.3, $u_i \in R(p, q)$ for $i \in I \setminus I(q)$. Since $u \in R(p, q)$, we have $u_{I(q)} \in R(p, q)$. Therefore, there is a neighbourhood $V \subseteq T$ of $q$ such that the function $s_V(u_{I(q)})$ is constant in some neighbourhood $U_3$ of $p$.

There is a neighbourhood $U_3$ of $p$ such that $a_i(D_i \cap U_3) \subseteq V$ for all $i \in I(q)$. For $x \in X$, put $J(x) = \{i \in I(q) : x \in D_i\}$. For $x \in U_2 \cap U_3$, we have $k_{J(q)}(x) = s_V(u_{I(q)})(x) = s_V(u_{I(q)})(p) = k_{I(q)}$, i.e., $J(x) \in E$.

Set $U = U_1 \cap U_2 \cap U_3$. Take $x \in U$. We have $G(u_{I(q)})(x) = b_{J(x)}(x) \in W$ because $J(x) \in E$. \hfill \Box

6.7. Lemma. There exists a homomorphism $g: L(X, T) \rightarrow C(X, T)$ such that $g(\langle a \rangle) = a$ for all $a \in C(X, T)$.

Proof. We have $G(\langle a \rangle) = a$ for all $a \in T^X$. Since $G(K \cap R) \subseteq C(X, T)$ (by Lemma 6.6) and the abelian group $L(X, T)/(K \cap R)$ is free (by Lemma 6.5), there is a homomorphism $g: L(X, T) \rightarrow C(X, T)$ such that $g(\langle a \rangle) = G(u)$ for $u \in K \cap R$. For $a \in C(X, T)$, we have $\langle a \rangle \in K$ (because $\langle a \rangle = e(X, a)$) and $\langle a \rangle \in R$ (by Lemma 6.1). We get $g(\langle a \rangle) = G(\langle a \rangle) = a$. \hfill \Box
6.8. Corollary. Suppose that \((*)\) \([X, T]\) is an abelian group with respect to pointwise addition\(^2\). Then the invariant \(\text{id} : [X, T] \to [X, T]\) is straight.

Proof. By Lemma 6.7, there is a homomorphism \(g : L(X, T) \to C(X, T)\) such that \(g(\langle a \rangle) = a\) for all \(a \in C(X, T)\). Consider the homomorphism \(F : L(X, T) \to [X, T], u \mapsto [g(u)]\). For \(a \in C(X, T)\), we have \([a] = [g(\langle a \rangle)] = F(\langle a \rangle)\). \(\square\)

§ 7. Sufficiency in Theorem 1.1

The proof of sufficiency in Theorem 1.1 relies on Corollary 6.8. If the group \(M\) is divisible, it is easy to use Lemma 7.1 instead (then the stuff of §§ 5, 6 is needless).

7.1. Lemma (cf. [10, Lemma 1.2]). Let \(X\) and \(T\) be spaces and \(T\) have a structure of an abelian group such that \((*)\) the sets \(C(X, T)\) and \([X, T]\) become abelian groups with respect to pointwise addition\(^3\). Let \(M\) be a divisible abelian group and \(f : [X, T] \to M\) be a homomorphism. Then \(f\) is a straight invariant.

Proof. Consider the homomorphism \(G : L(X, T) \to T^X, G(u)(x) = r(u(\langle x \rangle)), x \in X, u \in L(X, T)\), where \(r = \text{id}^+: \langle T \rangle \to T\). Let \(D \subseteq L(X, T)\) be the subgroup generated by the homomorphisms \(\langle a \rangle, a \in C(X, T)\). Clearly, \(G(\langle a \rangle) = a\) for \(a \in C(X, T)\). Therefore, \(G(D) \subseteq C(X, T)\). Consider the homomorphism \(F_0 : D \to M, u \mapsto f([G(u)])\). Since \(M\) is divisible, there is a homomorphism \(F : L(X, T) \to M\) such that \(F|_D = F_0\). For \(a \in C(X, T)\), we have \(f([a]) = f([G(\langle a \rangle)]) = F_0(\langle a \rangle) = F(\langle a \rangle)\). \(\square\)

7.2. Claim. Let \(U\) and \(V\) be simplicial sets. Then the main invariant \(h : [|U|, |V|] \to [|S|U|, |S|V|]\) is straight.

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
|[U|, |V| & \xrightarrow{h} & [S|U|, S|V|] \\
|c_V|^{|U|} \downarrow & & \downarrow E \\
|[U|, |\langle V \rangle|] & \downarrow & [S|U|, S|\langle V \rangle|]
\end{array}
\]

where \(E\) is the isomorphism from Lemma 4.1. By Corollary 6.8, the invariant \(\text{id} : [|U|, |\langle V \rangle|] \to [|U|, |\langle V \rangle|]\) is straight. Therefore, by Lemma 3.1, the invariant \(|c_V|^{|U|}\) is straight. Since \(E\) is an isomorphism, \(h\) is also straight. \(\square\)

7.3. Proposition. Let \(X\) be a space and \(Y\) be a valid space. Then the main invariant \(h : [X, Y] \to [S|X|, S|Y|]\) is straight.

\(^2\)See footnote 1.  
\(^3\)See footnote 1.
Proof. There are homology equivalences $r: |U| \to X$ and $s: Y \to |V|$, where $U$ and $V$ are simplicial sets. Consider the commutative diagram

$$
\begin{array}{ccc}
[X,Y] & \xrightarrow{h} & [SX, SY] \\
\downarrow{k} & & \downarrow{l} \\
[[U],[V]] & \xrightarrow{\tilde{h}} & [S[U], S[V]],
\end{array}
$$

where $\tilde{h}$ is the main invariant and the map $k$ and the isomorphism $l$ are induced by the pair $(r, s)$. By Claim 7.2, $\tilde{h}$ is straight. By Lemma 3.1, the invariant $\tilde{h} \circ k$ is straight. Since $h = l^{-1} \circ \tilde{h} \circ k$, $h$ is also straight.

§ 8. The superposition $Z: \langle \text{Si}(U, V) \rangle_0 \to \text{Si}(U, \langle V \rangle_0)$

For a set $X$, let $\langle X \rangle_0 \subseteq \langle X \rangle$ be the kernel of the homomorphism $\langle X \rangle \to \mathbb{Z}$, $<x> \mapsto 1$. We apply the functor $\langle ? \rangle_0$ to simplicial sets degreewise.

Let $U$ and $V$ be simplicial sets. The canonical simplicial map $c = c_V: V \to \langle V \rangle$ gives rise to the map $c^U_\# : \text{Si}(U, V) \to \text{Si}(U, \langle V \rangle)$ and the homomorphism $(c^U_\#)^+ : \langle \text{Si}(U, V) \rangle \to \text{Si}(U, \langle V \rangle)$. We have the commutative diagram

$$
\begin{array}{ccc}
\langle \text{Si}(U, V) \rangle_0 & \xrightarrow{Z} & \text{Si}(U, \langle V \rangle_0) \\
\downarrow{\langle c^U_\# \rangle^+} & & \downarrow{} \\
\langle \text{Si}(U, V) \rangle & \xrightarrow{(c^U_\#)^+} & \text{Si}(U, \langle V \rangle),
\end{array}
$$

where the vertical arrows are induced by the canonical inclusion $\langle ? \rangle_0 \to \langle ? \rangle$ and $Z$ is a new homomorphism called the superposition.

§ 9. Surjectivity of the superposition

Our aim here is Lemma 9.1. We follow [10, §§ 12, 13].

**Extension of simplicial maps.** For $n \geq 0$, let $\Delta^n$ be the combinatorial standard $n$-simplex (a simplicial set) and $\partial \Delta^n$ be its boundary.

Let $W$ be a contractible fibrant simplicial set. For each $n \geq 0$, choose a map $e_n: \text{Si}(\partial \Delta^n, W) \to \text{Si}(\Delta^n, W)$ such that $e_n(q)|_{\partial \Delta^n} = q$ for any $q \in \text{Si}(\partial \Delta^n, W)$.

Let $U$ be a simplicial set. For each simplicial subset $A \subseteq U$, we introduce the map $E_A: \text{Si}(A, W) \to \text{Si}(U, W)$, $x \mapsto t$, where $t|_A = x$ and $t \circ p = e_n(t \circ p|_{\partial \Delta^n})$ for the characteristic map $p: \Delta^n \to U$ of each non-degenerate simplex outside $A$. Clearly,

1. $E_A(x)|_A = x$;
2. $E_A(x)|_B = E_{A \cap B}(x|_{A \cap B})|_B$. 


where $A, B \subseteq U$ are simplicial subsets and $x \in \text{Si}(A, W)$.

**The ring $\langle Q \rangle$ and its identity $I$.** Let $Q$ be the system of simplicial subsets of $U$ consisting of all subsets isomorphic to $\Delta^n$, $n \geq 0$, and the empty subset. Suppose that the simplicial set $U$ is polyhedral, i.e., $Q$ is its cover closed under intersection, and compact, i.e., generated by a finite number of simplices. $Q$ is finite.

We introduce multiplication in $\langle Q \rangle$ by putting $\langle A \rangle \langle B \rangle = \langle A \cap B \rangle$ for $A, B \in Q$. The ring $\langle Q \rangle$ has an identity $I$. Indeed, the homomorphism $e: \langle Q \rangle \to \mathbb{Z}^Q$, $e(\langle A \rangle) = \begin{cases} 1 & \text{if } A \supseteq B, \\ 0 & \text{otherwise,} \end{cases}$ $A, B \in Q$, is an isomorphism (“an upper unitriangular matrix”) preserving multiplication. Therefore, $I = e^{-1}(1)$ is an identity.

**The homomorphism $K: \text{Si}(U, \langle W \rangle)_0 \to \langle \text{Si}(U, W) \rangle_0$.** For a simplicial set $T$, let $Z_T: \langle \text{Si}(T, W) \rangle_0 \to \text{Si}(T, \langle W \rangle)_0$ be the superposition. For simplicial sets $T, A$, let $r_T: \text{Si}(T, \langle W \rangle)_0 \to \text{Si}(A, \langle W \rangle)_0$ be the restriction maps. $s_A$ is a homomorphism. If $T = U$, we omit the corresponding sub/superscript in this notation.

Note that $Z_A$ is an isomorphism for $A \in Q$. Consider the map $k: Q \to \text{Hom}(\text{Si}(U, \langle W \rangle)_0, \langle \text{Si}(U, W) \rangle_0)$, $A \mapsto (E_A)_0 \circ Z_A^{-1} \circ s_A$:

$$k(A): \text{Si}(U, \langle W \rangle)_0 \xrightarrow{s_A} \text{Si}(A, \langle W \rangle)_0 \xrightarrow{Z_A^{-1}} \langle \text{Si}(A, W) \rangle_0 \xrightarrow{(E_A)_0} \langle \text{Si}(U, W) \rangle_0.$$

Put $K = k^+(I)$.

**9.1. Lemma.** The diagram

$$
\begin{array}{ccc}
\langle \text{Si}(U, W) \rangle_0 & \xrightarrow{K} & \langle \text{Si}(U, W) \rangle_0 \\
\downarrow{Z} & & \downarrow{Z} \\
\text{Si}(U, \langle W \rangle)_0 & \xrightarrow{id} & \text{Si}(U, \langle W \rangle)_0
\end{array}
$$

is commutative.

**Proof.** Take $A, B \in Q$. We have the commutative diagram

$$
\begin{array}{ccc}
\text{Si}(A, \langle W \rangle)_0 & \xrightarrow{Z_A^{-1}} & \langle \text{Si}(A, W) \rangle_0 \\
\downarrow{s_A} & & \downarrow{(E_A)_0} \\
\text{Si}(U, \langle W \rangle)_0 & \xrightarrow{Z_U^{-1}} & \langle \text{Si}(U, W) \rangle_0
\end{array}
$$

$$
\begin{array}{ccc}
\text{Si}(B, \langle W \rangle)_0 & \xrightarrow{(r_B)_0} & \langle \text{Si}(B, W) \rangle_0 \\
\downarrow{(r_B)_0} & & \downarrow{(r_B)_0} \\
\text{Si}(C, \langle W \rangle)_0 & \xrightarrow{Z_C^{-1}} & \langle \text{Si}(C, W) \rangle_0
\end{array}
$$

$$
\begin{array}{ccc}
\text{Si}(U, \langle W \rangle)_0 & \xrightarrow{s_C} & \text{Si}(C, \langle W \rangle)_0 \\
\downarrow{(r_B)_0} & & \downarrow{(r_B)_0} \\
\text{Si}(C, \langle W \rangle)_0 & \xrightarrow{Z_C^{-1}} & \langle \text{Si}(C, W) \rangle_0
\end{array}
$$
where \( C = A \cap B \) (commutativity of the “pentagon” follows from the property (2) of the family \( E \)). Therefore, \( (r_B)_0 \circ k(A) = (r_B)_0 \circ k(A \cap B) \). Therefore, \( (r_B)_0 \circ \langle I \rangle = (r_B)_0 \circ k^+(X < B \ra) \) for \( X \in \langle Q \rangle \). We have \( (r_B)_0 \circ k = (r_B)_0 \circ k^+(I \ra) = (r_B)_0 \circ k^+(B \ra) = (r_B)_0 \circ k(B) = (r_B)_0 \circ (E_B)_0 \circ Z_B^{-1} \circ s_B = Z_B^{-1} \circ s_B \), because \( r_B \circ E_B = \text{id} \) by property (1) of the family \( E \).

We get \( s_B \circ Z \circ K = Z_B \circ (r_B)_0 \circ K = s_B \). Since \( B \) is arbitrary, \( Z \circ K = \text{id} \). 

\[ \square \]

§ 10. A cocartesian square

Let \( U \) be a compact polyhedral simplicial set and \( V \) be a fibrant simplicial set. The canonical simplicial map \( c = c_V : V \to (V) \) induces the maps \( c^U_{\#} : \text{Si}(U, V) \to \text{Si}(U, (V)) \) and \( c^U_{\ast} : [U, V] \to [U, (V)] \). Consider the commutative square of abelian groups and homomorphisms

\[
\begin{array}{ccc}
\langle \text{Si}(U, V) \rangle & \xrightarrow{(c^U_{\#})^+} & \text{Si}(U, (V)) \\
\langle [U, V] \rangle & \xrightarrow{(c^U_{\ast})^+} & [U, (V)]
\end{array}
\]

where \( p = \langle ? \rangle : \text{Si}(U, V) \to [U, V] \) and \( q = ? \) (the projections).

10.1. Lemma. This square is cocartesian.

Proof. Since \( p \) and \( q \) are epimorphisms, it suffices to show that \( \text{Ker} q = (c^U_{\#})^+ \text{Ker} (p) \).

Suppose we have a decomposition \( V = \prod_{i \in I} V_i \).

Consider the commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i \in I} \langle \text{Si}(U, V_i) \rangle & \xrightarrow{\bigoplus_{i \in I} (c^U_i)^+_{\#}} & \bigoplus_{i \in I} \text{Si}(U, (V_i)) \\
\bigoplus_{i \in I} \langle [U, V_i] \rangle & \xrightarrow{\bigoplus_{i \in I} (c^U_i)^+_{\ast}} & \bigoplus_{i \in I} [U, (V_i)]
\end{array}
\]

where \( c_i, p_i, \) and \( q_i \) are similar to \( c, p, \) and \( q \) (respectively) and the slanting arrows are induced by the inclusions \( V_i \to V \). Since \( U \) is compact, \( E \) and \( e \) are...
isomorphisms. Therefore, it suffices to show that \( \text{Ker } q_i = ((c_i)_*)^+ (\text{Ker } p_i) \) for each \( i \in I \). This reduction allows us to assume that \( V \) is 0-connected.

Consider the commutative diagram

\[
\begin{array}{ccc}
\langle \text{Si}(U, V) \rangle_0 & \xrightarrow{Z} & \text{Si}(U, \langle V \rangle_0) \\
\downarrow \quad \langle p \rangle_0 & & \downarrow \quad \langle q \rangle_0 \\
\langle [U, V] \rangle_0 & \xrightarrow{\langle c_U \rangle_*} & [U, \langle V \rangle] \\
\end{array}
\]

where \( q_0 = [?] \) (the projection), \( Z \) is the superposition, \( z \) is the homomorphism such that the outer square is commutative, \( I \) and \( i \) are the inclusion homomorphisms, and \( j : \langle V \rangle_0 \rightarrow \langle V \rangle \) is the inclusion simplicial homomorphism. Clearly, \( \text{Ker } q = j^* (\text{Ker } q_0) \). Therefore, it suffices to show that \( \text{Ker } q_0 = Z (\text{Ker } p_0) \).

Since \( V \) is fibrant and 0-connected, there is a surjective simplicial map \( f : W \rightarrow V \), where \( W \) is a contractible fibrant simplicial set. Consider the commutative diagram

\[
\begin{array}{ccc}
\langle \text{Si}(U, W) \rangle_0 & \xrightarrow{\tilde{Z}} & \text{Si}(U, \langle W \rangle_0) \\
\downarrow \quad \langle (f)_0 \rangle & & \downarrow \quad \langle (f)_+ \rangle \\
\langle [U, V] \rangle_0 & \xrightarrow{z} & [U, \langle V \rangle] \\
\end{array}
\]

where the map \( f^* : \text{Si}(U, W) \rightarrow \text{Si}(U, V) \) and the simplicial homomorphism \( (f)_0 : \langle W \rangle_0 \rightarrow \langle V \rangle_0 \) are induced by \( f \) and \( \tilde{Z} \) is the superposition. Since \( (f)_0 \) is surjective, it is a fibration. Therefore, \( \text{Ker } q_0 \subseteq \text{Im } (f^*)_0 \). By Lemma 9.1, \( \tilde{Z} \) is surjective. Since \( W \) is contractible, \( \text{Im } (f^*)_0 \subseteq \text{Ker } p_0 \). Therefore, \( \text{Ker } q_0 \subseteq Z (\text{Ker } p_0) \). The reverse inclusion is obvious.

\[ \square \]

\section{11. The homomorphism \( P : \text{Si}(U, \langle V \rangle) \rightarrow L([U], [V]) \)}

For \( n \geq 0 \), let \( \Delta^n \) be the geometric standard \( n \)-simplex and \( \Delta^n \) be its interior. For a simplicial set \( U \) and a point \( z \in \Delta^n \), there is a canonical map \( z_U : U_n \rightarrow |U| \). The map \( \Delta^n \times U_n \rightarrow |U|, (z, u) \mapsto z_U(u) \), is the canonical pairing of geometric realization.
Let $U$ and $V$ be simplicial sets. We define a homomorphism $\tilde{P} : \text{Si}(U, \langle V \rangle) \to \text{Hom}(\langle U \rangle, \langle V \rangle)$. For $t \in \text{Si}(U, \langle V \rangle)$ and $x \in \langle U \rangle$, $x = z_U(u)$, where $z \in \Delta^n$ and $u \in U_n \ (n \geq 0)$, put $\tilde{P}(t)(\langle x \rangle) = \langle z_V \rangle(t_n(u))$:

$$u \in U_n \xrightarrow{t_n} \langle V \rangle_n = \langle V_n \rangle \xrightarrow{\langle z_V \rangle} \langle |V| \rangle.$$

$\tilde{P}$ is well-defined.

Suppose that $U$ is compact.

11.1. Lemma. $\text{Im} \tilde{P} \subseteq L(\langle U \rangle, \langle |V| \rangle)$.

Proof. Let $U^\times_n \subseteq U_n \ (n \geq 0)$ be the set of non-degenerate simplices. For $u \in U^\times_n \ (n \geq 0)$, we define a homomorphism $I_u : \langle V_n \rangle \to L(\langle U \rangle, \langle |V| \rangle)$. For $v \in V_n$, $x \in |U|$, put

$$I_u(\langle x \rangle)(\langle v \rangle) = \begin{cases} \langle z_V(v) \rangle & \text{if } x = z_U(u) \text{ for } z \in \Delta^n, \\ 0 & \text{otherwise.} \end{cases}$$

This equality is preserved if we replace $\langle v \rangle$ by $w \in \langle V_n \rangle$ and $\langle z_V(v) \rangle$ by $\langle z_V \rangle(\langle w \rangle)$. It suffices to show that

$$\tilde{P}(t) = \sum_{n \geq 0, u \in U^\times_n} I_u(t_n(u)), \quad t \in \text{Si}(U, \langle V \rangle).$$

Evaluating each side at $\langle x \rangle$, $x = z_U(u)$, where $z \in \Delta^n$ and $u \in U^\times_n \ (n \geq 0)$, we get $\langle z_V \rangle(t_n(u))$.

Lemma 11.1 allows us to introduce the homomorphism $P : \text{Si}(U, \langle V \rangle) \to L(\langle U \rangle, \langle |V| \rangle)$, $P(t) = \tilde{P}(t)$.

11.2. Lemma. The diagram

$$
\begin{array}{ccc}
\text{Si}(U, V) & \xrightarrow{c^U_V} & \text{Si}(U, \langle V \rangle) \\
\downarrow{?} & & \downarrow{P} \\
C(\langle U \rangle, \langle V \rangle) & \xrightarrow{\langle P \rangle} & L(\langle U \rangle, \langle |V| \rangle),
\end{array}
$$

where $c = c_V : V \to \langle V \rangle$ is the canonical simplicial map, is commutative.

Proof. For $s \in \text{Si}(U, V)$ and $x \in \langle U \rangle$, $x = z_U(u)$, where $z \in \Delta^n$ and $u \in U_n \ (n \geq 0)$, we have $(P \circ c^U_V)(\langle s \rangle)(\langle x \rangle) = P(c \circ s)(\langle x \rangle) = \langle z_V \rangle((c \circ s)_n(u)) = \langle z_V(s_n(u)) \rangle = \langle |s| \rangle(z_U(u)) = \langle |s| \rangle(\langle x \rangle) = \langle |s| \rangle(\langle z_V \rangle(\langle x \rangle)).$

§ 12. Necessity in Theorem 1.1

12.1. Claim. Let $U$ be a compact polyhedral simplicial set, $V$ be a fibrant simplicial set, $h : [\langle U \rangle, \langle V \rangle] \to [S[U], S[V]]$ be the main invariant, $M$ be an abelian group, and $f : [\langle U \rangle, \langle V \rangle] \to M$ be a straight invariant. Then there exists a homomorphism $d : [S[U], S[V]] \to M$ such that $f = d \circ h$. 

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Proof. Since $f$ is straight, there is a homomorphism $F: L([U], |V|) \to M$ such that $f([a]) = F([a])$ for $a \in C([U], |V|)$. Consider the diagram of abelian groups and homomorphisms

\[
\begin{array}{ccc}
\langle C([U], [V]) \rangle & \xrightarrow{k^+} & L([U], |V|) \\
\langle |U|, |V| \rangle & \xrightarrow{\langle p \rangle} & \langle |U|, |V| \rangle \\
\langle |U|, |V| \rangle & \xrightarrow{\langle q \rangle} & \langle |U|, |V| \rangle \\
\langle |U|, |V| \rangle & \xrightarrow{\langle r \rangle} & \langle |U|, |V| \rangle \\
\end{array}
\]

Here the inner square is as in § 10, $r = [?] : C([U], |V|) \to |U|, |V|$ (the projection), $k = (\langle \rangle) : C([U], |V|) \to L([U], |V|)$, $I = [?] : Si(U, V) \to C([U], |V|)$ (the geometric realization map), $i: |U, V| \to |U|, |V|, [s] \to |s|$, and $P$ is as in § 11. By Lemma 11.2, the upper trapezium is commutative. The solid arrows are defined and form a commutative subdiagram. Since the inner square is cartesian by Lemma 10.1, the dashed arrow $d$ is well-defined by the condition of commutativity of the diagram.

Consider the diagram

\[
\begin{array}{ccc}
\langle |U, V| \rangle & \xrightarrow{(c_+)^+} & |U, V| \\
\langle |U|, |V| \rangle & \xrightarrow{f^+} & \langle |U|, |V| \rangle \\
\langle |U, V| \rangle & \xrightarrow{e} & \langle |U|, |V| \rangle \\
\langle |U|, |V| \rangle & \xrightarrow{h^+} & \langle |U|, |V| \rangle \\
\end{array}
\]

where $e$ is the isomorphism from Lemma 4.1 and $d = \tilde{d} \circ e^{-1}$. The square is commutative by Lemma 4.1. We have $\tilde{d} \circ (c_+)^+ = f^+ \circ (i)$. Since $V$ is fibrant, $i$ is a bijection, and thus $\langle i \rangle$ is an isomorphism. We get $f^+ = d \circ h^+$ (so the diagram is commutative). Therefore, $f = d \circ h$.

12.2. Proposition. Let $X$ be finitary valid space, $Y$ be a space, $h: [X, Y] \to [SX, SY]$ be the main invariant, $M$ be an abelian group, and $f: [X, Y] \to M$ be a straight invariant. Then there exists a homomorphism $d: [SX, SY] \to M$ such that $f = d \circ h$.

Proof. There are a homotopy equivalence $r: X \to |U|$ and a weak homotopy equivalence $s: |V| \to Y$, where $U$ is a compact polyhedral simplicial set and $V$
is a fibrant simplicial set. We construct the commutative diagram

\[
\begin{array}{ccc}
[U, V] & \xrightarrow{k} & [SU, SV] \\
\downarrow{k} & & \downarrow{l} \\
[X, Y] & \xrightarrow{h} & [SX, SY].
\end{array}
\]

Here the bijection \( k \) and the isomorphism \( l \) are induced by the pair \((r, s)\) and \( \tilde{h} \) is the main invariant. The square is commutative. By Lemma 3.1, the invariant \( f = f \circ k \) is straight. By Claim 12.1, there is a homomorphism \( d \) such that \( f = d \circ \tilde{h} \). Set \( d = d \circ l^{-1} \). Since \( k \) is a bijection, we get \( f = d \circ h \) (so the diagram is commutative). \( \square \)

§ 13. Three counterexamples

The Hawaiian ear-ring. Let us show that the hypothesis of validity of \( Y \) in Theorem 1.1 and Proposition 7.3 is essential. Let \( X \) be the one-point compactification of the ray \( \mathbb{R}_+ = (0, \infty) \) (a circle) and \( Y \) be that of the space \( \mathbb{R}_+ \setminus \mathbb{N} \) (the Hawaiian ear-ring [3, Example 1.25]). We define a map \( m \in C(X, Y) \) by putting

\[
m(x) = \left[ \frac{x + 1}{2} \right] + (-1)^{\lfloor x/2 \rfloor} \{ -x \}
\]

for \( x \in \mathbb{R}_+ \setminus \mathbb{N} \). Here \([t]\) and \( \{t\}\) are the integral and the fractional (respectively) parts of a number \( t \in \mathbb{R} \). The element of \( \pi_1(Y, \infty) \) represented by the loop \( m \) is the (reasonably understood) infinite product of commutators

\[
\prod_{p=0}^{\infty} [u_{2p}, u_{2p+1}],
\]

where \( u_q \) is the element realized by the closure of the interval \((q, q + 1)\). Let \( e \in H_1(X) \) be the standard generator. As in [4, p. 76], we get that the element \( m_*(e) \in H_1(Y) \) has infinite order. Therefore, there is a homomorphism \( k: H_1(Y) \rightarrow \mathbb{Q} \) such that \( k(m_*(e)) = 1 \). We define a homomorphism \( d: [SX, SY] \rightarrow \mathbb{Q} \) by putting \( d([v]) = k(v_*(e)) \) for a morphism \( v: SX \rightarrow SY \). Let \( h: [X, Y] \rightarrow [SX, SY] \) be the main invariant. We show that the invariants \( d \circ h \) and \( h \) are not straight.

For \( y \in Y \) and \( i = 0, 1 \), put \( y_{(i)} \in Y \) equal to \( \infty \) if \( i = 1 \) and to \( y \) otherwise. For \( i, j = 0, 1 \), we define a map \( r_{ij} \in C(Y, Y) \). For \( y \in \mathbb{R}_+ \setminus \mathbb{N} \), we put \( r_{ij}(y) \) equal to \( y_{(j)} \) if \( [y] \) is odd and to \( y_{(i)} \) otherwise. For elements \( z_{ij}, i, j = 0, 1 \), of an abelian group, put \( v_{ij}z_{ij} = z_{00} - z_{10} - z_{01} + z_{11} \). Clearly, \( v_{ij}(r_{ij}) = 0 \) in \( L(Y, Y) \). Put \( a_{ij} = r_{ij} \circ m \in C(X, Y) \). We get \( v_{ij}(a_{ij}) = 0 \) in \( L(X, Y) \).

Therefore, \( v_{ij} f([a_{ij}]) = 0 \) for any straight invariant \( f \). We show that this is
false for the invariant $d \circ h$. We have $a_{00} = m$; the map $a_{11}$ is constant. It is easy to see that the maps $a_{10}$ and $a_{01}$ are null-homotopic (this “follows formally” from the presentation ($\ast$) and the equalities $r_{10} \ast (u_{2p}) = r_{01} \ast (u_{2p+1}) = 1$). We get $\vee_{ij} (d \circ h)([a_{ij}]) = (d \circ h)([m]) = k(m_*(e)) = 1$.

Using [1, Theorem 2], one can make the spaces $X$ and $Y$ simply-connected in this example.

**The Warsaw circle.** Let us show that the hypothesis of validity of $X$ in Theorem 1.1 and Proposition 12.2 is essential. Let $X$ be the Warsaw circle [3, Exercise 7 in § 1.3] and $Y$ be the unit circle in $\mathbb{C}$. $Y$ is a topological abelian group. The group $[X, Y]$ is non-zero by [3, Exercise 7 in § 1.3, Proposition 1.30] and torsion-free by [6, Theorem 1 in § 56-III]. Therefore, there is a non-zero homomorphism $f: [X, Y] \to \mathbb{Q}$. By Lemma 7.1, $f$ is a straight invariant. Since $X$ is weakly homotopy equivalent to a point [3, Exercise 10 in § 4.1] and $Y$ is 0-connected, the main invariant $h: [X, Y] \to [SX, SY]$ is constant. Therefore there exists no homomorphism $d: [SX, SY] \to \mathbb{Q}$ such that $f = d \circ h$. □

**An infinite discrete space.** Let us show that the hypothesis of finitarity of $X$ in Theorem 1.1 and Proposition 12.2 is essential (see also § 14).

Note that, for an infinite set $X$, the subgroup $B(X) \subseteq \mathbb{Z}^X$ is not a direct summand because the group $\mathbb{Z}^X$ is reduced and the group $\mathbb{Z}^X/B(X)$ is divisible and non-zero.

Let $X$ and $Y$ be discrete spaces, $X$ infinite and $Y = \{y_0, y_1\}$. Introduce the function $k: Y \to \mathbb{Z}$, $y_i \mapsto i$, $i = 0, 1$. Consider the invariant $f: [X, Y] \to B(X)$, $[a] \mapsto k \circ a$, $a \in C(X, Y)$.

The invariant $f$ is straight because, for the homomorphism $F: L(X, Y) \to B(X)$, $F(u)(x) = k^+(u(<x>))$, $x \in X$, $u \in L(X, Y)$, we have $f([a]) = F([a])$, $a \in C(X, Y)$.

Let $h: [X, Y] \to [SX, SY]$ be the main invariant. We show that there exists no homomorphism $d: [SX, SY] \to B(X)$ such that $f = d \circ h$. Assume that there is such a $d$.

Consider the homomorphism $l: \mathbb{Z}^X \to \text{Hom}(\langle X \rangle, \langle Y \rangle)$, $l(v)(<x>) = v(x)(<y_1>, \ldots, <y_0>)$, $x \in X$, $v \in \mathbb{Z}^X$. We have $l(f([a])) = \langle a \rangle - \langle a_0 \rangle$, $a \in C(X, Y)$, where $a_0 : X \to Y$, $x \mapsto y_0$. Clearly, there is an isomorphism $e: \text{Hom}(\langle X \rangle, \langle Y \rangle) \to [SX, SY]$ such that $e(\langle a \rangle) = h([a])$, $a \in C(X, Y)$. Consider the composition

$$r: \mathbb{Z}^X \xrightarrow{l} \text{Hom}(\langle X \rangle, \langle Y \rangle) \xrightarrow{e} [SX, SY] \xrightarrow{d} B(X).$$

For $a \in C(X, Y)$, we have $r(f([a])) = (d \circ e \circ l \circ f)([a]) = d(e(\langle a \rangle) - \langle a_0 \rangle) = d(h([a]) - h([a_0])) = f([a]) - f([a_0]) = f([a])$. Since the elements $f([a])$, $a \in C(X, Y)$, generate $B(X)$, we get $r|_{B(X)} = \text{id}$, which is impossible. □

§ 14. Invariants of maps $\mathbb{R}P^\infty \to \mathbb{R}P^\infty$

Here we show that the hypothesis of finitarity of $X$ in Theorem 1.1 and Proposition 12.2 is essential even if $M$ is divisible. (Possibly, if $M$ is divisible and/or $Y$ is (simply-)connected, the hypothesis of finitarity of $X$ can be replaced...
by the weaker one that \( X \) is weakly homotopy equivalent to a finite-dimensional CW-complex.)

Let \( X \) and \( Y \) be spaces. A set \( E \subseteq X \) is called \( Y \)-representative if any maps \( a, b \in C(X, Y) \) equal on \( E \) are homotopic. \( X \) is called \( Y \)-unitary if any finite cover of \( X \) contains a \( Y \)-representative set.

14.1. Lemma. Let \( M \) be a divisible group. If \( X \) is \( Y \)-unitary, then any invariant \( f: [X, Y] \to M \) is straight.

Proof. Introduce the maps \( r = [?]: C(X, Y) \to [X, Y] \) (the projection) and \( k = (?): C(X, Y) \to L(X, Y) \). We seek a homomorphism \( F \) giving the commutative diagram

\[
\begin{array}{ccc}
(C(X, Y)) & \xrightarrow{k^+} & L(X, Y) \\
\langle r \rangle & \downarrow & \downarrow F \\
\langle [X, Y] \rangle & \xrightarrow{f^+} & M.
\end{array}
\]

Since \( M \) is divisible, it suffices to show that \( \text{Ker} \, k^+ \subseteq \text{Ker} \langle r \rangle \). Take an element \( w \in \text{Ker} \, k^+ \). We show that \( w \in \text{Ker} \langle r \rangle \). There are a finite set \( I \), a map \( l: I \to C(X, Y) \), and an element \( v \in \langle I \rangle \) such that \( \langle l \rangle(v) = w \). Put \( \alpha_i = l(i), \, i \in I \). For an equivalence \( d \) on \( I \), let \( p_d: I \to I/d \) be the projection. Let \( N \) be the set of equivalences \( d \) on \( I \) such that \( \langle p_d \rangle(v) = 0 \) in \( \langle I/d \rangle \).

Take \( x \in X \). Consider the equivalence \( d(x) = \{ (i, j) : a_i(x) = a_j(x) \} \) on \( I \). We show that \( d(x) \in N \). We have the commutative diagrams

\[
\begin{array}{ccc}
I & \xrightarrow{t} & C(X, Y) \\
\downarrow p_d(x) & & \downarrow e_x \\
I/d(x) & \xrightarrow{l_x} & Y,
\end{array} \quad \begin{array}{ccc}
\langle C(X, Y) \rangle & \xrightarrow{k^+} & L(X, Y) \\
\langle e_x \rangle & \downarrow & \downarrow h_x \\
\langle Y \rangle & & \langle Y \rangle,
\end{array}
\]

where the map \( l_x \) is defined by the condition of commutativity of the diagram, \( e_x \) is the map of evaluation at \( x \), and \( h_x \) is the homomorphism of evaluation at \( \langle x \rangle \). We get \( \langle l_x \rangle((p_d(x))(v)) = \langle e_x \rangle((l)(v)) = \langle e_x \rangle(w) = h_x(k^+(w)) = 0 \). Since \( l_x \) is injective, we get \( \langle p_d(x) \rangle(v) = 0 \), which is what we promised.

For an equivalence \( d \) on \( I \), put \( E_d = \{ x \in X : (i, j) \in d \Rightarrow a_i(x) = a_j(x) \} \). Since \( x \in E_d(x) \) for any \( x \in X \), the family \( E_d, d \in N \), is a cover of \( X \). Since \( X \) is \( Y \)-unitary, \( E_d \) is \( Y \)-representative for some \( d \in N \). For \( (i, j) \in d \), the maps \( a_i \) and \( a_j \) are equal on \( E_d \) and thus homotopic. Therefore, there is a map \( m \) giving the commutative diagram

\[
\begin{array}{ccc}
I & \xrightarrow{t} & C(X, Y) \\
\downarrow p_d & & \downarrow r \\
I/d & \xrightarrow{m} & [X, Y],
\end{array}
\]

We get \( \langle r \rangle(w) = \langle r \rangle((l)(v)) = \langle m \rangle((p_d)(v)) = 0 \) because \( d \in N \).  \( \square \)
Hereafter, let $X$ and $Y$ be homeomorphic to $\mathbb{R}P^\infty$.

14.2. Lemma. $X$ is $Y$-unitary.

Proof. Let $H^*$ be the $\mathbb{Z}_2$-cohomology. Let $g \in H^1X$ and $h \in H^1Y$ be the non-zero classes.

We show that $(\ast)$ a set $E \subseteq X$ is $Y$-representative if $g|_U \neq 0$ for any neighbourhood $U$ of $E$. If maps $a, b \in C(X,Y)$ are equal on $E$, they are homotopic on some neighbourhood $U$ of $E$. Then $a^*(h)|_U = b^*(h)|_U$. Since $g|_U \neq 0$, the homomorphism $g|_U : H^1X \to H^1U$ is injective. Therefore, $a^*(h) = b^*(h)$. Since $Y$ is a $K(\mathbb{Z}_2, 1)$ space, $a$ and $b$ are homotopic, as needed.

We show that $X$ is $Y$-unitary. Assume that $X = E_1 \cup \ldots \cup E_n$, where the sets $E_i$ are not $Y$-representative. By $(\ast)$, each $E_i$ has a neighbourhood $U_i$ with $g|_{U_i} = 0$. Since $U_1 \cup \ldots \cup U_n = X$, we get $g^n = 0$, which is false. \qed

We have $[X,Y] = \{u_0, u_1\}$, where $u_0$ is the class of a constant map and $u_1$ is that of a homeomorphism. Consider the invariant $f : [X,Y] \to \mathbb{Q}$, $u_i \mapsto i$, $i = 0, 1$. By Lemmas 14.2 and 14.1, $f$ is straight. Let $h : [X,Y] \to [SX,SY]$ be the main invariant. Using the isomorphism

$$[SX,SY] \to \prod_{i \in \mathbb{Z}} \text{Hom}(H_iX,H_iY), \quad [v] \mapsto v_*,$$

we get $2h(u_0) = 2h(u_1)$. Therefore, there exists no homomorphism $d : [SX,SY] \to \mathbb{Q}$ such that $f = d \circ h$. \qed

§ 15. $K$-straight invariants

Let $K$ be a unital ring. $K$-modules are unital.

**$K$-module $L_K(X,Y)$.** For a set $X$, let $\langle X \rangle_K$ be the (free) $K$-module with the basis $X^2 \subseteq \langle X \rangle_K$ endowed with the bijection $X \to X^2_K$, $x \mapsto \langle x \rangle_K$. For sets $X$ and $Y$, let $L_K(X,Y) \subseteq \text{Hom}_K(\langle X \rangle_K,\langle Y \rangle_K)$ be the $K$-submodule generated by the $K$-homomorphisms $\varphi$ such that $\varphi(X^2) \subseteq Y^2 \cup \{0\}$. A map $a : X \to Y$ induces a $K$-homomorphism $(a)_K \in L_K(X,Y)$, $(a)_K(\langle x \rangle_K) = \langle a(x) \rangle_K$.

**$K$-straight invariants.** Let $X$ and $Y$ be spaces and $M$ be a $K$-module. An invariant $f : [X,Y] \to M$ is called $K$-straight if there exists a $K$-homomorphism $\tilde{F} : L_K(X,Y) \to M$ such that $f([a]) = \tilde{F}((a)_K)$ for all $a \in C(X,Y)$.

15.1. Proposition. An invariant $f : [X,Y] \to M$ is $K$-straight if and only if it is straight.

Proof is given in § 16.

**The $K$-main invariant $\tilde{h} : [X,Y] \to [S_KX,S_KY]_K$.** Let $S_KX$ be the $K$-complex of singular chains of $X$ with coefficients in $K$ and $[S_KX,S_KY]_K$ be the $K$-module of $K$-homotopy classes of $K$-morphisms $S_KX \to S_KY$. For $a \in C(X,Y)$, let $S_Ka : S_KX \to S_KY$ be the induced $K$-morphism and $[S_Ka]_K \in [S_KX,S_KY]_K$ be its $K$-homotopy class. The invariant $\tilde{h} : [X,Y] \to [S_KX,S_KY]_K$, $[a] \mapsto [S_Ka]_K$, is called $K$-main.
15.2. Theorem. Suppose that $X$ is valid and finitary and $Y$ is valid. An invariant $f: [X, Y] \to M$ is $K$-straight if and only if there exists a $K$-homomorphism $\tilde{d}: [S_KX, S_KY]_K \to M$ such that $f = \tilde{d} \circ \tilde{h}$.

Proof is given in § 16. For $K = \mathbb{Z}$, this is Theorem 1.1.

§ 16. $K$-straight invariants: proofs

Let $X$ and $Y$ be sets. We define a homomorphism $e: L(X, Y) \to L_K(X, Y)$.

For $u \in L(X, Y)$, let $e(u)$ be the $K$-homomorphism giving the commutative diagram

$$
\begin{array}{c}
\langle X \rangle \\
\downarrow i_X \quad \downarrow i_Y \\
\langle X \rangle_K \quad \langle Y \rangle_K
\end{array}
\xrightarrow{e(u)}
\begin{array}{c}
\langle Y \rangle \\
\downarrow i_X \quad \downarrow i_Y \\
\langle X \rangle_K \quad \langle Y \rangle_K
\end{array}
$$

where $i_X$ is the homomorphism $<x> \mapsto <x>_K$ and $i_Y$ is similar.

For an abelian group $A$, a $K$-module $M$, and a homomorphism $t: A \to M$, we introduce the $K$-homomorphism $t^{(K)}: K \otimes A \to M$, $1 \otimes a \mapsto t(a)$.

16.1. Lemma. $e^{(K)}: K \otimes L(X, Y) \to L_K(X, Y)$ is a $K$-isomorphism.

Proof. For $w \in (Y)_K$ and $y \in Y$, let $w/y \in K$ be the coefficient of $<y>_K$ in $w$. For $v \in L_K(X, Y)$ and $k \in K \setminus \{0\}$, we introduce the homomorphism $v_k \in L(X, Y)$,

$$v_k(<x>) = \sum_{y \in Y: x(y) = y} <y>, \quad x \in X.$$

It is not difficult to verify that the map $d: L_K(X, Y) \to K \otimes L(X, Y)$,

$$d(v) = \sum_{k \in K \setminus \{0\}} k \otimes v_k,$$

is a $K$-homomorphism. Using this, we get $e^{(K)} \circ d = \text{id}$ and $d \circ e^{(K)} = \text{id}$. \qed

Proof of Proposition 15.1. Necessity. Let $f$ be $K$-straight. There is a $K$-homomorphism $\hat{F}: L_K(X, Y) \to M$ such that $\hat{F}([a]) = \hat{F}([a]_K)$, $a \in C(X, Y)$.

Consider the homomorphism $F = \hat{F} \circ e$:
The diagram is commutative. We get $f([a]) = F((a))$, $a \in C(X, Y)$. Therefore, $f$ is straight.

Sufficiency. Let $f$ be straight. There is a homomorphism $F: L(X, Y) \to M$ such that $f([a]) = F((a))$, $a \in C(X, Y)$. By Lemma 16.1, $e(K)$ is a $K$-isomorphism. Consider the homomorphism $\tilde{F} = F(e(K))^{-1}$:

\[
\begin{array}{ccc}
C(X, Y) & \xrightarrow{(7)_K} & L_K(X, Y) \\
\downarrow & & \downarrow \\
[X, Y] & \xrightarrow{\tilde{F}} & M.
\end{array}
\]

The diagram is commutative. We get $f([a]) = \tilde{F}((a)_K)$, $a \in C(X, Y)$. Therefore, $f$ is $K$-straight. □

**The homomorphism $I$:** $[SX, SY] \to [S_K X, S_K Y]_K$. Let $X$ and $Y$ be spaces. A morphism $v: SX \to SY$ induces a $K$-morphism

$$S_K X = K \otimes SX \xrightarrow{id \otimes v} K \otimes SY = S_K Y.$$ Consider the homomorphism $I: [SX, SY] \to [S_K X, S_K Y]_K$, $[v] \mapsto [id \otimes v]_K$.

**16.2. Lemma.** If the group $H_\bullet(X)$ is finitely generated, then the $K$-homomorphism

$$I(K): K \otimes [SX, SY] \to [S_K X, S_K Y]_K$$

is a $K$-split $K$-monomorphism, i.e., there exists a $K$-homomorphism $R: [S_K X, S_K Y]_K \to K \otimes [SX, SY]$ such that $R \circ I(K) = id$.

This is a variant of the universal coefficient theorem, cf. [12, Theorems 5.2.8 and 5.5.10]. □

**Proof of Theorem 15.2.** We have $\tilde{h} = I \circ h$, where $h: [X, Y] \to [SX, SY]$ is the main invariant. By Proposition 7.3, $h$ is straight. Therefore, $\tilde{h}$ is straight. By Proposition 15.1, $\tilde{h}$ is $K$-straight.

This gives the sufficiency. Necessity. Let $f$ be $K$-straight. By Proposition 15.1, $f$ is straight. By Proposition 12.2, there is a homomorphism $d: [SX, SY] \to M$ such that $f = d \circ h$. By Lemma 16.2, there is a $K$-homomorphism $\tilde{d}$ such that $\tilde{d} \circ I(K) = d(K)$:
The diagram is commutative. In particular, $f = \tilde{d} \circ \tilde{h}$.

References


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