Homotopy similarity of maps. Maps of the circle

S. S. Podkorytov

We describe the relation of r-similarity and finite-order invariants on the homotopy set $[S^1, Y] = \pi_1(Y)$.

§ 1. Introduction

This paper continues [4]. We adopt notation and conventions thereof. Here we are mainly interested in the set $[S^1, Y] = \pi_1(Y)$; in Part I, however, we consider a more general case. Let X and Y be cellular spaces, X compact. Let X be equipped with maps $\mu : X \to X \vee X$ (comultiplication) and $\nu : X \to X$ (coinversion). The set Y^X carries the operations

$$(a, b) \mapsto (a \# b : X \xrightarrow{\mu} X \vee X \xrightarrow{a \triangledown b} Y)$$

and

$$a \mapsto (a^\dagger : X \xrightarrow{\nu} X \xrightarrow{a} Y).$$

We suppose that the set $[X, Y]$ is a group with the identity $1 = [\lhd^X_Y]$, the multiplication

$$[a][b] = [a \# b]$$

and the inversion

$$[a]^{-1} = [a^\dagger].$$

Under these assumptions, we call $(X, \mu, \nu; Y)$ an *admissible couple*.

Put

$$[X, Y]^{(r+1)} = \{\, \boldsymbol{a} \in [X, Y] \mid 1 \overset{r}{\sim} \boldsymbol{a} \,\}.$$

We get the filtration

$$[X, Y] = [X, Y]^{(1)} \supseteq [X, Y]^{(2)} \supseteq \dots .$$

We prove that the subsets $[X, Y]^{(r+1)}$ are normal subgroups and form an N-series (Theorems 4.1 and 4.3). The equivalence holds

$$\boldsymbol{a} \overset{r}{\sim} \boldsymbol{b} \quad \Leftrightarrow \quad \boldsymbol{a}^{-1}\boldsymbol{b} \in [X, Y]^{(r+1)}$$

(Theorem 4.2).

In Part III, we concentrate on the case $X = S^1$ (with the standard μ and ν), when $[X, Y] = \pi_1(Y)$. We prove that

$$\pi_1(Y)^{(r+1)} = \gamma^{r+1}\pi_1(Y)$$

(Theorem 11.2). Here, as usual,

$$G = \gamma^1 G \supseteq \gamma^2 G \supseteq \dots$$

is the lower central series of a group G.

For a homotopy invariant (i. e., a function) $f : \pi_1(Y) \to L$, where L is an abelian group, its order $\operatorname{ord} f \in \{-\infty, 0, 1, \dots, \infty\}$ is defined (see [4, § 1]). We prove that $\operatorname{ord} f = \deg f$ (Theorem 12.2). Recall that, for a function $f : G \to L$, where G is a group, its degree $\deg f$ is defined (see § 12).

Do invariants of order at most r distinguish elements of $\pi_1(Y)$ that are not r-similar? In general, no. For $r \geqslant 3$, there is a group G and an element $g \in G \setminus \gamma^{(r+1)} G$ such that, for any abelian group L and function $f : G \to L$ of degree at most r, one has $f(1) = f(g)$ (see [5] for $r = 3$ and [3, Ch. 2]). Take a cellular space Y with $\pi_1(Y) = G$. Then, by Theorems 11.2 and 12.2, the homotopy classes 1 and g in $\pi_1(Y)$ are not r-similar, but cannot be distinguished by invariants of order at most r.

In Part II, which does not depend on the rest of the paper, we prove group-theoretic Theorem 9.1, which we need for the proof of the above-mentioned Theorem 11.2.

Part I

In this part, we discuss operations over coherent ensembles of maps between arbitrary spaces (§§ 2 and 3) and give our results concerning an arbitrary admissible couple (§ 4).

§ 2. Compositions

Let X, Y, X', and Y' be spaces and $k : X' \to X$ and $h : Y \to Y'$ be maps. Introduce the homomorphisms

$$k^{\#} : \langle Y^X \rangle \to \langle Y^{X'} \rangle, \qquad <a> \mapsto <a \circ k>,$$

and

$$h_{\#} : \langle Y^X \rangle \to \langle Y'^X \rangle, \qquad <a> \mapsto <h \circ a>.$$

2.1. Lemma. *We have*

$$k^{\#}(\langle Y^X \rangle^{(r+1)}) \subseteq \langle Y^{X'} \rangle^{(r+1)} \quad \textit{and} \quad h_{\#}(\langle Y^X \rangle^{(r+1)}) \subseteq \langle Y'^X \rangle^{(r+1)}.$$

Proof. Take an ensemble $A \in \langle Y^X \rangle^{(r+1)}$.

To show that $k^{\#}(A) \in \langle Y^{X'} \rangle^{(r+1)}$, we take $T' \in \mathcal{F}_r(X')$ and check that $k^{\#}(A)|_{T'} = 0$. We have the commutative diagram

$$\begin{array}{ccccc}
k^{\#}(A) & \langle Y^{X'} \rangle & \xleftarrow{\;k^{\#}\;} & \langle Y^X \rangle & A \\
 & \big\downarrow {\scriptstyle ?|_{T'}} & & \big\downarrow {\scriptstyle ?|_{k(T')}} & \\
k^{\#}(A)|_{T'} & \langle Y^{T'} \rangle & \xleftarrow{\;q^{\#}\;} & \langle Y^{k(T')} \rangle, & A|_{k(T')} = 0
\end{array}$$

where $q = k|_{T' \to k(T')}$. Since $k(T') \in \mathcal{F}_r(X)$, we have $A|_{k(T')} = 0$. By the diagram, $k^{\#}(A)|_{T'} = 0$.

To show that $h_{\#}(A) \in \langle {Y'}^X \rangle^{(r+1)}$, we take $T \in \mathcal{F}_r(X)$ and check that $h_{\#}(A)|_T = 0$. We have the commutative diagram

$$\begin{array}{ccccc} A & \langle Y^X \rangle & \xrightarrow{h_{\#}} & \langle {Y'}^X \rangle & h_{\#}(A) \\ & {\scriptstyle ?|_T}\downarrow & & \downarrow{\scriptstyle ?|_T} & \\ 0{=}A|_T & \langle Y^T \rangle & \xrightarrow{h_{\#}} & \langle {Y'}^T \rangle. & h_{\#}(A)|_T \end{array}$$

We have $A|_T = 0$. By the diagram, $h_{\#}(A)|_T = 0$. □

2.2. Corollary. *Let $a, b \in Y^X$ satisfy $a \overset{r}{\sim} b$. Then $a \circ k \overset{r}{\sim} b \circ k$ in ${X'}^Y$ and $h \circ a \overset{r}{\sim} h \circ b$ in $X^{Y'}$.*

Proof. There is an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<} a_i {>},$$

where $a_i \sim a$, such that $A \overset{r}{=} {<}b{>}$. By Lemma 2.1, $k^{\#}(A) \overset{r}{=} {<}b \circ k{>}$ and $h_{\#}(A) \overset{r}{=} {<}h \circ b{>}$. Since all the maps of $k^{\#}(A)$ are homotopic to $a \circ k$, we get $a \circ k \overset{r}{\sim} b \circ k$. Since all the maps of $h_{\#}(A)$ are homotopic to $h \circ b$, we get $h \circ a \overset{r}{\sim} h \circ b$. □

§ 3. Joining coherent ensembles

Let X_1, X_2, and Y be spaces. Introduce the homomorphism

$$(\veebar) : \langle Y^{X_1} \rangle \otimes \langle Y^{X_2} \rangle \to \langle Y^{X_1 \vee X_2} \rangle, \qquad {<}a{>} \otimes {<}b{>} \mapsto {<}a \veebar b{>}.$$

3.1. Lemma. *For $p, q \geqslant 0$, we have*

$$(\veebar)(\langle Y^{X_1} \rangle^{(p)} \otimes \langle Y^{X_2} \rangle^{(q)}) \subseteq \langle Y^{X_1 \vee X_2} \rangle^{(p+q)}.$$

Proof. Take $A \in \langle Y^{X_1} \rangle^{(p)}$ and $B \in \langle Y^{X_2} \rangle^{(q)}$. We show that $(\veebar)(A \otimes B) \in \langle Y^{X_1 \vee X_2} \rangle^{(p+q)}$. Take $T \in \mathcal{F}_{p+q-1}(X_1 \vee X_2)$. We check that $(\veebar)(A \otimes B)|_T = 0$. We have $T = T_1 \vee T_2$ for some finite subspaces $T_i \subseteq X_i$, $i = 1, 2$. We have the commutative diagram

$$\begin{array}{ccccc} A{\otimes}B & \langle Y^{X_1} \rangle \otimes \langle Y^{X_2} \rangle & \xrightarrow{(\veebar)} & \langle Y^{X_1 \vee X_2} \rangle & (\veebar)(A{\otimes}B) \\ & {\scriptstyle ?|_{T_1} \otimes ?|_{T_1}}\downarrow & & \downarrow{\scriptstyle ?|_T} & \\ A|_{T_1}{\otimes}B|_{T_2} & \langle Y^{T_1} \rangle \otimes \langle Y^{T_2} \rangle & \xrightarrow{(\veebar)} & \langle Y^T \rangle. & (\veebar)(A{\otimes}B)|_T \end{array}$$

We have $T_1 \in \mathcal{F}_{p-1}(X_1)$ or $T_2 \in \mathcal{F}_{q-1}(X_2)$. Thus $A|_{T_1} = 0$ or $B|_{T_2} = 0$. By the diagram, $(\veebar)(A \otimes B)|_T = 0$. □

§ 4. Similarity for an admissible couple

Let $(X, \mu, \nu; Y)$ be an admissible couple.

4.1. Theorem. *$[X, Y]^{(r+1)} \subseteq [X, Y]$ is a subgroup.*

Proof. To show that $[X, Y]^{(r+1)}$ is closed under multiplication, we take $a, b \in Y^X$ such that $\daleth \overset{r}{\sim} a$ and $\daleth \overset{r}{\sim} b$ and check that $\daleth \overset{r}{\sim} a \# b$. There are ensembles $D, E \in \langle Y^X \rangle$,

$$D = \sum_i u_i {<} d_i {>} \quad \text{and} \quad E = \sum_j v_j {<} e_j {>},$$

where $d_i \sim \daleth$ and $e_j \sim \daleth$, such that $D \overset{r}{=} {<} a {>}$ and $E \overset{r}{=} {<} b {>}$. Consider the maps $a \,\underline{\nabla}\, b, d_i \,\underline{\nabla}\, e_j : X \vee X \to Y$ and the ensemble $F \in \langle Y^{X \vee X} \rangle$,

$$F = \sum_{i,j} u_i v_j {<} d_i \,\underline{\nabla}\, e_j {>}.$$

We have

$$\begin{aligned} {<} a \,\underline{\nabla}\, b {>} - F &= (\underline{\nabla})({<} a {>} \otimes {<} b {>}) - (\underline{\nabla})(D \otimes E) = \\ &= (\underline{\nabla})(({<} a {>} - D) \otimes {<} b {>}) + (\underline{\nabla})(D \otimes ({<} b {>} - E) \in \langle Y^{X \vee X} \rangle^{(r+1)}, \end{aligned}$$

where $\in$ holds by Lemma 3.1. Since all the maps of F are null-homotopic, we get $\daleth \overset{r}{\sim} a \,\underline{\nabla}\, b$. Since $a \# b = (a \,\underline{\nabla}\, b) \circ \mu$, Corollary 2.2 yields $\daleth \overset{r}{\sim} a \# b$.

Take $a \in Y^X$ such that $\daleth \overset{r}{\sim} a$. Since $a^\dagger = a \circ \nu$, Corollary 2.2 yields $\daleth \overset{r}{\sim} a^\dagger$. Thus $[X, Y]^{(r+1)}$ is closed under inversion. □

4.2. Theorem. *For $\boldsymbol{a}, \boldsymbol{b} \in [X, Y]$, we have*

$$\boldsymbol{a} \overset{r}{\sim} \boldsymbol{b} \quad \Leftrightarrow \quad \boldsymbol{a}^{-1} \boldsymbol{b} \in [X, Y]^{(r+1)}. \tag{1}$$

Proof. It suffices to check the implication

$$a \overset{r}{\sim} b \quad \Rightarrow \quad c \# a \overset{r}{\sim} c \# b$$

for $a, b, c \in Y^X$. Given an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<} a_i {>},$$

where $a_i \sim a$, such that $A \overset{r}{=} {<} b {>}$, consider the ensemble $F \in \langle Y^{X \vee X} \rangle$,

$$F = \sum_i u_i {<} c \,\underline{\nabla}\, a_i {>}.$$

We have

$${<} c \,\underline{\nabla}\, b {>} - F = (\underline{\nabla})({<} c {>} \otimes ({<} b {>} - A)) \in \langle Y^{X \vee X} \rangle^{(r+1)},$$

where $\in$ holds by Lemma 3.1. Thus $F \overset{r}{=} {<} c \,\underline{\nabla}\, b {>}$. Since $c \,\underline{\nabla}\, a_i \sim c \,\underline{\nabla}\, a$, we get $c \underline{\nabla} a \overset{r}{\sim} c \underline{\nabla} b$. Taking composition with μ, we get $c \# a \overset{r}{\sim} c \# b$ by Corollary 2.2. □

Theorems 4.1 and 4.2 imply that the relaton $\overset{r}{\sim}$ on $[X,Y]$ is an equivalence, which is a special case of [4, Theorem 8.1] (note that we did not use it here).

One can prove similarly that

$$\boldsymbol{a} \overset{r}{\sim} \boldsymbol{b} \quad \Leftrightarrow \quad \boldsymbol{b}\boldsymbol{a}^{-1} \in [X,Y]^{(r+1)}. \tag{2}$$

It follows from (1) and (2) that the subgroup $[X,Y]^{(r+1)} \subseteq [X,Y]$ is normal. This is a special case of the following theorem.

Let $[\![\ ,\ ]\!]$ denote the group commutator.

4.3. Theorem. *Put $M^s = [X,Y]^{(s)} \subseteq [X,Y]$. Then $[\![M^p, M^q]\!] \subseteq M^{p+q}$.*

Proof. Introduce the map

$$\zeta : X \xrightarrow{\mu^{(3)}} X \vee X \vee X \vee X \xrightarrow{(\mathrm{in}_1 \circ \nu) \underline{\triangledown} (\mathrm{in}_2 \circ \nu) \underline{\triangledown} \mathrm{in}_1 \underline{\triangledown} \mathrm{in}_2} X \vee X,$$

where

$$\mu^{(3)} : X \xrightarrow{\mu} X \vee X \xrightarrow{\mu \vee \mathrm{id}_X} X \vee X \vee X \xrightarrow{\mu \vee \mathrm{id}_X \vee \mathrm{id}_X} X \vee X \vee X \vee X$$

(4-fold comultiplication). For $a, b \in Y^X$, we have

$$[(a \,\underline{\triangledown}\, b) \circ \zeta] = [\![[a],[b]]\!] \tag{3}$$

in the group $[X,Y]$.

Take $a, b \in Y^X$ such that $\urcorner \overset{p-1}{\sim} a$ and $\urcorner \overset{q-1}{\sim} b$. We show that $\urcorner \overset{p+q-1}{\sim} (a \,\underline{\triangledown}\, b) \circ \zeta$.

There are ensembles $D, E \in \langle Y^X \rangle$,

$$D = \sum_i u_i <d_i> \quad \text{and} \quad E = \sum_j v_j <e_j>,$$

where $d_i \sim \urcorner$ and $e_j \sim \urcorner$, such that $D \overset{p-1}{=} <a>$ and $E \overset{q-1}{=} <b>$. Consider the ensemble $F \in \langle Y^{X \vee X} \rangle$,

$$F = \sum_i u_i <d_i \,\underline{\triangledown}\, b> + \sum_j v_j <a \,\underline{\triangledown}\, e_j> - \sum_{i,j} u_i v_j <d_i \,\underline{\triangledown}\, e_j>.$$

We have

$$<a \,\underline{\triangledown}\, b> - F = (\underline{\triangledown})((<a> - D) \otimes (<b> - E)) \in \langle Y^{X \vee X} \rangle^{(p+q)},$$

where $\in$ holds by Lemma 3.1. Thus $F \overset{p+q-1}{=} <a \underline{\triangledown} b>$. By Lemma 2.1, $\zeta^{\#}(F) \overset{p+q-1}{=} <(a \,\underline{\triangledown}\, b) \circ \zeta>$. By (3), all the maps of $\zeta^{\#}(F)$ are null-homotopic. Thus we get $\urcorner \overset{p+q-1}{\sim} (a \,\underline{\triangledown}\, b) \circ \zeta$. ∎

Part II

In this part, which is algebraic and does not depend on the rest of the paper, we prove Theorem 9.1.

§ 5. Cultured sets

Let E be a set. Consider the $\mathbb{Q}$-algebra $\mathbb{Q}^E$ of functions $E \to \mathbb{Q}$. A *culture* on E is a filtration $\Phi = (\Phi_s)_{s \geqslant 0}$ of $\mathbb{Q}^E$ by $\mathbb{Q}$-submodules

$$\Phi_0 \subseteq \Phi_1 \subseteq \ldots \subseteq \mathbb{Q}^E$$

such that

$$1 \in \Phi_0 \quad \text{and} \quad \Phi_s \Phi_t \subseteq \Phi_{s+t}.$$

A set equipped with a culture is called a *cultured set*. The culture of a cultured set E is denoted by Φ^E.

A way to define a culture on a set E is to choose a collection of pairs (u_i, s_i), where $u_i \in \mathbb{Q}^E$ is a function and $s_i \geqslant 1$ is a number called the *weight*, and to let Φ_s be spanned by all products $u_{i_1} \ldots u_{i_p}$ $(p \geqslant 0)$ with $s_{i_1} + \ldots + s_{i_p} \leqslant s$. We define the cultured set

$$\mathbb{Q}^m_{s_1 \ldots s_m} \tag{4}$$

as $\mathbb{Q}^m$ with the culture given the collection (ξ_i, s_i), $i \in (m)$, where $\xi_i : \mathbb{Q}^m \to \mathbb{Q}$ is the ith coordinate. Hereafter, we put $(m) = \{1, \ldots, m\}$. The cultured set

$$\mathbb{Z}^m_{s_1 \ldots s_m}$$

is defined similarly. We put $\mathbb{Q}_s = \mathbb{Q}^1_s$ and $\mathbb{Z}_s = \mathbb{Z}^1_s$.

A function $g : E \to F$ between cultured sets is called a *cultural morphism* if the induced algebra homomorphism $h^\# : \mathbb{Q}^F \to \mathbb{Q}^E$ satisfies $g^\#(\Phi^F_s) \subseteq \Phi^E_s$ for all s. A function

$$g : \mathbb{Q}^m_{s_1 \ldots s_m} \to \mathbb{Q}^n_{t_1 \ldots t_n}$$

is a cultural morphism if and only if it has the form

$$g(x_1, \ldots, x_m) = (P_j(x_1, \ldots, x_m))_{j \in (n)},$$

where P_j is a rational polynomial of degree at most t_j with respect to its arguments having weights $s_1, \ldots, s_m$. Cultural maps

$$\mathbb{Z}^m_{s_1 \ldots s_m} \to \mathbb{Z}^n_{t_1 \ldots t_n}$$

are characterized similarly (their coordinate polynomials need not have integer coefficients).

Cultured sets and cultural morphisms form a category with products. We have

$$\mathbb{Q}^m_{s_1 \ldots s_m} \times \mathbb{Q}^n_{t_1 \ldots t_n} = \mathbb{Q}^{m+n}_{s_1 \ldots s_m t_1 \ldots t_n} \quad \text{and} \quad \mathbb{Z}^m_{s_1 \ldots s_m} \times \mathbb{Z}^n_{t_1 \ldots t_n} = \mathbb{Z}^{m+n}_{s_1 \ldots s_m t_1 \ldots t_n}.$$

A cultural morphism $g : E \to F$ is called a *cultural immersion* if $g^\#(\Phi^F_s) = \Phi^E_s$ for all s. Then a function $f : D \to E$, where D is a cultured set, is a cultural morphism if the composition

$$D \xrightarrow{f} E \xrightarrow{g} F$$

is. If the composition

$$E \xrightarrow{g} F \xrightarrow{h} G$$

of two cultural morphisms is a cultural immersion, then g is.

§ 6. The truncated free algebra $\boldsymbol{A}/\boldsymbol{A}^{(r+1)}$

Consider the algebra $\boldsymbol{A}$ of rational polynomials in non-commuting variables $T_1, \dots, T_n$. It is graded in the standard way,

$$\boldsymbol{A} = \bigoplus_{s \geqslant 0} \boldsymbol{A}_s.$$

Introduce the ideals $\boldsymbol{A}^{(s)} \subseteq \boldsymbol{A}$,

$$\boldsymbol{A}^{(s)} = \bigoplus_{t \geqslant s} \boldsymbol{A}_t.$$

We fix $r \geqslant 0$ and consider the algebra $\boldsymbol{A}/\boldsymbol{A}^{(r+1)}$. Let $\overline{T}_i \in \boldsymbol{A}/\boldsymbol{A}^{(r+1)}$ be the image of T_i. An element $w \in \boldsymbol{A}/\boldsymbol{A}^{(r+1)}$ has the form

$$w = \sum_{s \geqslant 0,\, i_1, \dots, i_s} w^{(s)}_{i_1 \dots i_s} \overline{T}_{i_1} \dots \overline{T}_{i_s},$$

where $w^{(s)}_{i_1 \dots i_s} \in \mathbb{Q}$ are uniquely defined for $s \leqslant r$ and arbitrary for greater s. Introduce the group

$$\mathcal{U} = 1 + \boldsymbol{A}^{(1)}/\boldsymbol{A}^{(r+1)} \subseteq (\boldsymbol{A}/\boldsymbol{A}^{(r+1)})^{\times}$$

with the filtration by subgroups

$$\mathcal{U} = \mathcal{U}^{(1)} \supseteq \mathcal{U}^{(2)} \supseteq \dots$$

where

$$\mathcal{U}^{(s)} = 1 + \boldsymbol{A}^{(s)}/\boldsymbol{A}^{(r+1)}, \qquad s \leqslant r+1,$$

and $\mathcal{U}^{(s)} = 1$ for $s \geqslant r+1$. We have $[\mathcal{U}^{(s)}, \mathcal{U}^{(t)}] \subseteq \mathcal{U}^{(s+t)}$. In particular, $\gamma^s \mathcal{U} \subseteq \mathcal{U}^{(s)}$.

We equip the set $\mathcal{U}$ with the culture given by the collection of pairs

$$(\xi^{(s)}_{i_1 \dots i_s}, s), \qquad 1 \leqslant s \leqslant r, \quad i_1, \dots, i_s \in (n),$$

where

$$\xi^{(s)}_{i_1 \dots i_s} : \mathcal{U} \to \mathbb{Q}, \qquad u \mapsto u^{(s)}_{i_1 \dots i_s} : \mathcal{U} \to \mathbb{Q}$$

for

$$u = \sum_{t \geqslant 0,\, j_1, \dots, j_t} u^{(t)}_{j_1 \dots j_t} \overline{T}_{j_1} \dots \overline{T}_{j_t}$$

with $u^{(0)} = 1$. Clearly, the cultured set $\mathcal{U}$ is a special case of (4).

For a group G, let $M_G : G \times G \to G$ be the multiplication.

6.1. Lemma. *The function*

$$M_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$$

is a cultural morphism.

Proof. Given $u, v \in \mathcal{U}$,

$$u = \sum_{s\geqslant 0,\, i_1,\dots,i_s} u^{(s)}_{i_1\dots i_s} \overline{T}_{i_1} \dots \overline{T}_{i_s} \quad \text{and} \quad v = \sum_{t\geqslant 0,\, j_1,\dots,j_s} v^{(t)}_{j_1\dots j_t} \overline{T}_{j_1} \dots \overline{T}_{j_t}$$

with $u^{(0)} = v^{(0)} = 1$, we have

$$uv = \sum_{\substack{s\geqslant 0,\, i_1,\dots,i_s,\\ t\geqslant 0,\, j_1,\dots,j_t}} u^{(s)}_{i_1\dots i_s} v^{(t)}_{j_1\dots j_t} \overline{T}_{i_1} \dots \overline{T}_{i_s} \overline{T}_{j_1} \dots \overline{T}_{j_t}.$$

We consider $u^{(s)}_{i_1\dots i_s}$ and $v^{(t)}_{j_1\dots j_t}$ with $s, t > 0$ here as variables of weights s and t, respectively. In the last expression, the monomial in $\overline{T}_i$ has degree $s+t$, and its coefficient is $u^{(s)}_{i_1\dots i_s} v^{(t)}_{j_1\dots j_t}$ and thus has degree $s+t$. Thus the total coefficient of each monomial in $\overline{T}_i$ of some degree $z > 0$ in this series is a polynomial in $u^{(s)}_{i_1\dots i_s}$ and $v^{(t)}_{j_1\dots j_t}$ of degree at most z. □

6.2. Lemma. *For $u \in \mathcal{U}^{(s)}$, the function*

$$E_u : \mathbb{Z}_s \to \mathcal{U}, \qquad x \to u^x,$$

is a cultural morphism. Moreover, it extends to a cultural morphism

$$\widehat{E}_u : \mathbb{Q}_s \to \mathcal{U}.$$

Proof. We have $u = 1 + w$ in $\boldsymbol{A}/\boldsymbol{A}^{(r+1)}$ for some $w \in \boldsymbol{A}^{(s)}/\boldsymbol{A}^{(r+1)}$,

$$w = \sum_{t\geqslant s,\, i_1,\dots,i_t} w^{(t)}_{i_1\dots i_t} \overline{T}_{i_1} \dots \overline{T}_{i_t}.$$

For $x \in \mathbb{Z}$, we have

$$E_u(x) = u^x = (1+w)^x = \sum_{p\geqslant 0} \binom{x}{p} w^p =$$

$$= \sum_{p\geqslant 0} \sum_{\substack{t_1\geqslant s,\, i_{11},\dots,i_{1t_1},\\ \dots,\\ t_p\geqslant s,\, i_{p1},\dots,i_{pt_p}}} w^{(t_1)}_{i_{11},\dots,i_{1t_1}} \dots w^{(t_p)}_{i_{p1},\dots,i_{pt_p}} \binom{x}{p} \cdot$$

$$\cdot \overline{T}_{i_{11}} \dots \overline{T}_{i_{1t_1}} \dots \overline{T}_{i_{p1}} \dots \overline{T}_{i_{pt_p}}.$$

Consider x here as a variable of weight s. In the last expression, the monomial in $\overline{T}_i$ has degree $t_1 + \ldots + t_p$, which is at least ps. Its coefficient is a rational multiple of $\binom{x}{p}$ and thus a polynomial in x of degree at most ps. Thus the total coefficient of each monomial in $\overline{T}_i$ of some degree $z > 0$ in this series is a polynomial in x of degree at most z.

The extension $\widehat{E}_u$ exists automatically. □

§ 7. The free nilpotent group $\mathcal{N}$

Recall that we fix numbers n and r. Let $\boldsymbol{F}$ be the free group on the generators $Z_1, \ldots, Z_n$. Consider the free nilpotent group $\mathcal{N} = \boldsymbol{F}/\gamma^{r+1}\boldsymbol{F}$. Put $\mathcal{N}^{(s)} = \gamma^s\mathcal{N} \subseteq \mathcal{N}$, $s \geqslant 1$.

Following Magnus, consider the homomorphism

$$\rho : \mathcal{N} \to \mathcal{U}, \qquad \overline{Z}_i \mapsto 1 + \overline{T}_i.$$

Hereafter, the bar denotes the projection to the proper quotient group. The homomorphism ρ exists because $\gamma^{r+1}\mathcal{U} = 1$, The quotient $\mathcal{N}^{(s)}/\mathcal{N}^{(s+1)}$ is abelian and finitely generated. Since $\rho(\mathcal{N}^{(s)}) \subseteq \gamma^s\mathcal{U} \subseteq \mathcal{U}^{(s)}$, there is a homomorphism

$$\sigma^{(s)} : \mathcal{N}^{(s)}/\mathcal{N}^{(s+1)} \to \boldsymbol{A}_s$$

such that

$$\rho(h) \equiv 1 + \sigma^{(s)}(\overline{h}) \pmod{\boldsymbol{A}^{(s+1)}}, \qquad h \in \mathcal{N}^{(s)}.$$

By Magnus [2] (see also [6, Part I, Ch. IV, Theorem 6.3]), $\mathcal{N}^{(s)} = \rho^{-1}(\mathcal{U}^{(s)})$. It follows that $\sigma^{(s)}$ are injective and $\mathcal{N}^{(s)}/\mathcal{N}^{(s+1)}$ are torsion-free and thus free abelian. It follows that there is a filtration

$$\mathcal{N} = \mathcal{N}^1 \supseteq \mathcal{N}^2 \supseteq \ldots \supseteq \mathcal{N}^q \supseteq \mathcal{N}^{q+1} = 1$$

such that $\mathcal{N}^{(s)} = \mathcal{N}^{j_s}$ for some $1 = j_1 \leqslant \ldots \leqslant j_{r+1} = q+1$ and $\mathcal{N}^j/\mathcal{N}^{j+1}$ are infinite cyclic. For $j \leqslant n+1$, we choose $\mathcal{N}^j$ be the subgroup generated by $\overline{Z}_j, \ldots, \overline{Z}_n$ and $\mathcal{N}^{(2)}$. Put $s_j = \max\{\, s \mid j_s \leqslant j \,\}$, $j \in (q)$. Clearly, $1 \leqslant s_1 \leqslant \ldots \leqslant s_q \leqslant r$, $s_1 = \ldots = s_n = 1$, and $\mathcal{N}^j \subseteq \mathcal{N}^{(s_j)}$. The subgroups $\mathcal{N}^j \subseteq \mathcal{N}$ are normal.

For each $j \in (q)$, choose an element $b_j \in \mathcal{N}^j$ such that $\overline{b}_j$ generates $\mathcal{N}^j/\mathcal{N}^{j+1}$. In doing so, we put

$$b_j = \overline{Z}_j, \qquad j \in (n).$$

The collection $(b_1, \ldots, b_q)$ is a "Mal′cev basis" [1, 4.2.2]. For $j \in (q+1)$, the function

$$\beta^j : \mathbb{Z}^{q-j+1} \to \mathcal{N}^j, \qquad (x_j, \ldots, x_q) \mapsto b_j^{x_j} \ldots b_q^{x_q},$$

is bijective. We put

$$\beta = \beta^1 : \mathbb{Z}^q \to \mathcal{N}.$$

The elements $\sigma^{(s_j)}(\overline{b}_j) \in \boldsymbol{A}$ are linearly independent.

Any group G carries the *immanent* culture Φ with Φ_s consisting of all functions $G \to \mathbb{Q}$ of degree at most s (see § 12). If $\mathcal{N}$ is equipped with its immanent culture,

$$\beta : \mathbb{Z}^q_{s_1 \ldots s_q} \to \mathcal{N}$$

becomes a culture isomorphism. The proof is omitted.

7.1. Lemma. *The composition*

$$\eta^j : \mathbb{Z}^{q-j+1}_{s_j \ldots s_q} \xrightarrow{\beta^j} \mathcal{N}^j \xrightarrow{\rho^j} \mathcal{U},$$

where $\rho^j = \rho|_{\mathcal{N}^j}$, *is a cultural immersion.*

Introduce the projections

$$\mathbf{p} : \mathbb{Q}^{q-j+1} \to \mathbb{Q}, \qquad (x_j \dots, x_q) \mapsto x_j,$$

and

$$\mathbf{R} : \mathbb{Q}^{q-j+1} \to \mathbb{Q}^{q-j}, \qquad (x_j \dots, x_q) \mapsto (x_{j+1}, \dots, x_q).$$

Proof. We show that η^j is a cultural morphism by backward induction on j. For $j = q + 1$, the assertion is trivial. Take $j \leqslant q$. Since $b^j \in \mathcal{N}^{(s_j)}$, we have $\rho(b^j) \in \mathcal{U}^{(s_j)}$. We have the decomposition

$$\eta^j : \mathbb{Z}^{q-j+1}_{s_j \dots s_q} = \mathbb{Z}_{s_j} \times \mathbb{Z}^{q-j}_{s_{j+1} \dots s_q} \xrightarrow{E_{\rho(b_j)} \times \eta^{j+1}} \mathcal{U} \times \mathcal{U} \xrightarrow{M_{\mathcal{U}}} \mathcal{U},$$

where $E_{\rho(b_j)} : x \mapsto \rho(b_j)^x$. Here cultural morphisms are $E_{\rho(b_j)}$ by Lemma 6.2, η^{j+1} by the induction hypothesis, and $M_{\mathcal{U}}$ by Lemma 6.1. Thus η^j is a cultural morphism.

For each $j \in (q)$, choose a linear functional $\phi_j : \boldsymbol{A}_{s_j} \to \mathbb{Q}$ such that $\phi_j(\sigma^{(s_j)}(b_k))$ equals 1 for $k = j$ and 0 for all other k with $s_k = s_j$. Given $j \leqslant q + 1$ and $x = (x_j, \dots, x_q) \in \mathbb{Z}^{q-j+1}$, we have

$$\eta^j(x) = \rho(\beta^j(x)) = \rho(b_j^{x_j} \dots b_q^{x_q}) = \rho(b_j^{x_j}) \dots \rho(b_q^{x_q}).$$

Assume $j \leqslant q$. Then

$$\eta^j(x) \equiv 1 + \sum_{k \geqslant j : s_k = s_j} x_k \sigma^{(s_j)}(b_k) \pmod{\boldsymbol{A}^{(s_j+1)}}$$

in $\boldsymbol{A}/\boldsymbol{A}^{(r+1)}$ and

$$\eta^j(x) = \rho(b_j)^{x_j} \eta^{j+1}(\mathbf{R}(x)).$$

Note that, for any linear functional $F : \boldsymbol{A}_{s_j} \to \mathbb{Q}$, the composition

$$F! : \mathcal{U} \xrightarrow{\text{in}} \boldsymbol{A}/\boldsymbol{A}^{(r+1)} \xrightarrow{\text{pr}} \boldsymbol{A}_{s_j} \xrightarrow{F} \mathbb{Q}_{s_j}$$

is a cultural morphism. For $c \in \mathbb{Q}$, we have

$$(c\phi_j)!\,(\eta^j(x)) = cx_j, \qquad x = (x_j, \dots, x_q) \in \mathbb{Z}^{q-j+1}.$$

We show that η^j is a cultural immersion by constructing a cultural morphism

$$\theta^j : \mathcal{U} \to \mathbb{Q}^{q-j+1}_{s_j \dots s_q}$$

such that $\theta^j \circ \eta^j$ is the inclusion

$$\mathbb{Z}^{q-j+1}_{s_j \dots s_q} \to \mathbb{Q}^{q-j+1}_{s_j \dots s_q}.$$

Backward induction on j. Let θ^{q+1} be the unique function $\mathcal{U} \to \mathbb{Q}^0$. Take $j \leqslant q$. Introduce the cultural morphism

$$L : \mathcal{U} \xrightarrow{(-\phi_j)!} \mathbb{Q}_{s_j} \xrightarrow{\widehat{E}_{\rho(b_j)}} \mathcal{U}$$

where $\widehat{E}_{\rho(b_j)}$ is given by Lemma 6.2. Given $x = (x_j, \dots, x_q) \in \mathbb{Z}^{q-j+1}$, we have $L(\eta^j(x)) = \rho(b_j)^{-x_j}$. Introduce the cultural morphism

$$\theta' : \mathcal{U} \xrightarrow{L \underline{\overline{\times}} \mathrm{id}} \mathcal{U} \times \mathcal{U} \xrightarrow{M_{\mathcal{U}}} \mathcal{U} \xrightarrow{\theta^{j+1}} \mathbb{Q}^{q-j}_{s_{j+1} \dots s_q}.$$

Hereafter, $\underline{\overline{\times}}$ combines two morphisms with one source into a morphism to the product of their targets. We have

$$\theta'(\eta^j(x)) = \theta^{j+1}(L(\eta^j(x))\eta^j(x)) = \theta^{j+1}(\rho(b_j)^{-x_j}\eta^j(x)) =$$
$$= \theta^{j+1}(\eta^{j+1}(\mathbf{R}(x))) = \mathbf{R}(x).$$

Put

$$\theta^j : \mathcal{U} \xrightarrow{\phi_j! \underline{\overline{\times}} \theta'} \mathbb{Q}_{s_j} \times \mathbb{Q}^{q-j}_{s_{j+1} \dots s_q} = \mathbb{Q}^{q-j+1}_{s_j \dots s_q}.$$

We get

$$\theta^j(\eta^j(x)) = (\phi_j!(\eta^j(x)), \theta'(\eta^j(x))) = (x_j, \mathbf{R}(x)) = x. \qquad \square$$

7.2. Lemma. *Define a function m^j by the commutative diagram*

$$\begin{array}{ccc}
\mathbb{Z}^{q-j+1}_{s_j \dots s_q} \times \mathbb{Z}^{q-j+1}_{s_j \dots s_q} & \xrightarrow{m^j} & \mathbb{Z}^{q-j+1}_{s_j \dots s_q} \\
\downarrow{\scriptstyle \beta^j \times \beta^j} & & \downarrow{\scriptstyle \beta^j} \\
\mathcal{N}^j \times \mathcal{N}^j & \xrightarrow{M_{\mathcal{N}^j}} & \mathcal{N}^j.
\end{array}$$

Then m^j is a cultural morphism. It extends to a cultural morphism

$$\widehat{m}^j : \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \times \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \to \mathbb{Q}^{q-j+1}_{s_j \dots s_q}.$$

The coordinate polynomials of m^j are known as the "multiplication polynomials" [1, 4.2.2].

Proof. We have the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}^{q-j+1}_{s_j \dots s_q} \times \mathbb{Z}^{q-j+1}_{s_j \dots s_q} & \xrightarrow{m^j} & \mathbb{Z}^{q-j+1}_{s_j \dots s_q} \\
\downarrow{\scriptstyle \beta^j \times \beta^j} & & \downarrow{\scriptstyle \beta^j} \\
\mathcal{N}^j \times \mathcal{N}^j & \xrightarrow{M_{\mathcal{N}^j}} & \mathcal{N}^j \\
\downarrow{\scriptstyle \rho^j \times \rho^j} & & \downarrow{\scriptstyle \rho^j} \\
\mathcal{U} \times \mathcal{U} & \xrightarrow{M_{\mathcal{U}}} & \mathcal{U}.
\end{array}$$

Here $M_{\mathcal{U}}$ is a cultural morphism by Lemma 6.1. It follows from Lemma 7.1 that the composition in the left column is a cultural morphism. Since the composition in the right column is a cultural immersion by Lemma 7.1, m^j is a cultural morphism.

The extension $\widehat{m}^j$ exists automatically. $\square$

7.3. Lemma. *Given an element $h \in \mathcal{N}^j$, define a function e_h^j by the commutative diagram*

$$\begin{array}{ccc} & & \mathbb{Z}^{q-j+1}_{s_j \dots s_q} \\ & \overset{e_h^j}{\nearrow} & \Big\downarrow \beta^j \\ \mathbb{Z}_{s_j} & \xrightarrow{x \mapsto h^x} & \mathcal{N}^j. \end{array}$$

Then e_h^j is a cultural morphism. It extends to a cultural morphism

$$\widehat{e}_h^j : \mathbb{Q}_{s_j} \to \mathbb{Q}^{q-j+1}_{s_j \dots s_q}.$$

The coordinate polynomials of e_h^j are a specialization of the "exponentiation polynomials" [1, 4.2.2].

Proof. The composition

$$\mathbb{Z}_{s_j} \xrightarrow{e_h^j} \mathbb{Z}^{q-j+1}_{s_j \dots s_q} \xrightarrow{\beta^j} \mathcal{N}^j \xrightarrow{\rho^j} \mathcal{U}$$

sends x to $\rho(h)^x$ and thus coincides with $E_{\rho(h)}$, which is a cultural morphism by Lemma 6.2. Since $\rho^j \circ \beta^j$ here is a cultural immersion by Lemma 7.1, e_h^j is a cultural morphism.

The extension $\widehat{e}_h^j$ exists automatically. $\square$

§ 8. Cultural view of a subgroup of $\mathcal{N}$

Let $K \subseteq \mathcal{N}$ be a subgroup. For each $j \in (q)$, the image of $\mathcal{N}^j \cap K$ in the quotient $\mathcal{N}^j/\mathcal{N}^{j+1}$ is generated by $\overline{b}_j^{d_j}$ for some $d_j \geqslant 0$.

8.1. Lemma. *There exists a cultural morphism $f : \mathbb{Q}^q_{s_1 \dots s_q} \to \mathbb{Q}^q_{s_1 \dots s_q}$ such that*

$$\beta^{-1}(K) = f^{-1}(d_1\mathbb{Z} \times \ldots \times d_q\mathbb{Z})$$

as subsets of $\mathbb{Q}^q$.

The morhism f constructed in the proof is a cultural isomorphism, its jth coordinate f_j depends on the first j coordinates of the argument only, and, for $x \in \mathbb{Z}^q$, $f_j(x) \in \mathbb{Z}$ if $f_k(x) \in d_k\mathbb{Z}$ for all $k < j$. The proof of these properties is omitted.

Proof. For each $j \in (q+1)$, we construct a cultural morphism $f^j : \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \to \mathbb{Q}^{q-j+1}_{s_j \dots s_q}$ such that

$$(\beta^j)^{-1}(\mathcal{N}^j \cap K) = (f^j)^{-1}(d_j\mathbb{Z} \times \ldots \times d_q\mathbb{Z})$$

as subsets of $\mathbb{Q}^{q-j+1}$. Backward induction on j. Let $f^{q+1} : \mathbb{Q}^0 \to \mathbb{Q}^0$ be the unique function. Take $j \leqslant q$.

Case $d_j = 0$. Then $\mathcal{N}^j \cap K \subseteq \mathcal{N}^{j+1}$. Put

$$f^j : \mathbb{Q}^{q-j+1}_{s_j \dots s_q} = \mathbb{Q}_{s_j} \times \mathbb{Q}^{q-j}_{s_{j+1} \dots s_q} \xrightarrow{\mathrm{id} \times f^{j+1}} \mathbb{Q}_{s_j} \times \mathbb{Q}^{q-j}_{s_{j+1} \dots s_q} = \mathbb{Q}^{q-j+1}_{s_j \dots s_q}.$$

Take $x = (x_j, \dots, x_q) \in \mathbb{Q}^{q-j+1}$. We have

$$(x \in \mathbb{Z}^{q-j+1},\ \beta^j(x) \in \mathcal{N}^j \cap K) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (x \in 0 \times \mathbb{Z}^{q-j},\ \beta^{j+1}(\mathbf{R}(x)) \in \mathcal{N}^{j+1} \cap K) \quad \Leftrightarrow$$
$$\Leftrightarrow \quad (x_j = 0,\ f^{j+1}(\mathbf{R}(x)) \in d_{j+1}\mathbb{Z} \times \dots \times d_q\mathbb{Z}) \quad \Leftrightarrow$$
$$\Leftrightarrow \quad f^j(x) \in 0 \times d_{j+1}\mathbb{Z} \times \dots \times d_q\mathbb{Z}.$$

Case $d_j \neq 0$. Choose an element $k \in \mathcal{N}^j \cap K$ such that $\overline{k} = \overline{b}_j^{d_j}$ in $\mathcal{N}^j/\mathcal{N}^{j+1}$. Consider the cultural morphisms

$$l : \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \xrightarrow{-\mathbf{p}/d_j} \mathbb{Q}_{s_j} \xrightarrow{\widehat{e}^j_k} \mathbb{Q}^{q-j+1}_{s_j \dots s_q},$$

where $-\mathbf{p}/d_j : (x_j, \dots, x_q) \mapsto -x_j/d_j$ and $\widehat{e}^j_k$ is given by Lemma 7.3, and

$$f' : \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \xrightarrow{l \underline{\times} \mathrm{id}} \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \times \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \xrightarrow{\widehat{m}^j} \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \xrightarrow{\mathbf{R}} \mathbb{Q}^{q-j}_{s_{j+1} \dots s_q} \xrightarrow{f^{j+1}} \mathbb{Q}^{q-j}_{s_{j+1} \dots s_q},$$

where $\widehat{m}^j$ is given by Lemma 7.2. Put

$$f^j : \mathbb{Q}^{q-j+1}_{s_j \dots s_q} \xrightarrow{\mathbf{p} \underline{\times} f'} \mathbb{Q}_{s_j} \times \mathbb{Q}^{q-j}_{s_{j+1} \dots s_q} = \mathbb{Q}^{q-j+1}_{s_j \dots s_q}.$$

Take $x = (x_j, \dots, x_q) \in d_j\mathbb{Z} \times \mathbb{Z}^{q-j}$. Then

$$k^{-x_j/d_j} \beta^j(x) \in \mathcal{N}^{j+1}$$

and

$$k^{-x_j/d_j} \beta^j(x) = \beta^j(y),$$

where $y \in \mathbb{Z}^{q-j+1}$,

$$y = \widehat{m}^j(\widehat{e}^j_k(-x_j/d_j), x).$$

Thus $\mathbf{p}(y) = 0$ and

$$k^{-x_j/d_j} \beta^j(x) = \beta^j(y) = \beta^{j+1}(\mathbf{R}(y)) = \beta^{j+1}(\mathbf{R}(\widehat{m}^j(\widehat{e}^j_k(-x_j/d_j), x))).$$

Take $x = (x_j, \dots, x_q) \in \mathbb{Q}^{q-j+1}$. Put $y = \widehat{m}^j(\widehat{e}^j_k(-x_j/d_j), x) \in \mathbb{Q}^{q-j+1}$ and $y' = \mathbf{R}(y) \in \mathbb{Q}^{q-j}$. We show that

$$\mathbf{p}(y) = 0 \tag{5}$$

and

$$x = \widehat{m}^j(\widehat{e}^j_k(x_j/d_j), y). \tag{6}$$

It suffices to consider the case $x \in d_j\mathbb{Z} \times \mathbb{Z}^{q-j}$. Then, as shown above, $y \in \mathbb{Z}^{q-j+1}$, $\mathbf{p}(y) = 0$, and

$$k^{-x_j/d_j}\beta^j(x) = \beta^j(y).$$

Thus

$$\beta^j(x) = k^{x_j/d_j}\beta^j(y),$$

which implies (6). It follows from (5) and (6) that

$$(x_j \in d_j\mathbb{Z},\ y' \in \mathbb{Z}^{q-j}) \quad \Rightarrow \quad x \in d_j\mathbb{Z} \times \mathbb{Z}^{q-j}.$$

We have $f'(x) = f^{j+1}(y')$ and

$$(x \in \mathbb{Z}^{q-j+1},\ \beta^j(x) \in \mathcal{N}^j \cap K) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (x \in d_j\mathbb{Z} \times \mathbb{Z}^{q-j},\ k^{-x_j/d_j}\beta^j(x) \in \mathcal{N}^{j+1} \cap K) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (x_j \in d_j\mathbb{Z},\ y' \in \mathbb{Z}^{q-j},\ \beta^{j+1}(y') \in \mathcal{N}^{j+1} \cap K) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (x_j \in d_j\mathbb{Z},\ f^{j+1}(y') \in d_{j+1}\mathbb{Z} \times \ldots \times d_q\mathbb{Z}) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad (x_j \in d_j\mathbb{Z},\ f'(x) \in d_{j+1}\mathbb{Z} \times \ldots \times d_q\mathbb{Z}) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad f^j(x) \in d_j\mathbb{Z} \times \ldots \times d_q\mathbb{Z}.$$

□

§ 9. Defining $\gamma^{r+1}G$ by equations and congruences

9.1. Theorem. *Let G be a group with elements $g_1, \ldots, g_n \in G$. Fix $r \geqslant 0$. Then, for some $q \geqslant 0$, there are rational polynomials $P_j(X_1, \ldots, X_n)$, $j \in (q)$, of degree at most r and integers $d_1, \ldots, d_q \geqslant 0$ such that, for any $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$,*

$$g_1^{x_1} \ldots g_n^{x_n} \in \gamma^{r+1}G \quad \Leftrightarrow \quad (P_j(x) \in d_j\mathbb{Z},\ j \in (q)).$$

The polynomials P_j constructed in the proof have the following integrality property: for $x \in \mathbb{Z}^n$, $P_j(x) \in \mathbb{Z}$ if $P_k(x) \in d_k\mathbb{Z}$ for $k < j$. The check is omitted.

Proof. We use the constructions of the previous sections of Part II for the given n and r. In particular, we let the required q be the size of the Mal′cev basis of $\mathcal{N}$, the free nilpotent group of rank n and class r. Consider the homomorphism

$$t : \mathcal{N} \to G/\gamma^{r+1}G, \qquad \overline{Z}_i \mapsto \overline{g}_i.$$

Put $K = \operatorname{Ker} t \subseteq \mathcal{N}$. By Lemma 8.1, there are integers $d_1, \ldots, d_q \geqslant 0$ and a cultural morphism $f : \mathbb{Q}^q_{s_1 \ldots s_q} \to \mathbb{Q}^q_{s_1 \ldots s_q}$ such that

$$\beta^{-1}(K) = f^{-1}(d_1\mathbb{Z} \times \ldots \times d_q\mathbb{Z})$$

as subsets of $\mathbb{Q}^q$. Define the required polynomials P_j by the equality

$$f(x_1, \ldots, x_n, 0, \ldots, 0) = (P_j(x))_{j \in (q)}, \qquad x = (x_1, \ldots, x_n) \in \mathbb{Q}^n.$$

Since $s_1 = \ldots = s_n = 1$, the degree of P_j is at most s_j, which does not exceed r. Given $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, we have

$$g_1^{x_1} \ldots g_n^{x_n} \bmod \gamma^{r+1}G = t(\overline{Z}_1^{x_1} \ldots \overline{Z}_n^{x_n}) = t(\beta(x_1, \ldots, x_n, 0, \ldots, 0))$$

in $G/\gamma^{r+1}G$ and thus

$$g_1^{x_1} \ldots g_n^{x_n} \in \gamma^{r+1}G \quad \Leftrightarrow \quad \beta(x_1, \ldots, x_n, 0, \ldots, 0) \in K \quad \Leftrightarrow$$

$$\Leftrightarrow \quad f(x_1, \ldots, x_n, 0, \ldots, 0) \in d_1\mathbb{Z} \times \ldots \times d_q\mathbb{Z} \quad \Leftrightarrow \quad (P_j(x) \in d_j\mathbb{Z},\ j \in (q)). \ \square$$

Part III

In this part, we consider the group $[S^1, Y] = \pi_1(Y)$.

§ 10. Managing an ensemble of maps $S^1 \to Y$

For $n \geqslant 0$ and a group G, introduce the function

$$M : G^n \to G, \qquad (g_1, \ldots, g_n) \mapsto g_1 \ldots g_n.$$

For $K \subseteq (n)$, let $\omega_K : G^n \to G^K$ be the projection.

10.1. Lemma. *Consider an ensemble $A \in \langle Y^{S^1} \rangle$,*

$$A = \sum_{i \in I} u_i {<} a_i {>},$$

such that $A \stackrel{r}{=} 0$. Then, for some $n \geqslant 1$, there exist elements $z_i \in \pi_1(Y)^n$, $i \in I$, such that $[a_i] = M(z_i)$ and the element

$$Z = \sum_{i \in I} u_i {<} z_i {>} \in \langle \pi_1(Y)^n \rangle \tag{7}$$

satisfies $\langle \omega_K \rangle (Z) = 0$ in $\langle \pi_1(Y)^K \rangle$ for all $K \subseteq (n)$ with $|K| \leqslant r$.

Proof. Take a finite subspace $D \subseteq S^1$ consisting of $n \geqslant 2$ points. It cuts S^1 into closed arcs B_k, $k \in (n)$. A continuous function $v : B_k \to Y$ with $v(\partial B_k) = \{\triangleleft\!\!\mid_Y\}$ has the (relative to ∂B_k) homotopy class $[v] \in \pi_1(Y)$. For a map $w : S^1 \to Y$ with $w(D) = \{\triangleleft\!\!\mid_Y\}$, we have

$$[w] = \prod_{k=1}^{n} [w|_{B_k}] \tag{8}$$

in $\pi_1(Y)$ (we assume that B_k are oriented and numbered properly).

By [4, Corollary 6.2], we may assume that $A \underset{\Gamma}{\stackrel{r}{=}} 0$ for some open cover Γ of S^1. We suppose that D is chosen dense enough so that each B_k is contained in some $G_k \in \Gamma$. Put

$$V = \bigvee_{i \in I} S^1.$$

Let U be the quotient of V by the identifications $\mathrm{in}_i(x) \approx \mathrm{in}_j(x)$ for $x \in B_k$ and $i, j \in I$ such that $a_i =|_{G_k} a_j$. U is a graph. Let $h : V \to U$ be the projection. Introduce the maps

$$e_i : S^1 \xrightarrow{\mathrm{in}_i} V \xrightarrow{h} U.$$

There is a map $q : U \to S^1$ such that $q \circ e_i = \mathrm{id}_{S^1}$. Put

$$b = \overline{\bigvee_{i \in I}} a_i : V \to Y.$$

There is a map $a : U \to Y$ such that $b = a \circ h$. Clearly, $a_i = a \circ e_i$. Put $\widetilde{D} = q^{-1}(D) \subseteq U$. $\widetilde{D}$ is a finite subspace. The map $a|_{\widetilde{D}}$ is null-homotopic because the inclusion $\widetilde{D} \to U$ is. Extending the homotopy, we get a map $\widehat{a} : U \to Y$ such that $\widehat{a} \sim a$ and $\widehat{a}(\widetilde{D}) = \{\daleth_Y\}$. Put $\widehat{a}_i = \widehat{a} \circ e_i : S^1 \to Y$. Clearly, $\widehat{a}_i \sim a_i$ and $\widehat{a}_i(D) = \{\daleth_Y\}$. Put

$$z_i = ([\widehat{a}_i|_{B_k}])_{k \in (n)} \in \pi_1(Y)^n.$$

We have

$$[a_i] = [\widehat{a}_i] \overset{(*)}{=} \prod_{k=1}^{n} [\widehat{a}_i|_{B_k}] = M(z_i),$$

where $(*)$ follows from (8). For $k \in (n)$ and $i, j \in I$, we have the implication

$$a_i =|_{G_k} a_j \quad \Rightarrow \quad [\widehat{a}_i|_{B_k}] = [\widehat{a}_j|_{B_k}] \tag{9}$$

because the premise implies that $e_i =|_{B_k} e_j$ and thus $\widehat{a}_i =|_{B_k} \widehat{a}_j$.

Consider the element $Z \in \langle \pi_1(Y)^n \rangle$ given by (7). Take $K \subseteq (n)$. Put

$$G(K) = \{\daleth_Y\} \cup \bigcup_{k \in K} G_k \subseteq S^1.$$

By (9), we have the implication

$$a_i =|_{G(K)} a_j \quad \Rightarrow \quad \omega_K(z_i) = \omega_K(z_j).$$

Suppose that $|K| \leqslant r$. Then $A|_{G(K)} = 0$ because $A \overset{r}{\underset{\Gamma}{=}} 0$. Thus $\langle \omega_K \rangle(Z) = 0$. $\square$

§ 11. Similarity on $\pi_1(Y)$

11.1. Lemma. *Let G be a group. Consider an element $Z \in \langle G^n \rangle$,*

$$Z = \sum_{i \in I} u_i \langle z_i \rangle,$$

where I has a distinguished element 0 and $u_0 = 1$. Suppose that $\langle \omega_K \rangle(Z) = 0$ for all $K \subseteq (n)$ with $|K| \leqslant r$ and $M(z_i) \in \gamma^{r+1} G$ for all $i \neq 0$. Then $M(z_0) \in \gamma^{r+1} G$.

Proof. We have $z_i = (z_{i1}, \dots, z_{in})$, where $z_{ik} \in G$. Take distinct $g_1, \dots, g_m \in G$ that include all the z_{ik}. We have

$$z_{ik} = \prod_{l=1}^{m} g_l^{[z_{ik}=g_l]}$$

and thus

$$M(z_i) = \prod_{k=1}^{n} \prod_{l=1}^{m} g_l^{[z_{ik}=g_l]}.$$

Hereafter, given a condition C, the integer $[C]$ is 1 under C and 0 otherwise. By Theorem 9.1, for some $q \geqslant 0$, there are rational polynomials $P_j(X)$, $X = (X_{kl})_{k\in(n),\, l\in(m)}$, $j \in (q)$, of degree at most r and integers $d_j \geqslant 0$ such that, for any collection $x = (x_{kl})_{k\in(n),\, l\in(m)}$, $x_{kl} \in \mathbb{Z}$, we have the equivalence

$$\prod_{k=1}^{n} \prod_{l=1}^{m} g_l^{x_{kl}} \in \gamma^{r+1} G \quad \Leftrightarrow \quad (P_j(x) \in d_j \mathbb{Z},\ j \in (q)).$$

Order the set $(n) \times (m)$ totally. We have

$$P_j(X) = \sum_{\substack{0\leqslant s\leqslant r,\\ (k_1,l_1)\leqslant\dots\leqslant(k_s,l_s)}} P^{(s)}_{jk_1l_1\dots k_sl_s} X_{k_1l_1} \dots X_{k_sl_s}$$

for some $P^{(s)}_{jk_1l_1\dots k_sl_s} \in \mathbb{Q}$. We have

$$\sum_{i\in I} u_i P_j(([z_{ik} = g_l])_{k\in(n),\, l\in(m)}) =$$
$$= \sum_{\substack{0\leqslant s\leqslant r,\\ (k_1,l_1)\leqslant\dots\leqslant(k_s,l_s)}} P^{(s)}_{jk_1l_1\dots k_sl_s} \sum_{i\in I} u_i [z_{ik_t} = g_{l_t},\ t \in (s)] \overset{(*)}{=} 0, \quad (10)$$

where $(*)$ holds because the inner sum is zero, which is because $\langle\omega_K\rangle(Z) = 0$ for $K = \{k_1, \dots, k_s\}$. Since $M(z_i) \in \gamma^{r+1}(G)$ for $i \neq 0$, we have

$$P_j(([z_{ik} = g_l])_{k\in(n),\, l\in(m)}) \in d_j \mathbb{Z} \quad (11)$$

for $i \neq 0$. Since $u_0 = 1$, it follows from (10) that (11) holds for $i = 0$ too. Thus $M(z_0) \in \gamma^{r+1}(G)$. □

11.2. Theorem. *Let Y be a cellular space. Then*

$$\pi_1(Y)^{(r+1)} = \gamma^{r+1}\pi_1(Y). \quad (12)$$

Proof. The inclusion $\supseteq$ in (12) follows from Theorem 4.3. To prove the inclusion $\subseteq$, we take $a \in Y^{S^1}$ such that $\urcorner \overset{r}{\sim} a$ and check that $[a] \in \gamma^{r+1}\pi_1(Y)$. There is an ensemble $D \in \langle Y^{S^1} \rangle$,

$$D = \sum_i u_u {<}d_i{>},$$

where $d_i \sim$ ⚑, such that $D \overset{r}{=} <\!a\!>$. By Lemma 10.1, for some $n \geqslant 1$, there are elements $z, w_i \in \pi_1(Y)^n$ such that $M(z) = [a]$ and $M(w_i) = 1$ in $\pi_1(Y)$ and, putting

$$W = \sum_i u_i <\!w_i\!> \in \langle \pi_1(Y)^n \rangle,$$

we have $\langle \omega_K \rangle (<\!z\!> - W) = 0$ for all $K \subseteq (n)$ with $|K| \leqslant r$. By Lemma 11.1, $M(z) \in \gamma^{r+1}\pi_1(Y)$, which is what we need. □

§ 12. Finite-order invariants on $\pi_1(Y)$

For a group G, $\langle G \rangle$ is its group ring. Let $[G] \subseteq \langle G \rangle$ be the augmentation ideal, i. e., the kernel of the ring homomorphism (called the augmentation) $\langle G \rangle \to \mathbb{Z}$, $<\!g\!> \mapsto 1$.

12.1. Lemma. *Let G be a group and $Z \in \langle G^n \rangle$ be an element such that $\langle \omega_K \rangle (Z) = 0$ in $\langle G^K \rangle$ for all $K \subseteq (n)$ with $K| \leqslant r$. Then $\langle M \rangle (Z) \in [G]^{r+1}$ $(\subseteq \langle G \rangle)$.*

Proof. For $K \subseteq (n)$, consider the function

$$\epsilon_K : G^K \to G^n, \qquad (g_k)_{k \in K} \mapsto (\widetilde{g}_k)_{k \in (n)},$$

where $\widetilde{g}_k$ equals g_k if $k \in K$ and 1 otherwise, the composition

$$\rho_K : G^n \xrightarrow{\omega_K} G^K \xrightarrow{\epsilon_K} G^n$$

and the homomorphism $S_K : \langle G^n \rangle \to \langle G^n \rangle$,

$$S_K = \sum_{L \subseteq K} (-1)^{|L|} \langle \rho_L \rangle.$$

If $K = \{k_1, \ldots, k_t\}$, $k_1 < \ldots < k_t$, then

$$(\langle M \rangle \circ S_K)(<\!(g_k)_{k \in (n)}\!>) = (1 - <\!g_{k_1}\!>) \ldots (1 - <\!g_{k_t}\!>)$$

in $\langle G \rangle$. Thus

$$\operatorname{Im}(\langle M \rangle \circ S_K) \subseteq [G]^{|K|}. \tag{13}$$

We have

$$\sum_{K \subseteq (n)} (-1)^{|K|} S_K = \sum_{K \subseteq (n)} (-1)^{|K|} \sum_{L \subseteq K} (-1)^{|L|} \langle \rho_L \rangle =$$
$$= \sum_{L \subseteq (n)} (-1)^{|L|} \Big(\sum_{K \subseteq (n) : K \supseteq L} (-1)^{|K|} \Big) \langle \rho_L \rangle.$$

The inner sum equals $(-1)^n [L = (n)]$. Thus

$$\sum_{K \subseteq (n)} (-1)^{|K|} S_K = \langle \rho_{(n)} \rangle = \mathrm{id}_{\langle G \rangle}.$$

For $L \subseteq (n)$, $|L| \leqslant r$, we have $\langle \rho_L \rangle(Z) = \langle \epsilon_L \rangle(\langle \omega_L \rangle(Z)) = 0$. Thus $S_K(Z) = 0$ if $|K| \leqslant r$. We get

$$Z = \sum_{K \subseteq (n)} (-1)^{|K|} S_K(Z) = \sum_{K \subseteq (n) : |K| \geqslant r+1} (-1)^{|K|} S_K(Z).$$

Thus

$$\langle M \rangle(Z) = \sum_{K \subseteq (n) : |K| \geqslant r+1} (-1)^{|K|} (\langle M \rangle \circ S_K)(Z).$$

By (13), $\langle M \rangle(Z) \in [G]^{r+1}$. □

A function $f : G \to L$, where L is an abelian group, gives rise to the homomorphism

$${}^+f : \langle G \rangle \to L, \qquad \text{<}g\text{>} \mapsto f(g).$$

We define $\deg f \in \{-\infty, 0, 1, \ldots, \infty\}$, the *degree* of f, as the infimum of $r \in \mathbb{Z}$ such that ${}^+f|_{[G]^{r+1}} = 0$ (adopting $[G]^s = \langle G \rangle$ for $s \leqslant 0$).

12.2. Theorem. *Let Y be a cellular space, L be an abelian group and $f : \pi_1(Y) \to L$ be a homotopy invariant (i. e., a function). Then* $\operatorname{ord} f = \deg f$

Proof. We suppose $f \neq 0$ omitting the converse case.

(1) Suppose that $\operatorname{ord} f \leqslant r$ $(r \geqslant 0)$. We show that $\deg f \leqslant r$. It suffices to check that

$${}^+f((1 - \text{<}[a_1]\text{>}) \ldots (1 - \text{<}[a_{r+1}]\text{>})) = 0$$

for any $a_1, \ldots, a_{r+1} \in Y^{S^1}$. Put $W = S^1 \vee \ldots \vee S^1$ ($r+1$ summands) and

$$q = a_1 \,\underline{\vee} \ldots \underline{\vee}\, a_{r+1} : W \to Y.$$

Let $p : S^1 \to W$ be the $(r+1)$-fold comultiplication and $\Lambda_d : W \to W$, $d \in \boldsymbol{\mathcal{E}}^{r+1}$, be as in [4, § 3]. Consider the ensemble $A \in \langle Y^{S^1} \rangle$,

$$A = \sum_{d \in \boldsymbol{\mathcal{E}}^{r+1}} (-1)^{|d|} \text{<}a(d)\text{>},$$

where

$$a(d) : S^1 \xrightarrow{p} W \xrightarrow{\Lambda(d)} W \xrightarrow{q} Y.$$

Clearly,

$$[a(d)] = [a_1]^{d_1} \ldots [a_{r+1}]^{d_{r+1}}$$

in $\pi_1(Y)$. By [4, Lemma 3.1], $A \overset{r}{=} 0$. We have

$${}^+f((1 - \text{<}[a_1]\text{>}) \ldots (1 - \text{<}[a_{r+1}]\text{>})) = \sum_{d \in \boldsymbol{\mathcal{E}}^{r+1}} (-1)^{|d|} f([a_1]^{d_1} \ldots [a_{r+1}]^{d_{r+1}}) =$$

$$= \sum_{d \in \boldsymbol{\mathcal{E}}^{r+1}} (-1)^{|d|} f([a(d)]) \overset{(*)}{=} 0,$$

where $(*)$ holds because $\operatorname{ord} f \leqslant r$.

(2) Suppose that $\deg f \leqslant r$ $(r \geqslant 0)$. We show that $\operatorname{ord} f \leqslant r$. Take an ensemble $A \in \langle Y^{S^1} \rangle$,

$$A = \sum_{i \in I} u_i {<} a_i {>},$$

such that $A \overset{r}{=} 0$. We should show that

$$\sum_{i \in I} u_i f([a_i]) = 0.$$

By Lemma 10.1, for some $n \geqslant 1$, there exist elements $z_i \in \pi_1(Y)^n$, $i \in I$, such that $[a_i] = M(z_i)$ and the element $Z \in \langle \pi_1(Y)^n \rangle$ given by (7) satisfies $\langle \omega_K \rangle (Z) = 0$ in $\langle \pi_1(Y)^K \rangle$ for all $K \subseteq (n)$ with $|K| \leqslant r$. We have

$$\sum_{i \in I} u_i f([a_i]) = {}^{+}f(\langle M \rangle (Z)).$$

By Lemma 12.1, $\langle M \rangle (Z) \in [G]^{r+1}$. Since $\deg f \leqslant r$, ${}^{+}f(\langle M \rangle (Z)) = 0$. □

References

[1] A. E. Clement, S. Majewicz, M. Zyman, The theory of nilpotent groups. Birkhäuser, 2017.

[2] W. Magnus, Über Beziehungen zwischen höheren Kommutatoren. J. reine angew. Math. **177** (1937), 105–115.

[3] R. Mikhailov, I. B. S. Passi, Lower central and dimension series of groups. Lect. Notes Math. 1952, Springer, 2009.

[4] S. S. Podkorytov, Homotopy similarity of maps, arXiv:2308.00859 (2023).

[5] E. Rips, On the fourth integer dimension subgroup, Isr. J. Math. **12** (1972), 342–346.

[6] J.-P. Serre, Lie algebras and Lie groups. W. A. Benjamin, 1965.

`ssp@pdmi.ras.ru`
`http://www.pdmi.ras.ru/~ssp`