Homotopy similarity of maps. Strong similarity

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Given pointed cellular spaces X and Y, X compact, and an integer $r \geqslant 0$, we define a relation $\overset{r}{\approx}$ on $[X, Y]$ and argue for the conjecture that it always coincides with the r-similarity $\overset{r}{\sim}$.

§ 1. Introduction

This paper continues [2]. We adopt notation and conventions thereof. Let X and Y be cellular spaces, X compact. For each $r \geqslant 0$, we define a relation $\overset{r}{\approx}$, called the strong r-similarity, on the set $[X, Y]$. We will need it in our next paper [4]. We conjecture that strong r-similarity always coincides with r-similarity $\overset{r}{\sim}$. It follows immediately from the definition that it implies r-similarity and gets nonstrictly stronger as r grows. We do not know whether the strong r-similarity is always an equivalence. We prove this in the case $X = \Sigma T$ (§ 8). The main results are as follows. Strong 1-similarity coincides with 1-similarity (Theorem 14.2). (We believe that 1-similarity can be given a homological characterization similar to that of homotopy invariants of order at most 1 [1].) If $X = S^1$, the strong r-similarity coincides with the r-similarity (§ 24). All $(r+1)$-fold Whitehead products are strongly r-similar to zero (Theorem 27.2).

§ 2. Definition of strong similarity

Augmentation. For a set V, introduce the homomorphism

$$\epsilon : \langle V\rangle \to \mathbb{Z}, \qquad {<}v{>} \mapsto 1,$$

the *augmentation*. An ensemble $S \in \langle V\rangle$ is called *affine* if $\epsilon(S) = 1$.

Hash product. Given a wedge

$$T = \bigvee_{i\in(m)} T_i$$

(hereafter, $(m) = \{1, \dots, m\}$) and a space Z, we have the $\mathbb{Z}$-multilinear operation

$$\mathop{\#}_{i\in(m)} : \prod_{i\in(m)} \langle Z^{T_i}\rangle \to \langle Z^T\rangle, \qquad \mathop{\#}_{i\in(m)} {<}v_i{>} = {<}\overline{\underline{\bigvee}}_{i\in(m)} v_i{>}.$$

Simplex and its faces. Fix a nonempty finite set E. Let $\mathcal{P}_\times(E)$ be the set of nonempty subsets $F \subseteq E$. Let $\mathcal{A}(E)$ be the set of subsets $A \subseteq \mathcal{P}_\times(E)$ such that all $F \in A$ are disjoint (*layouts*).

Let ΔE be the simplex spanned by E. For $F \in \mathcal{P}_\times(E)$, $\Delta F \subseteq \Delta E$ is a face. For $A \in \mathcal{A}(E)$, put

$$\Delta[A] = \bigcup_{F \in A} \Delta F \subseteq \Delta E.$$

Fissile ensembles. For $A \in \mathcal{A}(E)$, we have

$$\Delta[A]_+ = \bigvee_{F \in A} (\Delta F)_+.$$

Hereafter, $U_+ = U \sqcup \{\P\}$. Given a space Z, we call an ensemble $S \in \langle Z^{(\Delta E)_+} \rangle$ *fissile* if, for each $A \in \mathcal{A}(E)$,

$$S|_{\Delta[A]_+} = \mathop{\#}_{F \in A} S|_{(\Delta F)_+} \tag{1}$$

in $\langle Z^{\Delta[A]_+} \rangle$.

An ensemble of the form $<\!v\!>$ is fissile. A fissile ensemble is affine (take $A = \varnothing$ in the definition). An affine ensemble S is fissile if it satisfies (1) for all A with $|A| = 2$. Given a space $\widetilde{Z} \supseteq Z$, we have $\langle \widetilde{Z}^{(\Delta E)_+} \rangle \supseteq \langle Z^{(\Delta E)_+} \rangle$; the ensemble S is fissile as an element of $\langle \widetilde{Z}^{(\Delta E)_+} \rangle$ if and only if it is fissile as an element of $\langle Z^{(\Delta E)_+} \rangle$.

Spaces of maps. Let X and Y be cellular spaces, X compact. Then Y^X is the space of maps $X \to Y$; its basepoint is the constant map $\P^X_Y$. Given a map $a : X \to Y$, we define the spaces (Y^X, a) as Y^X with the basepoint moved to a and Y^X_a as the basepoint path component of (Y^X, a). For an unpointed space U, introduce the function

$$\theta^U_a : Y^X \to (Y^X, a)^{U_+}, \qquad \theta^U_a(d)(u) = d \quad (u \in U), \quad \theta^U_a(d)(\P) = a.$$

The filtration $\langle (Y^X, a)^T \rangle^{(s)}_X$. Let T be a space. The function

$$\square^X : (Y^X, a)^T \to Y^{T \times X}, \qquad \square^X(v)(t, x) = v(t)(x),$$

induces the homomorphism

$$\langle \square^X \rangle : \langle (Y^X, a)^T \rangle \to \langle Y^{T \times X} \rangle.$$

The filtration of $\langle Y^{T \times X} \rangle$ (see [2]) induces a filtration of $\langle (Y^X, a)^T \rangle$:

$$\langle (Y^X, a)^T \rangle^{(s)}_X = \langle \square^X \rangle^{-1}(\langle Y^{T \times X} \rangle^{(s)}).$$

Strong similarity. Let X and Y be as above and $a, b : X \to Y$ be maps. We adopt the inclusion $\langle (Y^X_a)^T \rangle \subseteq \langle (Y^X, a)^T \rangle$. We say that a is *strongly r-similar* to b,

$$a \overset{r}{\approx} b,$$

if, for any nonempty finite set E, there exists a fissile ensemble $S \in \langle (Y^X_a)^{(\Delta E)_+} \rangle$ such that

$$<\!\theta^{\Delta E}_a(b)\!> - S \in \langle (Y^X, a)^{(\Delta E)_+} \rangle^{(r+1)}_X.$$

We have $a \overset{r}{\approx} a$ (put $S = {<}\theta_a^{\Delta E}(a){>}$). Clearly, $a \overset{r}{\approx} b$ implies $a \overset{r}{\sim} b$ (take $E = \{\bullet\}$). We prove below (Theorem 6.2) that the relation $\overset{r}{\approx}$ is homotopy invariant.

§ 3. On the filtration $\langle (Y^X, a)^T \rangle_X^{(s)}$

3.1. Lemma. *Let X, Y, T and $\widetilde{T}$ be cellular spaces, X, T and $\widetilde{T}$ compact, and $a : X \to Y$ and $k : \widetilde{T} \to T$ be maps. Then the homomorphism*

$$\langle (Y^X, a)^k \rangle : \langle (Y^X, a)^T \rangle \to \langle (Y^X, a)^{\widetilde{T}} \rangle$$

takes $\langle (Y^X, a)^T \rangle_X^{(s)}$ to $\langle (Y^X, a)^{\widetilde{T}} \rangle_X^{(s)}$.

Proof. We have the commutative diagram

$$\begin{array}{ccc} \langle (Y^X, a)^T \rangle & \xrightarrow{\langle \square^X \rangle} & \langle Y^{T \times X} \rangle \\ {\scriptstyle \langle (Y^X, a)^k \rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle Y^{k \times \mathrm{id}_X} \rangle} \\ \langle (Y^X, a)^{\widetilde{T}} \rangle & \xrightarrow{\langle \square^X \rangle} & \langle Y^{\widetilde{T} \times X} \rangle. \end{array}$$

By the definition of $\langle (Y^X, a)^T \rangle_X^{(s)}$, $\langle \square^X \rangle$ takes it to $\langle Y^{T \times X} \rangle^{(s)}$. By [3, Lemma 2.1], $\langle Y^{k \times \mathrm{id}_X} \rangle$ takes the latter to $\langle Y^{\widetilde{T} \times X} \rangle^{(s)}$. By commutativity of the diagram, $\langle (Y^X, a)^k \rangle$ takes $\langle (Y^X, a)^T \rangle_X^{(s)}$ to $\langle \square^X \rangle^{-1}(\langle Y^{\widetilde{T} \times X} \rangle^{(s)})$, which is $\langle (Y^X, a)^{\widetilde{T}} \rangle_X^{(s)}$ by the definition of the latter. $\square$

The case $a = \triangleleft_Y^X$.

3.2. Lemma. *Let X, Y, and $\widetilde{X}$ be cellular spaces, X and $\widetilde{X}$ compact, and $k : \widetilde{X} \to X$ be a surjective map. Then the homomorphism*

$$\langle Y^k \rangle : \langle Y^X \rangle \to \langle Y^{\widetilde{X}} \rangle$$

satisfies

$$\langle Y^X \rangle^{(s)} = \langle Y^k \rangle^{-1}(\langle Y^{\widetilde{X}} \rangle^{(s)}). \tag{2}$$

Proof. By [3, Lemma 2.1], $\langle Y^k \rangle$ preserves the filtration, which yields the inclusion $\subseteq$ in (2). Check the inclusion $\supseteq$. Take $V \in \langle Y^k \rangle^{-1}(\langle Y^{\widetilde{X}} \rangle^{(s)})$ and show that $V \in \langle Y^X \rangle^{(s)}$. Take $R \in \mathcal{F}_{s-1}(X)$. We should check that $V|_R = 0$. We have $R = k(Q)$ for some $Q \in \mathcal{F}_{s-1}(\widetilde{X})$. Since $\langle Y^k \rangle(V) \in \langle Y^{\widetilde{X}} \rangle^{(s)}$, we have $\langle Y^k \rangle(V)|_Q = 0$. We have the commutative diagram

$$\begin{array}{ccccc} V & \langle Y^X \rangle & \xrightarrow{?|_R} & \langle Y^R \rangle & V|_R \\ & {\scriptstyle \langle Y^k \rangle}\big\downarrow & & \big\downarrow{\scriptstyle \langle Y^h \rangle} & \\ \langle Y^k \rangle(V) & \langle Y^{\widetilde{X}} \rangle & \xrightarrow{?|_Q} & \langle Y^Q \rangle, & 0 \end{array}$$

where $h = k|_{Q \to R}$. Since h is surjective, $\langle Y^h \rangle$ is injective. Thus $V|_R = 0$. $\square$

Let X, Y and T be cellular spaces, X and T compact. Let

$$\widehat{\Box}^X : (Y^X)^T \to Y^{T\wedge X}$$

be the standard bijection. Consider the homomorphism

$$\langle \widehat{\Box}^X \rangle : \langle (Y^X)^T \rangle \to \langle Y^{T\wedge X} \rangle.$$

Lemma 3.3. *One has*

$$\langle (Y^X)^T \rangle_X^{(s)} = \langle \widehat{\Box}^X \rangle^{-1} (\langle Y^{T\wedge X} \rangle^{(s)}).$$

Proof. We have the commutative diagram

$$\begin{array}{ccc} \langle (Y^X)^T \rangle & \xrightarrow{\langle \widehat{\Box}^X \rangle} & \langle Y^{T\wedge X} \rangle \\ & {\scriptstyle \langle \Box^X \rangle} \searrow & \downarrow {\scriptstyle \langle Y^k \rangle} \\ & & \langle Y^{T\times X} \rangle, \end{array}$$

where $k : T \times X \to T \wedge X$ is the projection. By definition,

$$\langle (Y^X)^T \rangle_X^{(s)} = \langle \Box^X \rangle^{-1} (\langle Y^{T\times X} \rangle^{(s)}).$$

By Lemma 3.2,

$$\langle Y^{T\wedge X} \rangle^{(s)} = \langle Y^k \rangle^{-1} (\langle Y^{T\times X} \rangle^{(s)}).$$

The desired equality follows. □

§ 4. Primitive transforms

Let Z and $\widetilde{Z}$ be spaces and $g : Z \to \widetilde{Z}$ be a map. For a compact cellular space T, we have the map $g^T : Z^T \to \widetilde{Z}^T$.

Lemma 4.1. *Let E be a nonempty finite set. Consider the homomorphism*

$$\langle g^{(\Delta E)_+} \rangle : \langle Z^{(\Delta E)_+} \rangle \to \langle \widetilde{Z}^{(\Delta E)_+} \rangle.$$

Then, for any fissile ensemble $S \in \langle Z^{(\Delta E)_+} \rangle$, the ensemble $\langle g^{(\Delta E)_+} \rangle (S)$ is fissile.

Proof. Take $A \in \mathcal{A}(E)$. We have the commutative diagram

$$
\begin{array}{ccccccc}
(S)_{F\in A} & & \stackrel{(1)}{\longmapsto} & & & & ((\langle g^{(\Delta E)_+}\rangle(S))_{F\in A} \\
\Big\downarrow{\scriptstyle (2)} & \prod\limits_{F\in A}\langle Z^{(\Delta E)_+}\rangle & \xrightarrow{\prod_{F\in A}\langle g^{(\Delta E)_+}\rangle} & \prod\limits_{F\in A}\langle \widetilde{Z}^{(\Delta E)_+}\rangle & & & \Big\downarrow{\scriptstyle (3)} \\
 & \Big\downarrow{\scriptstyle \prod_{F\in A}(?|_{(\Delta F)_+})} & & \Big\downarrow{\scriptstyle \prod_{F\in A}(?|_{(\Delta F)_+})} & & & \\
(S|_{(\Delta F)_+})_{F\in A} & \prod\limits_{F\in A}\langle Z^{(\Delta F)_+}\rangle & \xrightarrow{\prod_{F\in A}\langle g^{(\Delta F)_+}\rangle} & \prod\limits_{F\in A}\langle \widetilde{Z}^{(\Delta F)_+}\rangle & & & ((\langle g^{(\Delta E)_+}\rangle(S)|_{(\Delta F)_+})_{F\in A} \\
\Big\downarrow{\scriptstyle (4)} & \Big\downarrow{\scriptstyle \#_{F\in A}} & & \Big\downarrow{\scriptstyle \#_{F\in A}} & & & \Big\downarrow{\scriptstyle (5)} \\
S|_{\Delta[A]_+} & \langle Z^{\Delta[A]_+}\rangle & \xrightarrow{\langle g^{\Delta[A]_+}\rangle} & \langle \widetilde{Z}^{\Delta[A]_+}\rangle & & & \#_{F\in A}\langle g^{(\Delta E)_+}\rangle(S)|_{(\Delta F)_+} \\
\Big\uparrow{\scriptstyle (6)} & \Big\uparrow{\scriptstyle ?|_{\Delta[A]_+}} & & \Big\uparrow{\scriptstyle ?|_{\Delta[A]_+}} & & & \Big\uparrow{\scriptstyle (7)} \\
 & \langle Z^{(\Delta E)_+}\rangle & \xrightarrow{\langle g^{(\Delta E)_+}\rangle} & \langle \widetilde{Z}^{(\Delta E)_+}\rangle. & & & \\
S & & \stackrel{(8)}{\longmapsto} & & & & \langle g^{(\Delta E)_+}\rangle(S)
\end{array}
$$

The sending (4) is fissility of S. The sendings (1), (2), (3), (5), (6), and (8) are obvious. The sending (7) follows. It is fissility of $\langle g^{(\Delta E)_+}\rangle(S)$. $\square$

Primitive case. Let X, Y, $\widetilde{X}$, and $\widetilde{Y}$ be cellular spaces, X and $\widetilde{X}$ compact, and $g : Y^X \to \widetilde{Y}^{\widetilde{X}}$ be an unbased map (a *transform*). We suppose that the transform g is *primitive*: for each point $w \in \widetilde{X}$, there is a point $k(w) \in X$ and an unbased map $h^w : Y \to \widetilde{Y}$ such that

$$g(d)(w) = h^w(d(k(w))), \qquad d \in Y^X.$$

For a map $a : X \to Y$, we have the map $g : (Y^X, a) \to (\widetilde{Y}^{\widetilde{X}}, g(a))$.

Lemma 4.2. *For a map $a : X \to Y$ and a compact cellular space T, the homomorphism $\langle g^T\rangle$ takes $\langle (Y^X, a)^T\rangle^{(s)}_X$ to $\langle (\widetilde{Y}^{\widetilde{X}}, g(a))^T\rangle^{(s)}_{\widetilde{X}}$.*

Proof. We may assume that $k(\triangleleft_{\widetilde{X}}) = \triangleleft_X$ and $h^{\triangleleft_{\widetilde{X}}}(\triangleleft_Y) = \triangleleft_{\widetilde{Y}}$. We have the function

$$K = \mathrm{id} \times k : T \times \widetilde{X} \to T \times X.$$

For $Q \in \mathcal{F}_{s-1}(T \times \widetilde{X})$, we have $K(Q) \in \mathcal{F}_{s-1}(T \times X)$. We have the function

$$H : Y^{K(Q)} \to \widetilde{Y}^Q, \qquad H(u)(t, w) = h^w(u(K(t, w))),\ (t, w) \in Q,\ u \in Y^{K(Q)},$$

and the commutative diagram

$$
\begin{array}{ccccc}
\langle (Y^X, a)^T\rangle & \xrightarrow{\langle \square^X\rangle} & \langle Y^{T\times X}\rangle & \xrightarrow{?|_{K(Q)}} & \langle Y^{K(Q)}\rangle \\
\Big\downarrow{\scriptstyle \langle g^T\rangle} & & & & \Big\downarrow{\scriptstyle \langle H\rangle} \\
\langle (\widetilde{Y}^{\widetilde{X}}, g(a))^T\rangle & \xrightarrow{\langle \square^{\widetilde{X}}\rangle} & \langle \widetilde{Y}^{T\times \widetilde{X}}\rangle & \xrightarrow{?|_Q} & \langle \widetilde{Y}^Q\rangle.
\end{array}
$$

By the definition of $\langle (Y^X, a)^T \rangle_X^{(s)}$, it goes to zero under the composition in the upper row. Thus its image under $\langle g^T \rangle$ goes to zero under the composition in the lower row. Since Q was taken arbitrarily, this image is contained in $\langle (\widetilde{Y}^{\widetilde{X}}, g(a))^T \rangle_{\widetilde{X}}^{(s)}$. $\square$

Lemma 4.3. *Let $a, b : X \to Y$ be maps such that $a \overset{r}{\approx} b$. Then $g(a) \overset{r}{\approx} g(b)$.*

Proof. Take a finite set E. We have a fissile ensemble $S \in \langle (Y_a^X)^{(\Delta E)_+} \rangle$ such that
$$ {<}\theta_a^{\Delta E}(b){>} - S \in \langle (Y^X, a)^{(\Delta E)_+} \rangle_X^{(r+1)}. $$
Consider the homomorphism
$$ \langle g^{(\Delta E)_+} \rangle : \langle (Y^X, a)^{(\Delta E)_+} \rangle \to \langle (\widetilde{Y}^{\widetilde{X}}, g(a))^{(\Delta E)_+} \rangle. $$
Since
$$ \theta_{g(a)}^{\Delta E}(g(b)) = g^{(\Delta E)_+}(\theta_a^{\Delta E}(b)), $$
we have
$$ {<}\theta_{g(a)}^{\Delta E}(g(b)){>} - \langle g^{(\Delta E)_+} \rangle (S) = \langle g^{(\Delta E)_+} \rangle ({<}\theta_a^{\Delta E}(b){>} - S), $$
which belongs to $\langle (\widetilde{Y}^{\widetilde{X}}, g(a))^{(\Delta E)_+} \rangle_{\widetilde{X}}^{(r+1)}$ by Lemma 4.2. By Lemma 4.1, the ensemble $\langle g^{(\Delta E)_+} \rangle (S)$ is fissile. Since g is continuous, it takes Y_a^X to $\widetilde{Y}_{g(a)}^{\widetilde{X}}$. Thus
$$ \langle g^{(\Delta E)_+} \rangle (S) \in \langle (\widetilde{Y}_{g(a)}^{\widetilde{X}})^{(\Delta E)_+} \rangle. $$
We are done. $\square$

§ 5. Compositions and smash products

Compositions. Let X, Y, $\widetilde{X}$, and $\widetilde{Y}$ be cellular spaces, X and $\widetilde{X}$ compact.

Corollary 5.1. *Let $k : \widetilde{X} \to X$ and $h : Y \to \widetilde{Y}$ be maps and $a, b : X \to Y$ be maps such that $a \overset{r}{\approx} b$. Then $a \circ k \overset{r}{\approx} b \circ k$ in $Y^{\widetilde{X}}$ and $h \circ a \overset{r}{\approx} h \circ b$ in $\widetilde{Y}^X$.*

Proof. The transforms
$$ Y^X \to Y^{\widetilde{X}}, \qquad d \mapsto d \circ k, $$
and
$$ Y^X \to \widetilde{Y}^X, \qquad d \mapsto h \circ d, $$
are primitive. By Lemma 4.3, they preserve strong r-similarity. $\square$

Corollary 5.2. *Let $k : \widetilde{X} \to X$ and $h : Y \to \widetilde{Y}$ be maps and $a : X \to Y$ be a map such that $\daleth \overset{r}{\approx} a$. Then $\daleth \overset{r}{\approx} a \circ k$ in $Y^{\widetilde{X}}$ and $\daleth \overset{r}{\approx} h \circ a$ in $\widetilde{Y}^X$.*

Follows from Corollary 5.1.

Smash products. Let X, Y, and T be cellular spaces, X and T compact.

Corollary 5.3. *Let $a, b : X \to Y$ be maps such that $a \overset{r}{\approx} b$. Then the maps*

$$a \wedge \mathrm{id}_T, b \wedge \mathrm{id}_T : X \wedge T \to Y \wedge T$$

satisfy $a \wedge \mathrm{id}_T \overset{r}{\approx} b \wedge \mathrm{id}_T$.

Proof. The transform

$$Y^X \to (Y \wedge T)^{X \wedge T}, \qquad d \mapsto d \wedge \mathrm{id}_T,$$

is primitive. By Lemma 4.3, it preserves strong r-similarity. $\square$

Corollary 5.4. *Let $a : X \to Y$ be a map such that $\text{⚐} \overset{r}{\approx} a$. Then the map*

$$a \wedge \mathrm{id}_T : X \wedge T \to Y \wedge T$$

satisfies $\text{⚐} \overset{r}{\approx} a \wedge \mathrm{id}_T$.

Follows from Corollary 5.3.

§ 6. Homotopy invariance

Let X and Y be cellular spaces, X compact.

Lemma 6.1. *Let maps $a, b, \widetilde{a} : X \to Y$ satisfy*

$$\widetilde{a} \sim a \overset{r}{\approx} b.$$

Then $\widetilde{a} \overset{r}{\approx} b$.

Proof. By definition, the relation $\overset{r}{\approx}$ tolerates homotopy of its left argument. In detail. For an unbased space U, we have the bijection

$$e^U : (Y^X, a)^{U_+} \to (Y^X, \widetilde{a})^{U_+}, \qquad e^U(v) =|_U v, \quad e^U(v)(\text{⚐}) = \widetilde{a}.$$

Clearly,

$$e^U(\theta^U_a(d)) = \theta^U_{\widetilde{a}}(d), \qquad d \in Y^X. \tag{3}$$

Since $a \sim \widetilde{a}$, e^U takes $(Y^X_a)^{U_+}$ to $(Y^X_{\widetilde{a}})^{U_+}$.

The homomorphism

$$\langle e^U \rangle : \langle (Y^X, a)^{U_+} \rangle \to \langle (Y^X, \widetilde{a})^{U_+} \rangle$$

takes $\langle (Y^X, a)^{U_+} \rangle^{(s)}_X$ to $\langle (Y^X, \widetilde{a})^{U_+} \rangle^{(s)}_X$. Indeed, we have the commutative diagram

$$\begin{array}{ccccc}
\langle (Y^X, a)^{U_+} \rangle & & \xrightarrow{\quad\langle e^U \rangle\quad} & & \langle (Y^X, \widetilde{a})^{U_+} \rangle \\
\Big\downarrow{\scriptstyle \langle \square^X \rangle} & & & & \Big\downarrow{\scriptstyle \langle \square^X \rangle} \\
\langle Y^{U_+ \times X} \rangle & \xrightarrow{?\#<\widetilde{a}>} & \langle Y^{(U_+ \times X) \vee X} \rangle & \xrightarrow{\langle Y^k \rangle} & \langle Y^{U_+ \times X} \rangle,
\end{array}$$

where $k : U_+ \times X \to (U_+ \times X) \vee X$ is given by the rules

$$k(u, x) = \mathrm{in}_1(u, x), \qquad k(¶, x) = \mathrm{in}_2(x), \qquad u \in U,\ x \in X.$$

In the lower row, $\langle Y^{U_+ \times X} \rangle^{(s)}$ goes to $\langle Y^{(U_+ \times X) \vee X} \rangle^{(s)} \rangle$ by [3, Lemma 3.1] and then to $\langle Y^{U_+ \times X} \rangle^{(s)}$ by [3, Lemma 2.1]. This suffices by the definition of $\langle (Y^X, a)^{U_+} \rangle_X^{(s)}$ and $\langle (Y^X, \widetilde{a})^{U_+} \rangle_X^{(s)}$.

Take a nonempty finite set E. For $A \in \mathcal{A}(E)$ and a collection $S_F \in \langle (Y^X, a)^{(\Delta F)_+} \rangle$, $F \in A$, we have

$$\langle e^{\Delta[A]} \rangle (\mathop{\#}_{F \in A} S_F) = \mathop{\#}_{F \in A} e^{\Delta F}(S_F) \tag{4}$$

in $\langle (Y^X, \widetilde{a})^{\Delta[F]_+} \rangle$. We have a fissile ensemble $S \in \langle (Y_a^X)^{(\Delta E)_+} \rangle$ such that

$$<\theta_a^{\Delta E}(b)> - S \in \langle (Y_a^X)^{(\Delta E)_+} \rangle^{(r+1)}. \tag{5}$$

We get the ensemble $\langle e^{\Delta E} \rangle (S) \in \langle (Y_{\widetilde{a}}^X)^{(\Delta E)_+} \rangle$, which is fissile. Indeed, for $A \in \mathcal{A}(E)$, we have

$$\langle e^{\Delta E} \rangle (S)|_{\Delta[A]_+} \overset{(*)}{=} \langle e^{\Delta[A]} \rangle (S|_{\Delta[A]_+}) =$$

(since S is fissile)

$$= \langle e^{\Delta[A]} \rangle (\mathop{\#}_{F \in A} S|_{(\Delta F)_+}) =$$

(by (4))

$$= \mathop{\#}_{F \in A} \langle e^{\Delta F} \rangle (S|_{(\Delta F)_+}) \overset{(*)}{=} \mathop{\#}_{F \in A} \langle e^{\Delta E} \rangle (S)|_{(\Delta F)_+}$$

(the equalities $(*)$ hold by naturality of e^U with respect to U). We have

$$<\theta_{\widetilde{a}}^{\Delta E}(b)> - \langle e^{\Delta E} \rangle (S) = (\text{by } (3)) = \langle e^{\Delta E} \rangle (<\theta_a^{\Delta E}(b)> - S) \in \langle (Y_{\widetilde{a}}^X)^{(\Delta E)_+} \rangle^{(r+1)},$$

where $\in$ follows from (5) because $\langle e^{\Delta E} \rangle$ preserves the filtration. Thus $\widetilde{a} \overset{r}{\approx} b$. □

Theorem 6.2. *Let maps $a, b, \widetilde{a}, \widetilde{b} : X \to Y$ satisfy*

$$\widetilde{a} \sim a \overset{r}{\approx} b \sim \widetilde{b}.$$

Then $\widetilde{a} \overset{r}{\approx} \widetilde{b}$.

Proof. We crop Y and assume it compact. By [2, Corollary 4.2], we can continuously associate to each path $v : [0, 1] \to Y$ an unbased homotopy $E_t(v) : Y \to Y$, $t \in [0, 1]$, such that $E_0(v) = \mathrm{id}$ and $E_t(v)(v(0)) = v(t)$. Let $h_t : X \to Y$, $t \in [0, 1]$, be a homotopy such that $h_0 = b$ and $h_1 = \widetilde{b}$. For $x \in X$, introduce the path $v_x = h_?(x) : [0, 1] \to Y$. We have $v_x(0) = h_0(x) = b(x)$ and $v_x(1) = h_1(x) = \widetilde{b}(x)$. Introduce the homotopy

$$H_t : X \times Y \to Y,\ t \in [0, 1], \qquad H_t(x, y) = E_t(v_x)(y).$$

We have

$$H_0(x, y) = E_0(v_x)(y) = y$$

and

$$H_1(x, b(x)) = E_1(v_x)(b(x)) = E_1(v_x)(v_x(0)) = v_x(1) = \widetilde{b}(x).$$

Consider the primitive transforms

$$g_t : Y^X \to Y^X,\ t \in [0, 1], \qquad g_t(d)(x) = H_t(x, d(x)).$$

We have $d = g_0(d) \sim g_1(d)$, $d \in Y^X$, and $g_1(b) = \widetilde{b}$.

We have

$$\widetilde{a} \sim a \sim g_1(a) \overset{r}{\approx} g_1(b) = \widetilde{b},$$

where $\overset{r}{\approx}$ holds by Lemma 4.3. By Lemma 6.1, $\widetilde{a} \overset{r}{\approx} \widetilde{b}$. $\square$

Using Theorem 6.2, we define the relation of strong r-similarity on the set $[X, Y]$ by the rule

$$[a] \overset{r}{\approx} [b] \quad \Leftrightarrow \quad a \overset{r}{\approx} b.$$

§ 7. Joining ensembles

Let X_1, X_2, Y, and T be spaces, X_i and T compact. Consider the $\mathbb{Z}$-bilinear operation

$$\#_T : \langle (Y^{X_1})^T \rangle \times \langle (Y^{X_2})^T \rangle \to \langle (Y^{X_1 \vee X_2})^T \rangle, \qquad <v_1> \#_T <v_2> = <v>,$$
$$v(t) = v_1(t) \mathbin{\underline{\vee}} v_2(t) : X_1 \vee X_2 \to Y, \qquad t \in T.$$

Lemma 7.1. *Let E be a finite set and $S_i \in \langle (Y^{X_i})^{(\Delta E)_+} \rangle$, $i = 1, 2$, be fissile ensembles. Then the ensemble*

$$S_1 \#_{(\Delta E)_+} S_2 \in \langle (Y^{X_1 \vee X_2})^{(\Delta E)_+} \rangle$$

is fissile.

Proof. Take $A \in \mathcal{A}(E)$. We have the commutative diagram

$$\begin{array}{ccc}
\prod\limits_{F \in A} \big(\langle (Y^{X_1})^{(\Delta E)_+} \rangle \times \langle (Y^{X_2})^{(\Delta E)_+} \rangle \big) & \xrightarrow{\prod_{F \in A} \#_{(\Delta E)_+}} & \prod\limits_{F \in A} \langle (Y^{X_1 \vee X_2})^{(\Delta E)_+} \rangle \\
\Big\downarrow {\scriptstyle \prod_{F \in A} (?|_{(\Delta F)_+} \times ?|_{(\Delta F)_+})} & & \Big\downarrow {\scriptstyle \prod_{F \in A} ?|_{(\Delta F)_+}} \\
\prod\limits_{F \in A} \big(\langle (Y^{X_1})^{(\Delta F)_+} \rangle \times \langle (Y^{X_2})^{(\Delta F)_+} \rangle \big) & \xrightarrow{\prod_{F \in A} \#_{(\Delta F)_+}} & \prod\limits_{F \in A} \langle (Y^{X_1 \vee X_2})^{(\Delta F)_+} \rangle \\
\Big\downarrow {\scriptstyle \mathop{\#}_{F \in A} \times \mathop{\#}_{F \in A}} & & \Big\downarrow {\scriptstyle \mathop{\#}_{F \in A}} \\
\langle (Y^{X_1})^{\Delta[A]_+} \rangle \times \langle (Y^{X_2})^{\Delta[F]_+} \rangle & \xrightarrow{\#_{\Delta[A]_+}} & \langle (Y^{X_1 \vee X_2})^{\Delta[A]_+} \rangle \\
\Big\uparrow {\scriptstyle ?|_{\Delta[A]_+} \times ?|_{\Delta[A]_+}} & & \Big\uparrow {\scriptstyle ?|_{\Delta[A]_+}} \\
\langle (Y^{X_1})^{(\Delta E)_+} \rangle \times \langle (Y^{X_2})^{(\Delta E)_+} \rangle & \xrightarrow{\#_{(\Delta E)_+}} & \langle (Y^{X_1 \vee X_2})^{(\Delta E)_+} \rangle
\end{array}$$

with the sendings

$$\begin{array}{ccc}
((S_1,S_2))_{F\in A} & \xmapsto{(1)} & (S_1 \sharp_{(\Delta E)_+} S_2)_{F\in A} \\
\downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} \\
((S_1|_{(\Delta F)_+}, S_2|_{(\Delta F)_+}))_{F\in A} & & ((S_1 \sharp_{(\Delta E)_+} S_2)|_{(\Delta F)_+})_{F\in A} \\
\downarrow{\scriptstyle (4)} & & \downarrow{\scriptstyle (5)} \\
(S_1|_{\Delta[A]_+}, S_2|_{\Delta[A]_+}) & & \sharp_{F\in A} (S_1 \sharp_{(\Delta E)_+} S_2)|_{(\Delta F)_+} \\
\uparrow{\scriptstyle (6)} & & \uparrow{\scriptstyle (7)} \\
(S_1,S_2) & \xrightarrow{(8)} & S_1 \sharp_{(\Delta E)_+} S_2.
\end{array}$$

The sending (4) holds by fissility of S_1 and S_2. The sendings (1), (2), (3), (5), (6), and (8) are obvious. The sending (7) follows. Thus $S_1 \sharp_{(\Delta E)_+} S_2$ is fissile. □

Lemma 7.2. *We have*

$$\langle (Y^{X_1})^T \rangle^{(p)}_{X_1} \sharp_T \langle (Y^{X_2})^T \rangle^{(q)}_{X_2} \subseteq \langle (Y^{X_1 \vee X_2})^T \rangle^{(p+q)}_{X_1 \vee X_2}.$$

Proof. Take ensembles

$$Z_1 \in \langle (Y^{X_1})^T \rangle^{(p)}_{X_1}, \qquad Z_2 \in \langle (Y^{X_2})^T \rangle^{(q)}_{X_2}.$$

We have the commutative diagram

$$\begin{array}{ccc}
\langle (Y^{X_1})^T \rangle \times \langle (Y^{X_2})^T \rangle & \xrightarrow{\sharp_T} & \langle (Y^{X_1 \vee X_2})^T \rangle \\
{\scriptstyle \langle \widehat{\square}^{X_1} \rangle \times \langle \widehat{\square}^{X_2} \rangle} \downarrow & & \downarrow {\scriptstyle \langle \widehat{\square}^{X_1 \vee X_2} \rangle} \\
\langle Y^{T \wedge X_1} \rangle \times \langle Y^{T \wedge X_2} \rangle & \xrightarrow{\sharp} & \langle Y^{(T \wedge X_1) \vee (T \wedge X_2)} \rangle
\end{array}$$

(we used distributivity of smash product over wedge) and the sendings

$$\begin{array}{ccc}
(Z_1, Z_2) & \longmapsto & Z_1 \sharp_T Z_2 \\
\downarrow & & \downarrow \\
(\langle \widehat{\square}^{X_1} \rangle (Z_1), \langle \widehat{\square}^{X_2} \rangle (Z_2)) & \longmapsto & \langle \widehat{\square}^{X_1 \vee X_2} \rangle (Z_1 \sharp_T Z_2).
\end{array}$$

By Lemma 3.3,

$$(\langle \widehat{\square}^{X_1} \rangle (Z_1), \langle \widehat{\square}^{X_2} \rangle (Z_2)) \in \langle Y^{T \wedge X_1} \rangle^{(p)} \times \langle Y^{T \wedge X_2} \rangle^{(q)}.$$

Thus, by [3, Lemma 3.1],

$$\langle \widehat{\square}^{X_1 \vee X_2} \rangle (Z_1 \sharp_T Z_2) \in \langle Y^{(T \wedge X_1) \vee (T \wedge X_2)} \rangle^{(p+q)}.$$

Thus, by Lemma 3.3,

$$Z_1 \sharp_T Z_2 \in \langle (Y^{X_1 \vee X_2})^T \rangle^{(p+q)}. \qquad \square$$

Corollary 7.3. *Let maps $a_i : X_i \to Y$, $i = 1, 2$, satisfy $\daleth \overset{r}{\approx} a_i$. Then the map*

$$a_1 \veebar a_2 : X_1 \vee X_2 \to Y$$

satisfies $\daleth \overset{r}{\approx} a_1 \veebar a_2$.

Proof. Take a finite set E. We have fissile ensembles $S_i \in \langle (Y_{\daleth}^{X_i})^{(\Delta E)_+} \rangle$, $i = 1, 2$, such that

$$<\theta_{\daleth}^{\Delta E}(a_i)> - S_i \in \langle (Y^{X_i})^{(\Delta E)_+} \rangle_{X_i}^{(r+1)}.$$

By Lemma 7.1, the ensemble

$$S_1 \#_{(\Delta E)_+} S_2 \in \langle (Y_{\daleth}^{X_1 \vee X_2})^{(\Delta E)_+} \rangle$$

is fissile. We have

$$\begin{aligned} &<\theta_{\daleth}^{\Delta E}(a_1 \veebar a_2)> - S_1 \#_{(\Delta E)_+} S_2 = \\ &\qquad = <\theta_{\daleth}^{\Delta E}(a_1)> \#_{(\Delta E)_+} <\theta_{\daleth}^{\Delta E}(a_2)> - S_1 \#_{(\Delta E)_+} S_2 = \\ &\qquad\quad = (<\theta_{\daleth}^{\Delta E}(a_1)> - S_1) \#_{(\Delta E)_+} <\theta_{\daleth}^{\Delta E}(a_2)> + \\ &\qquad\quad + S_1 \#_{(\Delta E)_+} (<\theta_{\daleth}^{\Delta E}(a_2)> - S_2) \in \langle (Y^{X_1 \vee X_2})^{(\Delta E)_+} \rangle_{X_1 \vee X_2}^{(r+1)}, \end{aligned}$$

where $\in$ holds by Lemma 7.2. We are done. $\square$

§ 8. Strong similarity for an admissible couple

Let X and Y be cellular spaces, X compact. Let X be equipped with maps $\mu : X \to X \vee X$ (comultiplication) and $\nu : X \to X$ (coinversion). The set Y^X carries the operations

$$(a, b) \mapsto (a * b : X \xrightarrow{\mu} X \vee X \xrightarrow{a \veebar b} Y)$$

and

$$a \mapsto (a^\dagger : X \xrightarrow{\nu} X \xrightarrow{a} Y).$$

We suppose that $(X, \mu, \nu; Y)$ is an admissible couple in the sense of [3], that is, the set $[X, Y]$ is a group with the multiplication

$$[a][b] = [a * b],$$

the inversion

$$[a]^{-1} = [a^\dagger],$$

and the identity $1 = [\daleth_Y^X]$. We are mainly interested in the case of $X = \Sigma T$ with standard μ and ν.

We proceed parallelly to [3]. The subsets

$$[X, Y]^{((r+1))} = \{ \boldsymbol{a} \in [X, Y] \mid 1 \overset{r}{\approx} \boldsymbol{a} \}$$

form the filtration

$$[X, Y] = [X, Y]^{((1))} \supseteq [X, Y]^{((2))} \supseteq \dots .$$

Theorem 8.1. $[X,Y]^{((r+1))} \subseteq [X,Y]$ *is a normal subgroup.*

Proof. Take $a, b : X \to Y$, $\urcorner \stackrel{r}{\approx} a, b$. Check that $\urcorner \stackrel{r}{\approx} a * b$. We have the decomposition

$$a * b : X \xrightarrow{\mu} X \vee X \xrightarrow{a \underline{\vee} b} Y.$$

By Corollary 7.3, $\urcorner_Y^{X \vee X} \stackrel{r}{\approx} a \underline{\vee} b$. By Corollary 5.2, $\urcorner \stackrel{r}{\approx} a * b$.

Take $a : X \to Y$, $\urcorner \stackrel{r}{\approx} a$. Check that $\urcorner \stackrel{r}{\approx} a^\dagger$. We have the decomposition

$$a^\dagger : X \xrightarrow{\nu} X \xrightarrow{a} Y.$$

By Corollary 5.2, $\urcorner \stackrel{r}{\approx} a^\dagger$.

Take $a, b : X \to Y$, $\urcorner \stackrel{r}{\approx} a$. Check that $\urcorner \stackrel{r}{\approx} b^\dagger * (a * b)$. Consider the primitive transform

$$Y^X \to Y^X, \qquad d \mapsto b^\dagger * (d * b).$$

We have

$$\urcorner_Y^X \sim b^\dagger * (\urcorner_Y^X * b) \stackrel{r}{\approx} b^\dagger * (a * b),$$

where $\stackrel{r}{\approx}$ holds by Lemma 4.3. By Lemma 6.1, $\urcorner \stackrel{r}{\approx} b^\dagger * (a * b)$. □

We do not know whether the subgroups $[X,Y]^{((s))}$ form an N-series.

Theorem 8.2. *For* $\boldsymbol{a}, \boldsymbol{b} \in [X,Y]$*, we have*

$$\boldsymbol{a} \stackrel{r}{\approx} \boldsymbol{b} \quad \Leftrightarrow \quad \boldsymbol{a}^{-1}\boldsymbol{b} \in [X,Y]^{((r+1))}.$$

Proof. It suffices to check that, for maps $a, b, c : X \to Y$, $a \stackrel{r}{\approx} b$ implies $c * a \stackrel{r}{\approx} c * b$. This follows from Lemma 4.3 for the primitive transform

$$Y^X \to Y^X, \qquad d \mapsto c * d.$$ □

It follows from Theorems 8.1 and 8.2 that, for an admissible couple $(X, \mu, \nu; Y)$, the relation $\stackrel{r}{\approx}$ on $[X,Y]$ is an equivalence.

§ 9. Presheaves and extenders

Let P be a finite partially ordered set and C be a concrete category. (Concreteness is not essential; we assume it for convenience of notation only.) A cofunctor $S : P \to C$ is called a *presheaf.* For $p, q \in P$, $p \geqslant q$, we have the induced morphism

$$?|_q : S(p) \to S(q)$$

(the *restriction* morphism).

For a preasheaf $U : P \to \mathbf{Ab}$, we have the isomorphism

$$\nabla_P : \bigoplus_{p \in P} U(p) \to \bigoplus_{p \in P} U(p), \qquad \mathrm{in}_p(u) \mapsto \sum_{q \in P\lceil p \rceil} \mathrm{in}_q(u|_q), \qquad u \in U(p),\ p \in P.$$

Hereafter,

$$P\lceil p\rceil = \{\, q \in P \mid p \geqslant q \,\}$$

and

$$\mathrm{in}_q : U(q) \to \bigoplus_{p\in P} U(p)$$

are the canonical insertions.

Suppose that P has the infimum operation $\wedge$ and the greatest element $\top$. It follows that P is a lattice. We put $P^\times = P \setminus \{\top\}$. An *extender* λ for the preasheaf S is a collection of morphisms

$$\lambda_p^q : S(q) \to S(p), \qquad p, q \in P,\ p \geqslant q,$$

such that, for $p, q \in P$ and $s \in S(q)$,

$$\lambda_p^q(s)|_q = s \qquad \text{if } p \geqslant q$$

and

$$\lambda_\top^q(s)|_p = \lambda_p^{p\wedge q}(s|_{p\wedge q}).$$

In particular,

$$\lambda_p^q(s) = \lambda_\top^q(s)|_p.$$

(The extenders we deal with satisfy the identity $\lambda_p^q \circ \lambda_q^r = \lambda_p^r$. We neither check nor use this.)

Consider a preaheaf $U : P \to \mathbf{Ab}$ with an extender λ. The symbol $\overline{\underline{\bigoplus}}$ below denotes the homomorphism of a direct sum given by its restrictions to the summands.

Lemma 9.1. *For $q \in P$, the diagram*

$$\begin{array}{ccccc}
\bigoplus\limits_{p\in P} U(p) & \xleftarrow[\cong]{\nabla_P} & \bigoplus\limits_{p\in P} U(p) & \xrightarrow{\overline{\underline{\bigoplus}}_{p\in P} \lambda_\top^p} & U(\top) \\
\downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle R_q} & & \downarrow{\scriptstyle ?|_q} \\
\bigoplus\limits_{p\in P\lceil q\rceil} U(p) & \xleftarrow[\cong]{\nabla_{P\lceil q\rceil}} & \bigoplus\limits_{p\in P\lceil q\rceil} U(p) & \xrightarrow{\overline{\underline{\bigoplus}}_{p\in P\lceil q\rceil} \lambda_q^p} & U(q),
\end{array}$$

where R_q is the homomorphism defined by the rule

$$\mathrm{in}_p(u) \mapsto \mathrm{in}_{p\wedge q}(u|_{p\wedge q}),$$

is commutative.

Direct check. □

Lemma 9.2. *The homomorphism*

$$U(\top) \to \lim_{p\in P^\times} U(p), \qquad u \mapsto (u|_p)_{p\in P^\times},$$

is surjective.

Proof. Take a collection

$$(u_p)_{p\in P^\times} \in \lim_{p\in P^\times} U(p) \subseteq \bigoplus_{p\in P^\times} U(p).$$

Define a collection $(v_p)_{p\in P^\times}$ and a section u by the diagram

$$\begin{array}{ccccc} \bigoplus\limits_{p\in P^\times} U(p) & \xleftarrow[\cong]{\nabla_{P^\times}} & \bigoplus\limits_{p\in P^\times} U(p) & \xrightarrow{\overline{\bigoplus}_{p\in P^\times} \lambda^p_\top} & U(\top). \\ (u_p)_{p\in P^\times} & \longleftarrow\!\shortmid & (v_p)_{p\in P^\times} & \longmapsto & u \end{array}$$

Take $q \in P^\times$. We show that $u|_q = u_q$, which will suffice. In the diagram of Lemma 9.1, we have

$$\begin{array}{ccccc} (u_p)_{p\in P} & \xleftarrow{(1)}\!\shortmid & (v_p)_{p\in P} & \xmapsto{(2)} & u \\ {\scriptstyle (3)}\downarrow & & {\scriptstyle (4)}\downarrow & & \downarrow{\scriptstyle (5)} \\ (u_p)_{p\in P\lceil q\rceil} & \xleftarrow{(6)}\!\shortmid & \mathrm{in}_q(u_q) & \xmapsto{(7)} & u_q, \end{array}$$

where we put $u_\top = v_\top = 0$ in $U(\top)$. The sendings (1) and (2) follow from the construction of the collections. The sending (6) expresses the equalities $u_q|_p = u_p$, $p \in P\lceil q\rceil$, which hold by the definition of limit. The sending (3) is obvious. The sending (4) follows because the left square is commutative and $\nabla_{P\lceil q\rceil}$ is injective. The sending (7) is the equality $\lambda^q_q = \mathrm{id}$, which follows from the definition of extender. By commutativity of the right square, the sending (5) holds, which is what was to be checked. □

§ 10. The abstract fissilizer Φ^E on $\langle \underline{M}(E)\rangle$

Fix a nonempty finite set E. The set $\mathcal{P}_\times(E)$ is partially ordered by inclusion.

For $A, B \in \mathcal{A}(E)$, we say $A \geqslant B$ if, for each $G \in B$, there is $F \in A$ such that $F \supseteq G$. Such an F is unique; we denote it by $(A)G$. The set $\mathcal{A}(E)$ becomes a lattice with the infimum operation

$$A \wedge B = \{\, F \cap G \mid F \in A,\ G \in B \,\} \setminus \{\varnothing\}$$

and the greatest element $\top = \{E\}$.

Let **Mg** be the category of sets and $M : \mathcal{P}_\times(E) \to \mathbf{Mg}$ be a presheaf. We define a presheaf $\underline{M} : \mathcal{A}(E) \to \mathbf{Mg}$. For $A \in \mathcal{A}(E)$, put

$$\underline{M}(A) = \prod_{F\in A} M(F).$$

For $A, B \in \mathcal{A}(E)$, $A \geqslant B$, define the restriction function

$$\underline{M}(A) \to \underline{M}(B), \qquad \underline{m} \mapsto \underline{m}|_B,$$

by putting, for $\underline{m} = (m_F)_{F\in A}$,

$$\underline{m}|_B = (m_{(A)G}|_G)_{G\in B}.$$

Clearly, $\underline{M}(\{E\}) = M(E)$.

Taking composition with the functor $\langle ? \rangle : \mathbf{Mg} \to \mathbf{Ab}$, we get the presheaves

$$\mathcal{P}_\times(E) \to \mathbf{Ab}, \qquad F \mapsto \langle M(F)\rangle,$$

and

$$\mathcal{A}(E) \to \mathbf{Ab}, \qquad A \mapsto \langle \underline{M}(A)\rangle. \tag{6}$$

For $A \in \mathcal{A}(E)$, we have the $\mathbb{Z}$-multilinear operation

$$\mathop{\#}_{F\in A} : \prod_{F\in A} \langle M(F)\rangle \to \langle \underline{M}(A)\rangle, \qquad \mathop{\#}_{F\in A} {<}m_F{>} = {<}(m_F)_{F\in A}{>} \tag{7}$$

(cf. § 2). For $Q \in \langle M(E)\rangle$ and $A \in \mathcal{A}(E)$, put

$$Q^{\#}(A) = \mathop{\#}_{F\in A} Q|_F \in \langle \underline{M}(A)\rangle.$$

We call an ensemble $R \in \langle M(E)\rangle$ *fissile* if, for any layout $A \in \mathcal{A}(E)$,

$$R|_A = R^{\#}(A)$$

in $\langle \underline{M}(A)\rangle$.

We suppose that the presheaf $\underline{M}$ has an extender

$$\lambda_A^B : \underline{M}(B) \to \underline{M}(A), \qquad A, B \in \mathcal{A}(E),\ A \geqslant B.$$

Then the preasheaf (6) has the extender

$$\langle \lambda_A^B \rangle : \langle \underline{M}(B)\rangle \to \langle \underline{M}(A)\rangle, \qquad A, B \in \mathcal{A}(E),\ A \geqslant B.$$

For $Q \in \langle M(E)\rangle$, define an ensemble $\Phi^E(Q) \in \langle M(E)\rangle$ by the rule

$$\begin{array}{ccccc}
\bigoplus_{A\in\mathcal{A}(E)} \langle \underline{M}(A)\rangle & \xleftarrow[\cong]{\nabla_{\mathcal{A}(E)}} & \bigoplus_{A\in\mathcal{A}(E)} \langle \underline{M}(A)\rangle & \xrightarrow{\overline{\bigoplus}_{A\in\mathcal{A}(E)} \langle \lambda^A_{\{E\}}\rangle} & \langle M(E)\rangle. \\
Q^{\#} & \longleftarrow\!\!| & \nabla^{-1}_{\mathcal{A}(E)}(Q^{\#}) & |\!\!\longrightarrow & \Phi^E(Q)
\end{array}$$

We get a function (not a homomorphism)

$$\Phi^E : \langle M(E)\rangle \to \langle M(E)\rangle,$$

which we call the *fissilizer*.

Lemma 10.1. *For any ensemble $Q \in \langle M(E)\rangle$, the ensemble $\Phi^E(Q)$ is fissile.*

Proof. Take $A \in \mathcal{A}(E)$. We have the commutative diagram

$$
\begin{array}{ccccc}
Q^{\sharp} & \longleftarrow\!\dashv & \nabla^{-1}_{\mathcal{A}(E)}(Q^{\sharp}) & \longmapsto & \Phi^{E}(Q) \\
\bigoplus_{a\in\mathcal{A}(E)} \langle \underline{M}(a)\rangle & \xleftarrow[\cong]{\nabla_{\mathcal{A}(E)}} & \bigoplus_{a\in\mathcal{A}(E)} \langle \underline{M}(a)\rangle & \xrightarrow{\overline{\bigoplus}_{a\in\mathcal{A}(E)} \langle\lambda^{a}_{\{E\}}\rangle} & \langle M(E)\rangle \\
\downarrow \mathrm{pr} & & \downarrow R_A & & \downarrow ?|_A \\
\bigoplus_{a\in\mathcal{A}(E)\lceil A\rceil} \langle \underline{M}(a)\rangle & \xleftarrow[\cong]{\nabla_{\mathcal{A}(E)\lceil A\rceil}} & \bigoplus_{a\in\mathcal{A}(E)\lceil A\rceil} \langle \underline{M}(a)\rangle & \xrightarrow{\overline{\bigoplus}_{a\in\mathcal{A}(E)\lceil A\rceil} \langle\lambda^{a}_{A}\rangle} & \langle \underline{M}(A)\rangle \\
J_A \downarrow \cong & & J_A \downarrow \cong & & \cong \downarrow I_A \\
\bigotimes_{F\in A} \Big(\bigoplus_{b\in\mathcal{A}(F)} \langle \underline{M}(b)\rangle \Big) & \xleftarrow[\cong]{\bigotimes_{F\in A} \nabla_{\mathcal{A}(F)}} & \bigotimes_{F\in A} \Big(\bigoplus_{b\in\mathcal{A}(F)} \langle \underline{M}(b)\rangle \Big) & \xrightarrow{\bigotimes_{F\in A} (\overline{\bigoplus}_{b\in\mathcal{A}(F)} \langle\lambda^{b}_{\{F\}}\rangle)} & \bigotimes_{F\in A} \langle M(F)\rangle, \\
\bigotimes_{F\in A} Q|^{\sharp}_{F} & \longleftarrow\!\dashv & \bigotimes_{F\in A} \nabla^{-1}_{\mathcal{A}(F)}(Q|^{\sharp}_{F}) & \longmapsto & \bigotimes_{F\in A} \Phi^{F}(Q|_{F})
\end{array}
$$

where the upper half comes from Lemma 9.1, I_A is the isomorphism defined by the rule

$$<(m_F)_{F\in A}> \mapsto \bigotimes_{F\in A} <m_F>,$$

and J_A is the isomorphism defined by the rule

$$\mathrm{in}_a(<\underline{m}>) \mapsto \bigotimes_{F\in A} \mathrm{in}_{a\wedge\{F\}}(<\underline{m}|_{a\wedge\{F\}}>)$$

(note that $a\wedge\{F\} \in \mathcal{A}(F) \subseteq \mathcal{A}(E)$). Commutativity of the lower half is checked directly. The sendings in the upper row hold by the definition of Φ^E. The sendings in the lower row hold by the definition of $\Phi^F : \langle M(F)\rangle \to \langle M(F)\rangle$. The sending in the left column is checked directly. The sending in the right column follows. Since

$$I_A : \mathop{\#}_{F\in A} q_F \mapsto \bigotimes_{F\in A} q_F$$

for $q_F \in \langle M(F)\rangle$, $F \in A$, we get

$$\Phi^E(Q)|_A = \mathop{\#}_{F\in A} \Phi^F(Q|_F).$$

In particular, for $A = \{F\}$, this gives

$$\Phi^E(Q)|_F = \Phi^F(Q|_F).$$

Thus, for arbitrary A,

$$\Phi^E(Q)|_A = \mathop{\#}_{F\in A} \Phi^E(Q)|_F.$$

Thus $\Phi^E(Q)$ is fissile. $\square$

Let $N(A) \subseteq \langle \underline{M}(A) \rangle$, $A \in \mathcal{A}(E)$, be a collection of subgroups preserved by the restriction homomorphisms and the homomorphisms $\langle \lambda_A^B \rangle$.

Lemma 10.2. *Let an ensemble* $Q \in \langle M(E) \rangle$ *satisfy*

$$Q^{\sharp}(A) - Q|_A \in N(A)$$

for all $A \in \mathcal{A}(E)$*. Then*

$$\Phi^E(Q) - Q \in N(\{E\}).$$

Proof. We have the presheaf

$$\mathcal{A}(E) \to \mathbf{Ab}, \qquad A \mapsto \langle \underline{M}(A) \rangle / N(A),$$

with the induced restriction homomorphisms. We have the commutative diagram

$$\begin{array}{ccccc}
(Q|_A)_{A\in\mathcal{A}(E)} & \longleftarrow\!\!\!\mid & \mathrm{in}_{\{E\}}(Q) & \longmapsto & Q \\
Q^{\sharp} & \longleftarrow\!\!\!\mid & \nabla^{-1}_{\mathcal{A}(E)}(Q^{\sharp}) & \longmapsto & \Phi^E(Q) \\
\bigoplus\limits_{A\in\mathcal{A}(E)} \langle \underline{M}(A) \rangle & \xleftarrow[\cong]{\nabla_{\mathcal{A}(E)}} & \bigoplus\limits_{A\in\mathcal{A}(E)} \langle \underline{M}(A) \rangle & \xrightarrow{\overline{\bigoplus}_{A\in\mathcal{A}(E)} \langle \lambda^A_{\{E\}} \rangle} & \langle M(E) \rangle \\
\downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\
\bigoplus\limits_{A\in\mathcal{A}(E)} \langle \underline{M}(A) \rangle / N(A) & \xleftarrow[\cong]{\nabla_{\mathcal{A}(E)}} & \bigoplus\limits_{A\in\mathcal{A}(E)} \langle \underline{M}(A) \rangle / N(A) & \longrightarrow & \langle M(E) \rangle / N(\{E\}).
\end{array}$$

The upper line of sendings is obvious. The lower line of sendings holds by the definition of Φ^E. By hypothesis, the difference of the elements in the upper-left corner descends to zero. Since $\nabla_{\mathcal{A}(E)}$ in the lower row is an isomorphism, the difference of elements in the upper-right corner also descends to zero. □

§ 11. Topological and simplicial constructions

Topological cones. Take $s \in \{0, 1\}$. Given an unpointed space U, form the space

$$\mathrm{C}^s U = (U \times [0,1]) / (U \times \{s\}),$$

the *cone* over U. The innate basepoint (where $U \times \{s\}$ is projected) is called the *apex*. Using the "base" embedding

$$U \xrightarrow{u \mapsto (u, 1-s)} U \times [0,1] \xrightarrow{\mathrm{pr}} \mathrm{C}^s U,$$

we adopt the inclusion $U \subseteq \mathrm{C}^s U$ and the based one $U_+ \subseteq \mathrm{C}^s U$. A path of the form

$$[0,1] \xrightarrow{t \mapsto (u,t)} U \times [0,1] \xrightarrow{\mathrm{pr}} \mathrm{C}^s U$$

is called a *generating path*. For an unpointed subspace $V \subseteq U$, we have $\mathrm{C}^s V \subseteq \mathrm{C}^s U$.

Notation: $\check{\mathrm{C}} = \mathrm{C}^0$, $\hat{\mathrm{C}} = \mathrm{C}^1$.

Topological suspensions. For an unpointed space U, the *unreduced suspension* $\overline{\Sigma} U$ is the colimit of the diagram

$$\{0,1\} \xleftarrow{\mathrm{pr}} U \times \{0,1\} \xrightarrow{\mathrm{in}} U \times [0,1].$$

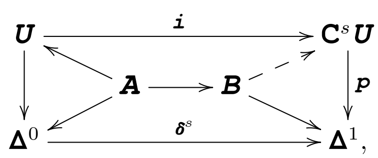
Let $s_{\overline{\Sigma} U} \in \overline{\Sigma} U$ be the point coming from $s \in \{0,1\}$. We appoint $0_{\overline{\Sigma} U}$ to be the basepoint of $\overline{\Sigma} U$.

We use also the usual reduced suspension Σ.

Unreduced Kan cones. Let $\boldsymbol{\Delta}^n$ be "the n-simplex", the simplicial set represented (as a cofunctor) by the object $[n]$ of the simplex category. Take $s \in \{0,1\}$. Let $\boldsymbol{\delta}^s : \boldsymbol{\Delta}^0 \to \boldsymbol{\Delta}^1$ be the morphism induced by the function $\delta^s : [0] \to [1]$, $0 \mapsto 1 - s$. Given a simplicial set $\boldsymbol{U}$, we define its *cone* $\mathbf{C}^s \boldsymbol{U}$. There is a unique (up to an isomorphism) Cartesian square

$$\begin{array}{ccc} \boldsymbol{U} & \xrightarrow{\boldsymbol{i}} & \mathbf{C}^s \boldsymbol{U} \\ \downarrow & & \downarrow{\scriptstyle \boldsymbol{p}} \\ \boldsymbol{\Delta}^0 & \xrightarrow{\boldsymbol{\delta}^s} & \boldsymbol{\Delta}^1 \end{array}$$

with the universal property expressed by the diagram



where the lower trapeze is assumed to be Cartesian. The morphism $\boldsymbol{\delta}^{1-s} : \boldsymbol{\Delta}^0 \to \boldsymbol{\Delta}^1$ lifts along $\boldsymbol{p}$ uniquely. This yields a morphism $\boldsymbol{\Delta}^0 \to \mathbf{C}^s \boldsymbol{U}$, which makes $\mathbf{C}^s \boldsymbol{U}$ a pointed simplicial set. The basepoint is called the *apex*. The morphism $\boldsymbol{i}$ is injective. Using it, we adopt the inclusion $\boldsymbol{U} \subseteq \mathbf{C}^s \boldsymbol{U}$ and the based one $\boldsymbol{U}_+ \subseteq \mathbf{C}^s \boldsymbol{U}$. We call $\boldsymbol{p}$ the *projection*.

All constructions are covariant/natural in $\boldsymbol{U}$. The functor $\mathbf{C}^s$ preserves injective morphisms. Using this, we adopt the inclusion $\mathbf{C}^s \boldsymbol{V} \subseteq \mathbf{C}^s \boldsymbol{U}$ for a simplicial subset $\boldsymbol{V} \subseteq \boldsymbol{U}$.

Notation: $\check{\mathbf{C}} = \mathbf{C}^0$, $\hat{\mathbf{C}} = \mathbf{C}^1$.

There is a unique natural map $r : \mathrm{C}^s |\boldsymbol{U}| \to |\mathbf{C}^s \boldsymbol{U}|$ such that the diagram

$$\begin{array}{ccc} & & \mathrm{C}^s |\boldsymbol{U}| \\ & \nearrow^{\mathrm{in}} & \downarrow{\scriptstyle r} \\ |\boldsymbol{U}| & \xrightarrow[\mathrm{in}\ (= |\boldsymbol{i}|)]{} & |\mathbf{C}^s \boldsymbol{U}| \end{array}$$

is commutative and each generating path of $\mathrm{C}^s|\boldsymbol{U}|$ is sent to an affine path in some simplex of $|\mathbf{C}^s\boldsymbol{U}|$. The map r is a homeomorphism. Using it, we adopt that $|\mathbf{C}^s\boldsymbol{U}| = \mathrm{C}^s|\boldsymbol{U}|$.

Reduced Kan cone. For a pointed simplicial set $\boldsymbol{T}$, introduce the pointed simplicial set $\check{\mathtt{c}}\boldsymbol{T} = \check{\mathbf{C}}\boldsymbol{T}/\check{\mathbf{C}}(⁊)$, where $(⁊) \subseteq \boldsymbol{T}$ is the simplicial subset generated by the basepoint $⁊ \in \boldsymbol{T}_0$ (so, $(⁊) \cong \boldsymbol{\Delta}^0$). We adopt the obvious inclusion $\boldsymbol{T} \subseteq \check{\mathtt{c}}\boldsymbol{T}$ and identification $\check{\mathtt{c}}(\boldsymbol{U}_+) = \check{\mathbf{C}}\boldsymbol{U}$. $\check{\mathtt{c}}$ is a functor; it preserves wedges.

Unreduced Kan suspension. For a simplicial set $\boldsymbol{U}$, introduce the pointed simplicial set $\hat{\boldsymbol{\Sigma}}\boldsymbol{U} = \hat{\mathbf{C}}\boldsymbol{U}/\boldsymbol{U}$. It has two vertices: the top $1_{\hat{\boldsymbol{\Sigma}}\boldsymbol{U}}$, which is the image of the apex of the cone $\hat{\mathbf{C}}\boldsymbol{U}$ under the projection $\hat{\mathbf{C}}\boldsymbol{U} \to \hat{\boldsymbol{\Sigma}}\boldsymbol{U}$, and the basepoint $0_{\hat{\boldsymbol{\Sigma}}\boldsymbol{U}}$ (where the base $\boldsymbol{U} \subseteq \hat{\mathbf{C}}\boldsymbol{U}$ is sent). We have

$$|\hat{\boldsymbol{\Sigma}}\boldsymbol{U}| = |\hat{\mathbf{C}}\boldsymbol{U}|/|\boldsymbol{U}| = \hat{\mathrm{C}}|\boldsymbol{U}|/|\boldsymbol{U}| = \overline{\Sigma}|\boldsymbol{U}|.$$

Thick simplex. For a set U, let $\mathbf{E}U$ be the simplicial set with $(\mathbf{E}U)_n = U^{[n]}$ $(= U^{n+1})$ and obvious structure functions.

For each $u \in U$, there is a unique retraction $\widetilde{\boldsymbol{\sigma}}_u : \check{\mathbf{C}}\hat{\mathbf{C}}\mathbf{E}U \to \hat{\mathbf{C}}\mathbf{E}U$ sending the apex to the vertex $u \in U = (\mathbf{E}U)_0 \subseteq (\hat{\mathbf{C}}\mathbf{E}U)_0$. Define retractions $\overline{\boldsymbol{\sigma}}_u$ and $\boldsymbol{\sigma}_u$ by the commutative diagram

$$\begin{array}{ccccc}
\check{\mathbf{C}}\hat{\mathbf{C}}\mathbf{E}U & \xrightarrow{\check{\mathbf{C}}\boldsymbol{q}} & \check{\mathbf{C}}\hat{\boldsymbol{\Sigma}}\mathbf{E}U & \xrightarrow{\boldsymbol{r}} & \check{\mathtt{c}}\hat{\boldsymbol{\Sigma}}\mathbf{E}U \\
\downarrow{\scriptstyle \widetilde{\boldsymbol{\sigma}}_u} & & \downarrow{\scriptstyle \overline{\boldsymbol{\sigma}}_u} & \swarrow{\scriptstyle \boldsymbol{\sigma}_u} & \\
\hat{\mathbf{C}}\mathbf{E}U & \xrightarrow{\boldsymbol{q}} & \hat{\boldsymbol{\Sigma}}\mathbf{E}U, & &
\end{array}$$

where $\boldsymbol{q}$ and $\boldsymbol{r}$ are projections. We call $\boldsymbol{\sigma}_u$ the *canonical contraction.*

Lemma 11.1. *Let $V \subseteq U$ be a subset. Then, for $u \in V$, the diagram*

$$\begin{array}{ccc}
\check{\mathtt{c}}\hat{\boldsymbol{\Sigma}}\mathbf{E}V & \longrightarrow & \check{\mathtt{c}}\hat{\boldsymbol{\Sigma}}\mathbf{E}U \\
\downarrow{\scriptstyle \boldsymbol{\sigma}_u} & & \downarrow{\scriptstyle \boldsymbol{\sigma}_u} \\
\hat{\boldsymbol{\Sigma}}\mathbf{E}V & \longrightarrow & \hat{\boldsymbol{\Sigma}}\mathbf{E}U,
\end{array}$$

where the horizontal arrows are induced by the inclusion $V \to U$, is commutative. □

If U is finite, let

$$\xi_U : |\mathbf{E}U| \to \Delta U \tag{8}$$

be the unbased map that sends, for each $u \in U$, the corresponding vertex $|u|$ of $|\mathbf{E}U|$ to the corresponding vertex $\langle u \rangle$ of ΔU and is affine on simplices. Hereafter, we put $\Delta\varnothing = \varnothing$.

Barycentric subdivision. Let K be an (abstract simplicial) complex. We order the set of simplices of K by reverse inclusion. Define the simplicial set $\boldsymbol{\beta}K$ as the nerve of this partially ordered set. For a subcomplex $L \subseteq K$, we have $\boldsymbol{\beta}L \subseteq \boldsymbol{\beta}K$.

There is a homeomorphism $|\boldsymbol{\beta} K| \to |K|$ that sends the vertex of $|\boldsymbol{\beta} K|$ corresponding to a simplex k of K to the barycentre of the simplex $|k| \subseteq |K|$ and takes each simplex of $|\boldsymbol{\beta} K|$ to some simplex of $|K|$ affinely. Using it, we adopt that $|\boldsymbol{\beta} K| = |K|$.

Canonical retractions. Given a complex K and a subcomplex $L \subseteq K$, we have $\check{\mathbf{C}}\boldsymbol{\beta} L \subseteq \check{\mathbf{C}}\boldsymbol{\beta} K$ and define the based morphism

$$\boldsymbol{\rho}^K_L : \check{\mathbf{C}}\boldsymbol{\beta} K \to \check{\mathbf{C}}\boldsymbol{\beta} L$$

as the retraction that sends all vertices outside $\check{\mathbf{C}}\boldsymbol{\beta} L$ to the apex of $\check{\mathbf{C}}\boldsymbol{\beta} L$. We call $\boldsymbol{\rho}^K_L$ the *canonical retraction*.

Lemma 11.2. *For two subcomplexes $L, M \subseteq K$, the diagram*

$$\begin{CD} \check{\mathbf{C}}\boldsymbol{\beta} L @>\mathtt{in}>> \check{\mathbf{C}}\boldsymbol{\beta} K \\ @V{\boldsymbol{\rho}^L_{L\cap M}}VV @VV{\boldsymbol{\rho}^K_M}V \\ \check{\mathbf{C}}\boldsymbol{\beta}(L \cap M) @>\mathtt{in}>> \check{\mathbf{C}}\boldsymbol{\beta} M \end{CD}$$

is commutative. □

§ 12. Canonical retractions in the cones $\check{\mathbf{C}}\boldsymbol{\beta}\Delta E$ and $\check{\mathrm{C}}\Delta E$

Fix a nonempty finite set E.

The simplex ΔE and its subcomplexes. Let the ΔE be the complex whose set of vertices is E and set of simplices is $\mathcal{P}_\times(E)$. For $F \in \mathcal{P}_\times(E)$, we have the subcomplex $\Delta F \subseteq \Delta E$. For $A \in \mathcal{A}(E)$, introduce the subcomplex $\Delta[A] \subseteq \Delta E$,

$$\Delta[A] = \bigcup_{F \in A} \Delta F.$$

For $A, B \in \mathcal{A}(E)$, we have

$$A \geqslant B \;\Rightarrow\; \Delta[A] \supseteq \Delta[B]$$

and $\Delta[A \wedge B] = \Delta[A] \cap \Delta[B]$. Moreover, $\Delta[\{E\}] = \Delta E$.

For $A, B \in \mathcal{A}(E)$, $A \geqslant B$, we have the canonical retraction

$$\boldsymbol{\rho}^A_B = \boldsymbol{\rho}^{\Delta[A]}_{\Delta[B]} : \check{\mathbf{C}}\boldsymbol{\beta}\Delta[A] \to \check{\mathbf{C}}\boldsymbol{\beta}\Delta[B].$$

Corollary 12.1. *For two layouts $A, B \in \mathcal{A}(E)$, the diagram*

$$\begin{CD} \check{\mathbf{C}}\boldsymbol{\beta}\Delta[A] @>\mathtt{in}>> \check{\mathbf{C}}\boldsymbol{\beta}\Delta E \\ @V{\boldsymbol{\rho}^A_{A\wedge B}}VV @VV{\boldsymbol{\rho}^{\{E\}}_B}V \\ \check{\mathbf{C}}\boldsymbol{\beta}\Delta[A \wedge B] @>\mathtt{in}>> \check{\mathbf{C}}\boldsymbol{\beta}\Delta[B] \end{CD}$$

is commutative.

Follows from Lemma 11.2. □

Geometric realization. We adopt the obvious identification $|\Delta E| = \Delta E$. For $F \in \mathcal{P}_\times(E)$, $|\Delta F| = \Delta F$ as subsets of ΔE. For $A \in \mathcal{A}(E)$, $|\Delta[A]| = \Delta[A]$ in the same sense. For $A, B \in \mathcal{A}(E)$, $A \geqslant B$, we have $\Delta[A] \supseteq \Delta[B]$ and the retraction ρ^A_B,

$$\begin{array}{ccc} \check{\mathrm{C}}\Delta[A] & \xrightarrow{\rho^A_B} & \check{\mathrm{C}}\Delta[B] \\ \| & & \| \\ |\check{\boldsymbol{\mathsf{C}}}\boldsymbol{\beta}\Delta[A]| & \xrightarrow{|\boldsymbol{\rho}^A_B|} & |\check{\boldsymbol{\mathsf{C}}}\boldsymbol{\beta}\Delta[B]|. \end{array}$$

We call ρ^A_B the *canonical retraction*, too.

Corollary 12.2. *For two layouts $A, B \in \mathcal{A}(E)$, the diagram*

$$\begin{array}{ccc} \check{\mathrm{C}}\Delta[A] & \xrightarrow{\mathrm{in}} & \check{\mathrm{C}}\Delta E \\ {\scriptstyle \rho^A_{A\wedge B}}\downarrow & & \downarrow{\scriptstyle \rho^{\{E\}}_B} \\ \check{\mathrm{C}}\Delta[A\wedge B] & \xrightarrow{\mathrm{in}} & \check{\mathrm{C}}\Delta[B] \end{array}$$

is commutative.

Follows from Corollary 12.1. □

§ 13. The fissilizer Φ^E on $\langle (Y^X_a)^{\check{\mathrm{C}}\Delta E}\rangle$

Fix a space Z and a finite set E. Consider the presheaf $M : \mathcal{P}_\times(E) \to \mathbf{Mg}$, $F \mapsto Z^{\check{\mathrm{C}}\Delta F}$ (with the obvious restriction functions). For $A \in \mathcal{A}(E)$, we have

$$\check{\mathrm{C}}\Delta[A] = \check{\mathrm{C}}\big(\bigcup_{F\in A} \Delta F\big) = \bigvee_{F\in A} \check{\mathrm{C}}\Delta F.$$

We identify the presheaf $\underline{M} : \mathcal{A}(E) \to \mathbf{Mg}$ (see § 10) with the presheaf $A \mapsto Z^{\check{\mathrm{C}}\Delta[A]}$ by the chain of equalities/obvious identifications

$$\underline{M}(A) = \prod_{F\in A} M(F) = \prod_{F\in A} Z^{\check{\mathrm{C}}\Delta F} = Z^{\bigvee_{F\in A}\check{\mathrm{C}}\Delta F} = Z^{\check{\mathrm{C}}\Delta[A]}.$$

The operation (7) in our case coincides with the operation

$$\mathop{\#}_{F\in A} : \prod_{F\in A} \langle Z^{\check{\mathrm{C}}\Delta F}\rangle \to \langle Z^{\check{\mathrm{C}}\Delta[A]}\rangle,$$

which we have by § 2. We will need the following formulas:

$$\epsilon\big(\mathop{\#}_{F\in A} Q_F\big) = \prod_{F\in A} \epsilon(Q_F) \tag{9}$$

and

$$\big(\mathop{\#}_{F\in A} Q_F\big)\big|_{\check{\mathrm{C}}\Delta G} = \big(\prod_{F\in A\setminus\{G\}} \epsilon(Q_F)\big)Q_G. \tag{10}$$

For $A, B \in \mathcal{A}(E)$, $A \geqslant B$, let $\lambda_A^B : \underline{M}(B) \to \underline{M}(A)$ be the function

$$Z^{\rho_B^A} : Z^{\check{\mathrm{C}}\Delta[B]} \to Z^{\check{\mathrm{C}}\Delta[A]},$$

where $\rho_B^A : \check{\mathrm{C}}\Delta[A] \to \check{\mathrm{C}}\Delta[B]$ is the canonical retraction. It follows from Corollary 12.2 that the functions λ_A^B form an extender. By § 10, we get the fissilizer

$$\Phi^E : \langle Z^{\check{\mathrm{C}}\Delta E}\rangle \to \langle Z^{\check{\mathrm{C}}\Delta E}\rangle.$$

Corollary 13.1. *For any ensemble $Q \in \langle Z^{\check{\mathrm{C}}\Delta E}\rangle$, the ensemble $\Phi^E(Q)$ is fissile.*

Follows from Lemma 10.1. □

We set $Z = Y_a^X$, where X and Y are cellular spaces, X compact, and $a : X \to Y$ is a map. For a space T, we have the inclusion $\langle (Y_a^X)^T\rangle \subseteq \langle (Y^X, a)^T\rangle$.

An ensemble $Q \in \langle (Y_a^X)^{\check{\mathrm{C}}\Delta E}\rangle$ is called (X, r)-*almost fissile* if, for any layout $A \in \mathcal{A}(E)$,

$$\mathop{\#}_{F\in A} Q|_{\check{\mathrm{C}}\Delta F} - Q|_{\check{\mathrm{C}}\Delta[A]} \in \langle (Y^X, a)^{\check{\mathrm{C}}\Delta[A]}\rangle_X^{(r+1)}.$$

Lemma 13.2. *Any affine ensemble $Q \in \langle (Y_a^X)^{\check{\mathrm{C}}\Delta E}\rangle$ is $(X, 1)$-almost fissile.*

Proof. Take $A \in \mathcal{A}(E)$. Consider the quantity $D \in \langle (Y_a^X)^{\check{\mathrm{C}}\Delta[A]}\rangle$,

$$D = \mathop{\#}_{F\in A} Q|_{\check{\mathrm{C}}\Delta F} - Q|_{\check{\mathrm{C}}\Delta[A]}.$$

We should show that $D \in \langle (Y^X, a)^{\check{\mathrm{C}}\Delta[A]}\rangle_X^{(2)}$. Consider the homomorphism

$$\langle \Box^X\rangle : \langle (Y^X, a)^{\check{\mathrm{C}}\Delta[A]}\rangle \to \langle Y^{\check{\mathrm{C}}\Delta[A]\times X}\rangle.$$

We should show that $\langle \Box^X\rangle(D) \in \langle Y^{\check{\mathrm{C}}\Delta[A]\times X}\rangle^{(2)}$. Take $R \in \mathcal{F}_1(\check{\mathrm{C}}\Delta[A] \times X)$. We check that $\langle \Box^X\rangle(D)|_R = 0$. We are in (at least) one of the two following cases.

Case 0: $R = \{\urcorner\}$. We have

$$\epsilon(\langle \Box^X\rangle(D)) = \epsilon(D) = \qquad \text{(using (9))} \qquad = \prod_{F\in A} \epsilon(Q|_{\check{\mathrm{C}}\Delta F}) - \epsilon(Q|_{\check{\mathrm{C}}\Delta[A]}) =$$

$$= \prod_{F\in A} \epsilon(Q) - \epsilon(Q) = \qquad \text{(since } \epsilon(Q) = 1) \qquad = 0,$$

which suffices in this case.

Case 1: $R \subseteq \check{C}\Delta G \times X$ for some $G \in A$. It suffices to check that $\langle \square^X \rangle(D)|_{\check{C}\Delta G \times X} = 0$. We have the commutative diagram

$$\begin{array}{ccc} \langle (Y^X, a)^{\check{C}\Delta[A]} \rangle & \xrightarrow{\langle \square^X \rangle} & \langle Y^{\check{C}\Delta[A] \times X} \rangle \\ {\scriptstyle ?|_{\check{C}\Delta G}} \downarrow & & \downarrow {\scriptstyle ?|_{\check{C}\Delta G \times X}} \\ \langle (Y^X, a)^{\check{C}\Delta G} \rangle & \xrightarrow{\langle \square^X \rangle} & \langle Y^{\check{C}\Delta G \times X} \rangle. \end{array}$$

Thus it suffices to check that $D|_{\check{C}\Delta G} = 0$. We have

$$D|_{\check{C}\Delta G} = \qquad \text{(using (10))} \qquad = \Big(\prod_{F \in A \setminus \{G\}} \epsilon(Q|_{\check{C}\Delta F}) \Big) Q|_{\check{C}\Delta G} - Q|_{\check{C}\Delta G} =$$

$$\text{(since } \epsilon(Q|_{\check{C}\Delta F}) = \epsilon(Q) = 1) \qquad = 0. \qquad \square$$

Corollary 13.3. *Let $Q \in \langle (Y_a^X)^{\check{C}\Delta E} \rangle$ be an (X, r)-almost fissile ensemble. Then*

$$\Phi^E(Q) - Q \in \langle (Y^X, a)^{\check{C}\Delta E} \rangle_X^{(r+1)}.$$

Proof. For $A \in \mathcal{A}(E)$, introduce the subgroup

$$N(A) = \langle (Y_a^X)^{\check{C}\Delta[A]} \rangle \cap \langle (Y^X, a)^{\check{C}\Delta[A]} \rangle_X^{(r+1)} \subseteq \langle (Y_a^X)^{\check{C}\Delta[A]} \rangle = \langle \underline{M}(A) \rangle.$$

By Lemma 3.1, this family is preserved by the restriction homomorphisms of the presheaf $A \mapsto \langle \underline{M}(A) \rangle$ and the homomorphisms $\langle \lambda_A^B \rangle$. Since the ensemble Q is (X, r)-almost fissile, it satisfies the hypothesis of Lemma 10.2. Thus $\Phi^E(Q) - Q \in N(\{E\})$, as required. $\square$

Given maps $a, b : X \to Y$, let us say that a is *firmly r-similar to b*, $a \overset{r}{\approxeq} b$, if, for any nonempty finite set E, there is a fissile ensemble $R \in \langle (Y_a^X)^{\check{C}\Delta E} \rangle$ such that

$$<\theta_a^{\Delta E}(b)> - R|_{(\Delta E)_+} \in \langle (Y^X, a)^{(\Delta E)_+} \rangle_X^{(r+1)}. \tag{11}$$

Lemma 13.4. *Let $a, b : X \to Y$ be maps. Then $a \overset{r}{\approxeq} b$ implies $a \overset{r}{\approx} b$.*

We do not know whether the converse holds.

Proof. Take a nonempty finite set E. We have a fissile ensemble $R \in \langle (Y_a^X)^{\check{C}\Delta E} \rangle$ satisfying (11). We seek a fissile ensemble $S \in \langle (Y_a^X)^{(\Delta E)_+} \rangle$ such that

$$<\theta_a^{\Delta E}(b)> - S \in \langle (Y^X, a)^{(\Delta E)_+} \rangle_X^{(r+1)}. \tag{12}$$

Put $S = R|_{(\Delta E)_+}$.

For a layout $A \in \mathcal{A}(E)$, we have

$$S|_{\Delta[A]_+} = R|_{\check{C}\Delta[A]}|_{\Delta[A]_+} = \qquad \text{(since } R \text{ is fissile)} \qquad = \Big(\mathop{\#}_{F \in A} R|_{\check{C}\Delta F} \Big)\Big|_{\Delta[A]_+} =$$

$$\text{(by naturality of } \#) \qquad = \mathop{\#}_{F \in A} R|_{(\Delta F)_+} = \mathop{\#}_{F \in A} S|_{(\Delta F)_+}.$$

Thus S is fissile.

The condition (12) is just the equality (11). $\square$

Proposition 13.5. *Let $a, b : X \to Y$ be maps. Suppose that, for any finite set E, there is an (X, r)-almost fissile ensemble $Q^E \in \langle (Y^X_a)^{\check{C}\Delta E} \rangle$ such that*

$$<\!\theta_a^{\Delta E}(b)\!> - \, Q^E|_{(\Delta E)_+} \in \langle (Y^X, a)^{(\Delta E)_+} \rangle_X^{(r+1)}. \tag{13}$$

Then $a \overset{r}{\approx} b$ and, moreover, $a \overset{r}{\approxeq} b$.

Proof. Take a nonempty finite set E. Put $Q = Q^E$ and $R = \Phi^E(Q) \in \langle (Y^X_a)^{\check{C}\Delta E} \rangle$. By Corollary 13.1, R is fissile. By Corollary 13.3,

$$R - Q \in \langle (Y^X, a)^{\check{C}\Delta E} \rangle_X^{(r+1)}.$$

By Lemma 3.1,

$$R|_{(\Delta E)_+} - Q|_{(\Delta E)_+} \in \langle (Y^X, a)^{(\Delta E)_+} \rangle_X^{(r+1)}.$$

Using (13), we get

$$<\!\theta_a^{\Delta E}(b)\!> - \, R|_{(\Delta E)_+} \in \langle (Y^X, a)^{(\Delta E)_+} \rangle_X^{(r+1)}.$$

Thus $a \overset{r}{\approxeq} b$. By Lemma 13.4, $a \overset{r}{\approx} b$. □

§ 14. Strong 1-similarity

Let X and Y be spaces and $a : X \to Y$ be a map.

Lemma 14.1. *Let U be an unpointed space. Then the homomorphism*

$$\langle \theta_a^U \rangle : \langle Y^X \rangle \to \langle (Y^X, a)^{U_+} \rangle$$

takes $\langle Y^X \rangle^{(s)}$ to $\langle (Y^X, a)^{U_+} \rangle_X^{(s)}$.

Proof. Introduce the map

$$p : U_+ \times X \to X \vee X, \qquad (u, x) \mapsto \mathrm{in}_1(x), \ (\daleth, x) \mapsto \mathrm{in}_2(x).$$

$(u \in U, \, x \in X)$. We have the commutative diagram

$$\begin{array}{ccc}
\langle Y^X \rangle & \xrightarrow{\langle \theta_a^U \rangle} & \langle (Y^X, a)^{U_+} \rangle \\
\Big\downarrow{\scriptstyle \langle e \rangle} & & \Big\downarrow{\scriptstyle \langle \Box^X \rangle} \\
\langle Y^{X \vee X} \rangle & \xrightarrow{\langle Y^p \rangle} & \langle Y^{U_+ \times X} \rangle,
\end{array}$$

where $e = ? \,\underline{\vee}\, a : Y^X \to Y^{X \vee X}$.

We show that $\langle e \rangle$ sends $\langle Y^X \rangle^{(s)}$ to $\langle Y^{X \vee X} \rangle^{(s)}$. Indeed, $\langle e \rangle$ equals the composition

$$\langle Y^X \rangle \xrightarrow{? \otimes <a>} \langle Y^X \rangle \otimes \langle Y^X \rangle \xrightarrow{\langle \underline{\vee} \rangle} \langle Y^{X \vee X} \rangle$$

(see [3, § 3] for ($\underline{\triangledown}$)). Here $\langle Y^X\rangle^{(s)}$ goes to $\langle Y^X\rangle^{(s)} \otimes \langle Y^X\rangle^{(0)}$, which goes to $\langle Y^{X\vee X}\rangle^{(s)}$ by [3, Lemma 3.1].

The homomorphism $\langle Y^p\rangle$ takes $\langle Y^{X\vee X}\rangle^{(s)}$ to $\langle Y^{U_+\times X}\rangle^{(s)}$ by [3, Lemma 2.1]. Thus, by the diagram, $\langle \theta_a^U\rangle$ takes $\langle Y^X\rangle^{(s)}$ to $\langle \square^X\rangle^{-1}(\langle Y^{U_+\times X}\rangle^{(s)})$, which is $\langle (Y^X, a)^{U_+}\rangle_X^{(r+1)}$ by definition. $\square$

Theorem 14.2. *Let $a, b : X \to Y$ be maps such that $a \overset{1}{\sim} b$. Then $a \overset{1}{\approx} b$.*

Proof. We have an ensemble $A \in \langle Y_a^X\rangle$,

$$A = \sum_i u_i \langle a_i \rangle,$$

such that $\langle b\rangle - A \in \langle Y^X\rangle^{(2)}$. For each i, choose a path $h_i : [0,1] \to Y_a^X$ from a to a_i and consider the composition

$$q_i : \check{C}\Delta E \xrightarrow{\text{projection}} [0,1] \xrightarrow{h_i} Y_a^X.$$

Consider the ensemble $Q \in \langle (Y_a^X)^{\check{C}\Delta E}\rangle$,

$$Q = \sum_i u_i \langle q_i \rangle.$$

We have

$$\epsilon(Q) = \epsilon(A) = (\text{since } \langle b\rangle - A \in \langle Y^X\rangle^{(1)}) = \epsilon(\langle b\rangle) = 1.$$

By Lemma 13.2, Q is $(X,1)$-almost fissile. Clearly, $q_i|_{(\Delta E)_+} = \theta_a^{\Delta E}(a_i)$. Thus $Q|_{(\Delta E)_+} = \langle \theta_a^{\Delta E}\rangle(A)$. We get

$$\langle \theta_a^{\Delta E}(b)\rangle - Q|_{(\Delta E)_+} = \langle \theta_a^{\Delta E}\rangle(\langle b\rangle - A) \in \langle (Y^X, a)^{(\Delta E)_+}\rangle^{(2)},$$

where $\in$ holds by Lemma 14.1. By Proposition 13.5, $a \overset{1}{\approx} b$. $\square$

§ 15. Two identities

Let A and I be finite sets. Let $\mathcal{P}(I)$ be the set of subsets of I. Consider the set $\mathcal{P}(I)^A$ of functions $k : A \to \mathcal{P}(I)$. For $k \in \mathcal{P}(I)^A$, put

$$U(k) = \bigcup_{a\in A} k(a) \in \mathcal{P}(I).$$

Let $\mathcal{R}(A, I)$ be the set of $k \in \mathcal{P}(I)^A$ such that $U(k) = I$ (covers).

Lemma 15.1. *In the group $\langle \mathcal{P}(I)\rangle^{\otimes A}$, the equality holds*

$$\sum_{J\in\mathcal{P}(I)} (-1)^{|I|-|J|} \bigotimes_{a\in A} \langle J\rangle = \sum_{k\in\mathcal{R}(A,I)} \bigotimes_{a\in A} \Big(\sum_{J\in\mathcal{P}(k(a))} (-1)^{|k(a)|-|J|} \langle J\rangle \Big).$$

Proof. We have

$$\sum_{J\in\mathcal{P}(I)}(-1)^{|I|-|J|}\bigotimes_{a\in A}\Big(\sum_{K\in\mathcal{P}(J)}{<}K{>}\Big)=$$
$$=\sum_{J\in\mathcal{P}(I)}(-1)^{|I|-|J|}\sum_{\substack{k\in\mathcal{P}(I)^A:\\ U(k)\subseteq J}}\bigotimes_{a\in A}{<}k(a){>}=$$
$$=\sum_{k\in\mathcal{P}(I)^A}\Big(\sum_{\substack{J\in\mathcal{P}(I):\\ J\supseteq U(k)}}(-1)^{|I|-|J|}\Big)\bigotimes_{a\in A}{<}k(a){>}\overset{(*)}{=}\sum_{k\in\mathcal{R}(A,I)}\bigotimes_{a\in A}{<}k(a){>},$$

where $(*)$ holds because the inner sum on the left equals 1 if $U(k)=I$ and 0 otherwise. The set $\mathcal{P}(I)$ is partially ordered by inclusion. We have the isomorphism

$$\nabla^{-1}_{\mathcal{P}(I)}:\langle\mathcal{P}(I)\rangle\to\langle\mathcal{P}(I)\rangle$$

(see § 9), under which

$$\sum_{K\in\mathcal{P}(J)}{<}K{>}\mapsto{<}J{>},\qquad J\in\mathcal{P}(I),$$

and

$${<}K{>}\mapsto\sum_{J\in\mathcal{P}(K)}(-1)^{|K|-|J|}{<}J{>},\qquad K\in\mathcal{P}(I).$$

Applying it to each factor of the summands in the left and right sides of the calculation, we get the required equality. □

Put $\mathcal{P}^\times(I)=\mathcal{P}(I)\setminus\{I\}$. We adopt the inclusion $\mathcal{P}^\times(I)^A\subseteq\mathcal{P}(I)^A$. Let $\mathcal{R}'(A,I)$ be the set of $k\in\mathcal{P}^\times(I)^A$ such that $U(k)=I$.

Lemma 15.2. *In the group $\langle\mathcal{P}^\times(I)\rangle^{\otimes A}$, the equality holds*

$$\bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}^\times(I)}(-1)^{|I|-1-|J|}{<}J{>}\Big)-\sum_{J\in\mathcal{P}^\times(I)}(-1)^{|I|-1-|J|}\bigotimes_{a\in A}{<}J{>}=$$
$$=\sum_{k\in\mathcal{R}'(A,I)}\bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}(k(a))}(-1)^{|k(a)|-|J|}{<}J{>}\Big).$$

Proof. We use the inclusion $\langle\mathcal{P}^\times(I)\rangle^{\otimes A}\subseteq\langle\mathcal{P}(I)\rangle^{\otimes A}$. Put

$$T(k)=\bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}(k(a))}(-1)^{|k(a)|-|J|}{<}J{>}\Big),\qquad k\in\mathcal{P}(I)^A.$$

We have

$$\sum_{k\in\mathcal{P}(I)^A}T(k)=\bigotimes_{a\in A}\Big(\sum_{K\in\mathcal{P}(I)}\sum_{J\in\mathcal{P}(K)}(-1)^{|K|-|J|}{<}J{>}\Big)=$$
$$=\bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}(I)}\Big(\sum_{\substack{K\in\mathcal{P}(I):\\ K\supseteq J}}(-1)^{|K|-|J|}\Big){<}J{>}\Big)\overset{(*)}{=}\bigotimes_{a\in A}{<}I{>},\quad(14)$$

where $(*)$ holds because the inner sum on the left equals 1 if $J = I$ and 0 otherwise. We have also

$$\sum_{k\in\mathcal{P}^\times(I)^A} T(k) = \bigotimes_{a\in A}\Big(\sum_{K\in\mathcal{P}^\times(I)} \sum_{J\in\mathcal{P}(K)} (-1)^{|K|-|J|}{<}J{>}\Big) =$$

$$= \bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}^\times(I)} \Big(\sum_{\substack{K\in\mathcal{P}^\times(I):\\ K\supseteq J}} (-1)^{|K|-|J|}\Big){<}J{>}\Big) = \bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}^\times(I)} (-1)^{|I|-1-|J|}{<}J{>}\Big). \tag{15}$$

Note that

$$\mathcal{R}(A,I) \supseteq \mathcal{R}'(A,I), \qquad \mathcal{P}(I)^A \supseteq \mathcal{P}^\times(I)^A,$$

and

$$\mathcal{R}(A,I) \setminus \mathcal{R}'(A,I) = \mathcal{P}(I)^A \setminus \mathcal{P}^\times(I)^A$$

as subsets of $\mathcal{P}(I)^A$. Thus

$$\sum_{k\in\mathcal{R}'(A,I)} T(k) = \sum_{k\in\mathcal{R}(A,I)} T(k) - \sum_{k\in\mathcal{P}(I)^A} T(k) + \sum_{k\in\mathcal{P}^\times(I)^A} T(k) =$$

(by Lemma 15.1 and equalities (14) and (15))

$$= \sum_{J\in\mathcal{P}(I)} (-1)^{|I|-|J|} \bigotimes_{a\in A} {<}J{>} - \bigotimes_{a\in A} {<}I{>} + \bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}^\times(I)} (-1)^{|I|-1-|J|}{<}J{>}\Big) =$$

$$= -\sum_{J\in\mathcal{P}^\times(I)} (-1)^{|I|-1-|J|} \bigotimes_{a\in A} {<}J{>} + \bigotimes_{a\in A}\Big(\sum_{J\in\mathcal{P}^\times(I)} (-1)^{|I|-1-|J|}{<}J{>}\Big),$$

as required. □

§ 16. Chained monoids

Let P be a monoid. Then $\langle P\rangle$ is its monoid ring. We call the monoid P *chained* if $\langle P\rangle$ is equipped with a chain of left ideals $\langle P\rangle^{[s]}$,

$$\langle P\rangle = \langle P\rangle^{[0]} \supseteq \langle P\rangle^{[1]} \supseteq \dots .$$

Given a finite set I, we consider $\mathcal{P}(I)$ as a monoid with respect to intersection and chain it by letting $\langle \mathcal{P}(I)\rangle^{[s]}$ be the subgroup generated by elements

$$\omega_J = \sum_{K\in\mathcal{P}(J)} (-1)^{|J|-|K|}{<}K{>},$$

where $J \in \mathcal{P}(I)$, $|J| \geqslant s$.

§ 17. The filtration $\langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle^{[s]}$

Let P be a chained monoid. Let $\boldsymbol{T}$ and $\boldsymbol{Z}$ be pointed simplicial sets. Let $\boldsymbol{Z}^{\boldsymbol{T}}$ denote the set of based morphisms $\boldsymbol{T} \to \boldsymbol{Z}$. Let P act on $\boldsymbol{Z}$ (on the left; preserving the basepoint). For an element $p \in P$, let $p_{(\boldsymbol{Z})} : \boldsymbol{Z} \to \boldsymbol{Z}$ be its action. (We will use this notation for all actions.) The set $\boldsymbol{Z}^{\boldsymbol{T}}$ carries the induced action of P. Thus the abelian group $\langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle$ becomes a (left) module over $\langle P \rangle$. We define a filtration

$$\langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle = \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle^{[0]} \supseteq \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle^{[1]} \supseteq \dots .$$

Let $\boldsymbol{T}^j$, $j \in (n)$, be pointed simplicial sets and

$$\boldsymbol{f} : \boldsymbol{T} \to \bigvee_{j \in (n)} \boldsymbol{T}^j$$

be a based morphism. We have the $\mathbb{Z}$-multilinear operation

$$\mathop{\#}_{j \in (n)} : \prod_{j \in (n)} \langle \boldsymbol{Z}^{\boldsymbol{T}^j} \rangle \to \langle \boldsymbol{Z}^{\bigvee_{j \in (n)} \boldsymbol{T}^j} \rangle, \qquad \mathop{\#}_{j \in (n)} <\boldsymbol{v}^j> = < \overline{\bigvee_{\underline{j \in (n)}}} \boldsymbol{v}^j >,$$

and the homomorphism

$$\langle \boldsymbol{Z}^{\boldsymbol{f}} \rangle : \langle \boldsymbol{Z}^{\bigvee_{j \in (n)} \boldsymbol{T}^j} \rangle \to \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle.$$

Take ensembles $v_j \in \langle P \rangle^{[s_j]} \langle \boldsymbol{Z}^{\boldsymbol{T}^j} \rangle$, $j \in (n)$, and consider the ensemble $v \in \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle$,

$$v = \langle \boldsymbol{Z}^{\boldsymbol{f}} \rangle \big(\mathop{\#}_{j \in (n)} v^j \big). \tag{16}$$

We call v a *block* of *rank* $s_1 + \dots + s_n$. We let $\langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle^{[s]} \subseteq \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle$ be the subgroup generated by all blocks of rank at least s. One easily sees that it is a submodule.

Lemma 17.1. *Let $\widetilde{\boldsymbol{T}}$ be a pointed simplicial set and $\boldsymbol{k} : \widetilde{\boldsymbol{T}} \to \boldsymbol{T}$ be a based simplicial morphism. Then the homomorphism*

$$\langle \boldsymbol{Z}^{\boldsymbol{k}} \rangle : \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle \to \langle \boldsymbol{Z}^{\widetilde{\boldsymbol{T}}} \rangle$$

takes $\langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle^{[s]}$ to $\langle \boldsymbol{Z}^{\widetilde{\boldsymbol{T}}} \rangle^{[s]}$. □

Lemma 17.2. *Let $\widetilde{\boldsymbol{Z}}$ be a pointed simplicial set acted on by P and $\boldsymbol{h} : \boldsymbol{Z} \to \widetilde{\boldsymbol{Z}}$ be a P-equivariant based simplicial morphism. Then the homomorphism*

$$\langle \boldsymbol{h}^{\boldsymbol{T}} \rangle : \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle \to \langle \widetilde{\boldsymbol{Z}}^{\boldsymbol{T}} \rangle$$

takes $\langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle^{[s]}$ to $\langle \widetilde{\boldsymbol{Z}}^{\boldsymbol{T}} \rangle^{[s]}$. □

The cone $\check{\mathbf{c}}\boldsymbol{Z}$ carries the induced action of P. We have the function

$$\check{\mathbf{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}} : \boldsymbol{Z}^{\boldsymbol{T}} \to (\check{\mathbf{c}}\boldsymbol{Z})^{\check{\mathbf{c}}\boldsymbol{T}}, \qquad \boldsymbol{v} \mapsto \check{\mathbf{c}}\boldsymbol{v}.$$

Lemma 17.3. *The homomorphism*

$$\langle \check{\mathtt{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}} \rangle : \langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle \to \langle (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{T}} \rangle$$

takes $\langle \boldsymbol{Z}^{\boldsymbol{T}} \rangle^{[s]}$ *to* $\langle (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{T}} \rangle^{[s]}$.

Proof. It suffices to show that $\langle \check{\mathtt{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}} \rangle$ sends any block to a block of the same rank. Consider the block (16). Since $v^j \in \langle P \rangle^{[s_j]} \langle \check{\mathtt{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}^j} \rangle$ and the functions

$$\check{\mathtt{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}^j} : \boldsymbol{Z}^{\boldsymbol{T}^j} \to (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{T}^j}$$

preserve the action of P, we have

$$\langle \check{\mathtt{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}^j} \rangle (v^j) \in \langle P \rangle^{[s_j]} \langle (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{T}^j} \rangle.$$

Let

$$\mathbf{in}^k : \boldsymbol{T}^k \to \bigvee_{j \in (n)} \boldsymbol{T}^j$$

be the canonical insertions. We have the commutative diagram

$$\begin{array}{ccc}
 & & \bigvee_{j\in(n)} \check{\mathtt{c}}\boldsymbol{T}^j \\
 & \overset{\boldsymbol{g}}{\nearrow} & \Big\downarrow {\scriptstyle \boldsymbol{e} := \underline{\overline{\bigvee}}_{j\in(n)} \check{\mathtt{c}}\,\mathbf{in}^j} \\
\check{\mathtt{c}}\boldsymbol{T} & \xrightarrow[\check{\mathtt{c}}\boldsymbol{f}]{} & \check{\mathtt{c}}(\bigvee_{j\in(n)} \boldsymbol{T}^j),
\end{array}$$

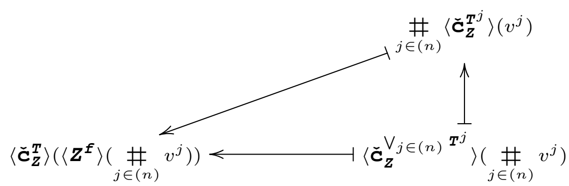
where $\boldsymbol{e}$ is an isomorphism (since $\check{\mathtt{c}}$ preserves wedges) and $\boldsymbol{g}$ is the unique lift of $\check{\mathtt{c}}\boldsymbol{f}$. For arbitrary based morphisms $\boldsymbol{v}^j : \boldsymbol{T}^j \to \boldsymbol{Z}$, we have the commutative diagram with sendings

$$\begin{array}{ccc}
 & & (\check{\mathtt{c}}\boldsymbol{Z})^{\bigvee_{j\in(n)} \check{\mathtt{c}}\boldsymbol{T}^j} \\
 & \overset{(\check{\mathtt{c}}\boldsymbol{Z})^{\boldsymbol{g}}}{\swarrow} & \Big\uparrow {\scriptstyle (\check{\mathtt{c}}\boldsymbol{Z})^{\boldsymbol{e}}} \\
(\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{T}} & \xleftarrow[(\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{f}}]{} & (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}(\bigvee_{j\in(n)} \boldsymbol{T}^j)},
\end{array}
\qquad
\begin{array}{ccc}
 & & \underline{\overline{\bigvee}}_{j\in(n)} \check{\mathtt{c}}\boldsymbol{v}^j \\
 & \swarrow & \uparrow \\
\check{\mathtt{c}}(\boldsymbol{Z}^{\boldsymbol{f}}(\underline{\overline{\bigvee}}_{j\in(n)} \boldsymbol{v}^j)) & \longleftarrow & \check{\mathtt{c}}(\underline{\overline{\bigvee}}_{j\in(n)} \boldsymbol{v}^j).
\end{array}$$

Thus we have the commutative diagram

$$\begin{array}{ccc}
 & & \langle (\check{\mathtt{c}}\boldsymbol{Z})^{\bigvee_{j\in(n)} \check{\mathtt{c}}\boldsymbol{T}^j} \rangle \\
 & \overset{\langle (\check{\mathtt{c}}\boldsymbol{Z})^{\boldsymbol{g}} \rangle}{\swarrow} & \Big\uparrow {\scriptstyle \langle \langle \check{\mathtt{c}}\boldsymbol{Z})^{\boldsymbol{e}} \rangle} \\
\langle (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{T}} \rangle & \xleftarrow[\langle (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}\boldsymbol{f}} \rangle]{} & \langle (\check{\mathtt{c}}\boldsymbol{Z})^{\check{\mathtt{c}}(\bigvee_{j\in(n)} \boldsymbol{T}^j)} \rangle
\end{array}$$

and the sendings



for our (and arbitrary) ensembles v^j. We get

$$\langle \check{\mathbf{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}}\rangle(v) = \langle \check{\mathbf{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}}\rangle\big(\langle \boldsymbol{Z}^{\boldsymbol{f}}\rangle\big(\mathop{\#}_{j\in(n)} v^j\big)\big) = \langle (\check{\mathbf{c}}\boldsymbol{Z})^{\boldsymbol{g}}\rangle\big(\mathop{\#}_{j\in(n)} \langle \check{\mathbf{c}}_{\boldsymbol{Z}}^{\boldsymbol{T}^j}\rangle(v^j)\big),$$

as promised. □

Lemma 17.4. *Let $\boldsymbol{T}^i$, $i \in (m)$, be pointed simplicial sets and $v^i \in \langle \boldsymbol{Z}^{\boldsymbol{T}^i}\rangle^{[s_i]}$ be ensembles. Then*

$$\mathop{\#}_{i\in(m)} v^i \in \langle \boldsymbol{Z}^{\bigvee_{i\in(m)} \boldsymbol{T}^i}\rangle^{[s_1+\ldots+s_m]}.$$ □

Fissile and almost fissile ensembles. Let E be a nonempty finite set. An ensemble $q \in \langle \boldsymbol{Z}^{\check{\mathbf{C}}\boldsymbol{\beta}\Delta E}\rangle$ is called *fissile* if, for any layout $A \in \mathcal{A}(E)$,

$$q|_{\check{\mathbf{C}}\boldsymbol{\beta}\Delta[A]} = \mathop{\#}_{F\in A} q|_{\check{\mathbf{C}}\boldsymbol{\beta}\Delta F}$$

in $\langle \boldsymbol{Z}^{\check{\mathbf{C}}\boldsymbol{\beta}\Delta[A]}\rangle$ (cf. §§ 2, 10). It is called r-*almost fissile* if, for any layout $A \in \mathcal{A}(E)$,

$$\mathop{\#}_{F\in A} q|_{\check{\mathbf{C}}\boldsymbol{\beta}\Delta F} - q|_{\check{\mathbf{C}}\boldsymbol{\beta}\Delta[A]} \in \langle \boldsymbol{Z}^{\check{\mathbf{C}}\boldsymbol{\beta}\Delta[A]}\rangle^{[r+1]}$$

(cf. § 13).

§ 18. The wedge $\boldsymbol{W}(I)$

Fix a finite set I. Consider the pointed simplicial set

$$\boldsymbol{W}(I) = \bigvee_{J\in\mathcal{P}(I)} \hat{\mathbf{\Sigma}}\mathbf{E}(I \setminus J).$$

Let

$$\mathbf{in}_J : \hat{\mathbf{\Sigma}}\mathbf{E}(I \setminus J) \to \boldsymbol{W}(I)$$

be the canonical insertions. The *lead* vertex

$$\top_{\boldsymbol{W}(I)} = (\mathbf{in}_I)_0(1_{\hat{\mathbf{\Sigma}}\mathbf{E}\varnothing}) \in \boldsymbol{W}(I)_0$$

is isolated. $\boldsymbol{W}(I)$ has the pointed simplicial subsets

$$\boldsymbol{W}^{\times}(I) = \bigvee_{J\in\mathcal{P}^{\times}(I)} \hat{\mathbf{\Sigma}}\mathbf{E}(I \setminus J)$$

and

$$\boldsymbol{W}^L(I) = \bigvee_{J\in\mathcal{P}(L)} \hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus J), \qquad L\in\mathcal{P}^\times(I).$$

For $J, K \in \mathcal{P}(I)$, $J \supseteq K$, let

$$\boldsymbol{\tau}_K^J : \hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus J) \to \hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus K)$$

be the morphism induced by the inclusion $I\setminus J \to I\setminus K$.

Let the monoid $\mathcal{P}(I)$ act on $\boldsymbol{W}(I)$ by the rule

$$\begin{array}{ccc} \hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus J) & \xrightarrow{\boldsymbol{\tau}_{K\cap J}^J} & \hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus (K\cap J)) \\ {\scriptstyle \mathbf{in}_J}\downarrow & & \downarrow{\scriptstyle \mathbf{in}_{K\cap J}} \\ \boldsymbol{W}(I) & \xrightarrow{K_{(\boldsymbol{W}(I))}} & \boldsymbol{W}(I), \end{array}$$

$K\in\mathcal{P}(I)$. The simplicial subsets $\boldsymbol{W}^\times(I)$ and $\boldsymbol{W}^L(I)$ are $\mathcal{P}(I)$-invariant.

For $L\in\mathcal{P}^\times(I)$ and $i\in I\setminus L$, we define a retraction $\boldsymbol{\sigma}_i^L$ by the commutative diagram

$$\begin{array}{ccc} \check{\mathbf{C}}\hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus J) & \xrightarrow{\check{\mathbf{C}}\,\mathbf{in}_J^L} & \check{\mathbf{C}}\boldsymbol{W}^L(I) \\ {\scriptstyle \boldsymbol{\sigma}_i}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\sigma}_i^L} \\ \hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus J) & \xrightarrow{\mathbf{in}_J^L} & \boldsymbol{W}^L(I), \end{array}$$

where $\mathbf{in}_J^L$ are the canonical insertions and $\boldsymbol{\sigma}_i$ are the canonical contractions (see § 11). We call $\boldsymbol{\sigma}_i^L$ the canonical contraction, too. It follows from Lemma 11.1 that $\boldsymbol{\sigma}_i^L$ is $\mathcal{P}(I)$-equivariant.

Given a pointed simplicial set $\boldsymbol{T}$, introduce the *filling* function

$$\chi_{L,i}^{\boldsymbol{T}} : \boldsymbol{W}^L(I)^{\boldsymbol{T}} \to \boldsymbol{W}^L(I)^{\check{\mathbf{C}}\boldsymbol{T}}, \qquad \boldsymbol{v} \mapsto (\check{\mathbf{C}}\boldsymbol{T} \xrightarrow{\check{\mathbf{C}}\boldsymbol{v}} \check{\mathbf{C}}\boldsymbol{W}^L(I) \xrightarrow{\boldsymbol{\sigma}_i^L} \boldsymbol{W}^L(I)).$$

Since $\boldsymbol{\sigma}_i^L$ is a retraction,

$$\chi_{L,i}^{\boldsymbol{T}}(\boldsymbol{v})|_{\boldsymbol{T}} = \boldsymbol{v}. \tag{17}$$

§ 19. The module $\langle \boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}^{\triangle E}}\rangle$

Fix a finite set I. We consider the $\langle\mathcal{P}(I)\rangle$-modules $\langle \boldsymbol{W}(I)^{\boldsymbol{T}}\rangle$ for a number of pointed simplicial sets $\boldsymbol{T}$. For a $\mathcal{P}(I)$-invariant pointed simplicial subset $\boldsymbol{Z}\subseteq \boldsymbol{W}(I)$, the subgroup $\langle \boldsymbol{Z}^{\boldsymbol{T}}\rangle \subseteq \langle \boldsymbol{W}(I)^{\boldsymbol{T}}\rangle$ is a $\langle\mathcal{P}(I)\rangle$-submodule. If $\boldsymbol{Z}\subseteq\widetilde{\boldsymbol{Z}}$ for two such subsets, then $\langle \boldsymbol{Z}^{\boldsymbol{T}}\rangle^{[s]} \subseteq \langle \widetilde{\boldsymbol{Z}}^{\boldsymbol{T}}\rangle^{[s]}$ by Lemma 17.2.

Lemma 19.1. *For $L\in\mathcal{P}^\times(I)$, $i\in I\setminus L$, and a pointed simplicial set $\boldsymbol{T}$, the filling homomorphism*

$$\langle\chi_{L,i}^{\boldsymbol{T}}\rangle : \langle \boldsymbol{W}^L(I)^{\boldsymbol{T}}\rangle \to \langle \boldsymbol{W}^L(I)^{\check{\mathbf{C}}\boldsymbol{T}}\rangle$$

takes $\langle \boldsymbol{W}^L(I)^{\boldsymbol{T}}\rangle^{[s]}$ to $\langle \boldsymbol{W}^L(I)^{\check{\mathbf{C}}\boldsymbol{T}}\rangle^{[s]}$.

Proof. By construction of $\chi^{\boldsymbol{T}}_{L,i}$, we have the decomposition

$$\langle\chi^{\boldsymbol{T}}_{L,i}\rangle : \langle\boldsymbol{W}^L(I)^{\boldsymbol{T}}\rangle \xrightarrow{\langle\check{\mathbf{c}}^{\boldsymbol{T}}_{\boldsymbol{W}^L(I)}\rangle} \langle(\check{\mathbf{c}}\boldsymbol{W}^L(I))^{\check{\mathbf{c}}\boldsymbol{T}}\rangle \xrightarrow{\langle(\boldsymbol{\sigma}^L_i)^{\check{\mathbf{c}}\boldsymbol{T}}\rangle} \langle\boldsymbol{W}^L(I)^{\check{\mathbf{c}}\boldsymbol{T}}\rangle.$$

By Lemma 17.3, $\langle\check{\mathbf{c}}^{\boldsymbol{T}}_{\boldsymbol{W}^L(I)}\rangle$ takes $\langle\boldsymbol{W}^L(I)^{\boldsymbol{T}}\rangle^{[s]}$ to $\langle(\check{\mathbf{c}}\boldsymbol{W}^L(I))^{\check{\mathbf{c}}\boldsymbol{T}}\rangle^{[s]}$. Since $\boldsymbol{\sigma}^L_i$ is $\mathcal{P}(I)$-equivariant, $\langle(\boldsymbol{\sigma}^L_i)^{\check{\mathbf{c}}\boldsymbol{T}}\rangle$ takes the $\langle(\check{\mathbf{c}}\boldsymbol{W}^L(I))^{\check{\mathbf{c}}\boldsymbol{T}}\rangle^{[s]}$ to $\langle\boldsymbol{W}^L(I)^{\check{\mathbf{c}}\boldsymbol{T}}\rangle^{[s]}$ by Lemma 17.2. □

Fix a nonempty finite set E. For $F \in \mathcal{P}_\times(E)$ and $J \in \mathcal{P}(I)$, introduce the based morphism

$$\boldsymbol{\theta}^F_J : (\boldsymbol{\beta}\triangle F)_+ \to \boldsymbol{W}(I)$$

that sends $\boldsymbol{\beta}\triangle F$ to the vertex $\mathbf{in}_I(1_{\hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus J)})$.

Lemma 19.2. *For $F \in \mathcal{P}_\times(E)$ and $J \in \mathcal{P}(I)$,*

$$\sum_{K\in\mathcal{P}(J)} (-1)^{|J|-|K|} \lessdot\boldsymbol{\theta}^F_K\gtrdot \in \langle\boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+}\rangle^{[|J|]}.$$

Proof. Since

$$\boldsymbol{\theta}^F_K = K_{(\boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+})}(\boldsymbol{\theta}^F_J),$$

the ensemble in question equals $\omega_J\lessdot\boldsymbol{\theta}^F_J\gtrdot$ and thus belongs to $\langle\mathcal{P}(I)\rangle^{[|J|]}\langle\boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+}\rangle$, which is contained in $\langle\boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+}\rangle^{[|J|]}$ by the definition of the latter. □

Lemma 19.3. *There exist fissile ensembles*

$$p_J \in \langle\boldsymbol{W}^\times(I)^{\check{\mathbf{c}}\boldsymbol{\beta}\triangle E}\rangle, \qquad J \in \mathcal{P}^\times(I),$$

satisfying the following conditions for each $J \in \mathcal{P}^\times(I)$:

(1) *one has*

$$p_J|_{(\boldsymbol{\beta}\triangle E)_+} = \lessdot\boldsymbol{\theta}^E_J\gtrdot$$

in $\langle\boldsymbol{W}^\times(I)^{(\boldsymbol{\beta}\triangle E)_+}\rangle$;

(2) *one has*

$$\sum_{K\in\mathcal{P}(J)} (-1)^{|J|-|K|} p_K \in \langle\boldsymbol{W}^\times(I)^{\check{\mathbf{c}}\boldsymbol{\beta}\triangle E}\rangle^{[|J|]}.$$

Proof. We will construct ensembles

$$p^F_J \in \langle\boldsymbol{W}^J(I)^{\check{\mathbf{c}}\boldsymbol{\beta}\triangle F}\rangle, \qquad (F,J) \in \mathcal{P}_\times(I)\times\mathcal{P}^\times(I),$$

satisfying the following conditions (0^F_J), (1^F_J), and (2^F_J) for each pair $(F,J) \in \mathcal{P}_\times(I)\times\mathcal{P}^\times(I)$:

(0^F_J) one has

$$p^F_J|_{\check{\mathbf{c}}\boldsymbol{\beta}\triangle[B]} = \mathop{\#}_{G\in B} p^G_J$$

in $\langle \boldsymbol{\breve{W}}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]}\rangle$ for all $B \in \mathcal{A}(F)$;

(1^F_J) one has

$$p^F_J|_{(\boldsymbol{\beta}\triangle F)_+} = \langle \boldsymbol{\theta}^F_J \rangle$$

in $\langle \boldsymbol{\breve{W}}^J(I)^{(\boldsymbol{\beta}\triangle E)_+}\rangle$;

(2^F_J) one has

$$\sum_{K\in\mathcal{P}(J)} (-1)^{|J|-|K|} p^F_K \in \langle \boldsymbol{\breve{W}}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle^{[|J|]}.$$

Note that (0^F_J) implies

$$p^F_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle G} = p^G_J$$

for $G \in \mathcal{P}_\times(F)$. Thus (0^F_J) will yield

$$p^F_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = \mathop{\#}_{G\in B} p^F_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle G}$$

for all $B \in \mathcal{A}(F)$, which means that p^F_J is fissile. Thus it will remain to put $p_J = p^E_J$.

Induction on $(F, J) \in \mathcal{P}_\times(E) \times \mathcal{P}^\times(I)$. Take a pair (F, J). We assume that p^G_K are defined and the conditions (0^G_K)–(2^G_K) are satisfied for

$$(G, K) \in \mathcal{P}_\times(F) \times \mathcal{P}(J) \setminus \{(F, J)\}.$$

We construct p^F_J and check the conditions (0^F_J)–(2^F_J).

For $B \in \mathcal{A}(F)$, put

$$U(B) = \langle \boldsymbol{\breve{W}}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]}\rangle^{[|J|]}.$$

For $B, C \in \mathcal{A}(F)$, $B \geqslant C$, we have, by Lemma 17.1, the restriction homomorphism

$$?|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[C]} : U(B) \to U(C).$$

Thus we have a presheaf

$$U : \mathcal{A}(F) \to \mathbf{Ab}.$$

By Lemma 17.1, the canonical retractions

$$\boldsymbol{\rho}^B_C : \check{\mathbf{C}}\boldsymbol{\beta}\triangle[B] \to \check{\mathbf{C}}\boldsymbol{\beta}\triangle[C]$$

induce homomorphisms

$$\lambda^C_B = \langle \boldsymbol{\breve{W}}^J(I)^{\boldsymbol{\rho}^B_C}\rangle|_{U(C)\to U(B)} : U(C) \to U(B),$$

which form an extender for U, as follows from Corollary 12.1. For $B \in \mathcal{A}^\times(F) = \mathcal{A}(F) \setminus \{\{F\}\}$, introduce the ensemble $u_B \in \langle \boldsymbol{\breve{W}}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]}\rangle$,

$$u_B = \sum_{K\in\mathcal{P}(J)} (-1)^{|J|-|K|} \mathop{\#}_{G\in B} p^G_K.$$

By Lemma 15.1,

$$u_B = \sum_{l \in \mathcal{R}(B,J)} \mathop{\#}_{G \in B} \Big(\sum_{K \in \mathcal{P}(l(G))} (-1)^{|l(G)|-|K|} p^G_K \Big).$$

By $(2^G_{l(G)})$, the inner sum belongs to $\langle \boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle G}\rangle^{[|l(G)|]}$. Using Lemma 17.4 and the inequality

$$\sum_{G \in B} |l(G)| \geqslant |J|,$$

we get that the hash product and thus u_B belong to $\langle \boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]}\rangle^{[|J|]}$. We have got $u_B \in U(B)$. For $B, C \in \mathcal{A}^\times(F)$, $B \geqslant C$, and $K \in \mathcal{P}(J)$, we have

$$\Big(\mathop{\#}_{G \in B} p^G_K\Big)\Big|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[C]} = \qquad \text{(by naturality of } \#) \qquad = \mathop{\#}_{G \in B} p^G_K|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[C \wedge \{G\}]} =$$

$$\text{(by } (0^G_K)) \qquad = \mathop{\#}_{G \in B} \Big(\mathop{\#}_{H \in C \wedge \{G\}} p^H_K \Big) = \mathop{\#}_{H \in C} p^H_K.$$

It follows that $u_B|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[C]} = u_C$, that is,

$$(u_B)_{B \in \mathcal{A}^\times(F)} \in \lim_{B \in \mathcal{A}^\times(F)} U(B).$$

By Lemma 9.2, there exists an ensemble

$$u \in U(\{F\}) = \langle \boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle^{[|J|]} \tag{18}$$

such that

$$u|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = u_B, \qquad B \in \mathcal{A}^\times(F).$$

Consider the ensembles $q, r \in \langle \boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle$,

$$q = \sum_{K \in \mathcal{P}^\times(J)} (-1)^{|J|-1-|K|} p^F_K, \qquad r = q + u.$$

For $B \in \mathcal{A}^\times(F)$, we have

$$q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = \sum_{K \in \mathcal{P}^\times(J)} (-1)^{|J|-1-|K|} p^F_K|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = \qquad \text{(by } (0^F_K))$$

$$= \sum_{K \in \mathcal{P}^\times(J)} (-1)^{|J|-1-|K|} \mathop{\#}_{G \in B} p^G_K \tag{19}$$

and

$$r|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} + u|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} + u_B = \qquad \text{(by (19))}$$

$$= \sum_{K \in \mathcal{P}^\times(J)} (-1)^{|J|-1-|K|} \mathop{\#}_{G \in B} p^G_K + \sum_{K \in \mathcal{P}(J)} (-1)^{|J|-|K|} \mathop{\#}_{G \in B} p^G_K = \mathop{\#}_{G \in B} p^G_J. \tag{20}$$

We have

$$r|_{(\boldsymbol{\beta}\triangle F)_+} + \sum_{K\in\mathcal{P}^\times(J)} (-1)^{|J|-|K|} {<}\boldsymbol{\theta}^F_K{>} = \qquad \text{(by } (1^F_K))$$

$$= r|_{(\boldsymbol{\beta}\triangle F)_+} + \sum_{K\in\mathcal{P}^\times(J)} (-1)^{|J|-|K|} p^F_K|_{(\boldsymbol{\beta}\triangle F)_+} = r|_{(\boldsymbol{\beta}\triangle F)_+} - q|_{(\boldsymbol{\beta}\triangle F)_+} =$$

$$= u|_{(\boldsymbol{\beta}\triangle F)_+} \in \qquad \text{(by Lemma 17.1)} \qquad \in \langle \boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+}\rangle^{[|J|]}.$$

From this and Lemma 19.2,

$$ {<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+} \in \langle \boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+}\rangle^{[|J|]}. \tag{21}$$

Choose $i\in I\setminus J$. We have the filling homomorphism

$$\langle \chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}\rangle : \langle \boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+}\rangle \to \langle \boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle.$$

Put

$$p^F_J = r + \langle \chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}\rangle({<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+}).$$

Check of (0^F_J). For $B=\{F\}$, the condition is satisfied trivially. Take $B\in\mathcal{A}^\times(F)$. We have

$$r|_{(\boldsymbol{\beta}\triangle[B])_+} = \qquad \text{(by (20) and naturality of } \#) \qquad = \mathop{\#}_{G\in B} p^G_J|_{(\boldsymbol{\beta}\triangle G)_+} =$$

$$\text{(by } (1^G_J)) \qquad = \mathop{\#}_{G\in B} {<}\boldsymbol{\theta}^G_J{>} = {<}\boldsymbol{\theta}^F_J{>}|_{(\boldsymbol{\beta}\triangle[B])_+}. \tag{22}$$

By naturality of $\chi^{\boldsymbol{T}}_{J,i}$ with respect to $\boldsymbol{T}$, we have the commutative diagram

$$\begin{array}{ccc} \boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle F)_+} & \xrightarrow{\chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}} & \boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F} \\ {\scriptstyle ?|_{(\boldsymbol{\beta}\triangle[B])_+}}\downarrow & & \downarrow{\scriptstyle ?|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]}} \\ \boldsymbol{W}^J(I)^{(\boldsymbol{\beta}\triangle[B])_+} & \xrightarrow{\chi^{(\boldsymbol{\beta}\triangle[B])_+}_{J,i}} & \boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]}. \end{array}$$

We get

$$p^F_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = r|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} + \langle \chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}\rangle({<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+})|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} =$$

$$\text{(by the diagram)} \qquad = r|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} + \langle \chi^{(\boldsymbol{\beta}\triangle[B])_+}_{J,i}\rangle({<}\boldsymbol{\theta}^F_J{>}|_{(\boldsymbol{\beta}\triangle[B])_+} - r|_{(\boldsymbol{\beta}\triangle[B])_+}) =$$

$$\text{(by (22))} \qquad = r|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[B]} = \qquad \text{(by (20))} \qquad = \mathop{\#}_{G\in B} p^G_J.$$

Check of (1^F_J). We have

$$p^F_J|_{(\boldsymbol{\beta}\triangle F)_+} - r|_{(\boldsymbol{\beta}\triangle F)_+} = \langle \chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}\rangle({<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+})|_{(\boldsymbol{\beta}\triangle F)_+} =$$

$$\text{(by (17))} \qquad = {<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+}.$$

Thus $p^F_J|_{(\boldsymbol{\beta}\triangle F)_+} = {<}\boldsymbol{\theta}^F_J{>}$.

Check of (2^F_J). It follows from (21) by Lemma 19.1, that

$$\langle\chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}\rangle({<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+}) \in \langle\boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle^{[|J|]}. \tag{23}$$

We have

$$\sum_{K\in\mathcal{P}(J)}(-1)^{|J|-|K|}p^F_K = p^F_J - q = r + \langle\chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}\rangle({<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+}) - q =$$

$$= u + \langle\chi^{(\boldsymbol{\beta}\triangle F)_+}_{J,i}\rangle({<}\boldsymbol{\theta}^F_J{>} - r|_{(\boldsymbol{\beta}\triangle F)_+}) \in \langle\boldsymbol{W}^J(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle^{[|J|]},$$

where $\in$ follows from (18) and (23). □

Corollary 19.4. *There exists an* $(|I|-1)$*-almost fissile ensemble* $q \in \langle\boldsymbol{W}^\times(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle$ *such that*

$${<}\boldsymbol{\theta}^E_I{>} - q|_{(\boldsymbol{\beta}\triangle E)_+} \in \langle\boldsymbol{W}(I)^{(\boldsymbol{\beta}\triangle E)_+}\rangle^{[|I|]}.$$

Proof. Lemma 19.3 gives fissile ensembles $p_J \in \langle\boldsymbol{W}^\times(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle$ satisfying the conditions (1) and (2) thereof. Put

$$q = \sum_{J\in\mathcal{P}^\times(I)}(-1)^{|I|-1-|J|}p_J.$$

Check that q is $(|I|-1)$-almost fissile. Take $A \in \mathcal{A}(E)$. We have

$$\mathop{\#}_{F\in A} q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F} - q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]} = \mathop{\#}_{F\in A}\Big(\sum_{J\in\mathcal{P}^\times(I)}(-1)^{|I|-1-|J|}p_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\Big) -$$

$$-\sum_{J\in\mathcal{P}^\times(I)}(-1)^{|I|-1-|J|}p_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]} = \qquad \text{(since } p_J \text{ are fissile)}$$

$$= \mathop{\#}_{F\in A}\Big(\sum_{J\in\mathcal{P}^\times(I)}(-1)^{|I|-1-|J|}p_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\Big) - \sum_{J\in\mathcal{P}^\times(I)}(-1)^{|I|-1-|J|}\mathop{\#}_{F\in A}p_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F} =$$

(by Lemma 15.2)
$$= \sum_{k\in\mathcal{R}'(A,I)}\mathop{\#}_{F\in A}\Big(\sum_{J\in\mathcal{P}(k(F))}(-1)^{|k(F)|-|J|}p_J|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\Big) =$$

$$= \sum_{k\in\mathcal{R}'(A,I)}\mathop{\#}_{F\in A}\Big(\sum_{J\in\mathcal{P}(k(F))}(-1)^{|k(F)|-|J|}p_J\Big)\Big|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}.$$

By condition (2), the inner sum of the last expression belongs to $\langle\boldsymbol{W}^\times(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle^{[|k(F)|]}$. By Lemma 17.1, its restriction to $\check{\mathbf{C}}\boldsymbol{\beta}\triangle F$ belongs to $\langle\boldsymbol{W}^\times(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle^{[|k(F)|]}$. Using Lemma 17.4 and the inequality

$$\sum_{F\in A}|k(F)| \geqslant |I|,$$

we get that the hash product and thus the whole expression belong to $\langle\boldsymbol{W}^\times(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]}\rangle^{[|I|]}$, as required.

We have

$$<\boldsymbol{\theta}_I^E> - q|_{(\boldsymbol{\beta}\triangle E)_+} = <\boldsymbol{\theta}_I^E> - \sum_{J\in\mathcal{P}^\times(I)} (-1)^{|I|-1-|J|} p_J|_{(\boldsymbol{\beta}\triangle E)_+} =$$

(by condition (1)) $$= <\boldsymbol{\theta}_I^E> - \sum_{J\in\mathcal{P}^\times(I)} (-1)^{|I|-1-|J|} <\boldsymbol{\theta}_J^E> =$$

$$= \sum_{J\in\mathcal{P}(I)} (-1)^{|I|-|J|} <\boldsymbol{\theta}_J^E> \in \qquad \text{(by Lemma 19.2)} \qquad \in \langle \boldsymbol{W}(I)^{(\boldsymbol{\beta}\triangle E)_+}\rangle^{[|I|]}.$$

□

§ 20. The filtration $\langle (Y^X)^T\rangle^{[s]}$

General case. We give a topological version of the definition of § 17. Let T and Z be spaces. Let a chained monoid P act on Z (preserving the basepoint). The set Z^T carries the induced action of P. Thus the abelian group $\langle Z^T\rangle$ becomes a module over $\langle P\rangle$. We define a filtration $\langle Z^T\rangle^{[s]}$. Let T^j, $j \in (n)$, be spaces and

$$f : T \to \bigvee_{j\in(n)} T^j$$

be a map. Take ensembles $V_j \in \langle P\rangle^{[s_j]}\langle Z^{T^j}\rangle$, $j \in (n)$, and consider the ensemble $V \in \langle Z^T\rangle$,

$$V = \langle Z^f\rangle\big(\mathop{\#}_{j\in(n)} V^j\big). \tag{24}$$

We call V a *block* of *rank* $s_1 + \ldots + s_n$. We let $\langle Z^T\rangle^{[s]} \subseteq \langle Z^T\rangle$ be the subgroup generated by all blocks of rank at least s. One easily sees that it is a submodule.

Lemma 20.1. *Let $\widetilde{Z}$ be a space acted on by P and $h : Z \to \widetilde{Z}$ be a P-equivariant map. Then the homomorphism*

$$\langle h^T\rangle : \langle Z^T\rangle \to \langle \widetilde{Z}^T\rangle$$

takes $\langle Z^T\rangle^{[s]}$ to $\langle \widetilde{Z}^T\rangle^{[s]}$. □

Lemma 20.2. *Let $\boldsymbol{T}$ and $\boldsymbol{Z}$ be pointed simplicial sets. Let P act on $\boldsymbol{Z}$ and thus on $|\boldsymbol{Z}|$. Consider the geometric realization function*

$$\gamma : \boldsymbol{Z}^{\boldsymbol{T}} \to |\boldsymbol{Z}|^{|\boldsymbol{T}|}, \qquad \boldsymbol{v} \mapsto |\boldsymbol{v}|,$$

and the homomorphism

$$\langle\gamma\rangle : \langle \boldsymbol{Z}^{\boldsymbol{T}}\rangle \to \langle |\boldsymbol{Z}|^{|\boldsymbol{T}|}\rangle.$$

Then $\langle\gamma\rangle$ takes $\langle \boldsymbol{Z}^{\boldsymbol{T}}\rangle^{[s]}$ to $\langle |\boldsymbol{Z}|^{|\boldsymbol{T}|}\rangle^{[s]}$. □

The case $Z = Y^X$. Let I be a finite set and Y be space acted on by the chained monoid $P = \mathcal{P}(I)$. We suppose that the action is *special*:

$$Y = \bigcup_{i \in I} \mathrm{Fix}\{i\}_{(Y)}.$$

Let X be a space. Consider the space $Z = Y^X$. It carries the induced action of $\mathcal{P}(I)$.

Lemma 20.3. *In the* $\langle \mathcal{P}(I) \rangle$*-module* $\langle Y^X \rangle$*, the inclusion holds*

$$\langle \mathcal{P}(I) \rangle^{[s]} \langle Y^X \rangle \subseteq \langle Y^X \rangle^{(s)}.$$

Proof. Take a map $a \in Y^X$ and a subset $J \in \mathcal{P}(I)$, $|J| \geqslant s$. The ensembles of the form $\omega_J{<}a{>}$ generate the subgroup $\langle \mathcal{P}(I) \rangle^{[s]} \langle Y^X \rangle$. Thus we should show that $\omega_J{<}a{>} \in \langle Y^X \rangle^{(s)}$. Take a subspace $R \in \mathcal{F}_{s-1}(X)$. We should check that $\omega_J{<}a{>}|_R = 0$ in $\langle Y^R \rangle$. Since the action is special, for each $x \in X$, there is $i_x \in I$ such that $a(x) \in \mathrm{Fix}\{i_x\}_{(Y)}$. Consider the subset

$$K = \{\, i_x \mid x \in R \setminus \{\urcorner\}\} \in \mathcal{P}(I).$$

Clearly, $|K| < s$. For $x \in R \setminus \{\urcorner\}$, we have

$$\begin{aligned} K_{(Y)}(a(x)) = K_{(Y)}(\{i_x\}_{(Y)}(a(x))) = \\ = (K \cap \{i_x\})_{(Y)}(a(x)) = \{i_x\}_{(Y)}(a(x)) = a(x). \end{aligned}$$

Thus $K_{(Y)} \circ a =|_R a$. Thus ${<}K{>}{<}a{>} =|_R {<}a{>}$ in $\langle Y^R \rangle$. Since $|K| < s$, we have $K \not\supseteq J$. It follows that $\omega_J{<}K{>} = 0$ in $\langle \mathcal{P}(I) \rangle$. We get

$$\omega_J{<}a{>} =|_R \omega_J{<}K{>}{<}a{>} = 0. \qquad \square$$

Lemma 20.4. *Let* T *be a space. Then*

$$\langle (Y^X)^T \rangle^{[s]} \subseteq \langle (Y^X)^T \rangle_X^{(s)}.$$

Proof. Take a block $V \in \langle (Y^X)^T \rangle$ of rank at least s. We should show that $V \in \langle (Y^X)^T \rangle_X^{(s)}$. Consider the homomorphism

$$\langle (Y^X)^T \rangle \xrightarrow{\langle \widehat{\square}^X \rangle} \langle Y^{T \wedge X} \rangle.$$

By Lemma 3.3, we should show that $\langle \widehat{\square}^X \rangle(V) \in \langle Y^{T \wedge X} \rangle^{(s)}$. We have the equality (24) for some spaces T^j, map f and ensembles $V^j \in \langle \mathcal{P}(I) \rangle^{[s_j]} \langle (Y^X)^{T^j} \rangle$, where $s_1 + \ldots s_n \geqslant s$. Since the function

$$\widehat{\square}^X : (Y^X)^{T^j} \to Y^{T^j \wedge X}$$

is $\mathcal{P}(I)$-equivariant, $\langle \widehat{\square}^X \rangle(V^j) \in \langle \mathcal{P}(I) \rangle^{[s_j]} \langle Y^{T^j \wedge X} \rangle$. By Lemma 20.3,

$$\langle \widehat{\square}^X \rangle(V^j) \in \langle Y^{T^j \wedge X} \rangle^{(s_j)}.$$

Consider the commutative diagram

$$\begin{array}{ccccc}
(V^j)_{j\in(n)} & \longmapsto & & \longrightarrow & (\langle\widehat{\square}^X\rangle(V^j))_{j\in(n)} \\
\Big\downarrow & \prod_{j\in(n)}\langle(Y^X)^{T^j}\rangle & \xrightarrow{\prod_{j\in(n)}\langle\widehat{\square}^X\rangle} & \prod_{j\in(n)}\langle Y^{T^j\wedge X}\rangle & \Big\downarrow \\
 & \Big\downarrow{\scriptstyle \#_{j\in(n)}} & & \Big\downarrow{\scriptstyle \#_{j\in(n)}} & \\
\#_{j\in(n)} V^j & \langle(Y^X)^{\bigvee_{j\in(n)} T^j}\rangle & \xrightarrow{\langle\widehat{\square}^X\rangle} & \langle Y^{\bigvee_{j\in(n)} T^j\wedge X}\rangle & \#_{j\in(n)}\langle\widehat{\square}^X\rangle(V^j) \\
\Big\downarrow & \Big\downarrow{\scriptstyle \langle(Y^X)^f\rangle} & & \Big\downarrow{\scriptstyle \langle Y^{f\wedge \mathrm{id}_X}\rangle} & \Big\downarrow{\scriptstyle (*)} \\
 & \langle(Y^X)^T\rangle & \xrightarrow{\langle\widehat{\square}^X\rangle} & \langle Y^{T\wedge X}\rangle. & \\
V & \longmapsto & & \longrightarrow & \langle\widehat{\square}^X\rangle(V)
\end{array}$$

(We used distributivity of smash product over wedge.) All the sendings are obvious except $(*)$, which follows by commutativity of the diagram. By [3, Lemma 3.1],

$$\#_{j\in(n)} \langle\widehat{\square}^X\rangle(V^j) \in \langle Y^{\bigvee_{j\in(n)} T^j\wedge X}\rangle^{(s)}.$$

By [3, Lemma 2.1], $\langle\widehat{\square}^X\rangle(V) \in \langle Y^{T\wedge X}\rangle^{(s)}$, as was to be shown. $\square$

§ 21. The wedge $V(I)$ and a $\mathcal{P}(I)$-equivariant map $h : V(I) \to Z$

Let I be a finite set. We give a topological version of $\boldsymbol{W}(I)$. Consider the space

$$V(I) = \bigvee_{J\in\mathcal{P}(I)} \overline{\Sigma}\Delta(I\setminus J).$$

Let

$$\mathrm{in}_J : \overline{\Sigma}\Delta(I\setminus J) \to V(I)$$

be the canonical insertions. $V(I)$ consists of the isolated *lead* point

$$\top_{V(I)} = \mathrm{in}_I(1_{\overline{\Sigma}\Delta\varnothing})$$

and the subspace

$$V^\times(I) = \bigvee_{J\in\mathcal{P}^\times(I)} \overline{\Sigma}\Delta(I\setminus J),$$

which is contractible.

For $J, K \in \mathcal{P}(I)$, $J \supseteq K$, let

$$\tau^J_K : \overline{\Sigma}\Delta(I\setminus J) \to \overline{\Sigma}\Delta(I\setminus K)$$

be the map induced by the inclusion $I\setminus J \to I\setminus K$.

Let the monoid $\mathcal{P}(I)$ act on $V(I)$ by the rule

$$\begin{array}{ccc} \overline{\Sigma}\Delta(I\setminus J) & \xrightarrow{\tau^J_{K\cap J}} & \overline{\Sigma}\Delta(I\setminus(K\cap J)) \\ \downarrow{\scriptstyle \mathrm{in}_J} & & \downarrow{\scriptstyle \mathrm{in}_{K\cap J}} \\ V(I) & \xrightarrow{K_{(V(I))}} & V(I), \end{array}$$

$K\in\mathcal{P}(I)$. The subspace $V^\times(I)$ is $\mathcal{P}(I)$-invariant.

For $J\in\mathcal{P}(I)$, we have the map

$$e_J : |\hat{\boldsymbol{\Sigma}}\mathbf{E}(I\setminus J)| = \overline{\Sigma}|\mathbf{E}(I\setminus J)| \xrightarrow{\overline{\Sigma}\xi_{I\setminus J}} \overline{\Sigma}\Delta(I\setminus J)$$

(see (8) for $\xi_{I\setminus J}$). These e_J form the map

$$e = \bigvee_{J\in\mathcal{P}(I)} e_J : |\boldsymbol{W}(I)| \to V(I). \tag{25}$$

It is $\mathcal{P}(I)$-equivariant, sends the point $|\top_{\boldsymbol{W}(I)}|$ to $\top_{V(I)}$, and takes the subspace $|\boldsymbol{W}^\times(I)|$ to $V^\times(I)$.

Lemma 21.1. *Let Z be a space acted on by $\mathcal{P}(I)$. Suppose that the basepoint path component $Z_\triangleleft\subseteq Z$ is weakly contractible. Let $\top_Z\in Z$ be a point such that*

$$K_{(Z)}(\top_Z)\in Z_\triangleleft$$

for all $K\in\mathcal{P}^\times(I)$. Then there exists a $\mathcal{P}(I)$-equivariant map $h: V(I)\to Z$ such that $h(\top_{V(I)})=\top_Z$.

Proof. We crop Z and assume that $Z=Z_\triangleleft\cup\{\top_Z\}$. We will construct maps

$$h^J : \overline{\Sigma}\Delta(I\setminus J)\to Z, \qquad J\in\mathcal{P}(I),$$

satisfying the following conditions (1) and (2^K_J) for $J,K\in\mathcal{P}(I)$, $J\subseteq K$:
(1) one has $h^I(1_{\overline{\Sigma}\Delta\varnothing})=\top_Z$;
(2^K_J) the diagram

$$\begin{array}{ccc} \overline{\Sigma}\Delta(I\setminus K) & \xrightarrow{h^K} & Z \\ \downarrow{\scriptstyle \tau^K_J} & & \downarrow{\scriptstyle J_{(Z)}} \\ \overline{\Sigma}\Delta(I\setminus J) & \xrightarrow{h^J} & Z \end{array}$$

is commutative.

Note that the condition (2^J_J) is the equality $J_{(Z)}\circ h^J=h^J$.

Induction on $J\in\mathcal{P}(I)$. We define the map h^I by the condition (1). The condition (2^I_I) is satisfied trivially. Take $J\in\mathcal{P}^\times(I)$. We assume that the maps

h^K are defined for $K \supsetneq J$ and the conditions (2_K^L) are satisfied for $L \supseteq K \supsetneq J$. We construct h^J and check (2_J^K) for $K \supseteq J$.

For $K \supsetneq J$, put

$$B_K = \operatorname{Im}(\overline{\Sigma}\Delta(I \setminus K) \xrightarrow{\tau_J^K} \overline{\Sigma}\Delta(I \setminus J)).$$

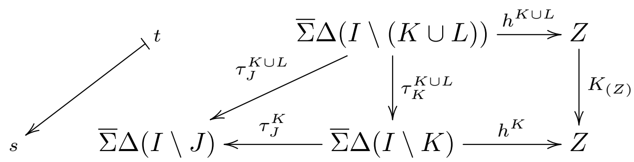
Since τ_J^K is an embedding, there is a map $f^K : B_K \to Z_{\triangleleft}$ such that

$$f^K(\tau_J^K(t)) = J_{(Z)}(h^K(t)), \qquad t \in \overline{\Sigma}\Delta(I \setminus K),$$

(we use here that $\operatorname{Im} J_{(Z)} \subseteq Z_{\triangleleft}$). We show that

$$f^K =|_{B_K \cap B_L} f^L$$

for $K, L \supsetneq J$. Take $s \in B_K \cap B_L$. Since $B_K \cap B_L = B_{K \cup L}$, we have $s = \tau_J^{K\cup L}(t)$ for some $t \in \overline{\Sigma}\Delta(I \setminus (K \cup L))$. We have the commutative diagram



(the square is commutative by $(2_K^{K\cup L})$). Using the diagram, we get

$$\begin{aligned} f^K(s) = f^K(\tau_J^{K\cup L}(t)) = f^K(\tau_J^K(\tau_K^{K\cup L}(t))) = J_{(Z)}(h^K(\tau_K^{K\cup L}(t))) = \\ = J_{(Z)}(K_{(Z)}(h^{K\cup L}(t))) = (J \cap K)_{(Z)}(h^{K\cup L}(t)) = J_{(Z)}(h^{K\cup L}(t)). \end{aligned}$$

Similarly, $f^L(s) = J_{(Z)}(h^{K\cup L}(t))$. Thus $f^K(s) = f^L(s)$, as promised.

We have

$$\bigcup_{K \supsetneq J} B_K = \overline{\Sigma}\partial\Delta(I \setminus J) \subseteq \overline{\Sigma}\Delta(I \setminus J),$$

where $\partial\Delta(I \setminus J)$ denotes the boundary of the simplex $\Delta(I \setminus J)$. Since B_K are closed, there is a map

$$f : \overline{\Sigma}\partial\Delta(I \setminus J) \to Z_{\triangleleft}$$

such that $f|_{B_K} = f^K$ for all $K \supsetneq J$. Since $\overline{\Sigma}\partial\Delta(I \setminus J)$ is the boundary of the ball $\overline{\Sigma}\Delta(I \setminus J)$ and $Z_{\triangleleft}$ is weakly contractible, f extends to a map

$$g : \overline{\Sigma}\Delta(I \setminus J) \to Z_{\triangleleft}.$$

We put

$$h^J(s) = J_{(Z)}(g(s)), \qquad s \in \overline{\Sigma}\Delta(I \setminus J).$$

Clearly, $J_{(Z)} \circ h^J = h^J$, which is the condition (2_J^J). We check the condition (2_J^K) for $K \supsetneq J$. For $t \in \overline{\Sigma}\Delta(I \setminus K)$, we have

$$\begin{aligned} h^J(\tau_J^K(t)) = J_{(Z)}(g(\tau_J^K(t))) = J_{(Z)}(f(\tau_J^K(t))) = J_{(Z)}(f^K(\tau_J^K(t))) = \\ = J_{(Z)}(J_{(Z)}(h^K(t))) = J_{(Z)}(h^K(t)), \end{aligned}$$

as required.

We join all the h^J into the desired h:

$$h = \overline{\bigvee_{J \in \mathcal{P}(I)}} h^J.$$

Since $\top_{V(I)} = \mathrm{in}_I(1_{\overline{\Sigma}\Delta\varnothing})$, we have

$$h(\top_{V(I)}) = h^I(1_{\overline{\Sigma}\Delta\varnothing}) = \qquad \text{(by (1))} \qquad = \top_Z.$$

To show that d is $\mathcal{P}(I)$-equivariant, we should check that, for $K, J \in \mathcal{P}(I)$, the diagram

$$\begin{array}{ccc} \overline{\Sigma}\Delta(I \setminus J) & \xrightarrow{h^J} & Z \\ {\scriptstyle \tau^J_{K\cap J}}\downarrow & & \downarrow{\scriptstyle K_{(Z)}} \\ \overline{\Sigma}\Delta(I \setminus (K \cap J)) & \xrightarrow{h^{K\cap J}} & Z \end{array}$$

is commutative. Indeed,

$$K_{(Z)} \circ h^J = \qquad \text{(by } (2^J_J)) \qquad = K_{(Z)} \circ J_{(Z)} \circ h^J =$$
$$= (K \cap J)_{(Z)} \circ h^J = \qquad \text{(by } (2^J_{K\cap J})) \qquad = h^{K\cap J} \circ \tau^J_{K\cap J}. \quad \square$$

§ 22. The realization $\Upsilon^{\boldsymbol{T}}_h : \boldsymbol{W}(I)^{\boldsymbol{T}} \to (Y^X)^{|\boldsymbol{T}|}$

Let X and Y be cellular spaces, X compact. Let I be a finite set and Y carry a special action of the monoid $\mathcal{P}(I)$. Let $h : V(I) \to Y^X$ be a $\mathcal{P}(I)$-equivariant map. Let $\boldsymbol{T}$ be a pointed simplicial set. Introduce the function

$$\Upsilon^{\boldsymbol{T}}_h : \boldsymbol{W}(I)^{\boldsymbol{T}} \to (Y^X)^{|\boldsymbol{T}|}, \qquad \boldsymbol{v} \mapsto (|\boldsymbol{T}| \xrightarrow{|v|} |\boldsymbol{W}(I)| \xrightarrow{e} V(I) \xrightarrow{h} Y^X),$$

(see (25) for e), the *realization.*

Lemma 22.1. *The function $\Upsilon^{\boldsymbol{T}}_h$ takes $\boldsymbol{W}^\times(I)^{\boldsymbol{T}}$ to $(Y^X_\triangleleft)^{|\boldsymbol{T}|}$.*

Proof. The map e takes $|\boldsymbol{W}^\times(I)|$ to $V^\times(I)$. Since $V^\times(I)$ is path connected, d takes it to $Y^X_\triangleleft$. $\square$

Consider the homomorphism

$$\langle \Upsilon^{\boldsymbol{T}}_h \rangle : \langle \boldsymbol{W}(I)^{\boldsymbol{T}} \rangle \to \langle (Y^X)^{|\boldsymbol{T}|} \rangle.$$

Lemma 22.2. *The homomorphism $\langle \Upsilon^{\boldsymbol{T}}_h \rangle$ takes $\langle \boldsymbol{W}(I)^{\boldsymbol{T}} \rangle^{[s]}$ to $\langle (Y^X)^{|\boldsymbol{T}|} \rangle^{(s)}_X$.*

Proof. We have the decomposition

$$\langle \Upsilon^{\boldsymbol{T}}_h \rangle : \langle \boldsymbol{W}(I)^{\boldsymbol{T}} \rangle \xrightarrow{\langle \gamma \rangle} \langle |\boldsymbol{W}(I)|^{|\boldsymbol{T}|} \rangle \xrightarrow{\langle (h \circ e)^{|\boldsymbol{T}|} \rangle} \langle (Y^X)^{|\boldsymbol{T}|} \rangle,$$

where $\gamma : \boldsymbol{W}(I)^{\boldsymbol{T}} \to |\boldsymbol{W}(I)|^{|\boldsymbol{T}|}$ is the geometric realization function. By Lemma 20.2, $\langle \gamma \rangle$ takes $\langle \boldsymbol{W}(I)^{\boldsymbol{T}} \rangle^{[s]}$ to $\langle |\boldsymbol{W}(I)|^{|\boldsymbol{T}|} \rangle^{[s]}$. By Lemma 20.1, $\langle (h \circ e)^{|\boldsymbol{T}|} \rangle$ takes $\langle |\boldsymbol{W}(I)|^{|\boldsymbol{T}|} \rangle^{[s]}$ to $\langle (Y^X)^{|\boldsymbol{T}|} \rangle^{[s]}$, which is contained in $\langle (Y^X)^{|\boldsymbol{T}|} \rangle^{(s)}_X$ by Lemma 20.4. $\square$

Lemma 22.3. *Let E be a nonempty finite set and $q \in \langle \boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle$ be an r-almost fissile ensemble. Then the ensemble $\langle \Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle(q) \in \langle (Y^X)^{\check{C}\Delta E}\rangle$ is (X, r)-almost fissile.*

Proof. Take $A \in \mathcal{A}(E)$. The diagram

$$
\begin{CD}
\prod\limits_{F\in A} \boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E} @>{\prod\limits_{F\in A}\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}}>> \prod\limits_{F\in A} (Y^X)^{\check{C}\Delta E}\\
@V{\prod\limits_{F\in A} ?|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}}VV @VV{\prod\limits_{F\in A} ?|_{\check{C}\Delta F}}V\\
\prod\limits_{F\in A} \boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F} @>{\prod\limits_{F\in A}\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}}>> \prod\limits_{F\in A} (Y^X)^{\check{C}\Delta F}\\
@V{\overline{\bigvee}_{F\in A}}VV @VV{\overline{\bigvee}_{F\in A}}V\\
\boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]} @>{\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]}}>> (Y^X)^{\check{C}\Delta[A]}\\
@A{?|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]}}AA @AA{?|_{\check{C}\Delta[A]}}A\\
\boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E} @>{\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}}>> (Y^X)^{\check{C}\Delta E}
\end{CD}
$$

is commutative because $\Upsilon_h^{\boldsymbol{T}}$ is natural with respect to $\boldsymbol{T}$. Thus the diagram

$$
\begin{CD}
\prod\limits_{F\in A} \langle\boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle @>{\prod\limits_{F\in A}\langle\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle}>> \prod\limits_{F\in A} \langle(Y^X)^{\check{C}\Delta E}\rangle\\
@V{\prod\limits_{F\in A} ?|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}}VV @VV{\prod\limits_{F\in A} ?|_{\check{C}\Delta F}}V\\
\prod\limits_{F\in A} \langle\boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle @>{\prod\limits_{F\in A}\langle\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F}\rangle}>> \prod\limits_{F\in A} \langle(Y^X)^{\check{C}\Delta F}\rangle\\
@V{\#_{F\in A}}VV @VV{\#_{F\in A}}V\\
\langle\boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]}\rangle @>{\langle\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]}\rangle}>> \langle(Y^X)^{\check{C}\Delta[A]}\rangle\\
@A{?|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]}}AA @AA{?|_{\check{C}\Delta[A]}}A\\
\langle\boldsymbol{W}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E} @>{\langle\Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle}>> \langle(Y^X)^{\check{C}\Delta E}\rangle
\end{CD}
$$

is also commutative. In it, we have

$$\begin{array}{ccc}
(q)_{F\in A} & \longmapsto & (Q)_{F\in A} \\
\downarrow & & \downarrow \\
(q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F})_{F\in A} & & (Q|_{\check{\mathrm{C}}\Delta F})_{F\in A} \\
\downarrow & & \downarrow \\
\#_{F\in A}\, q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F} & \overset{(1)}{\longmapsto} & \#_{F\in A}\, Q|_{\check{\mathrm{C}}\Delta F} \\
q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]} & \overset{(2)}{\longmapsto} & Q|_{\check{\mathrm{C}}\Delta[A]} \\
\uparrow & & \uparrow \\
q & \longmapsto & Q,
\end{array}$$

where $Q = \langle \Upsilon_h^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle E}\rangle(q)$. All the sendings are obvious except (1) and (2), which follow by commutativity of the diagram. Since q is r-almost fissile,

$$\#_{F\in A}\, q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle F} - q|_{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]} \in \langle \bar{\boldsymbol{W}}(I)^{\check{\mathbf{C}}\boldsymbol{\beta}\triangle[A]}\rangle^{[r+1]}.$$

By Lemma 22.2,

$$\#_{F\in A}\, Q|_{\check{\mathrm{C}}\Delta F} - Q|_{\check{\mathrm{C}}\Delta[A]} \in \langle (Y^X)^{\check{\mathrm{C}}\Delta[A]}\rangle_X^{(r+1)}.$$

Thus Q is (X, r)-almost fissile. □

§ 23. Brunnian loops in a wedge of circles

Fix a finite set I of cardinality $r+1$. Put $X = S^1$ and $Y = I_+ \wedge S^1$ (a wedge of $r+1$ circles). Let the monoid $\mathcal{P}(I)$ act on the space I_+ by putting

$$J_{(I_+)}(i) = \begin{cases} i & \text{if } i \in J, \\ \triangleleft & \text{otherwise,} \end{cases}$$

for $i \in I_+$, $J \in \mathcal{P}(I)$. This action induces one on Y. A map $b : X \to Y$ (a loop) is called *Brunnian* if the composition

$$X \xrightarrow{b} Y \xrightarrow{J_{(Y)}} Y$$

is null-homotopic for all $J \in \mathcal{P}^\times(I)$.

Lemma 23.1. *Let $b : X \to Y$ be a Brunnian loop. Then $\triangleleft \overset{r}{\approx} b$.*

Proof. Take a finite set E. Consider the loop space Y^X. It carries the induced action of the monoid $\mathcal{P}(I)$. The path component $Y^X_\triangleleft$ is weakly contractible. Since b is Brunnian, $J_{(Y^X)}(b)$ $(= J_{(Y)} \circ b) \in Y^X_\triangleleft$ for all $J \in \mathcal{P}^\times(I)$. Therefore, Lemma 21.1 yields a $\mathcal{P}(I)$-equivariant map $h : V(I) \to Y^X$ such that $d(\top) = b$. We get the realization homomorphism

$$\langle \Upsilon_h^{\check{\mathsf{C}}\boldsymbol{\beta}\triangle E} \rangle : \langle \boldsymbol{W}(I)^{\check{\mathsf{C}}\boldsymbol{\beta}\triangle E} \rangle \to \langle (Y^X)^{\check{\mathrm{C}}\triangle E} \rangle.$$

By Corollary 19.4, there is an r-almost fissile ensemble $q \in \langle \boldsymbol{W}(I)^{\check{\mathsf{C}}\boldsymbol{\beta}\triangle E} \rangle$ such that

$$<\boldsymbol{\theta}_\top^{\boldsymbol{\beta}\triangle E}> - q|_{(\boldsymbol{\beta}\triangle E)_+} \in \langle \boldsymbol{W}(I)^{(\boldsymbol{\beta}\triangle E)_+} \rangle^{[r]}. \tag{26}$$

Put $Q = \langle \Upsilon_h^{\check{\mathsf{C}}\boldsymbol{\beta}\triangle E} \rangle(q)$. By Lemma 22.1, $Q \in \langle (Y^X_{\triangleleft})^{\check{\mathrm{C}}\triangle E} \rangle$. By Lemma 22.3, Q is (X, r)-almost fissile. We have

$$\Upsilon_h^{\check{\mathsf{C}}\boldsymbol{\beta}\triangle E}(\boldsymbol{\theta}_I^E) = \text{by construction} = \theta_{\triangleleft}^{\triangle E}(h(\top)) = \theta_{\triangleleft}^{\triangle E}(b)$$

and

$$Q|_{(\triangle E)_+} = \langle \Upsilon_h^{\check{\mathsf{C}}\boldsymbol{\beta}\triangle E} \rangle(q)|_{(\triangle E)_+} = \text{by naturality of } \Upsilon = \langle \Upsilon_h^{(\boldsymbol{\beta}\triangle E)_+} \rangle(q|_{(\boldsymbol{\beta}\triangle E)_+}).$$

Thus

$$<\theta_b^{\triangle E}> - Q|_{(\triangle E)_+} = \langle \Upsilon_h^{(\boldsymbol{\beta}\triangle E)_+} \rangle(<\boldsymbol{\theta}_\top^{\boldsymbol{\beta}\triangle E}> - q|_{(\boldsymbol{\beta}\triangle E)_+}) \in \langle (Y^X)^{(\triangle E)_+} \rangle_X^{(r)},$$

where $\in$ follows from (26) by Lemma 22.2. By Proposition 13.5 $\triangleleft \overset{r}{\approx} b$. $\square$

§ 24. Loops in an arbitrary space

Nested commutators. A *nesting* t of *weight* $|t| \geqslant 1$ is either the atom $\bullet$ if $|t| = 1$, or a pair (t', t'') of nestings with $|t'| + |t''| = |t|$. Given elements $g_1, \dots, g_s$ of a group G, and a nesting t of weight s, the *t-nested commutator*

$$^t[\![g_i]\!]_{i=1}^s \in G$$

is defined to be either g_1 if $s = 1$, or

$$[\![{}^{t'}[\![g_i]\!]_{i=1}^{|t'|}, {}^{t''}[\![g_i]\!]_{i=|t'+|1}^s]\!]$$

if $t = (t', t'')$. Nested commutators of weight s in G generate $\gamma^s G$, the sth term of the lower central series of G.

Loops. Put $X = S^1$ and let Y be a cellular space. We consider the group $\pi_1(Y) = [X, Y]$ with the filtration $\pi_1(Y)^{((s))} = [X, Y]^{((s))}$ (see § 8).

Theorem 24.1. *One has*

$$\pi_1(Y)^{((s))} = \gamma^s \pi_1(Y).$$

Recall [3, Theorem 13.2]:

$$\pi_1(Y)^{(s)} = \gamma^s \pi_1(Y). \tag{27}$$

Thus, by Theorem 8.2 and [3, Theorem 4.2], the strong r-similarity on $\pi_1(Y)$ coincides with the r-similarity.

Proof. The inclusion $\pi_1(Y)^{((s))} \subseteq \gamma^s \pi_1(Y)$ follows from the comparisons $\pi_1(Y)^{((s))} \subseteq \pi_1(Y)^{(s)}$ (immediate from the definitions) and (27).

Check that $\gamma^s \pi_1(Y) \subseteq \pi_1(Y)^{((s))}$. Since $\pi_1(Y)^{((s))}$ is a subgroup (by Theorem 8.1), it suffices to show that, for any nesting t of weight s and any maps $a_1, \ldots, a_s : X \to Y$, one has

$$ {}^{t}[\![[a_i]]\!]_{i=1}^{s} \in \pi_1(Y)^{((s))}. $$

Put

$$ B_s = \bigvee_{i \in (s)} X \qquad \text{and} \qquad a = \overline{\bigvee_{i \in (s)}} a_i : B_s \to Y. $$

Let $\mathrm{in}_i : X \to B_s$ be the canonical insertions. Choose a loop $e : S^1 \to B_s$ with

$$ [e] = {}^{t}[\![[\mathrm{in}_i]]\!]_{i=1}^{s} $$

in $\pi_1(B_s)$. So ${}^{t}[\![[a_i]]\!]_{i=1}^{s} = [a \circ e]$. Clearly, the loop e is Brunnian. By Lemma 23.1, $[e] \in \pi_1(B_s)^{((s))}$. By Corollary 5.1, $[a \circ e] \in \pi_1(Y)^{((s))}$, as was to be shown. □

§ 25. Whitehead products

Whitehead product. Let T_i, $i = 1, 2$, be compact cellular spaces and

$$ T_i \xleftarrow{p_i} T_1 \times T_2 \xrightarrow{k} T_1 \wedge T_2 $$

be the projections. The map

$$ \Sigma(T_1 \times T_2) \xrightarrow{\Sigma k} \Sigma(T_1 \wedge T_2) $$

is homotopy right-invertible (because there is a canonical map r of the join $T_1 * T_2$ to $\Sigma(T_1 \times T_2)$ such that $\Sigma k \circ r$ is a homotopy equivalence). Let Y be a space. Given homotopy classes $\boldsymbol{a}_i \in [\Sigma T_i, Y]$, $i = 1, 2$, consider the homotopy classes

$$ \boldsymbol{a}_i \circ \Sigma p_i : \Sigma(T_1 \times T_2) \xrightarrow{\Sigma p_i} \Sigma T_i \xrightarrow{\boldsymbol{a}_i} Y, \qquad i = 1, 2, $$

and their commutator

$$ [\![\boldsymbol{a}_1 \circ \Sigma p_1, \boldsymbol{a}_2 \circ \Sigma p_2]\!] \in [\Sigma(T_1 \times T_2), Y]. $$

The Whitehead product

$$ \lfloor \boldsymbol{a}_1, \boldsymbol{a}_2 \rceil \in [\Sigma(T_1 \wedge T_2), Y] $$

is uniquely defined by (homotopy) commutativity of the diagram

$$ \begin{array}{ccc} \Sigma(T_1 \times T_2) & \xrightarrow{[\![\boldsymbol{a}_1 \circ \Sigma p_1, \boldsymbol{a}_2 \circ \Sigma p_2]\!]} & Y \\ {\scriptstyle \Sigma k} \downarrow & \nearrow {\scriptstyle \lfloor \boldsymbol{a}_1, \boldsymbol{a}_2 \rceil} & \\ \Sigma(T_1 \wedge T_2). & & \end{array} $$

see [5, Section 7.8].

Nested Whitehead products. Let T_i, $i \in (s)$, be compact cellular spaces and

$$T_i \xleftarrow{p_i} T_1 \times \ldots \times T_s \xrightarrow{k} T_1 \wedge \ldots \wedge T_s$$

be the projections.

Lemma 25.1. *The map*

$$\Sigma(T_1 \times \ldots \times T_s) \xrightarrow{\Sigma k} \Sigma(T_1 \wedge \ldots \wedge T_s)$$

is homotopy right-invertible.

Proof. Induction on s. If $s = 1$, k is the identity. Take $s > 1$. Put

$$T' = T_1 \times \ldots \times T_{s-1}, \qquad Z' = T_1 \wedge \ldots \wedge T_{s-1}.$$

Let

$$T' \times T_s \xrightarrow{K} T' \wedge T_s \qquad \text{and} \qquad T' \xrightarrow{k'} Z'$$

be the projections. We have the decomposition

$$\Sigma k : \Sigma(T' \times T_s) \xrightarrow{\Sigma K} \Sigma(T' \wedge T_s) \xrightarrow{\Sigma(k' \wedge \mathrm{id}_{T_s})} \Sigma(Z' \wedge T_s),$$

where ΣK is right-invertible (as noted above) and the second arrow is because it coincides with

$$\Sigma T' \wedge T_s \xrightarrow{\Sigma k' \wedge \mathrm{id}_{T_s}} \Sigma Z' \wedge T_s,$$

which is right-invertible because $\Sigma k'$ is by the induction hypothesis. $\square$

Let Y be space, and $\boldsymbol{a}_i \in [\Sigma T_i, Y]$ be homotopy classes. Given a nesting t of weight s, define the t-nested Whitehead product

$${}^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s \in \lfloor \Sigma(T_1 \wedge \ldots \wedge T_s), Y\rceil$$

by induction on s putting

$${}^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s = \boldsymbol{a}_1$$

for $s = 1$ and

$${}^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s = \lfloor {}^{t'}\lfloor \boldsymbol{a}_i \rceil_{i=1}^{|t'|}, {}^{t''}\lfloor \boldsymbol{a}_i \rceil_{i=|t'|+1}^s \rceil$$

for $t = (t', t'')$.

Lemma 25.2. *For a nesting t of weght s, the diagram*

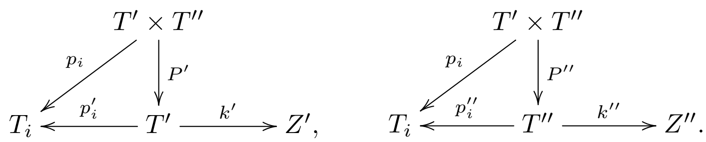
$$\begin{array}{ccc} \Sigma(T_1 \times \ldots \times T_s) & \xrightarrow{\boldsymbol{c} := {}^t[\![\boldsymbol{a}_i \circ \Sigma p_i]\!]_{i=1}^s} & Y \\ \Big\downarrow{\scriptstyle \Sigma k} & \nearrow{\scriptstyle \boldsymbol{w} := {}^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s} & \\ \Sigma(T_1 \wedge \ldots \wedge T_s) & & \end{array}$$

is (homotopy) commutative.

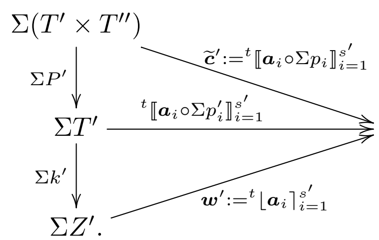
Proof. Induction on s. If $s = 1$, Σk is the identity and $\boldsymbol{c} = \boldsymbol{w} = \boldsymbol{a}_1$. Take $s > 1$. We have $t = (t', t'')$. Put $s' = |t'|$, $s'' = |t''|$, and

$$T' = T_1 \times \ldots \times T_{s'}, \qquad T'' = T_{s'+1} \times \ldots \times T_s,$$
$$Z' = T_1 \wedge \ldots \wedge T_{s'}, \qquad Z'' = T_{s'+1} \wedge \ldots \wedge T_s,$$

We have the commutative diagrams of projections



Consider the diagram

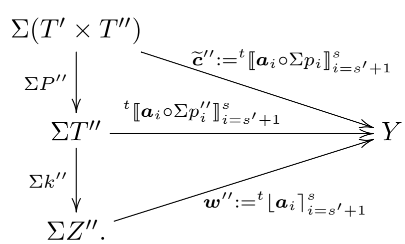


(28)

The upper triangle is commutative because the function

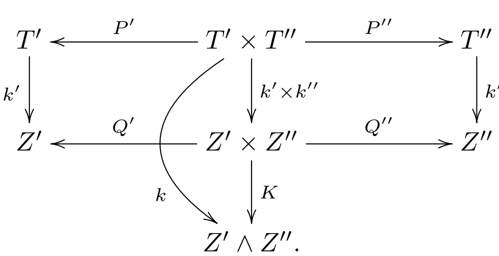
$$[\Sigma T', Y] \to [\Sigma(T' \times T''), Y]$$

induced by $\Sigma P'$ is a homomorphism and sends $\boldsymbol{a}_i \circ \Sigma p'_i$ to $\boldsymbol{a}_i \circ \Sigma p_i$. The lower triangle is commutative by the induction hypothesis. Similarly, we have the commutative diagram

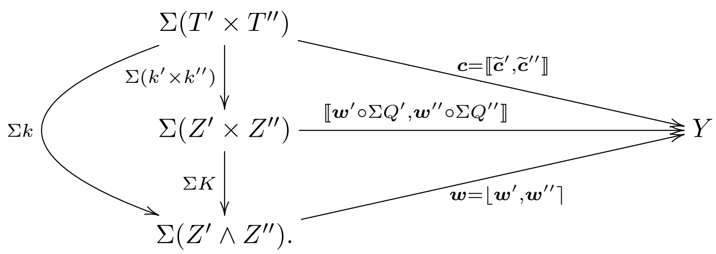


(29)

We have the commutative diagram of projections



Consider the diagram



The upper triangle is commutative because the function

$$[\Sigma(Z' \times Z''), Y] \to [\Sigma(T' \times T''), Y]$$

induced by $\Sigma(k' \times k'')$ is a homomorphism under which

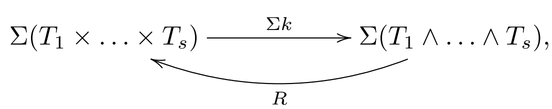
$$\boldsymbol{w}' \circ \Sigma Q' \mapsto \boldsymbol{w}' \circ \Sigma k' \circ \Sigma P' = \text{by diagram (28)} = \widetilde{\boldsymbol{c}}'$$

and

$$\boldsymbol{w}'' \circ \Sigma Q'' \mapsto \boldsymbol{w}'' \circ \Sigma k'' \circ \Sigma P'' = \text{by diagram (29)} = \widetilde{\boldsymbol{c}}''.$$

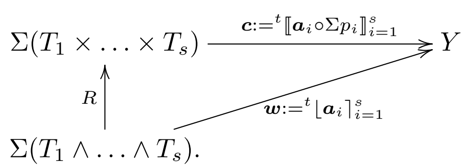
The lower triangle is commutative by the definition of Whitehead product. We are done. □

Corollary 25.3. *Let R be a homotopy right-inverse of Σk:*



$\Sigma k \circ R \sim \mathrm{id}.$

Then, for any nesting t of weght s, the diagram



is (homotopy) commutative.

Proof. We have

$$\widetilde{\boldsymbol{c}} \circ R = (\text{by Lemma 25.2}) = \boldsymbol{w} \circ \Sigma k \circ R = (\text{since } \Sigma k \circ R \sim \mathrm{id}) = \boldsymbol{w}. \qquad \square$$

§ 26. Loops and Whitehead products

Consider the wedge

$$B_s = \bigvee_{i \in (s)} S^1.$$

Given a map $v : S^1 \to B_s$ (a loop) and a space T, introduce the map v^Σ:

$$\begin{array}{ccc} \Sigma T & \xrightarrow{v^\Sigma} & \bigvee_{i\in(s)} \Sigma T \\ \| & & \| \\ S^1 \wedge T & \xrightarrow{v\wedge \mathrm{id}_T} & B_s \wedge T. \end{array}$$

Let

$$\mathrm{in}_i : S^1 \to B_s \qquad \text{and} \qquad \mathrm{in}_i^T : \Sigma T \to \bigvee_{i\in(s)} \Sigma T$$

be the canonical insertions.

Lemma 26.1. *The function*

$$\pi_1(B_s) \to [T, \bigvee_{i\in(s)} \Sigma T], \qquad [v] \mapsto [v^\Sigma],$$

is a homomorhism, under which $[\mathrm{in}_i] \mapsto [\mathrm{in}_i^T]$. □

Let T_i, $i \in (s)$, be spaces and

$$T_i \xleftarrow{p_i} T_1 \times \ldots \times T_s \xrightarrow{k} T_1 \wedge \ldots \wedge T_s$$

be the projections. Let Y be a space and $a_i : \Sigma T_i \to Y$ be maps. We have the compositions

$$a_i \circ \Sigma p_i : \Sigma(T_1 \times \ldots \times T_s) \xrightarrow{\Sigma p_i} \Sigma T_i \xrightarrow{a_i} Y.$$

Lemma 26.2. *Let t be a nesting of weight s. Let $e : S^1 \to B_s$ be a loop with*

$$[e] = {}^t[\![[\mathrm{in}_i]]\!]_{i=1}^s$$

in $\pi_1(B_s)$. Then the diagram

$$\begin{array}{ccc} \Sigma(T_1 \times \ldots \times T_s) & \xrightarrow{e^\Sigma} & \bigvee_{i\in(s)} \Sigma(T_1 \times \ldots \times T_s) \\ & \searrow^{\boldsymbol{c} := {}^t[\![[a_i]\circ\Sigma p_i]\!]_{i=1}^s} & \downarrow{\scriptstyle A := \overline{\bigvee}_{i\in(s)} (a_i\circ\Sigma p_i)} \\ & & Y. \end{array}$$

is (homotopy) commutative.

Proof. Put $T = T_1 \times \ldots T_s$. By Lemma, the function

$$\pi_1(B_s) \to [T, \bigvee_{i\in(s)} \Sigma T], \qquad [v] \mapsto [v^\Sigma],$$

is a homomorhism, under which $[\mathrm{in}_i] \mapsto [\mathrm{in}_i^T]$. Thus

$$[e^\Sigma] = {}^t[\![[\mathrm{in}_i^T]]\!]_{i=1}^s.$$

The map A induces a homomorphism

$$[\Sigma T, Y] \to [\Sigma T, \bigvee_{i \in (s)} \Sigma T],$$

under which $[\mathrm{in}_i^T] \mapsto [a_i \circ \Sigma p_i]$ and thus

$$[e^\Sigma] = {}^t[\![[\mathrm{in}_i^T]]\!]_{i=1}^s \mapsto {}^t[\![[a_i] \circ \Sigma p_i]\!]_{i=1}^s = \boldsymbol{c},$$

which is what was to be shown. $\square$

By Lemma 25.1, we have the diagram

$$\Sigma(T_1 \times \ldots \times T_s) \xrightarrow{\Sigma k} \Sigma(T_1 \wedge \ldots \wedge T_s), \qquad \Sigma(T_1 \wedge \ldots \wedge T_s) \xrightarrow{R} \Sigma(T_1 \times \ldots \times T_s)$$

where $\Sigma k \circ R \sim \mathrm{id}$. Introduce the composition $M_R^v(a_i)_{i=1}^s$:

$$\begin{array}{ccc} \Sigma(T_1 \times \ldots \times T_s) & \xrightarrow{v^\Sigma} & \bigvee_{i \in (s)} \Sigma(T_1 \times \ldots \times T_s) \\ \uparrow R & & \downarrow A := \underline{\bigvee}_{i \in (s)} (a_i \circ \Sigma p_i) \\ \Sigma(T_1 \wedge \ldots \wedge T_s) & \xrightarrow{M_R^v(a_i)_{i=1}^s} & Y. \end{array}$$

Lemma 26.3. *Let t be a nesting of weight s. Let $e : S^1 \to B_s$ be a loop with*

$$[e] = {}^t[\![[\mathrm{in}_i]]\!]_{i=1}^s$$

in $\pi_1(B_s)$. Then

$$[M_R^e(a_i)_{i=1}^s] = {}^t\lfloor [a_i] \rceil_{i=1}^s$$

in $[\Sigma(T_1 \wedge \ldots \wedge T_s), Y]$.

Proof. Recall the homotopy class

$$\Sigma(T_1 \times \ldots \times T_s) \xrightarrow{\boldsymbol{c} := {}^t[\![[a_i] \circ \Sigma p_i]\!]_{i=1}^s} Y.$$

We have

$$[M_R^e(a_i)_{i=1}^s] = [A \circ e^\Sigma \circ R] = (\text{by Lemma 26.2}) = \boldsymbol{c} \circ R = (\text{by Corollary 25.3}) = {}^t\lfloor [a_i] \rceil_{i=1}^s.$$

$\square$

§ 27. Strong nullarity of Whitehead products

Let $a_i : \Sigma T_i \to Y$, etc., be as in the previous section.

Lemma 27.1. *Let $v : S^1 \to B_s$ be a loop such that $\text{ᑫ} \overset{r}{\approx} v$. Then*

$$\text{ᑫ} \overset{r}{\approx} M_R^v(a_i)_{i=1}^s.$$

Proof. We have

$$M_R^v(a_i)_{i=1}^s = A \circ v^\Sigma \circ R$$

(see the construction). By Corollary 5.4, $\urcorner \overset{r}{\approx} v^\Sigma$. By Corollary 5.2, $\urcorner \overset{r}{\approx} A \circ v^\Sigma \circ R$. □

Given a nesting t of weight s and homotopy classes $\boldsymbol{a}_i \in [\Sigma T_i, Y]$, $i \in (s)$, consider the t-nested Whitehead product

$$^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s \in [\Sigma(T_1 \wedge \dots T_s), Y].$$

Theorem 27.2. *One has*

$$^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s \in [\Sigma(T_1 \wedge \dots T_s), Y]^{((s))}.$$

Proof. For each i, choose a representative $a_i : \Sigma T_i \to Y$ of $\boldsymbol{a}_i$. Choose a loop $e : S^1 \to B_s$ with

$$[e] = {}^t[\![[\mathrm{in}_i]]\!]_{i=1}^s$$

in $\pi_1(B_s)$. Clearly, the loop e is Brunnian. By Lemma 23.1, $\urcorner \overset{s-1}{\approx} e$. By Lemma 27.1,

$$\urcorner \overset{s-1}{\approx} M_R^e(a_i)_{i=1}^s.$$

By Lemma 26.3,

$$[M_R^e(a_i)_{i=1}^s] = {}^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s.$$

Thus

$$^t\lfloor \boldsymbol{a}_i \rceil_{i=1}^s \in [\Sigma(T_1 \wedge \dots T_s), Y]^{((s))}.$$ □

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