Homotopy similarity of maps. Compositions

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We describe the behavior of the homotopy similarity relations and finite-order invariants under the function $[X, Y] \to [X, Z]$ induced by a map $Y \to Z$ strongly r-similar to the constant map.

§ 1. Introduction

This paper continues [1] and [2]. We adopt notation and conventions thereof. Let X, Y, and Z be cellular spaces, X and Y compact. We prove the following two theorems.

1.1. Theorem. *Let maps $a, a' : X \to Y$ satisfy $a \overset{p-1}{\sim} a'$ and a map $b : Y \to Z$ satisfy $b \overset{q-1}{\approx} \triangleleft^Y_Z$ ($p, q \geqslant 1$). Then the maps $b \circ a, b \circ a' : X \to Z$ satisfy*

$$b \circ a \overset{pq-1}{\sim} b \circ a'.$$

1.2. Theorem. *Let $b : Y \to Z$ be a map such that $b \overset{q-1}{\approx} \triangleleft^Y_Z$ ($q \geqslant 1$). Let L be an abelian group and $h : [X, Z] \to L$ be a homotopy invariant. Then the invariant*

$$f : [X, Y] \to L, \qquad [a] \mapsto h([b \circ a]),$$

satisfies

$$q \operatorname{ord} f \leqslant \operatorname{ord} h.$$

We conjecture that the assumption of strong $(q-1)$-similarity in these statements can be replaced by that of $(q-1)$-similarity.

§ 2. Coherence of an ensemble of compositions

Consider a partition of unity

$$\sum_{e \in E} \phi_e = 1,$$

where E is a nonempty finite set and $\phi_e : X \to [0, 1]$ are continuous functions, and the unbased map

$$\phi = (\phi_e)_{e \in E} : X \to \Delta E.$$

Introduce the function

$$\xi^\phi : Y^X \times (Z^Y)^{(\Delta E)} \to Z^X, \qquad \xi^\phi(a, s) : X \xrightarrow{\phi \,\underline{\times}\, a} \Delta E \times Y \xrightarrow{\square^Y(s)} Z$$

(equivalently, $\xi^{\phi}(a,s) : x \mapsto s(\phi(x))(a(x))$), and the homomorphism

$$(\xi^{\phi}) : \langle Y^X \rangle \otimes \langle (Z^Y)^{(\Delta E)} \rangle \to \langle Z^X \rangle, \qquad {<}a{>} \otimes {<}s{>} \mapsto {<}\xi^{\phi}(a,s){>}.$$

We fix a metric on X. A number $\lambda > 0$ is called a *Lebesgue number* of an open cover Γ of X if any set $V \subseteq X$ of diameter at most λ is contained in some $G \in \Gamma$ (according to [3, proof of theorem 3.3.14]).

2.1. Proposition. *Let Γ be an open cover of X with a Lebesgue number λ. Suppose that*

$$\operatorname{diam} \operatorname{supp} \phi_e \leqslant \epsilon, \qquad e \in E,$$

where $\epsilon > 0$ and $(pq-2)\epsilon \leqslant \lambda$ $(p, q \geqslant 1)$. Let an ensemble $A \in \langle Y^X \rangle$ satisfy

$$A \overset{p-1}{\underset{\Gamma}{\equiv}} 0.$$

Let $S \in \langle (Z^Y)^{(\Delta E)} \rangle$ be a fissile ensemble such that

$${<}\theta({\lhd}^{Y}_{Z}){>} - S \in \langle (Z^Y)^{(\Delta E)} \rangle^{(q)}_{Y}.$$

Then

$$(\xi^{\phi})(A \otimes S) \in \langle Z^X \rangle^{(pq)}.$$

Proof. Take a set $V \subseteq X$ with $|V| \leqslant pq - 1$. We should show that

$$(\xi^{\phi})(A \otimes S)|_V = 0$$

in $\langle Z^{(V)} \rangle$. Consider the equivalence on V generated by all pairs (x_1, x_2) with $\operatorname{dist}(x_1, x_2) \leqslant \epsilon$. Let V_i, $i \in I$, be the classes of this equivalence. Clearly, $\operatorname{diam} V_i \leqslant (pq-2)\epsilon$ and $\operatorname{dist}(V_i, V_j) > \epsilon$ for $i \neq j$. For $i \in I$, put

$$F_i = \{\, e \mid \phi_e|_{V_i} \neq 0 \,\} \subseteq E.$$

Clearly, $\phi(V_i) \subseteq \Delta F_i$. We have $F_i \neq \varnothing$ because $V_i \neq \varnothing$. We have $F_i \cap F_j = \varnothing$ for $i \neq j$ because $\operatorname{diam} \operatorname{supp} \phi_e \leqslant \epsilon$ and $\operatorname{dist}(V_i, V_j) > \epsilon$. Thus we have the layout

$$F_* = \{\, F_i \mid i \in I \,\} \subseteq \mathcal{A}(E).$$

Consider the homomorphisms

$$\rho_1 : \langle Y^X \rangle \to \bigotimes_{i \in I} \langle Y^{(V_i)} \rangle, \qquad {<}a{>} \mapsto \bigotimes_{i \in I} {<}a|_{V_i}{>},$$

and

$$\rho_2 : \langle (Z^Y)^{(\Delta E)} \rangle \to \bigotimes_{i \in I} \langle (Z^Y)^{(\Delta F_i)} \rangle, \qquad {<}s{>} \mapsto \bigotimes_{i \in I} {<}s|_{\Delta F_i}{>}.$$

From now on, let decorated ρ's denote similar homomorphisms. For $i \in I$, we have, similarly to ξ and ξ^{ϕ}, the function

$$\xi_i^{\phi} : Y^{(V_i)} \times (Z^X)^{(\Delta F_i)} \to Z^{(V_i)}, \qquad \xi^{\phi}(d,t) : x \mapsto t(\phi(x))(d(x)),$$

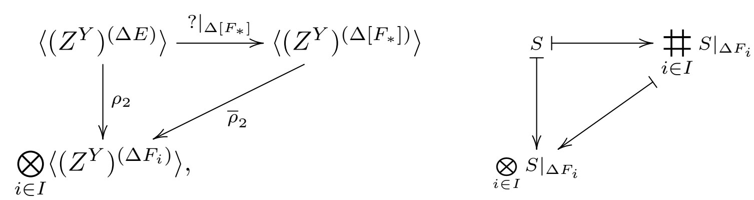
and the homomorphism

$$(\xi_i^{\phi}) : \langle Y^{(V_i)} \rangle \otimes \langle (Z^X)^{(\Delta F_i)} \rangle \to \langle Z^{(V_i)} \rangle, \qquad <d> \otimes <t> \mapsto <\xi_i^{\phi}(d,t)>.$$

We have

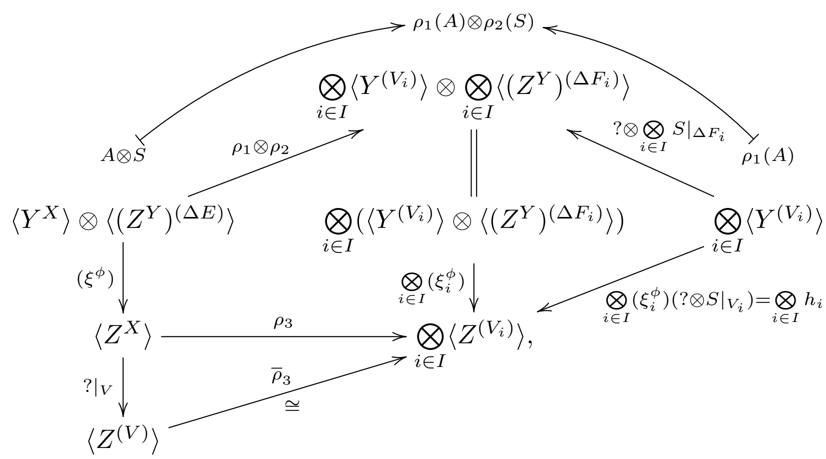
$$\rho_2(S) = \bigotimes_{i \in I} S|_{\Delta F_i}. \tag{1}$$

Indeed, consider the commutative diagram with sendings



The horizontal sending holds because S is fissile. The diagonal one is obvious. The vertical sending, which is (1), follows.

We have the commutative diagram with sendings



where

$$h_i = (\xi_i^{\phi})(? \otimes S_i) : \langle Y^{(S_i)} \rangle \xrightarrow{? \otimes S|_{\Delta F_i}} \langle Y^{(V_i)} \rangle \otimes \langle (Z^Y)^{(\Delta F_i)} \rangle \xrightarrow{(\xi_i^{\phi})} \langle Z^{(V_i)} \rangle.$$

Clearly, $\overline{\rho}_3$ is an isomorphism. The first sending is obvious. The second one follows from (1). We should show that $A \otimes S$ goes to zero under the composition in the left column. By the diagram, it suffices to show that

$$\Big(\bigotimes_{i \in I} h_i\Big)(\rho_1(A)) = 0. \tag{2}$$

Let $J \subseteq I$ consist of those i for which $|V_i| \geqslant q$. We have $|J| \leqslant p - 1$ because $|V| \leqslant pq - 1$.

Take $i \in I \setminus J$ and $d \in Y^{(V_i)}$. We show that

$$h_i(\text{<}d\text{>}) = \text{<}\triangleleft^{V_i}_{Z}\text{>}. \tag{3}$$

Consider the unbased maps

$$D : V_i \to \Delta E \times Y, \qquad x \mapsto (\phi(x), d(x)),$$

and $D' = D|_{V_i \to D(V_i)}$. We have the commutative diagram with sendings

$$\begin{array}{ccccc}
S & \text{<}\theta(\triangleleft^{Y}_{Z})\text{>} & \langle (Z^Y)^{(\Delta E)} \rangle & \xrightarrow{\ \square^Y\ } & \langle Z^{(\Delta E \times Y)} \rangle \\
\downarrow & \downarrow & \downarrow{\scriptstyle (\xi_i^{\phi})(\text{<}d\text{>}\otimes ?|_{\Delta F_i})} & & \downarrow{\scriptstyle ?|_{D(V_i)}} \\
h_i(\text{<}d\text{>}) & \text{<}\triangleleft^{V_i}_{Z}\text{>} & \langle Z^{(V_i)} \rangle & \xleftarrow{\ \langle Z^{(D')} \rangle\ } & \langle Z^{(D(V_i))} \rangle .
\end{array}$$

Commutativity is checked directly. The first sending holds by definition of h_i. The second one is obvious. Consider the difference $R = \text{<}\theta(\triangleleft^{Y}_{Z})\text{>} - S$. Since $R \in \langle (Z^Y)^{(\Delta E)} \rangle^{(q)}_{Y}$, we have $\square^Y(R) \in \langle (Z^Y)^{\Delta E \times Y} \rangle^{(q)}$. Since $|D(V_i)| \leqslant |V_i| \leqslant q-1$, we have $\square^Y(R)|_{D(V_i)} = 0$. The equality (3) follows by the diagram.

We have the commutative diagram

$$\begin{array}{ccccc}
\langle Y^X \rangle & \xrightarrow{\ \rho_1\ } & \bigotimes_{i\in I} \langle Y^{(V_i)} \rangle & \xrightarrow{\ \bigotimes_{i\in I} h_i\ } & \bigotimes_{i\in I} \langle Z^{(V_i)} \rangle \\
& \searrow^{\rho_1'} & \downarrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \sigma} \\
& & \bigotimes_{i\in J} \langle Y^{(V_i)} \rangle & \xrightarrow{\ \bigotimes_{i\in J} h_i\ } & \bigotimes_{i\in J} \langle Z^{(V_i)} \rangle ,
\end{array}$$

where π and σ are defined by the rules

$$\pi : \bigotimes_{i\in I} \text{<}d_i\text{>} \mapsto \bigotimes_{i\in J} \text{<}d_i\text{>}, \qquad d_i \in Y^{(V_i)}\ (i \in I),$$

$$\sigma : \bigotimes_{i\in J} \text{<}f_i\text{>} \mapsto \bigotimes_{i\in I} \text{<}f_i\text{>}, \qquad f_i \in Z^{(V_i)}\ (i \in I), \qquad f_i = \triangleleft^{V_i}_{Z} \text{ if } i \notin J.$$

Commutativity of the square follows from (3). We show that $\rho_1'(A) = 0$. By the diagram, (2) will follow.

For $i \in J$, there is $G_i \in \Gamma$ such that $V_i \subseteq G_i$ because $\operatorname{diam} V_i \leqslant (pq-2)\epsilon \leqslant \lambda$. Put

$$H = \{\triangleleft_X\} \cup \bigcup_{i\in J} G_i.$$

Since $|J| \leqslant p-1$, $H \in \Gamma(p-1)$. We have the commutative diagram

$$\begin{array}{ccc}
\langle Y^X \rangle & \xrightarrow{\ \rho_1'\ } & \bigotimes_{i\in J} \langle Y^{(V_i)} \rangle \\
\downarrow{\scriptstyle ?|_H} & \nearrow_{\widehat{\rho}_1'} & \\
\langle Y^H \rangle . & &
\end{array}$$

Since $A \overset{p-1}{\underset{\Gamma}{=}} 0$, we have $A|_H = 0$. By the diagram, $\rho'_1(A) = 0$. $\square$

§ 3. Exploiting Proposition 2.1

3.1. Corollary. *Let an ensemble* $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i{<}a_i{>},$$

satisfy $A \overset{p-1}{=}$ *and a map* $b : Y \to Z$ *satisfy* $b \overset{q-1}{\approx} \lhd^Y_Z$ $(p, q \geqslant 1)$*. Then there exists an ensemble* $C \in \langle Z^X \rangle$,

$$C = \sum_{i,j} u_i v_j{<}c_{ij}{>}, \tag{4}$$

where $c_{ij} \sim b \circ a_i$ *and*

$$\sum_j v_j = 1, \tag{5}$$

such that $C \overset{pq-1}{=} 0$.

Proof. By [1, Corollary 6.2], there is an ensemble $\widetilde{A} \in \langle Y^X \rangle$,

$$\widetilde{A} = \sum_i u_i{<}\widetilde{a}_i{>},$$

where $\widetilde{a}_i \sim a_i$, such that $\widetilde{A} \overset{p-1}{\underset{\Gamma}{=}} 0$ for some open cover Γ of X. Let λ be a Lebesgue number of Γ. Choose $\epsilon > 0$ such that $(pq - 2)\epsilon \leqslant \lambda$ and a partition of unity

$$\sum_{e \in E} \phi_e = 1,$$

where E is a nonempty finite set and $\phi_e : X \to [0, 1]$ are continuous functions such that

$$\operatorname{diam} \operatorname{supp} \phi_e \leqslant \epsilon, \qquad e \in E.$$

Form the unbased map

$$\phi = (\phi_e)_{e \in E} : X \to \Delta E.$$

Since $b \overset{q-1}{\approx} \lhd^Y_Z$, there is a fissile ensemble $S \in \langle (Z^Y_b)^{(\Delta E)} \rangle$,

$$S = \sum_j v_j{<}s_j{>},$$

such that $\theta(\lhd^Y_Z) - S \in \langle (Z^Y)^{(\Delta E)} \rangle^{(q)}_Y$. Put

$$C = (\xi^\phi)(\widetilde{A} \otimes S)$$

(see § 2 for (ξ^ϕ)). By Proposition 2.1, $C \overset{pq-1}{=} 0$. We have

$$C = \sum_{i,j} u_i v_j {<}\xi^\phi(\widetilde{a}_i, s_j){>}.$$

Since $\widetilde{a}_i \sim a_i$ and $s_j \in (Z_b^Y)^{(\Delta E)}$, we have $\xi^\phi(\widetilde{a}_i, s_j) \sim b \circ a_i$. The equality (5) holds because S is fissile and thus affine. □

Proof of Theorem 1.1. Since $a \overset{p-1}{\sim} a'$, we have

$$\sum_i u_i {<}a_i{>} \overset{p-1}{=} {<}a'{>}$$

in ${<}Y^X{>}$, where $a_i \sim a$. By Corollary 3.1,

$$\sum_{i,j} u_i v_j {<}c_{ij}{>} \overset{pq-1}{=} \sum_j v_j {<}c'_j{>}$$

in $\langle Z^Y \rangle$, where $c_{ij} \sim b \circ a_i \sim b \circ a$, $c'_j \sim b \circ a'$, and the equality (5) holds. By [1, Theorem 7.3], $b \circ a \overset{pq-1}{\sim} b \circ a'$. □

Proof of Theorem 1.2. Suppose that $\operatorname{ord} h \leqslant pq - 1$ for some integer $p \geqslant 1$. It suffices to show that $\operatorname{ord} f \leqslant p - 1$. Take an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<}a_i{>},$$

such that $A \overset{p-1}{=} 0$. Corollary 3.1 yields an ensemble $C \in \langle Z^X \rangle$ of the form (4) with $c_{ij} \sim b \circ a_i$ and v_j satisfying (5) such that $C \overset{pq-1}{=} 0$. We have

$$\sum_i u_i f([a_i]) = \sum_{i,j} u_i v_j h([b \circ a_i]) = \sum_{i,j} u_i v_j h([c_{ij}]) \overset{(*)}{=} 0,$$

where $(*)$ holds because $\operatorname{ord} h \leqslant pq - 1$ and $C \overset{pq-1}{=} 0$. Thus $\operatorname{ord} f \leqslant p - 1$. □

References

[1] S. S. Podkorytov, Homotopy similarity of maps, arXiv:2308.00859 (2023).

[2] S. S. Podkorytov, Homotopy similarity of maps. Strong similarity, `https://www.pdmi.ras.ru/~ssp/sim-2.pdf` (2025).

[3] E. H. Spanier, Algebraic topology. McGraw-Hill, 1966.

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