Homotopy similarity of maps

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Given based cellular spaces X and Y, X compact, we define a sequence of increasingly fine equivalences on the based-homotopy set $[X, Y]$.

§ 1. Introduction

Let X and Y be based cellular spaces (= CW-complexes), X compact. Let Y^X be the set of based continuous maps $X \to Y$ and $\langle Y^X \rangle$ be the free abelian group associated with Y^X. An element $A \in \langle Y^X \rangle$, an *ensemble*, has the form

$$A = \sum_i u_i{<}a_i{>}, \tag{1}$$

where $u_i \in \mathbf{Z}$ and $a_i \in Y^X$. Let $\mathcal{F}_r(X)$ be the set of *subspaces* (= subsets containing the basepoint) $T \subseteq X$ containing at most r points distinct from the basepoint. Introduce the subgroup

$$\langle Y^X \rangle^{(r+1)} = \{\, A : A|_T = 0 \text{ in } \langle Y^T \rangle \text{ for all } T \in \mathcal{F}_r(X) \,\} \subseteq \langle Y^X \rangle.$$

We have

$$\langle Y^X \rangle = \langle Y^X \rangle^{(0)} \supseteq \langle Y^X \rangle^{(1)} \supseteq \dots .$$

For ensembles $A, B \in \langle Y^X \rangle$, let

$$A \overset{r}{=} B$$

mean that $B - A \in \langle Y^X \rangle^{(r+1)}$.

For maps $a, b \in Y^X$, we say that a is *r-similar* to b,

$$a \overset{r}{\sim} b,$$

when there exists an ensemble $A \in \langle Y^X \rangle$ given by (1) with all $a_i \sim a$ ($\sim$ denotes based homotopy) such that $A \overset{r}{=} {<}b{>}$. A simple example is given in Section 3.

Our main results state that the relation $\overset{r}{\sim}$ is an equivalence (Theorem 8.1) and respects homotopy (Theorem 5.2). It follows that we get a sequence of increasingly fine equivalences on the based-homotopy set $[X, Y]$.

We conjecture that, for 0-connected Y, a map is r-similar to the constant map if and only if it lifts to the classifying space of the $(r+1)$th term of the lower central series of the loop group of Y.

A related notion is that of a homotopy invariant of finite degree [4, 5]. A function $f : [X, Y] \to L$, where L is an abelian group, is called an invariant of *degree* at most r when for any ensemble $A \in \langle Y^X \rangle$ given by (1) the congruence $A \overset{r}{=} 0$ implies

$$\sum_i u_i f([a_i]) = 0.$$

It is clear that $f([a]) = f([b])$ if $a \overset{r}{\sim} b$ and f has degree at most r. In § 11, we give an example of two maps that are not 2-similar but cannot be distinguished by invariants of degree at most 2. In the stable dimension range, invariants of degree at most r were characterized in a way similar to our conjecture about r-similarity [4].

The relation between r-similarity and finite-degree homotopy invariants is similar to that between n-equivalence and finite-degree invariants in knot theory [1, 2]. The example of § 11 is similar to that of [2, Remark 10.8].

§ 2. Preliminaries

By a *space* we mean a based space (unless the contrary is stated explicitly). The basepoint of a cellular space is a vertex. The basepoint of a space X is denoted by $\urcorner_X$ or $\urcorner$. A *subspace* contains the basepoint. A *cover* is a cover by subspaces. A *map* is a based continuous map. The constant map $X \to Y$ is denoted by $\urcorner_Y^X$ or $\urcorner$. A *homotopy* is a based homotopy.

For a subspace $Z \subseteq X$, $\mathrm{in} : Z \to X$ is the inclusion. A wedge of spaces comes with the insertions (= coprojections):

$$\mathrm{in}_k : X_k \to X_1 \vee \ldots \vee X_n.$$

Maps $a_k : X_k \to Y$ form the map

$$a_1 \,\underline{\triangledown}\, \ldots \,\underline{\triangledown}\, a_n : X_1 \vee \ldots \vee X_n \to Y.$$

This notation is also used for homotopy classes.

The formula $a \sim|_Z b$ means homotopy $a|_Z \sim b|_Z$. Similarly, equality of restrictions to a subset C is denoted by $=|_C$.

For a set E, the associated abelian group $\langle E \rangle$ is freely generated by the elements $<\!e\!>$, $e \in E$. A function $t : E \to F$ between two sets induces the homomorphism

$$\langle t \rangle : \langle E \rangle \to \langle F \rangle, \qquad <\!e\!> \mapsto <\!t(e)\!>.$$

For a cover Γ of a space X, we put

$$\Gamma(r) = \{\, \{\urcorner\} \cup G_1 \cup \ldots \cup G_s \subseteq X : G_1, \ldots, G_s \in \Gamma, 0 \leqslant s \leqslant r \,\}.$$

For ensembles $A, B \in \langle Y^X \rangle$, the formula

$$A \underset{\Gamma}{\overset{r}{=}} B$$

means that $A =|_W B$ in $\langle Y^W \rangle$ for all $W \in \Gamma(r)$.

Expressions with ? denote functions: for example, $?^2 : \mathbf{R} \to \mathbf{R}$ is the function $x \mapsto x^2$.

§ 3. A simple example

Put $\boldsymbol{\mathcal{E}} = \{0, 1\} \subseteq \mathbf{Z}$. Fix $r \geqslant 0$. For $d = (d_1, \dots, d_{r+1}) \in \boldsymbol{\mathcal{E}}^{r+1}$, put $|d| = d_1 + \dots + d_{r+1}$. Consider a wedge of spaces

$$W = U_1 \vee \dots \vee U_{r+1} \vee V.$$

Introduce the maps

$$\Lambda(d) = \lambda_1(d_1) \vee \dots \vee \lambda_{r+1}(d_{r+1}) \vee \mathrm{id}_V : W \to W, \qquad d \in \boldsymbol{\mathcal{E}}^{r+1},$$

where the map $\lambda_k(e) : U_k \to U_k$, for $e \in \boldsymbol{\mathcal{E}}$, is id if $e = 1$ and $\urcorner$ if $e = 0$.

3.1. Lemma. *Let X and Y be spaces and $p : X \to W$ and $q : W \to Y$ be maps. Consider the ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_{d \in \boldsymbol{\mathcal{E}}^{r+1}} (-1)^{|d|} {<}a(d){>},$$

where

$$a(d) : X \xrightarrow{p} W \xrightarrow{\Lambda(d)} W \xrightarrow{q} Y.$$

Then $A \overset{r}{=} 0$.

Proof. Take $T \in \mathcal{F}_r(X)$. There is a k such that $p(T) \cap \mathrm{in}_k(U_k) = \{\urcorner_W\}$. Then $a(d)|_T$ does not depend on d_k. We get

$$A|_T = \sum_{d \in \boldsymbol{\mathcal{E}}^{r+1}} (-1)^{|d|} {<}a(d)|_T{>} = 0. \qquad \square$$

Example. Consider the wedge

$$W = S^{n_1} \vee \dots \vee S^{n_{r+1}}$$

$(n_1, \dots, n_{r+1} \geqslant 1)$. Put $m = n_1 + \dots + n_{r+1} - r$ and let $p : S^m \to W$ be a map with

$$[p] = \lfloor \dots \lfloor [\mathrm{in}_1], [\mathrm{in}_2] \rceil, \dots, [\mathrm{in}_{r+1}] \rceil$$

(the iterated Whitehead product) in $\pi_m(W)$. We show that $\urcorner \overset{r}{\sim} p$. Consider the maps

$$a(d) : S^m \xrightarrow{p} W \xrightarrow{\Lambda(d)} W, \qquad d \in \boldsymbol{\mathcal{E}}^{r+1}.$$

Put $1_{r+1} = (1, \dots, 1) \in \boldsymbol{\mathcal{E}}^{r+1}$. By Lemma 3.1,

$$\sum_{d \in \boldsymbol{\mathcal{E}}^{r+1} \setminus \{1_{r+1}\}} (-1)^{r-|d|} {<}a(d){>} \overset{r}{=} {<}a(1_{r+1}){>}.$$

All $a(d)$ on the left side are homotopic to $\urcorner$. On the right, $a(1_{r+1}) = p$ because $\Lambda(1_{r+1}) = \mathrm{id}$. Thus $\urcorner \overset{r}{\sim} p$.

§ 4. Equipment of a cellular space

Let Y be a compact *unbased* cellular space. In this section, we turn off our convention that *maps* and *homotopies* preserve basepoints.

4.1. Lemma. *There exist homotopies*

$$q_t : Y^2 \to Y \quad \textit{and} \quad p_t : Y^2 \to [0,1], \qquad t \in [0,1],$$

such that

$$q_0(z,y) = y, \qquad q_t(z,z) = z, \qquad p_0(z,y) = 0, \qquad p_t(z,z) = t, \tag{2}$$

and, for any $(z,y) \in Y^2$ *and* $t \in [0,1]$*, one has*

$$p_t(z,y) = 0 \quad \textit{or} \quad q_t(z,y) = z. \tag{3}$$

Roughly speaking, the inclusions $\{z\} \to Y$, $z \in Y$, form a parametric cofibration.

Proof (after [6, Exemple on p. 490]). By [3, Corollary A.10], Y is an ENR. Embed it to $\mathbf{R}^n$ and choose its neighbourhood $U \subseteq \mathbf{R}^n$ and a retraction $r : U \to Y$. Choose $\epsilon > 0$ such that U includes all closed balls of radius ϵ with centres in Y. Consider the homotopy $l_t : (\mathbf{R}^n)^2 \to \mathbf{R}^n$, $t \in [0,1]$,

$$\begin{aligned} &l_t(z,y) = y + \min(\epsilon t/|z-y|, 1)(z-y), \qquad z \neq y, \\ &l_t(z,z) = z. \end{aligned}$$

Put

$$q_t(z,y) = r(l_t(z,y)) \quad \text{and} \quad p_t(z,y) = \max(t - |z-y|/\epsilon, 0). \qquad \square$$

4.2. Corollary. *One can continuously associate to each path* $v : [0,1] \to Y$ *a homotopy* $E_t(v) : Y \to Y$*,* $t \in [0,1]$*, such that* $E_0(v) = \mathrm{id}$ *and* $E_t(v)(v(0)) = v(t)$*.*

Proof. Using Lemma 4.1, put

$$E_t(v)(y) = \begin{cases} q_t(v(0), y) & \text{if } p_t(v(0), y) = 0, \\ v(p_t(v(0), y)) & \text{if } q_t(v(0), y) = v(0). \end{cases} \qquad \square$$

§ 5. Coherent homotopies

Let X and Y be cellular spaces, X compact.

5.1. Lemma. *Consider an ensemble* $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<} a_i {>},$$

and maps $b, \tilde{b} \in Y^X$, $b \sim \tilde{b}$. Then there exist maps $\tilde{a}_i \in Y^X$, $\tilde{a}_i \sim a_i$, such that the ensemble

$$\tilde{A} = \sum_i u_i {<}\tilde{a}_i{>}$$

has the following property: if $A =|_Z {<}b{>}$ for some subspace $Z \subseteq X$, then $\tilde{A} =|_Z {<}\tilde{b}{>}$.

Proof. We have a homotopy $h_t \in Y^X$, $t \in [0,1]$, such that $h_0 = b$ and $h_1 = \tilde{b}$. Replace Y by a compact cellular subspace that includes the images of all a_i and h_t.

For $x \in X$, introduce the path $v_x = h_?(x) : [0,1] \to Y$. We have $v_x(0) = b(x)$ and $v_x(1) = \tilde{b}(x)$. For a subspace $Z \subseteq X$, introduce the functions $e_t^Z : Y^Z \to Y^Z$, $t \in [0,1]$,

$$e_t^Z(d)(x) = E_t(v_x)(d(x)), \qquad x \in Z, \quad d \in Y^Z,$$

where E_t is given by Corollary 4.2. For $d \in Y^Z$, we have the homotopy $e_t^Z(d) \in Y^Z$, $t \in [0,1]$. The diagram

$$\begin{tikzcd}
Y^X \arrow[r, "e_t^X"] \arrow[d, "?|_Z"'] & Y^X \arrow[d, "?|_Z"] \\
Y^Z \arrow[r, "e_t^Z"] & Y^Z
\end{tikzcd}$$

is commutative. We have $e_0^Z = \mathrm{id}$ because

$$e_0^Z(d)(x) = E_0(v_x)(d(x)) = d(x).$$

We have $e_1^X(b) = \tilde{b}$ because

$$e_1^X(b)(x) = E_1(v_x)(b(x)) = E_1(v_x)(v_x(0)) = v_x(1) = \tilde{b}(x).$$

Put $\tilde{a}_i = e_1^X(a_i)$. Since $a_i = e_0^X(a_i)$, we have $\tilde{a}_i \sim a_i$. We have

$$({<}\tilde{b}{>} - \tilde{A})|_Z = \langle e_1^X \rangle ({<}b{>} - A)|_Z = \langle e_1^Z \rangle (({<}b{>} - A)|_Z).$$

Thus $A =|_Z {<}b{>}$ implies $\tilde{A} =|_Z {<}\tilde{b}{>}$. $\square$

5.2. Theorem. *Let maps $a, b, \tilde{a}, \tilde{b} \in Y^X$ satisfy*

$$\tilde{a} \sim a \overset{r}{\sim} b \sim \tilde{b}.$$

Then $\tilde{a} \overset{r}{\sim} \tilde{b}$.

Proof. By the definition of similarity, it suffices to show that $a \overset{r}{\sim} \tilde{b}$. We have an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<}a_i{>},$$

where $a_i \sim a$, such that $A \overset{r}{=} \langle b \rangle$. By Lemma 5.1, there is an ensemble $\tilde{A} \in \langle Y^X \rangle$,

$$\tilde{A} = \sum_i u_i \langle \tilde{a}_i \rangle,$$

where $\tilde{a}_i \sim a_i$, such that $\tilde{A} \overset{r}{=} \langle \tilde{b} \rangle$. Since $a_i \sim a$, we have shown that $a \overset{r}{\sim} \tilde{b}$. $\square$

§ 6. Underlaying a cover

Let X and Y be cellular spaces, X compact.

6.1. Lemma. *Consider an ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_i u_i \langle a_i \rangle.$$

Then there exist maps $\tilde{a}_i \in Y^X$, $\tilde{a}_i \sim a_i$, such that the ensemble

$$\tilde{A} = \sum_i u_i \langle \tilde{a}_i \rangle$$

has the following property: if $A|_Z = 0$ for some subspace $Z \subseteq X$, then $\tilde{A}|_V = 0$ for some neighbourhood $V \subseteq X$ of Z.

Proof. Replace Y by a compact cellular subspace that includes the images of all a_i. We will use the "equipment" (q_t, p_t) given by Lemma 4.1.

Let i that numbers a_i run over $1, \dots, n$. Define maps $a_i^k \in Y^X$, $1 \leqslant i \leqslant n$, $0 \leqslant k \leqslant n$, by the rules $a_i^0 = a_i$ and

$$a_i^k(x) = q_1(a_k^{k-1}(x), a_i^{k-1}(x)), \qquad x \in X, \tag{4}$$

for $k \geqslant 1$. Put $\tilde{a}_i = a_i^n$. We have $a_i^k \sim a_i^{k-1}$ because $a_i^k = h_1$ and $a_i^{k-1} = h_0$ for the homotopy $h_t \in Y^X$, $t \in [0,1]$,

$$h_t(x) = q_t(a_k^{k-1}(x), a_i^{k-1}(x)), \qquad x \in X.$$

Thus $\tilde{a}_i \sim a_i$.

Claim 1. If $a_i^{k-1} =|_Q a_j^{k-1}$ for some subspace $Q \subseteq X$, then $a_i^k =|_Q a_j^k$.

This follows from (4).

Claim 2. If $a_i^{i-1} =|_Q a_j^{i-1}$ for some subspace $Q \subseteq X$, then there exists a neighbourhood $W \subseteq X$ of Q such that $a_i^i =|_W a_j^i$.

Indeed, if $a_i^{i-1} =|_Q a_j^{i-1}$, then, by (2),

$$p_1(a_i^{i-1}(x), a_j^{i-1}(x)) = 1$$

for $x \in Q$. There exists a neighbourhood $W \subseteq X$ of Q such that

$$p_1(a_i^{i-1}(x), a_j^{i-1}(x)) > 0$$

for $x \in W$. Then, by (3),

$$q_1(a_i^{i-1}(x), a_j^{i-1}(x)) = a_i^{i-1}(x)$$

for $x \in W$. By (4),

$$a_i^i(x) = q_1(a_i^{i-1}(x), a_i^{i-1}(x)) = a_i^{i-1}(x)$$

(because $q_1(z, z) = z$ by (2)) and

$$a_j^i(x) = q_1(a_i^{i-1}(x), a_j^{i-1}(x)).$$

Thus $a_i^i(x) = a_j^i(x)$ for $x \in W$, as required.

Take a subspace $Z \subseteq X$.

Claim 3. If $a_i =|_Z a_j$, then there exists a neighbourhood $W \subseteq X$ of Z such that $\tilde{a}_i =|_W \tilde{a}_j$.

This follows from the construction of $\tilde{a}_i$ and the claims 1 and 2.

Consider the equivalence

$$R = \{\, (i, j) : a_i =|_Z a_j \,\}$$

on the set $I = \{1, \ldots, n\}$. It follows from the claim 3 that there exists a neighbourhood $V \subseteq X$ of Z such that $\tilde{a}_i =|_V \tilde{a}_j$ for all $(i, j) \in R$. We have the commutative diagram

$$\begin{tikzcd}
Y^Z & I \arrow[l, "a_i|_Z \mapsfrom i : l"'] \arrow[r, "d : i \mapsto \tilde{a}_i|_V"] \arrow[d, "\pi"] & Y^V \\
& I/R, \arrow[ul, "\bar{l}"] \arrow[ur, "\bar{d}"'] &
\end{tikzcd}$$

where π is the projection. The function $\bar{l}$ is injective. Consider the elements $U \in \langle I \rangle$,

$$U = \sum_i u_i {<}i{>},$$

and $\bar{U} = \langle \pi \rangle(U) \in \langle I/R \rangle$. We have

$$A|_Z = \langle l \rangle(U) = \langle \bar{l} \rangle(\bar{U}) \quad \text{and} \quad \tilde{A}|_V = \langle d \rangle(U) = \langle \bar{d} \rangle(\bar{U}).$$

If $A|_Z = 0$, then $\bar{U} = 0$ because $\langle \bar{l} \rangle$ is injective. Then $\tilde{A}|_V = 0$. □

6.2. Corollary. *Consider an ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_i u_i{<}a_i{>},$$

such that $A \overset{r}{=} 0$. Then there exist maps $\tilde{a}_i \in Y^X$, $\tilde{a}_i \sim a_i$, such that the ensemble

$$\tilde{A} = \sum_i u_i{<}\tilde{a}_i{>} \tag{5}$$

satisfies the condition $\tilde{A} \underset{\Gamma}{\overset{r}{=}} 0$ for some open cover Γ of X.

Proof. Since $A \overset{r}{=} 0$, we have $A =|_T 0$ for all $T \in \mathcal{F}_r(X)$. By Lemma 6.1, there are maps $\tilde{a}_i \in Y^X$, $\tilde{a}_i \sim a_i$, such that the ensemble $\tilde{A}$ given by (5) satisfies the condition $\tilde{A} =|_{V(T)} 0$ for some neighbourhood $V(T) \subseteq X$ of each $T \in \mathcal{F}_r(X)$. There is an open cover Γ of X such that every $W \in \Gamma(r)$ is included in $V(T)$ for some $T \in \mathcal{F}_r(X)$. Then $\tilde{A} =|_W 0$ for all $W \in \Gamma(r)$, that is, $\tilde{A} \underset{\Gamma}{\overset{r}{=}} 0$. □

6.3. Lemma. *Consider an ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_i u_i{<}a_i{>},$$

and a map $b \in Y^X$. Then there exist maps $\tilde{a}_i \in Y^X$, $\tilde{a}_i \sim a_i$, such that the ensemble

$$\tilde{A} = \sum_i u_i{<}\tilde{a}_i{>} \tag{6}$$

has the following property: if $A {=}|_Z {<}b{>}$ for some subspace $Z \subseteq X$, then $\tilde{A} {=}|_V {<}b{>}$ for some neighbourhood $V \subseteq X$ of Z.

Proof. Let Π be the set of subspaces $Z \subseteq X$ such that $A {=}|_Z {<}b{>}$. By Lemma 6.1, there are maps $\bar{a}_i, \bar{b} \in Y^X$, $\bar{a}_i \sim a_i$ and $\bar{b} \sim b$, such that the ensemble

$$\bar{A} = \sum_i u_i{<}\bar{a}_i{>}$$

satisfies the condition $\bar{A} =|_{V(Z)} {<}\bar{b}{>}$ for some neighbourhood $V(Z) \subseteq X$ of each $Z \in \Pi$. By Lemma 5.1, there are maps $\tilde{a}_i \in Y^X$, $\tilde{a}_i \sim \bar{a}_i$, such that the ensemble $\tilde{A}$ given by (6) satisfies the condition $\tilde{A} =|_{V(Z)} {<}b{>}$ for all $Z \in \Pi$. □

6.4. Corollary. *Consider an ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_i u_i{<}a_i{>},$$

and a map $b \in Y^X$. Suppose that $A \overset{r}{=} {<}b{>}$. Then there exist maps $\tilde{a}_i \in Y^X$, $\tilde{a}_i \sim a_i$, such that the ensemble

$$\tilde{A} = \sum_i u_i{<}\tilde{a}_i{>} \tag{7}$$

satisfies the condition $\tilde{A} \underset{\Gamma}{\overset{r}{=}} {<}b{>}$ for some open cover Γ of X.

This follows from Lemma 6.3 as Corollary 6.2 does from Lemma 6.1. $\square$

§ 7. Symmetric characterization of similarity

Let X and Y be cellular spaces, X compact.

7.1. Lemma. *Consider a cover Γ of X, an open subspace $G \in \Gamma$, a closed subspace $D \subseteq G$, and maps $a, b_0, b_1 \in Y^X$ such that $a \sim|_G b_0$, $b_0 \sim b_1 \operatorname{rel} X \setminus D$, and $a \overset{r-1}{\underset{\Gamma}{\sim}} b_0$ in the following sense: there is an ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_i u_i \lt a_i \gt,$$

where $a_i \sim a$, such that $A \overset{r-1}{\underset{\Gamma}{\equiv}} \lt b_0 \gt$. Then there exists an ensemble $C \in \langle Y^X \rangle$,

$$C = \sum_k w_k \lt c_k \gt,$$

where $c_k \sim a$, such that $C \overset{r}{=} \lt b_1 \gt - \lt b_0 \gt$.

Proof. There is a homotopy $h_t \in Y^X$, $t \in [0,1]$, such that $h_s = b_s$, $s = 0, 1$, and $h_t =|_{X \setminus D} b_0$. Choose a continuous function $\phi : X \to [0,1]$ such that $\phi|_E = 1$ and $\phi|_{X \setminus F} = 0$ for some subspaces $E, F \subseteq X$, E open, F closed, such that

$$D \subseteq E \subseteq F \subseteq G.$$

Let $p \in Y^G$ be a map such that $p \sim b_0|_G$. Choose a homotopy $K_t(p) \in Y^G$, $t \in [0,1]$, such that $K_0(p) = p$, $K_1(p) = b_0|_G$, and, moreover, $K_t(p) = b_0|_G$ if $p = b_0|_G$. Define a homotopy $L_t(p) \in Y^G$, $t \in [-1,1]$, by the rules

$$L_t(p)(x) = K_{\phi(x)(t+1)}(p)(x), \qquad x \in G,$$

for $t \in [-1, 0]$ and

$$L_t(p)(x) = \begin{cases} h_t(x) & \text{if } x \in E, \\ K_{\phi(x)}(p)(x) & \text{if } x \in G \setminus D \end{cases}$$

for $t \in [0,1]$. We have $L_{-1}(p) = p$, $L_s(p) =|_E b_s$, $s = 0, 1$, $L_0(p) =|_{G \setminus D} L_1(p)$, and $L_t(p) =|_{G \setminus F} p$. Moreover, $L_s(b_0|_G) = b_s|_G$, $s = 0, 1$.

Let $d \in Y^X$ be a map such that $d \sim|_G b_0$. Define a homotopy $l_t(d) \in Y^X$, $t \in [-1,1]$, by the rules $l_t(d) =|_G L_t(d|_G)$ and $l_t(d) =|_{X \setminus F} d$. We have $l_{-1}(d) = d$, $l_s(d) =|_E b_s$, $s = 0, 1$, $l_0(d) =|_{X \setminus D} l_1(d)$, and $l_t(d) =|_{X \setminus F} d$.

Since $a_i \sim a \sim|_G b_0$, the homotopies $l_t(a_i)$ are defined. Put

$$C = \sum_i u_i (\lt l_1(a_i) \gt - \lt l_0(a_i) \gt).$$

We have $l_s(a_i) \sim a_i \sim a$. It remains to show that $C \overset{r}{=} {<}b_1{>} - {<}b_0{>}$. Take $T \in \mathcal{F}_r(X)$. We check that

$$C =|_T {<}b_1{>} - {<}b_0{>}. \tag{8}$$

We are in one of the following three cases.

Case 1: $T \cap D = \{\triangleleft_X\}$. We have $l_0(a_i) =|_T l_1(a_i)$ and $b_0 =|_T b_1$. Thus both the sides of (8) are zero on T.

Case 2: $T \cap F = \{\triangleleft_X, x_*\}$, where $x_* \in E$ and $x_* \neq \triangleleft_X$. Put $Z = T \setminus \{x_*\}$. We have $Z \in \mathcal{F}_{r-1}(X)$ and $Z \cap F = \{\triangleleft_X\}$. Define functions $e_s : Y^Z \to Y^T$, $s = 0, 1$, by the rules $e_s(q)|_Z = q$ and $e_s(q)(x_*) = b_s(x_*)$. We have $e_s(b_0|_Z) = b_s|_T$ and $e_s(a_i|_Z) = l_s(a_i)|_T$. Thus

$$\big({<}b_0{>} - \sum_i u_i{<}a_i{>}\big)\big|_Z \xmapsto{\langle e_s\rangle} \big({<}b_s{>} - \sum_i u_i{<}l_s(a_i){>}\big)\big|_T.$$

Since $A \overset{r-1}{=} {<}b_0{>}$, the expression on the left is zero. Thus the one on the right is also zero, which implies (8).

For a finite space Z, let $\|Z\|$ be the cardinality of $Z \setminus \{\triangleleft\}$.

Case 3: $\|T \cap G\| \geqslant 2$. We have $T = W \cup Z$ for some subspaces $W, Z \subseteq X$ such that $W \cap Z = \{\triangleleft_X\}$, $W \subseteq G$, $Z \cap F = \{\triangleleft_X\}$, and $\|Z\| \leqslant r - 2$. Consider the subspace $M = G \cup Z \subseteq X$. Define functions $f_s : Y^M \to Y^T$, $s = 0, 1$. Take $q \in Y^M$. If $q \sim|_G b_0$, put $f_s(q) =|_W L_s(q|_G)$ and $f_s(q) =|_Z q$. Otherwise, put $f_s(q) = \triangleleft_Y^T$. We have $f_s(b_0|_M) = b_s|T$ and $f_s(a_i|_M) = l_s(a_i)|_T$. Thus

$$\big({<}b_0{>} - \sum_i u_i{<}a_i{>}\big)\big|_M \xmapsto{\langle f_s\rangle} \big({<}b_s{>} - \sum_i u_i{<}l_s(a_i){>}\big)\big|_T.$$

Since M is included in some element of $\Gamma(r-1)$ and $A \overset{r-1}{\underset{\Gamma}{=}} {<}b_0{>}$, the expression on the left is zero. Thus the one on the right is also zero, which implies (8). $\square$

7.2. Lemma. *Let $a, b, \tilde{b} \in Y^X$ be maps such that $a \overset{r-1}{\sim} b \sim \tilde{b}$ and (*) $a \sim|_S b$ for any $S \in \mathcal{F}_1(X)$. Then there exists an ensemble $C \in \langle Y^X\rangle$,*

$$C = \sum_k w_k{<}c_k{>},$$

where $c_k \sim a$, such that $C \overset{r}{=} {<}\tilde{b}{>} - {<}b{>}$.

The condition (*) is satisfied automatically if X or Y is 0-connected. It also follows from the condition $a \overset{r-1}{\sim} b$ if $r \geqslant 2$ (cf. the proof of Theorem 7.3).

Proof. There is an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<} a_i {>},$$

whre $a_i \sim a$, such that $A \overset{r-1}{=} {<}b{>}$. Using Corollary 6.4, replace each a_i by a homotopic map to get $A \underset{\Gamma}{\overset{r-1}{\equiv}} {<}b{>}$ for some open cover Γ of X.

Call a subspace $G \subseteq X$ *primitive* if the map $\mathrm{in} : G \to X$ is homotopic to the composition

$$G \xrightarrow{f} S \xrightarrow{\mathrm{in}} X$$

for some subspace $S \in \mathcal{F}_1(X)$ and map f. Since X is Hausdorff and locally contractible, for any open subspace $U \subseteq X$ and point $x \in U$, there exists a primitive open subspace $G \subseteq X$ such that $x \in G$ and $G \subseteq U$. We replace the cover Γ by its refinement consisting of primitive open subspaces. Then it follows from (*) that $a \sim|_G\, b$ for each $G \in \Gamma$.

Choose a finite partition of unity subordinate to Γ:

$$\sum_{j=1}^{m} \phi_j = 1,$$

where each $\phi_j : X \to [0,1]$ is a continuous function such that $\phi_j|_{X \setminus D_j} = 0$ for some closed subspace $D_j \subseteq X$ such that $D_j \subseteq G_j$ for some $G_j \in \Gamma$. Choose a homotopy $h_t \in Y^X$, $t \in [0,1]$, such that $h_0 = b$ and $h_1 = \tilde{b}$. Define maps $b_j \in Y^X$, $0 \leqslant j \leqslant m$, by the rule

$$b_j(x) = h_{\phi_1(x)+\ldots+\phi_j(x)}(x).$$

We have $b_0 = b$, $b_m = \tilde{b}$, and $b_{j-1} \sim b_j \operatorname{rel} X \setminus D_j$.

Take $j \geqslant 1$. Applying Lemma 5.1 to the congruence $A \underset{\Gamma}{\overset{r-1}{\equiv}} {<}b{>}$ and the homotopy $b \sim b_{j-1}$, we get an ensemble $A_j \in \langle Y^X \rangle$,

$$A_j = \sum_i u_i {<} a_{ji} {>},$$

where $a_{ji} \sim a_i$ ($\sim a$), such that $A_j \underset{\Gamma}{\overset{r-1}{\equiv}} {<}b_{j-1}{>}$. We have $a \sim|_{G_j}\, b \sim b_{j-1}$. By Lemma 7.1, there is an ensemble $C_j \in \langle Y^X \rangle$,

$$C_j = \sum_k w_{jk} {<} c_{jk} {>},$$

where $c_{jk} \sim a$, such that $C_j \overset{r}{=} {<}b_j{>} - {<}b_{j-1}{>}$.

We get

$$\sum_{j=1}^{m} C_j = {<}b_m{>} - {<}b_0{>} = {<}\tilde{b}{>} - {<}b{>}. \qquad \square$$

7.3. Theorem. *Consider maps $a, b \in Y^X$ and ensembles $A, B \in \langle Y^X \rangle$,*

$$A = \sum_i u_i \langle a_i \rangle \quad \text{and} \quad B = \sum_j v_j \langle b_j \rangle,$$

where

$$\sum_i u_i = \sum_j v_j = 1,$$

$a_i \sim a$, and $b_j \sim b$, such that $A \overset{r}{=} B$. Then $a \overset{r}{\sim} b$.

Proof. Induction on r. If $r \leqslant 0$, the assertion is trivial. Suppose $r \geqslant 1$.

For $S \in \mathcal{F}_1(X)$, we have $a \sim|_S b$ because

$$\langle [a|_S] \rangle = \sum_i u_i \langle [a_i|_S] \rangle = [\![A|_S]\!] = [\![B|_S]\!] = \sum_j v_j \langle [b_j|_S] \rangle = \langle [b|_S] \rangle$$

in $\langle [S, Y] \rangle$. Here $[\![?]\!] : \langle Y^S \rangle \to \langle [S, Y] \rangle$ is the homomorphism induced by the projection $[?] : Y^S \to [S, Y]$.

By induction hypothesis, $a \overset{r-1}{\sim} b$. Take j. Since $b \sim b_j$, Lemma 7.2 gives an ensemble $C_j \in \langle Y^X \rangle$,

$$C_j = \sum_k w_{jk} \langle c_{jk} \rangle,$$

where $c_{jk} \sim a$, such that $C_j \overset{r}{=} \langle b_j \rangle - \langle b \rangle$. We have

$$A - \sum_j v_j C_j \overset{r}{=} A - \sum_j v_j (\langle b_j \rangle - \langle b \rangle) = A - B + \langle b \rangle \overset{r}{=} \langle b \rangle,$$

which proves the assertion. □

§ 8. Similarity is an equivalence

Let X and Y be cellular spaces, X compact.

8.1. Theorem. *The relation $\overset{r}{\sim}$ on Y^X is an equivalence.*

This was conjectured by A. V. Malyutin.

Proof. Reflexivity is trivial. Symmetry follows from Theorem 7.3. It remains to prove transitivity.

Let maps $a, b, c \in Y^X$ satisfy $a \overset{r}{\sim} b \overset{r}{\sim} c$. There are ensembles $A, B \in \langle Y^X \rangle$,

$$A = \sum_i u_i \langle a_i \rangle, \quad \text{and} \quad B = \sum_j v_j \langle b_j \rangle,$$

where $a_i \sim a$ and $b_j \sim b$, such that $A \overset{r}{=} \langle b \rangle$ and $B \overset{r}{=} \langle c \rangle$. For each j, we have $b \sim b_j$ and, by Lemma 5.1, there is an ensemble $A_j \in \langle Y^X \rangle$,

$$A_j = \sum_i u_i \langle a_{ji} \rangle,$$

where $a_{ji} \sim a_i$ ($\sim a$), such that $A_j \overset{r}{=} {<}b_j{>}$. We have

$$\sum_j v_j A_j \overset{r}{=} \sum_j v_j {<}b_j{>} = B \overset{r}{=} {<}c{>}.$$

Thus $a \overset{r}{\sim} c$. $\square$

Using Theorem 5.2, we introduce the relation of r-similarity on $[X, Y]$:

$$[a] \overset{r}{\sim} [b] \quad \Leftrightarrow \quad a \overset{r}{\sim} b.$$

It follows from Theorem 8.1 that it is an equivalence.

§ 9. The Hopf invariant

Let X and Y be spaces. Let $e \in C^m(Y)$ and $f \in C^n(Y)$ $(m, n \geqslant 1)$ be (singular) cocycles and $g \in C^{m+n-1}(Y)$ be a cochain with $\delta g = ef$. Put

$$[X, Y]_{e,f} = \{\, \boldsymbol{a} : \boldsymbol{a}^*([e]) = 0 \text{ and } \boldsymbol{a}^*([f]) = 0 \text{ in } H^\bullet(X) \,\} \subseteq [X, Y]$$

and

$$Y^X_{e,f} = \{\, a : [a] \in [X, Y]_{e,f} \,\} \subseteq Y^X.$$

Given $a \in Y^X_{e,f}$, choose a cochain $p \in C^{m-1}(X)$ such that $\delta p = a^\#(e)$ and put

$$q = pa^\#(f) - a^\#(g) \in C^{m+n-1}(X).$$

Then $\delta q = 0$ and the class $[q] \in H^{m+n-1}(X)$ neither depends on the choice of p nor changes if a is replaced by a homotopic map. Putting $h([a]) = [q]$, we get the function

$$h : [X, Y]_{e,f} \to H^{m+n-1}(X),$$

which we call the *Hopf invariant* [7].

9.1. Lemma. *Let X_0 be a space and $t : X \to X_0$ be a map. We have the Hopf invariants*

$$h_0 : [X_0, Y]_{e,f} \to H^{m+n-1}(X_0) \quad \textit{and} \quad h : [X, Y]_{e,f} \to H^{m+n-1}(X).$$

Given $a_0 \in Y^{X_0}$, put $a = a_0 \circ t \in Y^X$. If $a_0 \in Y^{X_0}_{e,f}$, then $a \in Y^X_{e,f}$ and $h([a]) = t^*(h_0([a_0]))$ in $H^{m+n-1}(X)$. $\square$

9.2. Lemma. *Take elements $\boldsymbol{u} \in \pi_m(Y)$ and $\boldsymbol{v} \in \pi_n(Y)$. Put*

$$\Delta = \langle \boldsymbol{u}^*([e]), [S^m] \rangle \langle \boldsymbol{v}^*([f]), [S^n] \rangle + (-1)^{mn} \langle \boldsymbol{u}^*([f]), [S^m] \rangle \langle \boldsymbol{v}^*([e]), [S^n] \rangle \in \mathbf{Z}$$

(the last two Kronecker indices vanish unless $m = n$). Consider the Hopf invariant

$$h : [S^{m+n-1}, Y]_{e,f} \to H^{m+n-1}(S^{m+n-1})$$

and the Whitehead product $\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil \in \pi_{m+n-1}(Y) = [S^{m+n-1}, Y]$. Then $\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil \in [S^{m+n-1}, Y]_{e,f}$ and

$$\langle h(\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil), [S^{m+n-1}] \rangle = (-1)^{mn+m+n} \Delta.$$

Caution: the sign in the last equality is sensitive to certain conventions.

Proof (after [7, § 19]). We assume that $S^m \vee S^n \subseteq S^m \times S^n$ in the standard way. We have the commutative diagram

$$\begin{tikzcd}
S^{m+n-1} \arrow[r, "\phi"] \arrow[d, "\mathrm{in}"'] & S^m \vee S^n \arrow[d, "\mathrm{in}"] \\
D^{m+n} \arrow[r, "\chi"] & S^m \times S^n,
\end{tikzcd}$$

where $[\phi] = \lfloor [\mathrm{in}_1], [\mathrm{in}_2] \rceil$ in $\pi_{m+n-1}(S^m \vee S^n)$. We have the chain of homomorphisms and sendings

$$\begin{tikzcd}
H_{m+n-1}(S^{m+n-1}) & {[S^{m+n-1}]} \\
H_{m+n}(D^{m+n}, S^{m+n-1}) \arrow[u, "\partial"] \arrow[d, "(\chi,\phi)_*"'] & {[D^{m+n}]} \arrow[u, mapsto] \arrow[d, mapsto] \\
H_{m+n}(S^m \times S^n, S^m \vee S^n) & \mathrm{rel}_*([S^m \times S^n]) \\
H_{m+n}(S^m \times S^n). \arrow[u, "\mathrm{rel}_*"] & {[S^m \times S^n]} \arrow[u, mapsto]
\end{tikzcd} \tag{9}$$

Choose representatives $u : S^m \to Y$ and $v : S^n \to Y$ of $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. Consider the maps

$$a : S^{m+n-1} \xrightarrow{\phi} S^m \vee S^n \xrightarrow{w = u \underline{\vee} v} Y.$$

Clearly, $[a] = \lfloor \boldsymbol{u}, \boldsymbol{v} \rceil$ in $\pi_{m+n-1}(Y)$.

Choose cocycles $\hat{e} \in C^m(S^m \times S^n)$ and $\hat{f} \in C^n(S^m \times S^n)$ and a cochain $\hat{g} \in C^{m+n-1}(S^m \times S^n)$ such that

$$\hat{e}|_{S^m \vee S^n} = w^{\#}(e), \qquad \hat{f}|_{S^m \vee S^n} = w^{\#}(f), \quad \text{and} \quad \hat{g}|_{S^m \vee S^n} = w^{\#}(g).$$

We have

$$a^{\#}(e) = \phi^{\#}(w^{\#}(e)) = \phi^{\#}(\hat{e}|_{S^m \vee S^n}) = \chi^{\#}(\hat{e})|_{S^{m+n-1}}$$

in $C^m(S^{m+n-1})$. It follows that $a^*([e]) = 0$ in $H^m(S^{m+n-1})$ (which is automatic unless $n = 1$). Similarly, $a^*([f]) = 0$ in $H^n(S^{m+n-1})$. Thus $[a] \in [S^{m+n-1}, Y]_{e,f}$.

Let $z_k \in H^k(S^k)$ be the class with $\langle z_k, [S^k] \rangle = 1$. One easily sees that

$$[\hat{e}] = \langle \boldsymbol{u}^*([e]), [S^m] \rangle (z_m \times 1) + \langle \boldsymbol{v}^*([e]), [S^n] \rangle (1 \times z_n)$$

in $H^m(S^m \times S^n)$ and

$$[\hat{f}] = \langle \boldsymbol{v}^*([f]), [S^n]\rangle(1 \times z_n) + \langle \boldsymbol{u}^*([f]), [S^m]\rangle(z_m \times 1)$$

in $H^n(S^m \times S^n)$. Thus $[\hat{e}][\hat{f}] = \Delta(z_m \times z_n)$ in $H^{m+n}(S^m \times S^n)$ and

$$\langle [\hat{e}][\hat{f}], [S^m \times S^n]\rangle = (-1)^{mn}\Delta. \tag{10}$$

Choose a cochain $\tilde{p} \in C^{m-1}(D^{m+n})$ such that $\delta\tilde{p} = \chi^{\#}(\hat{e})$. Put

$$\tilde{q} = \tilde{p}\chi^{\#}(\hat{f}) - \chi^{\#}(\hat{g}) \in C^{m+n-1}(D^{m+n}).$$

Put

$$p = \tilde{p}|_{S^{m+n-1}} \in C^{m-1}(S^{m+n-1}) \quad \text{and} \quad q = \tilde{q}|_{S^{m+n-1}} \in C^{m+n-1}(S^{m+n-1}).$$

We have

$$\delta p = \delta\tilde{p}|_{S^{m+n-1}} = \chi^{\#}(\hat{e})|_{S^{m+n-1}} = \phi^{\#}(\hat{e}|_{S^m \vee S^n}) = \phi^{\#}(w^{\#}(e)) = a^{\#}(e)$$

and

$$\begin{aligned} q = p\chi^{\#}(\hat{f})|_{S^{m+n-1}} - \chi^{\#}(\hat{g})|_{S^{m+n-1}} = p\phi^{\#}(\hat{f}|_{S^m \vee S^n}) - \phi^{\#}(\hat{g}|_{S^m \vee S^n}) = \\ = p\phi^{\#}(w^{\#}(f)) - \phi^{\#}(w^{\#}(g)) = pa^{\#}(f) - a^{\#}(g). \end{aligned}$$

Thus $\delta q = 0$ and $h([a]) = [q]$.

We have

$$\delta\tilde{q} = \chi^{\#}(\hat{e})\chi^{\#}(\hat{f}) - \delta\chi^{\#}(\hat{g}) = \chi^{\#}(\hat{e}\hat{f} - \delta\hat{g}).$$

We have the chain of homomorphisms and sendings

$$\begin{tikzcd}
H^{m+n-1}(S^{m+n-1}) \arrow[d, "\delta"'] & {[q]} \arrow[d, mapsto] \\
H^{m+n}(D^{m+n}, S^{m+n-1}) & {[\chi^{\#}(\hat{e}\hat{f} - \delta\hat{g})]} \\
H^{m+n}(S^m \times S^n, S^m \vee S^n) \arrow[u, "(\chi,\phi)^*"] \arrow[d, "\mathrm{rel}^*"'] & {[\hat{e}\hat{f} - \delta\hat{g}]} \arrow[u, mapsto] \arrow[d, mapsto] \\
H^{m+n}(S^m \times S^n). & {[\hat{e}][\hat{f}]}
\end{tikzcd}$$

Collating it with (9) and using (10), we get

$$\langle [q], [S^{m+n-1}]\rangle = (-1)^{m+n}\langle [\hat{e}][\hat{f}], [S^m \times S^n]\rangle = (-1)^{mn+m+n}\Delta.$$

This is what we need because $h(\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil) = h([a]) = [q]$. $\square$

Let Γ be an open cover of X. Consider the differential graded ring $C^\bullet(\Gamma)$ of Γ-cochains of X (that is, functions on the set of singular simplices subordinate to Γ). The projection

$$?|_\Gamma : C^\bullet(X) \to C^\bullet(\Gamma)$$

is a morphism of differential graded rings; it induces an isomorphism of cohomology rings,

$$?|_\Gamma : H^\bullet(X) \to H^\bullet(\Gamma).$$

9.3. Lemma. *Given $a \in Y^X_{e,f}$, choose $\tilde{p} \in C^{m-1}(\Gamma)$ such that $\delta\tilde{p} = a^\#(e)|_\Gamma$ and put*

$$\tilde{q} = \tilde{p}a^\#(f)|_\Gamma - a^\#(g)|_\Gamma \in C^{m+n-1}(\Gamma).$$

Then $\delta\tilde{q} = 0$ and $h([a])|_\Gamma = [\tilde{q}]$ in $H^{m+n-1}(\Gamma)$. $\square$

We suppose that X and Y are cellular spaces and X is compact.

9.4. Theorem. *Consider an ensemble $A \in \langle Y^X \rangle$,*

$$A = \sum_i u_i{<}a_i{>},$$

where $a_i \in Y^X_{e,f}$, such that $A \overset{2}{=} 0$. Then

$$\sum_i u_i h([a_i]) = 0$$

in $H^{m+n-1}(X)$.

Thus h may be called a *partial* invariant of degree at most 2.

Proof. Using Corollary 6.2, replace a_i by homotopic maps so that $A \underset{\Gamma}{\overset{2}{=}} 0$ for some open cover Γ of X.

Let $B \subseteq C^m(\Gamma)$ be the subgroup generated by the coboundaries $a_i^\#(e)|_\Gamma$. It is free because finitely generated and torsion-free. Thus there is a homomorphiam $P : B \to C^{m-1}(\Gamma)$ such that $\delta P(b) = b$, $b \in B$. Put

$$\tilde{q}_i = P(a^\#(e)|_\Gamma)a^\#(f)|_\Gamma - a^\#(g)|_\Gamma \in C^{m+n-1}(\Gamma).$$

By Lemma 9.3, $\delta\tilde{q}_i = 0$ and

$$h([a_i])|_\Gamma = [\tilde{q}_i]$$

in $H^{m+n-1}(\Gamma)$.

Take a singular simplex $\sigma : \Delta^{m+n-1} \to G$, $G \in \Gamma$. Let

$$\sigma' : \Delta^{m-1} \to G \quad \text{and} \quad \sigma'' : \Delta^n \to G$$

be its front and back faces, respectively.

The group $\mathrm{Hom}(B, \mathbf{Q})$ is formed by homomorphisms $\langle ?, T\rangle$, where T runs over $C_m(\Gamma; \mathbf{Q})$, the group of rational Γ-chains in X. Thus there is a chain $T \in C_m(\Gamma; \mathbf{Q})$ such that

$$\langle P(b), \sigma'\rangle = \langle b, T\rangle, \qquad b \in B.$$

We have

$$T = \sum_k c_k \tau_k,$$

where $c_k \in \mathbf{Q}$ and $\tau_k : \Delta^m \to G_k$, $G_k \in \Gamma$. Thus

$$\langle P(a_i^{\#}(e)|_\Gamma), \sigma'\rangle = \langle a_i^{\#}(e)|_\Gamma, T\rangle = \sum_k c_k \langle a_i^{\#}(e)|_\Gamma, \tau_k\rangle.$$

We get

$$\langle \tilde{q}_i, \sigma\rangle = (-1)^{(m-1)n} \langle P(a_i^{\#}(e)|_\Gamma), \sigma'\rangle \langle a_i^{\#}(f)|_\Gamma, \sigma''\rangle - \langle a_i^{\#}(g)|_\Gamma, \sigma\rangle =$$
$$= (-1)^{(m-1)n} \sum_k c_k \langle a_i^{\#}(e)|_\Gamma, \tau_k\rangle \langle a_i^{\#}(f)|_\Gamma, \sigma''\rangle - \langle a_i^{\#}(g)|_\Gamma, \sigma\rangle =$$
$$= (-1)^{(m-1)n} \sum_k c_k \langle (a_i|_{G\cup G_k})^{\#}(e), \tau_k\rangle \langle (a_i|_{G\cup G_k})^{\#}(f), \sigma''\rangle - \langle (a_i|_G)^{\#}(g), \sigma\rangle.$$

We have found functions $R_k : Y^{G\cup G_k} \to \mathbf{Q}$ and $S : Y^G \to \mathbf{Q}$ such that

$$\langle \tilde{q}_i, \sigma\rangle = \sum_k R_k(a_i|_{G\cup G_k}) - S(a_i|_G)$$

for all i. Since $A \underset{\Gamma}{\overset{2}{=}} 0$, we have $A|_{G\cup G_k} = 0$ and $A|_G = 0$. Thus

$$\sum_i u_i \langle \tilde{q}_i, \sigma\rangle = 0.$$

Since σ was taken arbitrarily, we have

$$\sum_i u_i \tilde{q}_i = 0.$$

We get

$$\sum_i u_i h([a_i])|_\Gamma = \sum_i u_i [\tilde{q}_i] = 0.$$

Since restriction to Γ here is an isomorphism, we get

$$\sum_i u_i h([a_i]) = 0. \qquad \square$$

9.5. Corollary. *Let $a, b \in Y^X_{e,f}$ satisfy $a \overset{2}{\sim} b$. Then $h([a]) = h([b])$.*

Proof. There is an ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_i u_i {<} a_i {>},$$

where $a_i \sim a$, such that $A \overset{2}{=} {<}b{>}$. Since $A =|_{\{\triangleleft\}} {<}b{>}$, we have

$$\sum_i u_i = 1.$$

By Theorem 9.4,

$$\sum_i u_i h([a_i]) = h([b]).$$

Since $[a_i] = [a]$, we get $h([a]) = h([b])$. $\square$

§ 10. Maps of $S^p \times S^n$

This section does not depend of the rest of the paper. We recall a theorem of G. W. Whitehead about the fibration of free spheroids (Theorem 10.1) and deduce Lemma 10.3 about certain maps $S^p \times S^n \to Y$ (we need it in § 11).

We fix numbers $p, n \geqslant 1$ and a space Y. Let $\Omega^n Y$ be the space of maps $S^n \to Y$, as usual. Let

$$\epsilon : S^p \times S^n \to S^p \wedge S^n \to S^{p+n}$$

be the composition of the projection and the standard homeomorphism. For a map $w : S^{p+n} \to Y$, introduce the map

$$\nabla^n(w) : S^p \to \Omega^n Y, \qquad \nabla^n(w)(t)(z) = w(\epsilon(t, z)).$$

Introduce the isomorphism

$$\boldsymbol{\nabla}^n : \pi_{p+n}(Y) \to \pi_p(\Omega^n Y), \qquad [w] \mapsto [\nabla^n(w)].$$

Let

$$\mu : S^n \to S^n \vee S^n$$

be the standard comultiplication. Consider the usual multiplication

$$\Omega^n Y \times \Omega^n Y \xrightarrow{\#} \Omega^n Y, \qquad v_1 \# v_2 : S^n \xrightarrow{\mu} S^n \vee S^n \xrightarrow{v_1 \underline{\vee} v_2} Y.$$

For a map $v : S^n \to Y$, introduce the map

$$\tau_v : \Omega^n Y \xrightarrow{v\#?} (\Omega^n Y, v\#\triangleleft),$$

where the target is $\Omega^n Y$ with the specified new basepoint. It induces the isomorphism

$$\tau_{v\,*} : \pi_p(\Omega^n Y) \to \pi_p(\Omega^n Y, v\#\triangleleft).$$

Let $\Lambda^n Y$ be the space of *unbased* maps $S^n \to Y$. Consider the fibration

$$\rho : \Lambda^n Y \to Y, \qquad v \mapsto v(\triangleleft).$$

We have $\rho^{-1}(\triangleleft) = \Omega^n(Y)$.

10.1. Theorem (G. W. Whitehead). *For a map $v : S^n \to Y$, the composition*

$$\mathbf{\Gamma} : \pi_{p+1}(Y) \xrightarrow{\lfloor ?, [v] \rceil} \pi_{p+n}(Y) \xrightarrow{\boldsymbol{\nabla}^n} \pi_p(\Omega^n Y) \xrightarrow{\tau_{v\,*}} \pi_p(\Omega^n Y, v\#\triangleleft)$$

coincides up to a sign with the connecting homomorphism of the fibration ρ at the point $v\#\triangleleft \in \Omega^n Y$. Consequently, the composition

$$\pi_{p+1}(Y) \xrightarrow{\mathbf{\Gamma}} \pi_p(\Omega^n Y, v\#\triangleleft) \xrightarrow{\mathrm{in}_*} \pi_p(\Lambda^n Y, v\#\triangleleft)$$

is zero.

See [8, Theorem (3.2)] and [9, § 3]. □

For a map $v : S^n \to Y$, introduce the homomorphism

$$\mathbf{\Psi}_v : \pi_{p+n}(Y) \xrightarrow{\boldsymbol{\nabla}^n} \pi_p(\Omega^n Y) \xrightarrow{\tau_{v\,*}} \pi_p(\Omega^n Y, v\#\triangleleft) \xrightarrow{\mathrm{in}_*} \pi_p(\Lambda^n Y, v\#\triangleleft).$$

By Theorem 10.1,

$$\mathbf{\Psi}_v(\lfloor \boldsymbol{u}, [v] \rceil) = 0, \qquad \boldsymbol{u} \in \pi_{p+1}(Y). \tag{11}$$

For maps $v : S^n \to Y$ and $w : S^{p+n} \to Y$, introduce the map

$$\Psi_v(w) : S^p \xrightarrow{\nabla^n(w)} \Omega^n Y \xrightarrow{\tau_v} (\Omega^n Y, v\#\triangleleft) \xrightarrow{\mathrm{in}} (\Lambda^n Y, v\#\triangleleft).$$

Clearly,

$$[\Psi_v(w)] = \mathbf{\Psi}_v([w])$$

in $\pi_p(\Lambda^n Y, v\#\triangleleft)$.

Introduce the map

$$\Phi : S^p \times S^n \xrightarrow{\mathrm{id} \times \mu} S^p \times (S^n \vee S^n) \xrightarrow{\theta} S^n \vee S^{p+n}, \tag{12}$$

where

$$\theta : \quad (t, \mathrm{in}_1(z)) \mapsto \mathrm{in}_1(z), \quad (t, \mathrm{in}_2(z)) \mapsto \mathrm{in}_2(\epsilon(t,z)), \qquad t \in S^p, \quad z \in S^n.$$

For maps $v : S^n \to Y$ and $w : S^{p+n} \to Y$, introduce the map

$$\Xi(v, w) : S^p \times S^n \xrightarrow{\Phi} S^n \vee S^{p+n} \xrightarrow{v \overline{\underline{\vee}} w} Y. \tag{13}$$

For elements $\boldsymbol{v} \in \pi_n(Y)$ and $\boldsymbol{w} \in \pi_{p+n}(Y)$, put

$$\mathbf{\Xi}(\boldsymbol{v}, \boldsymbol{w}) = [\Xi(v, w)] \in [S^p \times S^n, Y], \tag{14}$$

where v and w are representatives of $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively.

For maps $v_0 : S^n \to Y$ and $V : S^p \to (\Lambda^n Y, v_0)$, introduce the map

$$V^\times : S^p \times S^n \to Y, \qquad (t, z) \mapsto V(t)(z).$$

For $\boldsymbol{V} \in \pi_p(\Lambda^n Y, v_0)$, put

$$\boldsymbol{V}^\times = [V^\times] \in [S^p \times S^n, Y],$$

where V is a representative of $\boldsymbol{V}$.

10.2. Lemma. *For maps $v : S^n \to Y$ and $w : S^{p+n} \to Y$, one has*

$$\Xi(v, w) = \Psi_v(w)^\times : S^p \times S^n \to Y.$$

Consequently,

$$\boldsymbol{\Xi}([v], [w]) = \boldsymbol{\Psi}_v([w])^\times$$

in $[S^p \times S^n, Y]$.

Proof. Take a point $(t, z) \in S^p \times S^n$. We have $\mu(z) = \mathrm{in}_k(\tilde{z})$ in $S^n \vee S^n$ for some $k \in \{1, 2\}$ and $\tilde{z} \in S^n$. We have

$$\begin{aligned} \theta(t, \mu(z)) = \theta(t, \mathrm{in}_k(\tilde{z})) = \qquad & (\text{if } k = 1) \qquad = \mathrm{in}_1(\tilde{z}), \\ & (\text{if } k = 2) \qquad = \mathrm{in}_2(\epsilon(t, \tilde{z})) \end{aligned}$$

in $S^n \vee S^{p+n}$. Thus

$$\begin{aligned} &\Xi(v, w)(t, z) = ((v \mathbin{\underline{\vee}} w) \circ \Phi)(t, z) = \\ &\qquad = ((v \mathbin{\underline{\vee}} w) \circ \theta \circ (\mathrm{id} \times \mu))(t, z) = (v \mathbin{\underline{\vee}} w)(\theta(t, \mu(z))) = \\ &\qquad\qquad (\text{if } k = 1) \qquad = (v \mathbin{\underline{\vee}} w)(\mathrm{in}_1(\tilde{z})) = v(\tilde{z}), \\ &\qquad\qquad (\text{if } k = 2) \qquad = (v \mathbin{\underline{\vee}} w)(\mathrm{in}_2(\epsilon(t, \tilde{z}))) = w(\epsilon(t, \tilde{z})). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\Psi_v(w)^\times(t, z) = \Psi_v(w)(t)(z) = \tau_v(\nabla^n(w)(t))(z) = \\ &\quad = (v \# \nabla^n(w)(t))(z) = (v \mathbin{\underline{\vee}} \nabla^n(w)(t))(\mu(z)) = (v \mathbin{\underline{\vee}} \nabla^n(w)(t))(\mathrm{in}_k(\tilde{z})) = \\ &\qquad\qquad (\text{if } k = 1) \qquad = v(\tilde{z}), \\ &\qquad\qquad (\text{if } k = 2) \qquad = \nabla^n(w)(t)(\tilde{z}) = w(\epsilon(t, \tilde{z})). \end{aligned}$$

The same. $\square$

10.3. Lemma. *For elements $\boldsymbol{u} \in \pi_{p+1}(Y)$, $\boldsymbol{v} \in \pi_n(Y)$, and $\boldsymbol{w} \in \pi_{p+n}(Y)$, one has*

$$\boldsymbol{\Xi}(\boldsymbol{v}, \lfloor \boldsymbol{u}, \boldsymbol{v} \rceil + \boldsymbol{w}) = \boldsymbol{\Xi}(\boldsymbol{v}, \boldsymbol{w})$$

in $[S^p \times S^n, Y]$.

Proof. Choose a representative $v : S^n \to Y$ of $\boldsymbol{v}$. By (11),

$$\boldsymbol{\Psi}_v(\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil + \boldsymbol{w}) = \boldsymbol{\Psi}_v(\boldsymbol{w})$$

in $\pi_p(\Lambda^n Y, v \# \urcorner)$. Applying Lemma 10.2 yields the desired equality. $\square$

For a map $w : S^{p+n} \to Y$, introduce the map

$$\xi(w) : S^p \times S^n \xrightarrow{\epsilon} S^{p+n} \xrightarrow{w} Y.$$

For an element $\boldsymbol{w} \in \pi_{p+n}(Y)$, put

$$\boldsymbol{\xi}(\boldsymbol{w}) = [\xi(w)] \in [S^p \times S^n, Y], \tag{15}$$

where w is a representative of $\boldsymbol{w}$.

10.4. Lemma. *For en element $\boldsymbol{w} \in \pi_{p+n}(Y)$, one has*

$$\boldsymbol{\Xi}(0, \boldsymbol{w}) = \boldsymbol{\xi}(\boldsymbol{w})$$

in $[S^p \times S^n, Y]$.

Proof. Choose a representative $w : S^{p+n} \to Y$ of $\boldsymbol{w}$. Consider the diagram

$$\begin{tikzcd}
S^p \times (S^n \vee S^n) \arrow[rrr, "\theta"] \arrow[dd, "\mathrm{id}\times(\triangleleft\underline{\vee}\mathrm{id})"'] & & & S^n \vee S^{p+n} \arrow[dl, "\triangleleft\underline{\vee}w"] \arrow[dd, "\triangleleft\underline{\vee}\mathrm{id}"] \\
& S^p \times S^n \arrow[ul, "\mathrm{id}\times\mu"'] \arrow[urr, "\Phi"] \arrow[r, bend left, "\Xi(\triangleleft,w)"] \arrow[r, bend right, "\xi(w)"'] \arrow[dl, "\mathrm{id}"] \arrow[drr, "\epsilon"] & Y & \\
S^p \times S^n \arrow[rrr, "\epsilon"'] & & & S^{p+n}. \arrow[ul, "w"']
\end{tikzcd}$$

Since the map

$$S^n \xrightarrow{\mu} S^n \vee S^n \xrightarrow{\triangleleft\underline{\vee}\mathrm{id}} S^n$$

is homotopic to the identity, the left triangle is homotopy commutative. The other empty triangles and the square are commutative. It follows that the parallel curved arrows are homotopic. $\square$

§ 11. Fineness of 2-similarity

Put $X = S^p \times S^n$ ($p \geqslant 1$, $n \geqslant 2$). Let Y be a space with elements $\boldsymbol{u} \in \pi_{p+1}(Y)$ and $\boldsymbol{v} \in \pi_n(Y)$. Consider the Whitehead product $\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil \in \pi_{p+n}(Y)$ and the homotopy classes

$$\boldsymbol{k}(t) = \boldsymbol{\xi}(t\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil) \in [X, Y], \qquad t \in \mathbf{Z}$$

(see (15)).

11.1. Lemma. *Let L be an abelian group and $f : [X, Y] \to L$ be an invariant of degree at most r. Then*

$$f(\boldsymbol{k}(r! + t)) = f(\boldsymbol{k}(t)), \qquad t \in \mathbf{Z}.$$

Proof (after [5, Lemma 1.5]). We will use the homotopy classes

$$\boldsymbol{K}(s, t) = \boldsymbol{\Xi}(s\boldsymbol{v}, t\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil) \in [X, Y], \qquad s, t \in \mathbf{Z}$$

(see (14)). By Lemma 10.4,

$$\boldsymbol{K}(0,t) = \boldsymbol{k}(t). \tag{16}$$

We have

$$\boldsymbol{K}(s, m+t) = \boldsymbol{K}(s,t) \qquad \text{if } s \mid m \tag{17}$$

because

$$\begin{aligned}\Xi(s\boldsymbol{v}, (m+t)\lfloor \boldsymbol{u}, \boldsymbol{v}\rceil) = \Xi(s\boldsymbol{v}, \lfloor (m/s)\boldsymbol{u}, s\boldsymbol{v}\rceil + t\lfloor \boldsymbol{u}, \boldsymbol{v}\rceil) = \\ \text{(by Lemma 10.3)} \qquad = \Xi(s\boldsymbol{v}, t\lfloor \boldsymbol{u}, \boldsymbol{v}\rceil).\end{aligned}$$

Consider the wedge of r copies of S^n and two copies of S^{p+n}

$$W = S^n \vee \ldots \vee S^n \vee S^{p+n} \vee S^{p+n}$$

and the maps

$$\Lambda(d) = \lambda_1(d_1) \vee \ldots \lambda_r(d_r) \vee \lambda_{r+1}(d_{r+1}) \vee \mathrm{id} : W \to W,$$

$d = (d_1, \ldots, d_{r+1}) \in \boldsymbol{\mathcal{E}}^{r+1}$, as in § 3. Put

$$\mu = \mu_1 \vee \mu_2 : S^n \vee S^{p+n} \to W,$$

where

$$\mu_1 : S^n \to S^n \vee \ldots \vee S^n \quad \text{and} \quad \mu_2 : S^{p+n} \to S^{p+n} \vee S^{p+n}$$

are the comultiplications. Choose a map $q : W \to Y$ with

$$[q] = \boldsymbol{v} \,\underline{\vee} \ldots \underline{\vee}\, \boldsymbol{v} \,\underline{\vee}\, r!\lfloor \boldsymbol{u}, \boldsymbol{v}\rceil \,\underline{\vee}\, t\lfloor \boldsymbol{u}, \boldsymbol{v}\rceil.$$

Consider the ensemble $A \in \langle Y^X \rangle$,

$$A = \sum_{d \in \boldsymbol{\mathcal{E}}^{r+1}} (-1)^{|d|} \langle a(d) \rangle,$$

where

$$a(d) : X \xrightarrow{\Phi} S^n \vee S^{p+n} \xrightarrow{\mu} W \xrightarrow{\Lambda(d)} W \xrightarrow{q} Y,$$

where Φ is as in (13). By Lemma 3.1, $A \overset{r}{=} 0$. Clearly,

$$[q \circ \Lambda(d) \circ \mu] = (d_1 + \ldots d_r)\boldsymbol{v} \,\underline{\vee}\, (d_{r+1} r! + t)\lfloor \boldsymbol{u}, \boldsymbol{v}\rceil$$

in $[S^n \vee S^{p+n}, Y]$. Thus, by the construction of $\boldsymbol{K}(s,t)$,

$$[a(d)] = \boldsymbol{K}(d_1 + \ldots d_r, d_{r+1} r! + t)$$

in $[X, Y]$. Thus, since f has degree at most r,

$$\sum_{d \in \boldsymbol{\mathcal{E}}^{r+1}} (-1)^{|d|} f(\boldsymbol{K}(d_1 + \ldots d_r, d_{r+1} r! + t)) = 0.$$

By (17), $\boldsymbol{K}(d_1 + \ldots d_r, d_{r+1} r! + t)$ does not depend on d_{r+1} if $(d_1, \ldots, d_r) \neq (0, \ldots, 0)$. Thus the corresponding summands cancel out. We get $f(\boldsymbol{K}(0,t)) - f(\boldsymbol{K}(0, r! + t)) = 0$. By (16), this is what we need. $\square$

Let classes $E \in H^{p+1}(Y)$ and $F \in H^n(Y)$ satisfy $EF = 0$ in $H^{p+n+1}(Y)$. Put, as in Lemma 9.2,

$$\Delta = \langle \boldsymbol{u}^*(E), [S^{p+1}]\rangle\langle \boldsymbol{v}^*(F), [S^n]\rangle + (-1)^{(p+1)n}\langle \boldsymbol{u}^*(F), [S^{p+1}]\rangle\langle \boldsymbol{v}^*(E), [S^n]\rangle \in \mathbf{Z}.$$

If $Y = S^{p+1} \vee S^n$ with $\boldsymbol{u} = [\mathrm{in}_1]$ and $\boldsymbol{v} = [\mathrm{in}_2]$, taking obvious E and F yields $\Delta = 1$. If $p = n - 1$ and $Y = S^n$ with $\boldsymbol{u} = \boldsymbol{v} = [\mathrm{id}]$, taking obvious equal E and F yields $\Delta = 1 + (-1)^n$.

11.2. Lemma. *If $\Delta \neq 0$, the classes $\boldsymbol{k}(t)$, $t \in \mathbf{Z}$, are pairwise not 2-similar.*

Proof. Choose cocycles $e \in C^{p+1}(Y)$ and $f \in C^n(Y)$ representing E and F, respectively. Choose a cochain $g \in C^{p+n}(Y)$ with $\delta g = ef$. Consider the corresponding Hopf invariants (see § 9)

$$h_0 : \pi_{p+n}(Y) \to H^{p+n}(S^{p+n}) \quad \text{and} \quad h : [X, Y]_{e,f} \to H^{p+n}(X).$$

By Lemma 9.2,

$$\langle h_0(\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil), [S^{p+n}]\rangle = (-1)^{pn+p+1}\Delta.$$

We have the decomposition

$$\boldsymbol{k}(t) : X \xrightarrow{\ \epsilon\ } S^{p+n} \xrightarrow{t[\mathrm{id}]} S^{p+n} \xrightarrow{\lfloor \boldsymbol{u}, \boldsymbol{v} \rceil} Y$$

(the wavy arrows denote homotopy classes). Clearly, $\boldsymbol{k}(t) \in [X, Y]_{e,f}$. Since the Brouwer degree of ϵ is 1 and that of $t[\mathrm{id}]$ is t, Lemma 9.1 yields

$$\langle h(\boldsymbol{k}(t)), [X]\rangle = (-1)^{pn+p+1}\Delta t.$$

By Corollary 9.5, the classes $\boldsymbol{k}(t)$, $t \in \mathbf{Z}$, are pairwise not 2-similar if $\Delta \neq 0$. $\square$

Moral. Suppose that $\Delta \neq 0$. The classes $\boldsymbol{k}(0)$ $(= [\urcorner])$ and $\boldsymbol{k}(2)$ in $[X, Y]$, which are not 2-similar by Lemma 11.2, cannot be distinguished by an invariant of degree at most 2 by Lemma 11.1. Recall that (X, Y) can be $(S^p \times S^n, S^{p+1} \vee S^n)$ for any $p \geqslant 1$ and $n \geqslant 2$ or $(S^{n-1} \times S^n, S^n)$ for even $n \geqslant 2$.

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