Steenrod operations and the diagonal morphism

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Abstract

We show how to find the Steenrod operations in $H^\bullet(X)$ (the coefficients in $\mathbb{F}_p$) given the diagonal morphism $d_\# : S_\bullet(X) \to S_\bullet(X^p)$ and the action of the cyclic group C_p on $S_\bullet(X^p)$. Our construction needs no other data such as Eilenberg–Zilber morphisms.

1. Introduction

Fix a prime p. Chains, cohomology, etc. have coefficients in $\mathbb{F}_p$. For a (topological) space X, $S_\bullet(X)$ is the singular chain complex and $d_X^p \colon X \to X^p$ is the diagonal map. The group C_p of cyclic permutations of the set $\{1, \dots, p\}$ acts on X^p (on the left) in the obvious way.

1.1. Theorem. *Let X and Y be spaces. Suppose we have a commutative diagram of complexes and morphisms*

$$\begin{array}{ccc} S_\bullet(X) & \xrightarrow{d^p_{X\,\#}} & S_\bullet(X^p) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ S_\bullet(Y) & \xrightarrow{d^p_{Y\,\#}} & S_\bullet(Y^p), \end{array}$$

where F preserves the C_p-action. Then the induced homomorphism $f^* \colon H^\bullet(Y) \to H^\bullet(X)$ preserves the action of the Steenrod algebra $\mathscr{A}_p$.

Proof is given in § 4. We do not assume F to be "f to the power p" in any sense, cf. [D, Satz 4.4]. To get an example of a pair (f, F) satisfying our hypotheses, we may take any linear combination of pairs of the form $(a_\#, (a^p)_\#)$, where $a \colon X \to Y$ is a continuous map and $a^p \colon X^p \to Y^p$ is its Cartesian power. In this case, the assertion of the theorem is obvious. Note that f^* here need not preserve the cup product.

Theorem 1.1 implies that the Steenrod operations in $H^\bullet(X)$ are encoded somehow in the diagonal morphism $d^p_{X\,\#}$ considered as a morphism from $S_\bullet(X)$ to some complex of $\mathbb{F}_p$-modules with a C_p-action. We show how to extract Steenrod operations from $d^p_{X\,\#}$, see Theorem 4.1. Note that Steenrod's construction of the operations requires knowing, in addition, the cross product morphism $\xi^p_X \colon S_\bullet(X)^{\otimes p} \to S_\bullet(X^p)$ (see § 6).

We conjecture that, conversely, the $\mathscr{A}_p$-module $H^\bullet(X)$ determines the diagonal morphism $d^p_{X\,\#}$ up to coherent quasi-isomorphisms.

Conjecture. *Let X and Y be spaces with finite $\mathbb{F}_p$-homology groups and $k\colon H^\bullet(Y) \to H^\bullet(X)$ be an isomorphism of graded $\mathscr{A}_p$-modules. Then there exists a commutative diagram of complexes and morphisms*

$$\begin{array}{ccc} S_\bullet(X) & \xrightarrow{d^p_{X\,\#}} & S_\bullet(X^p) \\ \downarrow f & & \downarrow F \\ S_\bullet(Y) & \xrightarrow{d^p_{Y\,\#}} & S_\bullet(Y^p), \end{array}$$

where F preserves the C_p-action, f and F are quasi-isomorphisms, and $f^* = k$.

Most of the paper is taken by known facts about Steenrod's construction. The only point we state as new, besides Theorems 1.1 and 4.1, is Corollary 9.2.

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2. Preliminaries

Fix a field $\boldsymbol{k}$. Convention: $\otimes = \otimes_{\boldsymbol{k}}$, $\mathrm{Hom} = \mathrm{Hom}_{\boldsymbol{k}}$. Complexes are over $\boldsymbol{k}$. $\boldsymbol{k}[n]_\bullet$ ($n \in \mathbb{Z}$) is the complex with $\boldsymbol{k}[n]_n = \boldsymbol{k}$ and the other terms zero.

Cohomology. For a complex $A_\bullet$, let $h^\bullet(A_\bullet)$ be the cohomology of the cochain complex $\mathrm{Hom}(A_\bullet, \boldsymbol{k})$. We naturally identify $h^n(A_\bullet)$ with the $\boldsymbol{k}$-module of homotopy classes of morphisms $A_\bullet \to \boldsymbol{k}[n]_\bullet$. For a morphism $f\colon A_\bullet \to \boldsymbol{k}[n]_\bullet$, we thus have $\{f\} \in h^n(A_\bullet)$.

Group actions. Let G be a group. A $\boldsymbol{k}G$-module is, clearly, a $\boldsymbol{k}$-module equipped with a ($\boldsymbol{k}$-linear) G-action. Given two $\boldsymbol{k}G$-modules, we equip their tensor product (over $\boldsymbol{k}$) with the diagonal G-action. We equip $\boldsymbol{k}$ with the trivial G-action.

Let $A_\bullet$ be a G-complex (i. e., a complex over $\boldsymbol{k}G$) and M be a $\boldsymbol{k}G$-module. For $n \in \mathbb{Z}$, we have the G-complex $M[n]_\bullet$. The G-homotopy class $\{f\}_G$ of a G-morphism $f\colon A_\bullet \to M[n]_\bullet$ is naturally identified with an element of the nth cohomology $\boldsymbol{k}$-module of the cochain complex $\mathrm{Hom}_G(A_\bullet, M)$.

Group cohomology. The bar resolution $R_\bullet(G)$ is a complex of free $\boldsymbol{k}G$-modules. $H^\bullet(G)$ is defined as the cohomology of the cochain complex $\mathrm{Hom}_G(R_\bullet(G), \boldsymbol{k})$.

The augmentation $\epsilon\colon R_\bullet(G) \to \boldsymbol{k}[0]_\bullet$ and the comultiplication $\Delta\colon R_\bullet(G) \to R_\bullet(G) \otimes R_\bullet(G)$ are G-quasi-isomorphisms. $H^\bullet(G)$ is a graded algebra with the unit $1 = \{\epsilon\}_G \in H^0(G)$ and the product $H^\bullet(G) \otimes H^\bullet(G) \to H^\bullet(G)$ induced by the composition of morphisms

$$\begin{array}{l} \mathrm{Hom}_G(R_\bullet(G), \boldsymbol{k}) \otimes \mathrm{Hom}_G(R_\bullet(G), \boldsymbol{k}) \\ \qquad\qquad \downarrow (1) \\ \mathrm{Hom}_G(R_\bullet(G) \otimes R_\bullet(G), \boldsymbol{k}) \xrightarrow{(2)} \mathrm{Hom}_G(R_\bullet(G), \boldsymbol{k}), \end{array}$$

where (1) is the obvious product and (2) is induced by Δ.

Equivariant cohomology. For a G-complex $A_\bullet$ and a $\boldsymbol{k}G$-module M, let $h^\bullet_G(A_\bullet, M)$ be the cohomology of the cochain complex $\operatorname{Hom}_G(R_\bullet(G) \otimes A_\bullet, M)$. If a G-morphism $f\colon A_\bullet \to B_\bullet$ is a quasi-isomorphism, then the induced homomorphism $h^\bullet_G(f, \mathrm{id})\colon h^\bullet_G(B_\bullet, M) \to h^\bullet_G(A_\bullet, M)$ is an isomorphism. We put $h^\bullet_G(A_\bullet) = h^\bullet_G(A_\bullet, \boldsymbol{k})$.

$h^\bullet_G(A_\bullet)$ is a graded $H^\bullet(G)$-module with the module structure $H^\bullet(G) \otimes h^\bullet_G(A_\bullet) \to h^\bullet_G(A_\bullet)$ induced by the composition of morphisms

$$\begin{array}{c}\operatorname{Hom}_G(R_\bullet(G), \boldsymbol{k}) \otimes \operatorname{Hom}_G(R_\bullet(G) \otimes A_\bullet, \boldsymbol{k}) \\ \downarrow{\scriptstyle (1)} \\ \operatorname{Hom}_G(R_\bullet(G) \otimes R_\bullet(G) \otimes A_\bullet, \boldsymbol{k}) \xrightarrow{(2)} \operatorname{Hom}_G(R_\bullet(G) \otimes A_\bullet, \boldsymbol{k}),\end{array}$$

where (1) is the obvious product and (2) is induced by Δ.

The case of a trivial action. Let $A_\bullet$ be a complex. Equip it with the trivial G-action. The obvious morphism

$$\operatorname{Hom}_G(R_\bullet(G), \boldsymbol{k}) \otimes \operatorname{Hom}(A_\bullet, \boldsymbol{k}) \xrightarrow{(1)} \operatorname{Hom}_G(R_\bullet(G) \otimes A_\bullet, \boldsymbol{k})$$

induces a graded $\boldsymbol{k}$-homomorphism

$$H^\bullet(G) \otimes h^\bullet(A_\bullet) \xrightarrow{(2)} h^\bullet_G(A_\bullet),$$

for which we use the notation

$$z \otimes u \mapsto z \times u.$$

The homomorphism (2) is an $H^\bullet(G)$-homomorphism. If G is finite, then (1) and (2) are isomorphisms.

3. The algebra $H^\bullet(C_p)$

Here $\boldsymbol{k} = \mathbb{F}_p$. The algebra $H^\bullet(C_p)$ has a standard basis. It consists of classes $e_i \in H^i(C_p)$, $i = 0, 1, \dots$. We have $e_i e_j = c_p(i,j) e_{i+j}$, where $c_p(i,j) \in \mathbb{F}_p$ is 1 if pij is even and 0 otherwise. See [M, proof of Proposition 9.1, p. 206]; cf. [SE, Ch. V, § 5].

4. Our construction of Steenrod operations

Here $\boldsymbol{k} = \mathbb{F}_p$. For each $k \geqslant 0$, we define the degree k operation $\Sigma^k \in \mathscr{A}_p$ by putting: $\Sigma^k = \mathrm{Sq}^k$ if $p = 2$ and

$$\Sigma^k = \begin{cases} (-1)^s P^s, & k = 2(p-1)s, \\ (-1)^{s+1} \beta P^s, & k = 2(p-1)s + 1, \\ 0 & \text{otherwise} \end{cases}$$

if $p \neq 2$. (These Σ^k differ from the common St^k in the sign.) For $u \in H^n(X)$, we have $\Sigma^k u = 0$ if $n + k > pn$. The operations Σ^k generate the algebra $\mathscr{A}_p$.

4.1. Theorem. *Let X be a space and $u \in H^n(X)$ ($n \geqslant 0$). Consider the C_p-morphism $d^p_{X\,\#}\colon S_\bullet(X) \to S_\bullet(X^p)$ (C_p acts on $S_\bullet(X)$ trivially) and the induced homomorphism*

$$h^{pn}_{C_p}(S_\bullet(X)) \xleftarrow{h^{pn}_{C_p}(d^p_{X\,\#})} h^{pn}_{C_p}(S_\bullet(X^p)).$$

Then: (a) there exists a unique class $w \in \operatorname{Im} h^{pn}_{C_p}(d^p_{X\,\#})$,

$$w = \sum_{i=0}^{pn} e_{pn-i} \times v_i, \qquad v_i \in H^i(X),$$

such that $v_i = 0$ for $i < n$ and $v_n = u$; (b) we have $v_{n+k} = \Sigma^k u$ for $n \leqslant n+k \leqslant pn$.

Proof is given in § 9. Theorem 4.1 is a way to find/construct the operations Σ^k.

Proof of Theorem 1.1. Take $u \in H^n(Y)$ ($n \geqslant 0$). We show that $\Sigma^k f^*(u) = f^*(\Sigma^k u)$ for $n \leqslant n + k \leqslant pn$.

Consider the commutative diagram

$$\begin{array}{ccc} h^{pn}_{C_p}(S_\bullet(X)) & \xleftarrow{h^{pn}_{C_p}(d^p_{X\,\#})} & h^{pn}_{C_p}(S_\bullet(X^p)) \\ {\scriptstyle h^{pn}_{C_p}(f)}\uparrow & & \uparrow{\scriptstyle h^{pn}_{C_p}(F)} \\ h^{pn}_{C_p}(S_\bullet(Y)) & \xleftarrow{h^{pn}_{C_p}(d^p_{Y\,\#})} & h^{pn}_{C_p}(S_\bullet(Y^p)). \end{array}$$

By Theorem 4.1 (a), there is a class $w \in \operatorname{Im} h^{pn}_{C_p}(d^p_{Y\,\#})$,

$$w = \sum_{i=0}^{pn} e_{pn-i} \times v_i, \qquad v_i \in H^i(Y),$$

such that $v_i = 0$ for $i < n$ and $v_n = u$. By naturality of the operation $\times$, we have

$$h^{pn}_{C_p}(f)(w) = \sum_{i=0}^{pn} e_{pn-i} \times f^*(v_i).$$

By commutativity of the diagram, $h^{pn}_{C_p}(f)(w) \in \operatorname{Im} h^{pn}_{C_p}(d^p_{X\,\#})$. We have $f^*(v_i) = 0$ for $i < n$ and $f^*(v_n) = f^*(u)$. By Theorem 4.1 (b) for Y and X, $v_{n+k} = \Sigma^k u$ and $f^*(v_{n+k}) = \Sigma^k f^*(u)$. The promised equality follows. $\square$

5. The equivariant power map $\boldsymbol{\theta_G^r}$

Let G be a group acting on the set $\{1, \dots, r\}$ $(r \geqslant 0)$. For $n \in \mathbb{Z}$, we define the G-module $\boldsymbol{k}^{(n)}$ as $\boldsymbol{k}$ on which an element $g \in G$ acts by multiplication by s^n, where s is the sign of the permutation corresponding to g.

Let $A_\bullet$ be a complex. G acts on the complex $A_\bullet^{\otimes r}$ by permuting factors and multiplying by ± 1 according to the Koszul convention.

Given a morphism $f\colon A_\bullet \to \boldsymbol{k}[n]_\bullet$ $(n \in \mathbb{Z})$, we introduce the G-morphism

$$f_G^r\colon R_\bullet(G) \otimes A_\bullet^{\otimes r} \xrightarrow{\epsilon \otimes f^{\otimes r}} \boldsymbol{k}[0]_\bullet \otimes \boldsymbol{k}[n]_\bullet^{\otimes r} \xrightarrow{(1)} \boldsymbol{k}^{(n)}[rn]_\bullet,$$

where (1) is the G-isomorphism given by $1 \otimes 1^{\otimes r} \mapsto 1$. We have $\{f\} \in h^n(A_\bullet)$ and $\{f_G^r\}_G \in h_G^{rn}(A_\bullet^{\otimes r}, \boldsymbol{k}^{(n)})$.

5.1. Lemma. *Let $f_0, f_1\colon A_\bullet \to \boldsymbol{k}[n]_\bullet$ be homotopic morphisms. Then the G-morphisms $(f_i)_G^r$, $i = 0, 1$, are G-homotopic.*

This follows from [M, Lemma 1.1 (ii)]; cf. [SE, Ch. VII, Lemma 2.2].

Proof. Let $L_\bullet$ be the complex with a basis consisting of $b \in L_{n-1}$ and $c_0, c_1 \in L_n$ and the differential given by $\partial c_i = (-1)^i b$. Define morphisms $s\colon \boldsymbol{k}[n]_\bullet \to L_\bullet$ and $t_i\colon L_\bullet \to \boldsymbol{k}[n]_\bullet$, $i = 0, 1$, by $s(1) = c_0 + c_1$ and $t_i(c_j) = \delta_{ij}$ (the Kronecker delta). We have $t_i \circ s = \mathrm{id}$. Thus, for each i, we have the commutative diagram

$$\begin{array}{ccc} & h_G^{rn}(L_\bullet^{\otimes r}, \boldsymbol{k}^{(n)}) & \\ {}^{s^\#}\swarrow & & \nwarrow^{t_i^\#} \\ h_G^{rn}(\boldsymbol{k}[n]_\bullet^{\otimes r}, \boldsymbol{k}^{(n)}) & \xleftarrow{\quad \mathrm{id} \quad} & h_G^{rn}(\boldsymbol{k}[n]_\bullet^{\otimes r}, \boldsymbol{k}^{(n)}) \end{array}$$

(we put $s^\# = h_G^{rn}(s^{\otimes r}, \mathrm{id})$, etc.). Since s is a quasi-isomorphism, $s^\#$ is an isomorphism. It follows that $t_0^\# = t_1^\#$. Put $e = \mathrm{id}\colon \boldsymbol{k}[n]_\bullet \to \boldsymbol{k}[n]_\bullet$. We have $\{(t_i)_G^r\}_G = t_i^\#(\{e_G^r\}_G)$. Thus $\{(t_0)_G^r\}_G = \{(t_1)_G^r\}_G$.

It follows from the definition of chain homotopy that there exists a morphism $F\colon A_\bullet \to L_\bullet$ such that $f_i = t_i \circ F$, $i = 0, 1$. Consider the homomorphism $F^\#\colon h_G^{rn}(L_\bullet^{\otimes r}, \boldsymbol{k}^{(n)}) \to h_G^{rn}(A_\bullet^{\otimes r}, \boldsymbol{k}^{(n)})$. We have $\{(f_i)_G^r\}_G = F^\#(\{(t_i)_G^r\}_G)$. Thus $\{(f_0)_G^r\}_G = \{(f_1)_G^r\}_G$. □

Lemma 5.1 allows us to introduce the map

$$\theta_G^r\colon h^n(A_\bullet) \to h_G^{rn}(A_\bullet^{\otimes r}, \boldsymbol{k}^{(n)}), \quad \{f\} \mapsto (-1)^{nr(r-1)/2}\{f_G^r\}_G.$$

It is not additive in general.

6. Steenrod's construction of Σ^k

Here $\boldsymbol{k} = \mathbb{F}_p$. Define the *Steenrod numbers* $a_p(n) \in \mathbb{F}_p$, $n \in \mathbb{N}$: put $a_p(n) = 1$ if $p = 2$ and $a_p(n) = (-1)^{qn(n+1)/2}(q!)^n$ if $p = 2q+1$. We have $a_p(n) \neq 0$.

Let X be a space. We have the cross product quasi-isomorphism $\xi_X^p \colon S_\bullet(X)^{\otimes p} \to S_\bullet(X^p)$, which preserves the action of C_p (in fact, of the whole symmetric group Σ_p). Consider the diagram of C_p-complexes

$$S_\bullet(X) \xrightarrow{d^p_{X\,\#}} S_\bullet(X^p) \xleftarrow{\xi^p_X} S_\bullet(X)^{\otimes p},$$

where the C_p-action in $S_\bullet(X)$ is trivial. Given a class $u \in H^n(X) = h^n(S_\bullet(X))$ ($n \geqslant 0$), we have the class $\theta^p_{C_p}(u) \in h^{pn}_{C_p}(S_\bullet(X)^{\otimes p})$ (note that $\mathbb{F}_p^{(n)} = \mathbb{F}_p$ as an $\mathbb{F}_p C_p$-module). Define the classes $\Psi(u)$ and $\Phi(u)$:

$$\Phi(u) \longleftarrow\!\shortmid \Psi(u) \longmapsto a_p(n)^{-1}\theta^p_{C_p}(u)$$

$$h^{pn}_{C_p}(S_\bullet(X)) \xleftarrow{h^{pn}_{C_p}(d^p_{X\,\#})} h^{pn}_{C_p}(S_\bullet(X^p)) \xrightarrow{h^{pn}_{C_p}(\xi^p_X)} h^{pn}_{C_p}(S_\bullet(X)^{\otimes p}).$$

They are well-defined since $h^{pn}_{C_p}(\xi^p_X)$ is an isomorphism.

We have

$$\Phi(u) = \sum_{i=0}^{pn} e_{pn-i} \times \phi_i(u)$$

for some $\phi_i(u) \in h^i(S_\bullet(X)) = H^i(X)$, $i = 0, \ldots, pn$.

6.1. Fact. *We have $\phi_i(u) = 0$ for $i < n$, $\phi_n(u) = u$, and $\phi_{n+k}(u) = \Sigma^k u$ for $n \leqslant n+k \leqslant pn$.*

This is the construction of Steenrod operations given in [M, §§ 2, 5, 7, 8], note Remarks 7.2, Theorem 7.9 (i), Propositions 8.1 and 2.3 (iv), and also formula (2) in the proof of Proposition 9.1 there. We used unnormalized chains, but we could use normalized ones equivalently, thus following [M]. Cf. also [SE, Ch. VII]. □

7. The transfer and the functor $\tilde{h}^\bullet_G$

Let G be a finite group. Put

$$M = \sum_{g \in G} g \in \boldsymbol{k}G.$$

Let $A_\bullet$ be a G-complex. The morphism $\epsilon \otimes M \colon R_\bullet(G) \otimes A_\bullet \to \boldsymbol{k}[0]_\bullet \otimes A_\bullet = A_\bullet$ induces a morphism $\mathrm{Hom}(A_\bullet, \boldsymbol{k}) \to \mathrm{Hom}_G(R_\bullet(G) \otimes A_\bullet, \boldsymbol{k})$. The induced $\boldsymbol{k}$-homomorphism on the cohomology $t \colon h^\bullet(A_\bullet) \to h^\bullet_G(A_\bullet)$ is called the *transfer* ([SE, Ch. V, § 7], cf. [B, Ch. III, § 9]).

Equip $h^\bullet(A_\bullet)$ with a (trivial) $H^\bullet(G)$-module structure using the $\boldsymbol{k}$-algebra homomorphism $H^\bullet(G) \to H^\bullet(1) = \boldsymbol{k}$ induced by the group inclusion $1 \to G$. Then t becomes an $H^\bullet(G)$-homomorphism (cf. [B, Ch. V, (3.8)]).

Put $\tilde{h}^\bullet_G(A_\bullet) = \operatorname{Coker} t$. This is an $H^\bullet(G)$-module. Obviously, $\tilde{h}^\bullet_G$ is a functor. It respects quasi-isomorphisms.

The following lemmas are immediate.

7.1. Lemma. *Let $A_\bullet$ be a complex. Consider the G-complex $B_\bullet = A_\bullet \otimes \boldsymbol{k}G$. Then $\tilde{h}^\bullet_G(B_\bullet) = 0$.* □

7.2. Lemma. *Let $A_\bullet$ be a complex. Equip it with the trivial G-action. Then the transfer $t\colon h^\bullet(A_\bullet) \to h^\bullet_G(A_\bullet)$ is given by $t(u) = |G| \times u$.* □

8. The functor $\tilde{h}^\bullet_{C_p}(?^{\otimes p})$ and the map $\tilde{\theta}^p_{C_p}$

Here we follow [H, Ch. II, proof of Theorem 3.7].

We have a prime p. We show that the functor $\tilde{h}^m_{C_p}(?^{\otimes p})$ takes sums to products. It follows that it is additive.

8.1. Lemma. *Let $A_{i\,\bullet}$, $i \in I$, be a family of complexes. Put*

$$A_\bullet = \bigoplus_{i\in I} A_{i\,\bullet}.$$

Let $k_i\colon A_{i\,\bullet} \to A_\bullet$ be the canonical morphisms. Then

$$\tilde{h}^m_{C_p}(A_\bullet^{\otimes p}) \xrightarrow{(\tilde{h}^m_{C_p}(k_i^{\otimes p}))_{i\in I}} \textstyle\prod_{i\in I} \tilde{h}^m_{C_p}(A_{i\,\bullet}^{\otimes p})$$

is an isomorphism for each $m \in \mathbb{Z}$.

Proof. Put $J = I^p \setminus \operatorname{Im} d$, where $d\colon I \to I^p$ is the diagonal map. C_p acts on J by permuting coordinates. Consider the complex

$$N_\bullet = \bigoplus_{(i_1,\ldots,i_p)\in J} A_{i_1\,\bullet} \otimes \ldots \otimes A_{i_p\,\bullet}.$$

Equip it with the obvious C_p-action. We have the C_p-isomorphism

$$A_\bullet^{\otimes p} \xleftarrow{((k_i^{\otimes p})_{i\in I},l)} \big(\textstyle\bigoplus_{i\in I} A_{i\,\bullet}^{\otimes p}\big) \oplus N_\bullet,$$

where $l = (k_{i_1} \otimes \ldots \otimes k_{i_p})_{(i_1,\ldots,i_p)\in J}\colon N_\bullet \to A_\bullet^{\otimes p}$. Obviously, the functor $\tilde{h}^m_{C_p}$ takes sums to products. Thus

$$\tilde{h}^m_{C_p}(A_\bullet^{\otimes p}) \xrightarrow{((\tilde{h}^m_{C_p}(k_i^{\otimes p}))_{i\in I},\tilde{h}^m_{C_p}(l))} \big(\textstyle\prod_{i\in I} \tilde{h}^m_{C_p}(A_{i\,\bullet}^{\otimes p})\big) \oplus \tilde{h}^m_{C_p}(N_\bullet)$$

is an isomorphism. It remains to show that $\tilde{h}^m_{C_p}(N_\bullet) = 0$.

Choose a section $s\colon J/C_p \to J$ and put $J' = \operatorname{Im} s \subseteq J$. Consider the complex

$$N'_\bullet = \bigoplus_{(i_1,\ldots,i_p)\in J'} A_{i_1\,\bullet} \otimes \ldots \otimes A_{i_p\,\bullet}.$$

Since p is prime, C_p acts on J freely. It follows that $N_\bullet \cong N'_\bullet \otimes \boldsymbol{k}C_p$. By Lemma 7.1, $\tilde{h}^m_{C_p}(N_\bullet) = 0$. $\square$

Set $\boldsymbol{k} = \mathbb{F}_p$. Given a complex $A_\bullet$ and a class $u \in h^n(A_\bullet)$ ($n \in \mathbb{Z}$), we define $\tilde{\theta}^p_{C_p}(u) \in \tilde{h}^p_{C_p}(A_\bullet^{\otimes p})$ as the image of $\theta^p_{C_p}(u) \in h^p_{C_p}(A_\bullet^{\otimes p})$ under the projection (note that $\mathbb{F}_p^{(n)} = \mathbb{F}_p$ as an $\mathbb{F}_pC_p$-module).

8.2. Lemma. *Let $A_\bullet$ be a complex. Then: (a) the map $\tilde{\theta}^p_{C_p}: h^n(A_\bullet) \to \tilde{h}^{pn}_{C_p}(A_\bullet^{\otimes p})$ is $\mathbb{F}_p$-linear; (b) the $H^\bullet(C_p)$-homomorphism*

$$H^\bullet(C_p) \otimes h^\bullet(A_\bullet) \to \tilde{h}^\bullet_{C_p}(A_\bullet^{\otimes p}), \quad 1 \otimes u \mapsto \tilde{\theta}^p_{C_p}(u),\ u \in h^n(A_\bullet),\ n \in \mathbb{Z},$$

is a (non-graded) isomorphism.

Proof. (a) The map $\tilde{\theta}^p_{C_p}: h^n(A_\bullet) \to \tilde{h}^{pn}_{C_p}(A_\bullet^{\otimes p})$ is natural in $A_\bullet$. Its source and target are additive in $A_\bullet$ (obvious for $h^n(A_\bullet)$, Lemma 8.1 for the $\tilde{h}^{pn}_{C_p}(A_\bullet^{\otimes p})$). It follows that it is additive and thus $\mathbb{F}_p$-linear.

(b) There exists a quasi-isomorphism

$$A_\bullet \xleftarrow{q=(q_i)_{i\in I}} \textstyle\bigoplus_{i\in I} E_{i\,\bullet},$$

where $E_{i\,\bullet} = \mathbb{F}_p[m_i]_\bullet$, $m_i \in \mathbb{Z}$. We have the commutative diagram

$$\begin{array}{ccc} h^n(A_\bullet) & \xrightarrow{(h^n(q_i))_{i\in I}} & \prod_{i\in I} h^n(E_{i\,\bullet}) \\ \downarrow{\scriptstyle \tilde{\theta}^p_{C_p}} & & \downarrow{\scriptstyle \prod_{i\in I}(\tilde{\theta}^p_{C_p}: h^n(E_{i\bullet})\to\tilde{h}^{pn}_{C_p}(E_{i\bullet}^{\otimes p}))} \\ \tilde{h}^{pn}_{C_p}(A_\bullet^{\otimes p}) & \xrightarrow{(\tilde{h}^{pn}_{C_p}(q_i^{\otimes p}))_{i\in I}} & \prod_{i\in I} \tilde{h}^{pn}_{C_p}(E_{i\,\bullet}^{\otimes p}). \end{array}$$

The functors h^n and $\tilde{h}^{pn}_{C_p}(?^{\otimes p})$ respect quasi-isomorphisms (obvious) and take sums to products (obvious for h^n, Lemma 8.1 for $\tilde{h}^{pn}_{C_p}(?^{\otimes p})$). Thus the horizontal arrows are isomorphisms. Tensoring by $H^\bullet(C_p)$ commutes with products since $H^k(C_p)$ are finite. Thus all reduces to the case $A_\bullet = \mathbb{F}_p[m]_\bullet$, where the assertion is verified immediately. $\square$

9. Back to topology

Here $\boldsymbol{k} = \mathbb{F}_p$. Let X be a space.

9.1. Lemma. *(a) The maps* $\Phi\colon H^n(X) \to h^{pn}_{C_p}(S_\bullet(X))$, $n \geqslant 0$, *are* $\mathbb{F}_p$*-linear.*
(b) The image of the $H^\bullet(C_p)$*-homomorphism*

$$h^\bullet_{C_p}(S_\bullet(X)) \xleftarrow{h^\bullet_{C_p}(d^p_{X\,\#})} h^\bullet_{C_p}(S_\bullet(X^p))$$

is $H^\bullet(C_p)$*-generated by classes of the form* $\Phi(u)$, $u \in H^n(X)$, $n \geqslant 0$.

This is known, see [H, Ch. II, proof of Theorem 3.7]. The assertion (a) follows also from Fact 6.1 and the well-known linearity of Σ^k. Our proof follows [H].

Proof. We have the commutative diagram

$$\begin{array}{ccccc}
\Phi(u) & \longleftarrow\!\shortmid & \Psi(u) & \longmapsto & a_p(n)^{-1}\theta^p_{C_p}(u) \\
\\
h^{pn}_{C_p}(S_\bullet(X)) & \xleftarrow{h^{pn}_{C_p}(d^p_{X\,\#})} & h^{pn}_{C_p}(S_\bullet(X^p)) & \xrightarrow{h^{pn}_{C_p}(\xi^p_X)} & h^{pn}_{C_p}(S_\bullet(X)^{\otimes p}) \\
\downarrow r & & \downarrow & & \downarrow \\
\tilde{h}^{pn}_{C_p}(S_\bullet(X)) & \xleftarrow{\tilde{h}^{pn}_{C_p}(d^p_{X\,\#})} & \tilde{h}^{pn}_{C_p}(S_\bullet(X^p)) & \xrightarrow{\tilde{h}^{pn}_{C_p}(\xi^p_X)} & \tilde{h}^{pn}_{C_p}(S_\bullet(X)^{\otimes p}),
\end{array}$$

where the vertical arrows are the projections. By Lemma 7.2, the transfer for $S_\bullet(X)$ is zero (cf. [SE, Ch. VII, proof of Lemma 4.1]). Thus r is an isomorphism. Since ξ^p_X is a C_p-quasi-isomorphism, $\tilde{h}^\bullet_{C_p}(\xi^p_X)$ is an isomorphism. By Lemma 8.2, the maps $\tilde{\theta}^p_{C_p}: H^n(X) \to \tilde{h}^{pn}_{C_p}(S_\bullet(X)^{\otimes p})$ are $\mathbb{F}_p$-linear and the $H^\bullet(C_p)$-module $\tilde{h}^\bullet_{C_p}(S_\bullet(X)^{\otimes p})$ is generated by classes of the form $\tilde{\theta}^p_{C_p}(u)$, $u \in H^n(X)$, $n \geqslant 0$. The desired assertions follow. □

9.2. Corollary. *Take a class* $w \in \operatorname{Im} h^m_{C_p}(d^p_{X\,\#})$ $(m \geqslant 0)$,

$$w = \sum_{i=0}^{m} e_{m-i} \times v_i, \qquad v_i \in H^i(X).$$

If $v_i = 0$ *for* $i \leqslant m/p$, *then* $w = 0$.

Proof. By Lemma 9.1 (a, b), we have

$$w = \sum_{k=0}^{[m/p]} e_{m-pk}\Phi(u_k) \tag{$*$}$$

for some $u_k \in H^k(X)$. Using the formulas

$$\Phi(u_k) = \sum_{i=0}^{pk} e_{pk-i} \times \phi_i(u_k)$$

and $e_i e_j = c_p(i,j) e_{i+j}$, we get

$$v_i = \sum_{k\colon i \leqslant pk \leqslant m} c_p(m-pk, pk-i)\phi_i(u_k).$$

We take successively $n = 0, \ldots, [m/p]$ and show that $u_n = 0$. On each step, $u_k = 0$ for $k < n$. By Fact 6.1, $\phi_n(u_n) = u_n$ and $\phi_n(u_k) = 0$ for $k > n$. Therefore, $v_n = c_p(m-pn, pn-n)u_n$. We have $c_p(m-pn, pn-n) = 1$ since $p(pn-n)$ is even. Thus $v_n = u_n$. By assumption, $v_n = 0$. Thus $u_n = 0$. □

A similar reasoning shows that the presentation $(*)$ is unique for any $w \in \operatorname{Im} h^m_{C_p}(d^p_{X\,\#})$. Thus we get a (non-graded) isomorphism $\operatorname{Im} h^\bullet_{C_p}(d^p_{X\,\#}) \cong H^\bullet(C_p) \otimes H^\bullet(X)$. It follows also that $\tilde{h}^\bullet_{C_p}(d^p_{X\,\#})$ is injective. This stuff is known, see [H, Ch. II, proof of Theorem 3.7].

Proof of Theorem 4.1. The uniqueness follows from Corollary 9.2. By Fact 6.1, $w = \Phi(u)$ satisfies the conditions of the parts (a) and (b). Thus we are done. □

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