Commutative algebras and representations of the category of finite sets

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Abstract

We prove that two finite-dimensional commutative algebras over an algebraically closed field are isomorphic if and only if they give rise to isomorphic representations of the category of finite sets and surjective maps.

Let Ω be the category whose objects are the sets $\underline{n} = \{1, \dots, n\}$, $n \in \mathbf{N}$ $(= \{0, 1, \dots\})$, and whose morphisms are surjective maps. Let $\boldsymbol{k}$ be a field. For a [commutative] algebra A [over $\boldsymbol{k}$ possibly without unity], let us define a functor $L_A \colon \Omega \to \boldsymbol{k}$-**Mod** (a "representation of Ω"). For $n \in \mathbf{N}$, set $L_A(\underline{n}) = A^{\otimes n}$. For a morphism $h \colon \underline{m} \to \underline{n}$, set $L_A(h) \colon x_1 \otimes \dots \otimes x_m \mapsto y_1 \otimes \dots \otimes y_n$, where

$$y_j = \prod_{i \in h^{-1}(j)} x_i.$$

The functor L_A is a variant of the Loday functor [1].

Theorem. *Let the field $\boldsymbol{k}$ be algebraically closed. Let A and B be finite-dimensional algebras. Suppose that the functors L_A and L_B are isomorphic. Then the algebras A and B are isomorphic.*

We do not know whether the assertion is true for infinite-dimensional algebras. It is false for the field $\mathbf{R}$ (which is not algebraically closed). Indeed, take the non-isomorphic algebras $A = \mathbf{R}[X]/(X^2 - 1)$ and $B = \mathbf{R}[Y]/(Y^2 + 1)$. We have the bases $\{X^e\}_{e=0,1}$ $(= \{1, X\})$ in A and $\{Y^e\}_{e=0,1}$ in B. The linear maps $s_n \colon A^{\otimes n} \to B^{\otimes n}$,

$$X^{e_1} \otimes \dots \otimes X^{e_n} \mapsto k_{e_1 + \dots + e_n} Y^{e_1} \otimes \dots \otimes Y^{e_n}, \qquad e_1, \dots, e_n = 0, 1,$$

where $k_m = (-1)^{[m/2]}$, form a functor isomorphism $s \colon L_A \to L_B$.

1. Preliminaries

Algebra of polynomials. If V is a vector space [over $\boldsymbol{k}$], then the symmetric group $\Sigma_n = \operatorname{Aut} \underline{n}$ acts [from the left] on $V^{\otimes n}$ by the rule $g(v_1 \otimes \dots \otimes v_n) = v_{g^{-1}(1)} \otimes \dots \otimes v_{g^{-1}(n)}$. The symmetric powers $S^n(V) = (V^{\otimes n})_{\Sigma_n}$ form the symmetric algebra

$$S(V) = \bigoplus_{n=0}^{\infty} S^n(V)$$

with the multiplication induced by the tensor one: $\overline{x}\,\overline{y} = \overline{x \otimes y}$, $x \in V^{\otimes m}$, $y \in V^{\otimes n}$ (the bar denotes the projection $V^{\otimes n} \to S^n(V)$).

For a vector space U, put $\boldsymbol{k}[U] = S(U^*)$. For $u \in U$, there is the evaluation map $\boldsymbol{k}[U] \to \boldsymbol{k}$, $f \mapsto f(u)$, which is the unital algebra homomorphism defined by the condition $v(u) = \langle v, u \rangle$ for $v \in U^* = S^1(U^*) \subseteq \boldsymbol{k}[U]$. For a polynomial $f \in \boldsymbol{k}[U]$ and a set $X \subseteq U$, there is the function $f \mid X \colon X \to \boldsymbol{k}$, $u \mapsto f(u)$. An ideal $P \subseteq \boldsymbol{k}[U]$ determines the set

$$Z(P) = \{\, u : f(u) = 0 \text{ for all } f \in P \,\} \subseteq U.$$

Symmetric tensors, isomorphism θ***.*** Put $D^n(U) = (U^{\otimes n})^{\Sigma_n}$,

$$\hat{D}(U) = \prod_{n=0}^{\infty} D^n(U).$$

The pairing

$$\boldsymbol{\langle} -, - \boldsymbol{\rangle} \colon (U^*)^{\otimes n} \times U^{\otimes n} \to \boldsymbol{k}, \tag{1}$$

$\boldsymbol{\langle} v_1 \otimes \ldots \otimes v_n, u_1 \otimes \ldots \otimes u_n \boldsymbol{\rangle} = \langle v_1, u_1 \rangle \ldots \langle v_n, u_n \rangle$, induces the pairing

$$\boldsymbol{(} -, - \boldsymbol{)} \colon S^n(U^*) \times D^n(U) \to \boldsymbol{k}, \tag{2}$$

$\boldsymbol{(}\overline{z}, w\boldsymbol{)} = \boldsymbol{\langle} z, w \boldsymbol{\rangle}$, where $w \in D^n(U) \subseteq U^{\otimes n}$, $z \in (U^*)^{\otimes n}$. Summing over $n \in \mathbf{N}$, we get a pairing

$$\boldsymbol{(} -, - \boldsymbol{)} \colon \boldsymbol{k}[U] \times \hat{D}(U) \to \boldsymbol{k}.$$

We have the linear map

$$\theta \colon \hat{D}(U) \to \boldsymbol{k}[U]^*, \quad \langle \theta(W), f \rangle = \boldsymbol{(} f, W \boldsymbol{)}.$$

If U is finite-dimensional, then the pairings (1) and (2) are perfect and θ is an isomorphism.

Functor T_A***.*** Let $\Sigma \subseteq \Omega$ be the subcategory of isomorphisms. We have $\Sigma = \Sigma_0 \sqcup \Sigma_1 \sqcup \ldots$. For a vector space A, we have the functor $T_A \colon \Sigma \to \boldsymbol{k}\text{-}\mathbf{Mod}$, $T_A(\underline{n}) = A^{\otimes n}$ (with the ordinary action of Σ_n). If A is an algebra, then $T_A = L_A|_\Sigma$.

Kronecker product, isomorphism κ***.*** If a group G acts on vector spaces X and Y, then it acts on $\mathrm{Hom}(X, Y)$ by the rule $(gt)(x) = g(t(g^{-1}x))$. We have $\mathrm{Hom}(X, Y)^G = \mathrm{Hom}_G(X, Y)$.

Let A and B be vector spaces. The Kronecker product $\mathrm{Hom}(A, B)^{\otimes n} \to \mathrm{Hom}(A^{\otimes n}, B^{\otimes n})$, $w \mapsto [w]$ (a notation), preserves the action of Σ_n and thus induces a linear map $D^n(B^A) \to \mathrm{Hom}_{\Sigma_n}(A^{\otimes n}, B^{\otimes n})$ (from now on, $B^A = \mathrm{Hom}(A, B)$). Since

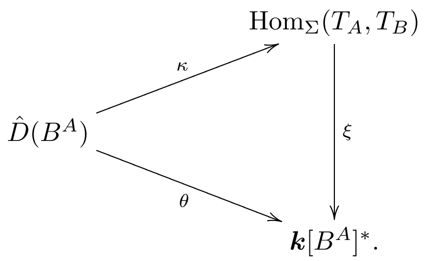
$$\mathrm{Hom}_\Sigma(T_A, T_B) = \prod_{n=0}^{\infty} \mathrm{Hom}_{\Sigma_n}(A^{\otimes n}, B^{\otimes n}),$$

these maps form a linear map

$$\kappa\colon \hat{D}(B^A) \to \mathrm{Hom}_\Sigma(T_A, T_B).$$

If A and B are finite-dimensional, then κ is an isomorphism.

Morphisms $T_A \to T_B$ and functionals on $\boldsymbol{k}[B^A]$, isomorphism ξ. For finite-dimensional vector spaces A and B we have the isomorphism ξ that fits in the commutative diagram



Example. A linear map $u\colon A \to B$ induces the functor morphism $T_u\colon T_A \to T_B$, $(T_u)_n = u^{\otimes n}$. Then $\langle \xi(T_u), f\rangle = f(u)$, $f \in \boldsymbol{k}[B^A]$.

Antisymmetrization. For a vector space V, we have the operator $\mathrm{alt}_n\colon V^{\otimes n} \to V^{\otimes n}$,

$$\mathrm{alt}_n(w) = \sum_{g \in \Sigma_n} \operatorname{sgn} g \; gw.$$

2. The determinant

Let A and B be vector spaces of equal finite dimension m. Put $U = B^A$. Choose bases $e_1, \ldots, e_m \in A$ and $f_1, \ldots, f_m \in B$. Put

$$E = \mathrm{alt}_m(e_1 \otimes \ldots \otimes e_m) \in A^{\otimes m}, \quad F = \mathrm{alt}_m(f_1 \otimes \ldots \otimes f_m) \in B^{\otimes m}.$$

We have the bases $\check{f}^1, \ldots, \check{f}^m \in B^*$, $\langle \check{f}^j, f_i\rangle = \delta_i^j$ (δ_i^j is the Kronecker delta) and $l_i^j \in U^*$, $i, j = 1, \ldots, m$, $\langle l_i^j, u\rangle = \langle \check{f}^j, u(e_i)\rangle$. Put

$$H = \sum_{g \in \Sigma_m} \operatorname{sgn} g \; l^1_{g^{-1}(1)} \otimes \ldots \otimes l^m_{g^{-1}(m)} \in (U^*)^{\otimes m}.$$

Then $\overline{H} \in \boldsymbol{k}[U]$ is the determinant, so

$$\overline{H}(u) = \det u, \qquad u \in U. \tag{3}$$

We have $\boldsymbol{\langle} \check{f}^1 \otimes \ldots \otimes \check{f}^m, [v](E)\boldsymbol{\rangle} = \boldsymbol{\langle} H, v\boldsymbol{\rangle}$, $v \in U^{\otimes m}$. Hence

$$\boldsymbol{\langle}(\check{f}^1 \otimes \ldots \otimes \check{f}^m)^{\otimes r}, [w](E^{\otimes r})\boldsymbol{\rangle} = \boldsymbol{\langle} H^{\otimes r}, w\boldsymbol{\rangle}, \qquad w \in U^{\otimes mr}, \quad r \in \mathbf{N}. \tag{4}$$

For $w \in D^{mr}(U)$, we have

$$[w](E^{\otimes r}) = \boldsymbol{(}\overline{H}^r, w\boldsymbol{)} F^{\otimes r}. \tag{5}$$

Indeed, $E^{\otimes r}$ belongs to the image of $\mathrm{alt}_m^{\otimes r}\colon A^{\otimes mr} \to A^{\otimes mr}$. The image of $\mathrm{alt}_m^{\otimes r}\colon B^{\otimes mr} \to B^{\otimes mr}$ is generated by $F^{\otimes r}$ since the image of $\mathrm{alt}_m\colon B^{\otimes m} \to B^{\otimes m}$ is generated by F. The map $[w]\colon A^{\otimes mr} \to B^{\otimes mr}$ preserves the action of Σ_{mr} and thus commutes with $\mathrm{alt}_m^{\otimes r}$. Therefore, $[w](E^{\otimes r}) = tF^{\otimes r}$ for some $t \in \boldsymbol{k}$. From (4), we get $t = \boldsymbol{\langle} H^{\otimes r}, w \boldsymbol{\rangle} = \boldsymbol{(}\overline{H}^r, w\boldsymbol{)}$.

For a morphism $s\colon T_A \to T_B$, we have

$$s_{mr}(E^{\otimes r}) = \langle \xi(s), \overline{H}^r \rangle F^{\otimes r}, \qquad r \in \mathbf{N}. \tag{6}$$

This follows from (5): if $s = \kappa(W)$, $W \in \hat{D}(U)$, then $s_{mr} = [W_{mr}]$ and $\langle \xi(s), \overline{H}^r \rangle = \boldsymbol{(}\overline{H}^r, W_{mr}\boldsymbol{)}$.

3. Homomorphisms $A \to B$ and morphisms $L_A \to L_B$

Let A and B be finite-dimensional algebras. Put $U = B^A$.

Multiplicativity ideal. Take $x, y \in A$ and $p \in B^*$. We have the linear form $I^p_{x,y} \in U^*$,

$$\langle I^p_{x,y}, u \rangle = \langle p, u(xy) \rangle, \qquad u \in U$$

(the multiplication in A is used) and the tensor $J^p_{x,y} \in (U^*)^{\otimes 2}$,

$$\boldsymbol{\langle} J^p_{x,y}, u \otimes v \boldsymbol{\rangle} = \langle p, u(x)v(y) \rangle, \qquad u, v \in U$$

(the multiplication in B is used). Put

$$g^p_{x,y} = \overline{J^p_{x,y}} - I^p_{x,y} \in \boldsymbol{k}[U].$$

We have

$$g^p_{x,y}(u) = \langle p, u(x)u(y) - u(xy) \rangle, \qquad u \in U.$$

Let $M \subseteq \boldsymbol{k}[U]$ be the ideal generated by the polynomials $g^p_{x,y}$, $x, y \in A$, $p \in B^*$.

Lemma 1. *The set $Z(M) \subseteq U$ coincides with the set of algebra homomorphisms $A \to B$.* □

Note that $\mathrm{Hom}_\Omega(L_A, L_B) \subseteq \mathrm{Hom}_\Sigma(T_A, T_B)$.

Lemma 2. *Let $s \in \mathrm{Hom}_\Sigma(T_A, T_B)$. Then the conditions $s \in \mathrm{Hom}_\Omega(L_A, L_B)$ and $\xi(s) \perp M$ are equivalent.*

Thus we establish an isomorphism $\mathrm{Hom}_\Omega(L_A, L_B) \to (\boldsymbol{k}[U]/M)^*$.

Proof. For $n \in \mathbf{N}$, define the morphism $\tau_n\colon \underline{n+2} \to \underline{n+1}$ by the rules $1 \mapsto 1$ and $i \mapsto i-1$, $i > 1$. The category Ω is obtained from Σ by adjunction of the morphisms τ_n. Therefore, the condition $s \in \mathrm{Hom}_\Omega(L_A, L_B)$ is equivalent to commutativity of the diagrams

$$\begin{array}{ccc}
A^{\otimes(n+2)} & \xrightarrow{\;s_{n+2}\;} & B^{\otimes(n+2)} \\
{\scriptstyle L_A(\tau_n)}\downarrow & & \downarrow{\scriptstyle L_B(\tau_n)} \\
A^{\otimes(n+1)} & \xrightarrow{\;s_{n+1}\;} & B^{\otimes(n+1)},
\end{array}$$

$n \in \mathbf{N}$. Consider the discrepancy

$$r_n = L_B(\tau_n) \circ s_{n+2} - s_{n+1} \circ L_A(\tau_n) \colon A^{\otimes(n+2)} \to B^{\otimes(n+1)}.$$

For $z \in A$, $q \in B^*$, we have the linear form $l_z^q \in U^*$, $\langle l_z^q, u\rangle = \langle q, u(z)\rangle$, $u \in U$. These forms generate U^*. For $n \in \mathbf{N}$, $x, y, z_1, \ldots, z_n \in A$, $p, q_1, \ldots, q_n \in B^*$, put

$$G_{x,y,z_1,\ldots,z_n}^{\,p,\ q_1,\ldots,q_n} = g_{x,y}^p l_{z_1}^{q_1} \ldots l_{z_n}^{q_n} \in \boldsymbol{k}[U].$$

These polynomials linearly generate M. Therefore, it suffices to prove that

$$\langle\!\langle p^\sim, r_n(x^\sim)\rangle\!\rangle = \langle \xi(s), G_{x,y,z_1,\ldots,z_n}^{\,p,\ q_1,\ldots,q_n}\rangle,$$

where $x^\sim = x \otimes y \otimes z_1 \otimes \ldots \otimes z_n$, $p^\sim = p \otimes q_1 \otimes \ldots \otimes q_n$.

We have

$$\langle\!\langle p^\sim, [w_1](L_A(\tau_n)(x^\sim))\rangle\!\rangle = \langle\!\langle I_{x,y}^p \otimes l^\sim, w_1\rangle\!\rangle, \qquad w_1 \in U^{\otimes(n+1)},$$
$$\langle\!\langle p^\sim, L_B(\tau_n)([w_2](x^\sim))\rangle\!\rangle = \langle\!\langle J_{x,y}^p \otimes l^\sim, w_2\rangle\!\rangle, \qquad w_2 \in U^{\otimes(n+2)},$$

where $l^\sim = l_{z_1}^{q_1} \otimes \ldots \otimes l_{z_n}^{q_n}$ (direct check). By construction,

$$G_{x,y,z_1,\ldots,z_n}^{\,p,\ q_1,\ldots,q_n} = \overline{J_{x,y}^p \otimes l^\sim} - \overline{I_{x,y}^p \otimes l^\sim}.$$

We have $s = \kappa(W)$ for some sequence $W \in \hat{D}(U)$, so $s_n = [W_n]$. We have

$$\begin{aligned}\langle\!\langle p^\sim, r_n(x^\sim)\rangle\!\rangle &= \langle\!\langle p^\sim, L_B(\tau_n)([W_{n+2}](x^\sim))\rangle\!\rangle - \langle\!\langle p^\sim, [W_{n+1}](L_A(\tau_n)(x^\sim))\rangle\!\rangle = \\ &= \langle\!\langle J_{x,y}^p \otimes l^\sim, W_{n+2}\rangle\!\rangle - \langle\!\langle I_{x,y}^p \otimes l^\sim, W_{n+1}\rangle\!\rangle = \\ &= \langle \theta(W), G_{x,y,z_1,\ldots,z_n}^{\,p,\ q_1,\ldots,q_n}\rangle = \langle \xi(s), G_{x,y,z_1,\ldots,z_n}^{\,p,\ q_1,\ldots,q_n}\rangle. \quad \square\end{aligned}$$

Proof of Theorem. Let $s \colon L_A \to L_B$ be a functor isomorphism. Then $s_1 \colon A \to B$ is an isomorphism of vector spaces. Put $m = \dim A = \dim B$. Choose bases in A and B. Let the tensors E, F and H be as in § 2. We seek an algebra homomorphism $u \colon A \to B$ with $\det u \neq 0$. Assume that there exists no such a homomorphism. Then, by (3) and Lemma 1, $\overline{H} \mid Z(M) = 0$. By Hilbert's Nullstellensatz, $\overline{H}^r \in M$ for some $r \in \mathbf{N}$. By (6) and Lemma 2, $s_{mr}(E^{\otimes r}) = \langle \xi(s), \overline{H}^r\rangle F^{\otimes r} = 0$. This is absurd since $E^{\otimes r} \neq 0$ and s_{mr} is an isomorphism of vector spaces. $\square$

Reference

[1] Hochschild homology, English Wikipedia entry.

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