Corrections of typos and small errors to the book “A Course in Metric Geometry” by D. Burago, Yu. Burago, and S. Ivanov

April 22, 2010

We are grateful to many mathematicians, especially M. Bonk and J. Lott, who have informed us of numerous misprints and inaccuracies in the book.

Misspelled words, missing articles and similar minor errors are not included in this list. The exclamation mark after page numbers indicates errors which affect larger (than a couple of lines) parts of the text. Some of such large corrections are included at the end of the table.

page	line	Description
Chapter 1		
13	−1	Add “finite” before “ε-net” and “(2ε)-net”.
21	−8	$B \in \mathfrak{B}_{m-1} \quad \longrightarrow \quad B \in \mathfrak{B}_{k-1}$
21	−2	sup $\quad \longrightarrow \quad \subset$
Chapter 2		
38	sections 2.4–2.5	Although we usually allow infinite distances, many statements here are obviously valid for finite metrics only. This condition has to be added where necessary.
42	4th line of section 2.4.4	Remove “locally-compact”. (This assumption is not essential and is never used.)
47	−8	$\lvert t - t_j \rvert < \varepsilon \quad \longrightarrow \quad \lvert t - t_j \rvert < \varepsilon / C$
48	2	$\frac{C}{4\varepsilon} \quad \longrightarrow \quad \frac{4C}{\varepsilon}$.
48	6	$\gamma_i(k/N) \quad \longrightarrow \quad \gamma_i(t)$.
53	−10	Remove “different”. (Ambiguity.)
54	Ex 2.6.4	$\ell \quad \longrightarrow \quad L \quad$ (two times).
Chapter 3		
65	12	Add “around a fixed point” after “rotations”.

70	Example 3.2.9	Exercise 3.2.10 and the next paragraph, the way how they are formulated, require a condition $k \le 1$. Indeed, it may happen that adding simplices as described can create a shorter path between two points in X.
71	-15	Replace "cardinality of an" by "cardinality of the endpoints of segments in the"
77	9	Should read: "... and with p_i being equivalent to q_{i-1} for all $i \ge 1$. ..." (This means essentially the same, but provides consistent notations for indices through the argument.)
77	10	$g_i(q_i) = p_{i+1} \longrightarrow g_i(p_i) = q_{i-1}$
77	-14	Omit "(q_i)" after the long composition of maps.
77	-10	$x \longrightarrow p$.
79	-8	of sets $\longrightarrow$ of open sets
85	1st par of Proof	$U_y \longrightarrow U_q$ (two times), connected components $\longrightarrow$ disjoint open sets
85	-14	$g_p^{-1} \longrightarrow g_p$ (two times)
85!		The proof of Theorem 3.4.18 is incomplete. We should have proved that the set V_p is open (required in the definition of a covering map) and the set $\overline{V}_p$ is closed (for the "compactness implies continuity" argument to work). Adding an assumption that every geodesic segment contained in our neighborhood is a shortest path makes the proof correct (and it may be even simplified). For all our applications such a weaker theorem is enough.
88	Ex. 3.6.2	The two latter product spaces contain $\longrightarrow$ The last product space contains
89	2nd par of Rem 3.6.3	One has to require that the restrictions of the norm to the rays $\{x_0, y > 0\}$ and $\{x > 0, y_0)\}$ are monotone.
91	14	$a = tx \longrightarrow a = rx$
93	Def 3.6.16	$t + s \le \pi \longrightarrow d(x,y) \le \pi$ $t + s \ge \pi \longrightarrow d(x,y) \ge \pi$
99	-1	$\ge \theta(a,c) \longrightarrow \ge \theta(a,b)$

Chapter 4

103	14	$g_0(t) \le g(t)$ (resp. $g_0(t) \ge g(t)$) $\longrightarrow$ $g_0(t) \ge g(t)$ (resp. $g_0(t) \le g(t)$)
103	Def 4.1.2	Same as above.
105	10	$c \longrightarrow p$
105	14	$\bar{c} \longrightarrow \bar{p}$
106	14	$\frac{\sqrt{3}}{2} < \frac{1}{2} \longrightarrow \frac{\sqrt{3}}{2} > \frac{1}{2}$
107	last line of Dfn 4.1.9	$\lvert\bar{d}\bar{b}\rvert = \lvert db\rvert \longrightarrow \lvert\bar{a}\bar{d}\rvert = \lvert ad\rvert$
113	-7	$\triangle abc \longrightarrow \triangle a'b'c'$
117	-11	Add "for $\triangle pq_0s$ and $\triangle sq_0r$" before "on different sides"
117	-6	The inequality should read: $\measuredangle p_0q_0s_0 + \measuredangle s_0q_0r_0 \le \pi$.

120	−14	$F_k\lambda \longrightarrow F_\lambda$

Chapter 5

144	−3	$\varphi\colon U \in \mathbb{R}^2 \to \mathbb{R}^3 \longrightarrow \varphi\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3$
146	11	$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$
148	7	a curve γ $\longrightarrow$ a naturally parameterized shortest path γ
148	12	a certain speed $\longrightarrow$ a certain (unit) speed
148	eq. (5.5)	$t \longrightarrow t_0$
150	eq. (5.10) and (5.11)	The equations should read:

$$E\frac{d^2x}{dt^2} + F\frac{d^2y}{dt^2} = -\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2\frac{\partial E}{\partial x} + \frac{dx}{dt}\frac{dy}{dt}\frac{\partial E}{\partial y} + \left(\frac{dy}{dt}\right)^2\left(\frac{\partial F}{\partial y} - \frac{1}{2}\frac{\partial G}{\partial x}\right)\right),$$

$$F\frac{d^2x}{dt^2} + G\frac{d^2y}{dt^2} = -\left(\left(\frac{dx}{dt}\right)^2\left(\frac{\partial F}{\partial x} - \frac{1}{2}\frac{\partial E}{\partial y}\right) + \frac{dx}{dt}\frac{dy}{dt}\frac{\partial G}{\partial x} + \frac{1}{2}\left(\frac{dy}{dt}\right)^2\frac{\partial G}{\partial y}\right).$$

151	3	The equation should read:

$$\frac{d}{dt}\left(E\left(\frac{dx}{dt}\right)^2 + 2F\frac{dx}{dt}\frac{dy}{dt} + G\left(\frac{dy}{dt}\right)^2\right) = 0.$$

153	19	Replace F by E in the first of two identical equations: "$0 = -\partial F/\partial x$".
158	−13	In fact, medians **do** meet in one point in the sphere and in the hyperbolic plane. We'd better choose another example...
160	3	The formula covers only orientation-preserving hyperbolic isometries. Isometries inverting orientation can be described as complex transformation of the form $T(z) = \frac{a\bar{z}+b}{c\bar{z}+b}$ where a, b, c, d are arbitrary reals with $ad - bc = -1$. The set of all isometries is the union of two these classes.
162	−8	Remove the minus sign before "ln".
165	−13	The first line of the long equation should read:

$$L(I \circ \gamma, a, b) = \int_a^b I_y(\gamma(t))^{-1}\left|\frac{dI(\gamma(t))}{dt}\right| dt$$

165	−2	of hyperbolic rigid motions $\longrightarrow$ of orientation-preserving hyperbolic rigid motions
173	−9	$0 > \alpha \geq \beta > \pi \longrightarrow \alpha$ and β (α is at the left vertical side and β is at the right one)
174	−12	area of a circle $\longrightarrow$ length of a circle
175	1	1-neighborhood $\longrightarrow$ 2-neighborhood
175	11	In both inequalities, replace the constant 2 by 4.
180	−13	$\dot{x}\cos\alpha = \dot{y}\sin\alpha \longrightarrow \dot{x}\sin\alpha = \dot{y}\cos\alpha$
180	−11	$(\cos\alpha, \sin\alpha, 0) \longrightarrow (-\sin\alpha, \cos\alpha, 0)$
183	−12	The argument "(p)" should be replaced by "$(0, 0, 0)$" everywhere in these formulae.

183	equations at the bottom of the page	These formulae make sense only for $\tau \geq 0$. To make them valid in general, replace all $\sqrt{\tau}$ by $\sqrt{\lvert\tau\rvert}$ and, in addition, multiply the exponents at φ_2 (but not at φ_1!) by the term $sign(\tau)$.
189	−16	$H_V \longrightarrow X_V$ (two times).
191	−11	$= (x, y, x_0 y) \longrightarrow = (x, y, z + x_0 y)$
195	−14	$T_\varphi(x)M \longrightarrow T_{\varphi(x)}M$ (two times).
204	−14 (item 2)	Add a sentence: "This means that f maps I^n to the parallelotope $P = [0, d_1] \times [0, d_2] \times \ldots [0, d_n]$."
205	−3, −4	$1/4\pi \longrightarrow 2/\pi$ (two times)
205	−1	$1/2\pi \longrightarrow 1/\pi$
206	−16	$\frac{1}{4\pi} \longrightarrow \frac{2}{\pi}$

Chapter 6

212!	pages 212–213	The formula (6.1) is incorrect just for the reason explained two lines below in the book—computing the derivative with respect to ε, one should take into account the variable nature of a Riemannian scalar product, just like in the subsequent computations of derivatives with respect to t! This leads to a missing term in (6.1) which is carried over through all computations on pages 212–213. As a result, the derived equations for geodesics ((5.10) and (5.11)) are incorrect (see the above comment for page 150).
213	−5	Replace ∇ by Δ
217	12–17	All notations of the form Γ^k_{ij} should be changed to $\Gamma_{ij,k}$.
218	eq.(6.4)	Another fundamentally incorrect formula. It is valid only at points where $E = G = 1$ and $F = 0$. Luckily, we never use it...
228	−13	$g(t)\neg 0 \longrightarrow g(t) \neq 0$ $g(t)) = \longrightarrow g(t) =$
229	eq.(6.13)	The first part of the equation (namely, the statement that the scalar product involving Y is zero) is correct but only *a posteriori*. It follows from the last equation of (6.13) because Y is proportional to N.
229	−10,−2	Replace all occurrences of $\frac{D}{dx}$ by $\frac{D}{dt}$ (4 times).
232	1	Add "at γ_0" after the formula.
232	2	V and T are unit vector fields $\longrightarrow$ V is a unit vector field
235	Lem 6.4.12	Remove piece: "For a unit vector $V \in T_q\Omega$,"
237	8	$[0, T] \longrightarrow]0, T[$.
237	Proof of Thm 6.5.1	The proof works only under the assumption $K_0 \leq 0$. (This is implicitly used in the equation on the last line of the page.) See page 8 below for another proof.

Chapter 7

248	−3	The fragment "indexconvergence!uniform" should be an index entry.

253	−16	for an $x \in S$ $\longrightarrow$ for every $x \in S$
253	−12	for an $x \in S_n$ $\longrightarrow$ for every $x \in S_n$
263	12	S_n converge to X $\longrightarrow$ S_n converge to S
269	Ex 7.5.11	Add an assumption that X is locally simply connected. Refer to Exercise 7.5.8 instead of 7.5.9.

Chapter 8

273	3	ε_n $\longrightarrow$ ε (two times)
278	−18	$f(X)$ and $g(Y')$ $\longrightarrow$ $f_1(X)$ and $f_2(Y')$
280	−15, Eq. (3)	$\|g_1\| \cdot \|g_2\|$ $\longrightarrow$ $\|g_1\| + \|g_2\|$
281	13	of x. $\longrightarrow$ of x (up to inner automorphisms).
283	7	$\|g_1\|_2 \cdots \|g_n\|_2$ $\longrightarrow$ $\|g_1\|_2 + \cdots + \|g_n\|_2$.
284	7	$= x_i$ $\longrightarrow$ $= y_i$
286	9–10	$\leq$ $\longrightarrow$ $\geq$ (two times)
288	13	$[cf]$ $\longrightarrow$ $[df]$
289	Thm 8.4.16	In the proof below we work with strictly intrinsic metrics. Using “almost shortest paths”, the reader can easily adopt the argument to the situation when shortest paths may fail to exist.
290	3	c $\longrightarrow$ C
290	4	M $\longrightarrow$ X
291	8	$d(a,b)$ $\longrightarrow$ $d(b,c)$
291	15	$d(a'_i, a'_{i+1})$ $\longrightarrow$ $d(b'_i, b'_{i+1})$
291	−9	$+6kL\frac{1}{k} =$ $\longrightarrow$ $+6L\frac{1}{k} =$
292!	9	2δ-neighborhood $\longrightarrow$ 3δ-neighborhood This correction obviously implies some changes to other constants too (like 4δ). Namely, one should replace 4δ by 6δ twice in Lemma 8.4.24 and do corresponding (obvious) changes in the proofs of Lemma 8.4.24 and of the Morse Lemma 8.4.20 on page 292. (A nice proof of the Morse Lemma also can be found in [BH]).
292	−15	smallest $\longrightarrow$ biggest
292	−11	ga $\longrightarrow$ γ
292	−7	Omit the last term $\geq \frac{n}{9}$
292	−6	Replace the line by “Since $R \gg \delta$ (namely, $R > k^2\delta > (20C+8)^2\delta > 400C\delta$), the last inequality implies $n < 9C$.”
297	17	$K\tilde{K}$ $\longrightarrow$ $\tilde{K}$
299	−4	$\leq \sum x_i \|e_i\| \leq N^2$ $\longrightarrow$ $\leq \sum \|x_i\| \|e_i\| \leq N$

300	-12	The formula does not follow from the preceding ones. Correction: replace the whole statement "Hence ..." by:

Let k be the integer such that $k \leq \|w\| < k+1$, then $\|w - kv\| \leq \varepsilon\|w\| + 1$, then

$$d(Mw) \leq d(kMv) + d(Mw - kMv) \leq (1+\varepsilon)kM + CM(\varepsilon\|w\| + 1),$$

hence $d(Mw)/\|Mw\| \leq 1 + \varepsilon + C\varepsilon + M/\|w\|$.

Chapter 9

308	10	$\sqrt{k} \longrightarrow 1/\sqrt{k}$
308	11	$1/\sqrt{k} \longrightarrow \pi/\sqrt{k}$
308	-17	$U - I \longrightarrow U_i$
313	1	$\leq r \longrightarrow < r$
313	2	$\leq r \longrightarrow < r$
313	12	paths starting $\longrightarrow$ parts in U starting
314	-18	three $\longrightarrow$ four
314	-16	$d \longrightarrow x$
314	-3	$\|a_0b_0\| = \|\bar{a}\bar{b}\| \longrightarrow \|a_0x_0\| = \|\bar{a}\bar{x}\|$
315	5	second $\measuredangle bxc \longrightarrow \measuredangle axc$
315	9	(Theorem 4.5.6) $\longrightarrow$ (for $k = 0$ it is Theorem 4.5.6)
315	16	$\bar{x}, \bar{x}' \longrightarrow \bar{x}, \bar{y}$
316	10	$C \in \mathbf{X} \longrightarrow C \subset \mathbf{X}$
316	Figure 9.1	Letters X_1 and X_2 should be interchanged
318	3	Hopf–Rinow Theorem $\longrightarrow$ simple implication $(i) \Rightarrow (iii)$ in the Hopf–Rinow Theorem (This implication does not require local compactness of X.)
318	7	Hopf-Rinow theorem $\longrightarrow$ the implication mentioned above.
320	-12	$d(x,y) \longrightarrow \|xy\|$
324	17	$\triangle pab \longrightarrow \widetilde{\triangle} pab$
324	18	$\|pa\| = \|O\tilde{a}\|$, $\|pb\| = \|O\tilde{b}\| \longrightarrow \|\bar{p}\bar{a}\| = \|O\tilde{a}\|$, $\|\bar{p}\bar{b}\| = \|O\tilde{b}\|$
324	19	$\measuredangle apb \geq \longrightarrow \measuredangle \bar{a}\bar{p}\bar{b} \geq$
333	13	convex $\longrightarrow$ 1-convex
336	-16	surface $T_1 \cup T_2 \longrightarrow$ surface $T_3 \cup T_4$
336	-12	$T_1 \cup T_2 \longrightarrow T_3 \cup T_4$
338	-2 -3	$p \longrightarrow q$ (3 times)
339	7	Pressmann $\longrightarrow$ Preissmann
340	1	the set of sums $\longrightarrow$ the set of positive sums
341	-2	$\mathbb{R}^{3n} \longrightarrow \mathbb{R}^{3N}$

341	-1	$\dots, x_n, y_n, z_n \quad \longrightarrow \quad \dots, x_N, y_N, z_N$
342	9	The formula should read $K((v_1, v_2, \dots, v_N), (v_1, v_2, \dots, v_N)) = \frac{1}{2}\sum_{i=1}^{N} m_i \langle v_i, v_i \rangle$,
348	-18	$a_{a+1} \quad \longrightarrow \quad a_{i+1}$ (two times)

Chapter 10

353	$-$ 12–13	The equation should read $\widetilde{\measuredangle} ba'd + \widetilde{\measuredangle} da'c + \widetilde{\measuredangle} ca'b \le \measuredangle ba'd + \measuredangle da'c + \measuredangle ca'b \le (\measuredangle ba'd + \measuredangle da'a) + (\measuredangle aa'c + \measuredangle ca'b) = 2\pi.$
358	10	dilatation $\longrightarrow$ distortion
361	12	of p. $\longrightarrow$ of q.
361	-7	in Step 2, $\longrightarrow$ in Step 2 and such that the angle condition fails just at c,
362	2, 11, -5	Figure 10.3 $\longrightarrow$ Figure 10.1 (three times)
365	15	$\measuredangle zy'z' = \measuredangle yz'y' = 0, \quad \longrightarrow \quad \measuredangle zz'y' = \measuredangle yy'z' = 0,$
366	2	$\ge k \quad \longrightarrow \quad \ge 1$
366	5	maximal $\longrightarrow$ minimal
366	Def 10.5.3	straight lines $\longrightarrow$ parallel straight lines
367	-10	$= \lvert\bar a\bar c\rvert + \lvert\bar a c\bar b\rvert = \lvert\bar a\bar c\rvert. \quad \longrightarrow \quad = \lvert\bar a\bar c\rvert + \lvert\bar c\bar b\rvert = \lvert\bar a\bar b\rvert.$
367	-8	$\lvert\bar a\bar c\rvert \quad \longrightarrow \quad \lvert\bar a\bar b\rvert$
369	5,6	$\gamma_q \quad \longrightarrow \quad \gamma_x$
369	6	through q. $\longrightarrow$ through x.
383	4	$-2\delta \quad \longrightarrow \quad -\dfrac{\varepsilon}{2}$
383	-1	Add "Note that $\lvert pa_2\rvert < \lvert a_2b_2\rvert$, $\lvert pb_2\rvert < \lvert a_2b_2\rvert$, so Lemma 10.8.13 is applicable to the 1-strainer (a_1, b_1) for p and points $q = a_2$, $q = b_2$." before "Then" .
384	6	$\widetilde{\measuredangle} b_1pb_2 \quad \longrightarrow \quad \widetilde{\measuredangle} b_1pa_2$
384	-17	$-2\delta \quad \longrightarrow \quad -\dfrac{\varepsilon}{2}$
384	-13	$+\lvert p_2\rvert \quad \longrightarrow \quad +\lvert pb_2\rvert$
386	7	$\lvert x_na_{i_0}\rvert < \lvert x_{n+1}a_{i_0}\rvert \quad \longrightarrow \quad \lvert x_na_{i_0}\rvert > \lvert x_{n+1}a_{i_0}\rvert$
386	16	$\widetilde{\measuredangle} a_ix_nx_{n+1} \quad \longrightarrow \quad \widetilde{\measuredangle} x_na_ix_{n+1}$
387	-13	$< \quad \longrightarrow \quad >$
391	-13	$t \to \infty \quad \longrightarrow \quad t \to 0$
392	9	$p_i\gamma_i(r) \quad \longrightarrow \quad p_i = \gamma_i(r)$
394	-10	$\lvert\xi\eta_i\rvert \quad \longrightarrow \quad \lvert\xi_i\eta_i\rvert$
400	-3	$\mathbb{C}^k \quad \longrightarrow \quad \mathbb{C}^{k+1}$
401	1	$(z_1, \dots, z_3) \quad \longrightarrow \quad (z_1, z_2, z_3)$
401	4	$\mathbb{CP}^1 \quad \longrightarrow \quad K_0(\mathbb{CP}^1)$

401 4 $\mathbb{CP}^2 \longrightarrow K_0(\mathbb{CP}^2)$

Large corrections

Proof of Theorem 6.5.1

First we prove the theorem under an additional assumption that Y_0 does not vanish on $]0, T[$. Let $g_0(t) = |Y_0(t)|$, $g_1(t) = |Y_1(t)|$. Then the functions g_0 and g_1 satisfy the equations

$$\ddot{g}_i(t) = -K_i(t) g_i(t) \quad \text{subject to} \quad g_i(0) = 0,\ \dot{g}_i(0) = 1$$

where the dot denotes the derivative with respect to t.

We want to prove that $g_1(t) \leq g_0(t)$ for all $t \in [0, T]$.

Consider a function $\varphi(t) = \frac{g_0(t)}{g_1(t)}$ defined on $]0, T[$. Observe that $\lim_{t\to 0} \varphi(t) = 1$ by the L'Hopital rule. We will prove that φ is (non-strictly) monotone increasing and hence $\varphi(t) \geq 1$ for all t.

To prove the monotonicity of φ, it suffices to verify that $\dot{\varphi}(t) \geq 0$ for all t. We have

$$\dot{\varphi}(t) = \frac{\dot{g}_0(t) g_1(t) - g_0(t) \dot{g}_1(t)}{g_1(t)^2}.$$

Denote the numerator of the last formula by $\psi(t)$. Since the denominator $g_1(t)^2$ is positive, we have to prove that $\psi(t) \geq 0$. Observe that $\psi(0) = 0$ because $g_0(0) = g_1(0) = 0$. So again it suffices to prove that $\dot{\psi}(t) \geq 0$ for all t. ¿From the equations for g_0 and g_1 one gets

$$\dot{\psi}(t) = \ddot{g}_0(t) g_1(t) - g_0(t) \ddot{g}_1(t) = (K_1(t) - K_0(t))\, g_0(t)\, g_1(0) \geq 0.$$

since $K_1(t) \geq K_0(t)$. Thus we have proved that $\varphi(t) \geq 1$ for all $t \in]0, T[$ and hence $g_1(t) \leq g_0(t)$ for all $t \in [0, T]$.

It remains to get rid of the assumption that Y_0 does not vanish. Suppose this is not the case, and let T_0 be the first point where Y_0 vanishes. Then the above argument applies to the interval $]0, T_0[$ instead of $]0, T[$, and we conclude that $|Y_0(T_0)| \geq |Y_1(T_0)| > 0$, contrary to the choice of T_0.