RIEMANNIAN TORI WITHOUT CONJUGATE POINTS ARE FLAT

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Introduction

The purpose of this paper is to prove the following

Theorem 1. *Let* Tor^n *be an* n*-dimensional torus with a Riemannian metric* d *which does not have conjugate points. Then* d *is a flat metric.*

This statement is known as the Hopf conjecture and it has been proved by E. Hopf ([Ho]) for the case $n = 2$.

The proof of Theorem 1 is contained in sections 1–5.

The main idea of our proof is that the limit norm of such a metric (see section 1.2) is a Euclidean norm. For two unit vectors p, q in a Banach space with its unit sphere having a unique supporting linear function $-B_p$ at p one can define something like inner product $\langle p, q\rangle = -B_p(q)$. To show that it actually is an inner product we prove that Euclidean norms possess some extremal property of integral type which makes them distinguishable among all Banach norms. Then we note that the functions $B_p(q)$ for the limit norm of our metric can be drawn from the infinitesimal inner product by the means of integral geometry, and we check the property above for the limit norm. This proves that the limit norm is Euclidean and our inequalities for the integrals turn out to be equalities almost everywhere. Then a rather simple additional argument shows that in this case our metric is flat.

Section 6 contains a brief discussion and the volume growth theorem.

The history of the subject will not be touched upon in this short paper.

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1. Rational Foliations

1.1. We consider a Riemannian metric $\tilde{d}$ on $\mathbf{R}^n$ which is a lift of some metric d on $\mathrm{Tor}^n = \mathbf{R}^n/\mathbf{Z}^n$. Thus $\tilde{d}$ is invariant under the action of $\mathbf{Z}^n$ by integer translations. We denote by UTTor^n and $\mathrm{UT}\mathbf{R}^n$ the unit tangent bundles for metrics d and $\tilde{d}$. By $\exp$, $\langle , \rangle$ we mean the exponential maps and inner products of the metrics d and $\tilde{d}$.

1.2. It is known (see [Bu1]) that there exists a Banach norm $\|\cdot\|$ on $\mathbf{R}^n$ (whose unit sphere we denote by F) and a constant c such that

$$\forall x, y \in \mathbf{R}^n \quad \big|\|x - y\| - \tilde{d}(x, y)\big| \leq c. \tag{1}$$

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Remark 1.3. The proof of (1) for a metric without conjugate points is much simpler than in the general case.

1.4. Hereafter, we assume that $\tilde{d}$ does not have conjugate points and hence the length of every geodesic segment is just the distance between its endpoints.

For $(x, v) \in \mathrm{UT}\mathbf{R}^n$ we define its direction $D(x, v)$ at infinity as

$$D(x, v) = \lim(\exp_x tv - x)/t, \qquad t \to \infty.$$

The function D is correctly defined for almost all $(x, v) \in \mathrm{UT}\mathbf{R}^n$ (by the Birkhoff ergodic theorem). It is clear that $\|D(x, v)\| = 1$ and thus D is a map to F.

1.5. We call a point $p \in F$ *rational* if $ap \in \mathbf{Z}^n$ for some positive $a \in \mathbf{R}$. Let $p \in F$ be rational and $ap \in \mathbf{Z}^n$, $a > 0$. It is known (see [Bus]) that there exists a $\mathbf{Z}^n$-invariant vector field $v_p : \mathbf{R}^n \to \mathrm{UT}\mathbf{R}^n$ such that its trajectories are geodesics of the direction p, i.e. $D(v_p(x)) = p$, and $\exp_x tv_p(x) = x + tp$ whenever $tp \in \mathbf{Z}^n$. Obviously v_p is smooth since $v_p(x) = a^{-1} \exp_x^{-1}(x + ap)$. We call this foliation determined by v_p a rational foliation of the direction p.

1.6. Let $\gamma(t) = \exp_0 tv_p(0)$ be a ray of our rational foliation. We denote by B_γ the Busemann function of the ray γ in $(\mathbf{R}^n, \tilde{d})$. Recall that $B_\gamma(y) = \lim(\tilde{d}(\gamma(t), y) - t)$, $t \to \infty$.

By B_p we denote the Busemann function of the ray tp in the Banach space $(\mathbf{R}^n, \|\cdot\|)$. Applying (1) and the periodicity of γ we have

$$\forall y \in \mathbf{R}^n \qquad |B_p(y) - B_\gamma(y)| \leq c \tag{2}$$

The Busemann function B_γ of every ray $\gamma(t) = \exp_{x_0} tv_p(x_0)$ of our rational foliation has the gradient field $-v_p$. Indeed, B_γ is a Lipschitz function with Lipschitz constant 1; hence it has a gradient g almost everywhere and $\langle g, g\rangle \leq 1$.

Then $\frac{d}{dt}\big|_0 B_\gamma(\exp_x tv_p(x)) = \langle g(x), v_p\rangle$ whenever $g(x)$ is defined.

However, $B_\gamma(\exp_x tv_p(x)) = B_\gamma(x) - t$ since $B_\gamma(\exp_x av_p(x)) = B_{\gamma(t+a)}(x) = B_\gamma(x) - a$, and B_γ is Lipschitz with constant 1. Hence $\langle g, v_p\rangle = -1$ (i.e. $g = -v_p$) almost everywhere and thus $g = -v_p$ since v_p is smooth. (Probably the equality $g = -v_p$ was already known to Busemann in the 50th and, certainly, to J. Heber ([H]) but the authors do not have an appropriate reference).

1.7. Thus the Busemann function of every ray of our rational foliation can be represented as $B_{\gamma_p} + \mathrm{const}$ where B_{γ_p} is the Busemann function of the ray $\gamma_p(t) = \exp_0 tv_p(0)$ that starts from the origin. Thus we obtain a $\mathbf{Z}^n$-invariant horospherical foliation. On the other hand, it follows from (2) that every horosphere for the Busemann function B_{γ_p} lies within bounded distance from some horosphere for the Busemann function B_p. Since every horosphere of a ray pt in the Banach space $(\mathbf{R}^n, \|\cdot\|)$ is a translation of the tangent cone to F at $-p$ we see that B_p is a linear function and F has unique supporting hyperplane at p. Then $-B_p$ is a unique linear support function to F at p, i. e. $-B_p(p) = 1$ and $-B_p(x) \leq 1$ for all $x \in F$.

The points of F that have unique supporting hyperplane we call *smooth points of* F.

2. Integral Geometry

2.1. We fix a rational point $p \in F$. Let $q = D(x,w)$, $(x,w) \in UT\mathbf{R}^n$ and $\gamma(t) = \exp_x tw$. We have

$$-\lim\ T^{-1}\int_0^T \langle \dot\gamma(t), v_p\rangle dt = \lim\ T^{-1}\big(B_{\gamma_p}(\gamma(T)) - B_{\gamma_p}(\gamma(0))\big) =$$
$$= \lim\ T^{-1}\big(B_p(\gamma(T)) - B_p(\gamma(0))\big) = B_p\big(\lim\ T^{-1}(\gamma(T)-\gamma(0))\big) = B_p(q).$$

Hence by Schwartz inequality, we have

$$\liminf\ T^{-1}\int_0^T \langle \dot\gamma(t), v_p\rangle^2 dt \geq (B_p(q))^2. \tag{3}$$

2.2. We fix some (measurable) lifting map $L : \mathrm{UTTor}^n \to \mathrm{UT}\mathbf{R}^n$ such that $d\pi \circ L = \mathrm{id}$ where π is the covering map $\pi : \mathbf{R}^n \to \mathrm{Tor}^n$.

Let mes be the normalized Liouville measure on UTTor^n and $m = D \circ L(\mathrm{mes})$ be the measure on F swept by the geodesic flow onto F. Setting

$$C : \mathrm{UTTor}^n \to \mathbf{R}, \qquad C(w) = \langle L(w), v_p\rangle^2 = \langle w, d\pi(v_p)\rangle^2$$

we obtain from (3) by applying the Birkhoff ergodic theorem to the function C that

$$\int_{\mathrm{UTTor}^n} C d\mathrm{mes} \geq \int_{\mathrm{UTTor}^n} (B_p(D\circ L(w)))^2 d\mathrm{mes} = \int_F (B_p(q))^2 dm(q)$$

where the latter equality holds just by the definition of the measure m.

2.3. Denote by vol and mes_x the normalized Riemannian volume on (Tor^n, d) and the normalized Riemannian volume on the sphere $(\mathrm{UT}_x\mathrm{Tor}^n, \langle,\rangle)$, correspondingly. We rewrite the integral $\int_{\mathrm{UTTor}^n} C d\mathrm{mes}$ as

$$\int_{\mathrm{UTTor}^n} C d\mathrm{mes} = \int_{\mathrm{Tor}^n} d\mathrm{vol} \int_{\mathrm{UT}_x\mathrm{Tor}^n} \langle w, d\pi(v_p)\rangle^2 d\mathrm{mes}_x(w) =$$
$$= \int_{S^{n-1}} \langle w, v\rangle^2_{\mathbf{R}^n} d\mu(w)$$

where S^{n-1} is the standard sphere in $\mathbf{R}^n$, $\langle,\rangle_{\mathbf{R}^n}$ is the standard inner product and μ is the standard (normalized) measure.

2.4. Obviously the latter integral does not depend on $v \in S^{n-1}$. Adding its value over $v_1, v_2, \ldots v_n$ where $\{v_i\}$ form an orthonormal basis in $(\mathbf{R}^n, \langle,\rangle_{\mathbf{R}^n})$, we obtain

$$n\int_{S^{n-1}} \langle w, v\rangle^2_{\mathbf{R}^n} d\mu(w) = \int_{S^{n-1}} \sum \langle w, v_i\rangle^2_{\mathbf{R}^n} d\mu(w) = 1.$$

Thus for every rational $p \in F$ we have

$$\int_F (B_p(q))^2 dm(q) \leq \frac{1}{n} \tag{4}$$

and hence (4) holds for every smooth point $p \in F$ (since B_p continuously depends on p at every smooth point).

3. D is Continuous at Vectors of Rational Foliations

Lemma 3.1. *Let $p \in F$ be a rational point and P be the supporting hyperplane for F at p. Then $P \cap F = \{p\}$.*

Proof. Reasoning by contradiction, assume $p \neq q \in P \cap F$. Thus $\|ap + bq\| = a + b$ for all $a, b \geq 0$.

Let $\delta > 0$ be such that for every geodesic γ such that $\|D(\dot\gamma) - p\| \geq \|p - q\|/2$ there exists $t > 0$ such that $1 - \langle \dot\gamma(t), v_p \rangle \geq \delta$. Such δ can be chosen since the derivatives of projectors along the rays of our rational foliation onto a horosphere of any ray of the foliation are uniformly bounded.

Let

$$0 < s = \min\{a + b - \tilde d(\exp_x av, \exp_x -bw) : \\ a, b \geq 1, (x, v), (x, w) \in UT\mathbf{R}^n, 1 - \langle v, w \rangle \geq \delta\}.$$

To show $s > 0$ one can substitute the condition $a = b = 1$ in place of $a, b \geq 1$ in the definition of s (not increasing s by the triangle inequality) and then apply the standard compactness reasonings.

Let $ap \in \mathbf{Z}^n$ be an integer vector, $a \geq 1$. Using the standard technique of rational approximations we construct sequences of rational $q_i \in F$ and positive $b_i \in \mathbf{R}$ such that $\lim q_i = q$, $b_i q_i \in \mathbf{Z}^n$ are integer vectors and $\lim b_i\|q - q_i\| = 0$.

Choose i such that $\|q - q_i\| < \|q - p\|/2$ and $b_i\|q - q_i\| < s/2$.

Let $r = q_i$ and $b = b_i$. For all $x \in \mathbf{R}^n$ we have

$$\tilde d(\exp_x av_p, \exp_x -bv_r) = \|br + ap\|$$

since $ap + br$ is an integer vector.

Take $x \in \mathbf{R}^n$ such that $1 - \langle v_p(x), v_r(x) \rangle \geq \delta$ (this is possible by the definition of δ). Then

$$a + b - \|br + ap\| \geq s$$

by the definition of s. On the other hand,

$$\|br + ap\| = \|bq + ap - b(q - r)\| \leq \|bq + ap\| + b\|q - r\|$$

Taking into account that $\|bq + ap\| = b + a$ and $b\|q - r\| < s/2$ we have

$$a + b - \|br + ap\| < s/2,$$

and this is a contradiction. □

Lemma 3.2. *D is continuous at vectors of rational foliations.*

Proof. This lemma is an immediate consequence of the previous lemma and Theorem 3.10 of [Ba1]. However we present a simple proof based on (1). Let $\gamma(t) = \exp_x tv_p(x)$ be a ray of our rational foliation and $N_\varepsilon = \{q \in F : \|p - q\| \leq \varepsilon\}$. We take arbitrary $1 > \varepsilon > 0$ and seek for a neighborhood U_ε of (x, v) such that $D(U_\varepsilon) \subset N_\varepsilon$. We shall choose U_ε such that every ray with the initial vector from U_ε remains close to the ray γ for very long time. Let $s = \min\{2 - \|p + q\| : q \in F, \|p - q\| = \varepsilon\}$. The previous lemma implies $s > 0$ (since F is smooth at p). Let $Tp \in \mathbf{Z}^n$ and $T > 100(1+10c)/(s\varepsilon)$. Choose U_ε such that $\tilde d(\gamma, \exp_y tw) < c$ for every $(y, w) \in U_\varepsilon$ and $t \leq 3T$. Reasoning by contradiction, assume $\|D(y, w) - p\| \geq \varepsilon$ for some $(y, w) \in U_\varepsilon$. Then $\|p - q\| = \varepsilon$ for some $t > T$ where $q = (A - B)/R$, $A = \gamma(T)$,

$B = \exp_y(T+t)w$, and $R = \|A - B\|$. We have $\tilde{d}(y, B) = T + \tilde{d}(\exp_y Tw, B) \geq T + R - 2c$, hence $\tilde{d}(x, B) \geq T + R - 3c$. On the other hand,

$$\begin{aligned}\tilde{d}(x, B) &\leq \|B - x\| + c = \|Tp + Rq\| + c \leq T\|p + q\| + (R - T) + c \leq \\ &\leq 2T - Ts + (R - T) + c < T + R - 100c,\end{aligned}$$

and this is a contradiction. □

4. Roundness of Banach Norm

4.1. Let F be the unit sphere of some norm $\|\cdot\|$ in $\mathbf{R}^n$. Recall that we call a point $p \in F$ *smooth* if F has unique supporting linear function at p; this linear function is denoted by $-B_p$. For a probabilistic measure m on F we define the *loss of roundness* $R(F, m)$ setting

$$R(F, m) = \sup\{r(p, F, m) : p \in F \text{ is smooth}\},$$

where

$$r(p, F, m) = \int_F (B_p(q))^2 dm(q).$$

Lemma 4.2. *For all F, m we have $R(F, m) \geq 1/n$. If $R(F, m) = 1/n$ and the support of m is dense in F then F is an ellipsoid and $r(p, F, m) = 1/n$ for every $p \in F$.*

Proof. Denote by A the space of non-negative quadratic forms $A = \{Q : \mathbf{R}^n \to \mathbf{R}\}$. For a $Q \in A$ by Ball_Q we denote the unit ball $\{x \in \mathbf{R}^n, Q(x) \leq 1\}$. Let

$$v(Q) = (\mathrm{vol}(\mathrm{Ball}_Q))^{-2}$$

where vol is the Lebesgue measure (if $\mathrm{vol}(\mathrm{Ball}_Q)$ is infinite we put $v(Q) = 0$).

Sublemma 4.3. *For a linear function $L : \mathbf{R}^n \to \mathbf{R}$ let $\|L\|_Q = \max\{L(x) : Q(x) = 1\}$. Then for every nondegenerated Q we have*

$$\left.\frac{d}{d\varepsilon}\right|_0 v((1-\varepsilon)Q + n\varepsilon L^2) = nv(Q)((\|L\|_Q)^2 - 1)$$

Proof. In coordinates $(x_1 = L/\|L\|_Q, x_2, \dots, x_n)$ orthonormal with respect to Q we have

$$\mathrm{Ball}_{(1-\varepsilon)Q + n\varepsilon L^2} = (1-\varepsilon)^{1/2}\mathrm{Ball}_{Q + \frac{n\varepsilon(\|L\|_Q)^2}{1-\varepsilon}x_1^2} = (1-\varepsilon)^{-1/2}H(\mathrm{Ball}_Q)$$

where $H(a_1, \cdots, a_n) = ((1 + n\varepsilon(\|L\|_Q)^2/(1-\varepsilon))^{-1/2}a_1, a_2, \dots, a_n)$.

Hence $v((1-\varepsilon)Q + n\varepsilon L^2) = (1-\varepsilon)^n(1 + n\varepsilon(\|L\|_Q)^2/(1-\varepsilon))v(Q)$ and thus

$$\left.\frac{d}{d\varepsilon}\right|_0 v((1-\varepsilon)Q + n\varepsilon L^2) = nv(Q)((\|L\|_Q)^2 - 1).$$

□

4.4. Let F^* be the set of all linear functions supporting F (not necessarily at smooth points), $A \supset A_F = \{nL^2 : L \in F^*\}$ and $\bar{A}_F$ be the convex hull of A_F. By the Caratheodory theorem every $Q \in \bar{A}_F$ can be represented as

$$Q = n \sum a_i L_i^2, \qquad i \le n(n+1)/2 + 1, \quad L_i \in F^*, \quad a_i \ge 0, \quad \sum a_i = 1.$$

We seek for a quadratic form $Q \in \bar{A}_F$ whose unit ball is inscribed into F. Let $Q = n \sum a_i L_i^2$ maximize the function v on $\bar{A}_F$. Then it follows from Sublemma that $Q(P) \ge 1$ for every supporting F hyperplane P and $\|L_i\|_Q = 1$ whenever $a_i \ne 0$. Thus the ball Ball_Q is inscribed into F and L_i are the supporting linear functions at some of the touch points $p_i \in F \cap \mathrm{Ball}_Q$. Obviously p_i are smooth points of F, and hence $Q = n \sum a_i (B_{p_i})^2$.

4.5. Reasoning by contradiction suppose that for every i we have

$$\int_F (B_{p_i}(q))^2 dm(q) \le 1/n$$

Hence

$$n \int_F \sum a_i (B_{p_i}(q))^2 dm(q) = \int_F Q(q) dm(q) \le 1.$$

However $Q(F) \ge 1$ since $Q(P) \ge 1$ for every supporting F hyperplane P (Ball_Q is inscribed into F); hence $m\{q \in F : Q(q) > 1\} = 0$. It means that $F = \{x \in \mathbf{R}^n : Q(x) = 1\}$ since the support of m is dense. Reasoning as in 2.4 one can easily show now that $r(p, F, m) = 1/n$ for all $p \in F$. □

Remark 4.6. The other proof of this lemma due to V. Bangert is based on duality being involutive for convex bodies. This elegant proof is just a few lines of computations in coordinates orthonormal with respect to some scalar product $\langle , \rangle$ and orthogonal with respect to a scalar product $\langle\langle , \rangle\rangle$ defined by $\langle\langle x, y \rangle\rangle = \int_F \langle x, z \rangle \langle y, z \rangle dm(z)$.

Remark 4.7. As pointed out by M. Gromov, the first inequality in the statement of the lemma is well known in terms of functional analysis. See [Gr] for details.

5. Proof of the Hopf Conjecture

5.1. It follows from Lemma 3.2 that the support of m is dense in F. Thus (4) and Lemma 4.2 imply that our limit norm $\|\cdot\|$ is actually a Euclidean norm and inequality (3) turns out to be an equality for almost all γ.

Lemma 5.2. *Let $f : \mathbf{R} \to \mathbf{R}$ be such that*

$$\lim\ T^{-1} \int_0^T f(x) dx = -s, \qquad \lim\ T^{-1} \int_0^T (f(x))^2 dx = s^2.$$

Then $\lim\ T^{-1} \int_0^T (f(x) + s)^2 dx = 0$.

Proof.

$$\begin{aligned} &\lim\ T^{-1} \textstyle\int_0^T (f(x) + s)^2 dx = \\ = \quad &\lim\ T^{-1} \textstyle\int_0^T (f(x))^2 dx + 2s \cdot \lim\ T^{-1} \int_0^T f(x) dx + \lim\ T^{-1} \int_0^T s^2 dx = \\ &= s^2 - 2s^2 + s^2 = 0. \end{aligned}$$

□

5.3. We fix a rational point $p \in F$. For almost every geodesic γ we have the equality (3) and hence by Lemma 5.2

$$\lim\ T^{-1} \int_0^T (\langle \dot\gamma(t), v_p\rangle + B_p(D(\dot\gamma(t))))^2 dt = 0.$$

Hence by the Birkhoff ergodic theorem we have $\langle w, v_p\rangle + B_p(D(w)) = 0$ almost everywhere. Hence for every rational $q \in F$ we have $\langle v_q, v_p\rangle = -B_p(q)$ because D is continuous at v_q. Now we just take n rational linearly independent points $p_i \in F$ and consider the Busemann functions of the rays $\exp_0 tv_{p_i}(0)$ as coordinate functions. In these coordinates the metric tensor of $\tilde d$ is constant. Hence d is a flat metric.

6. Concluding Remarks. Volume Growth Theorem

6.1. We continue using the notation of the previous sections, but we now return to metrics with conjugate points. Note that we have constructed a correspondence $d \to (\|\cdot\|, m)$ where (in the general case) m is a measure on the unit ball of $\|\cdot\|$. One can easily show that the condition of "no conjugate points" is *a priori* equivalent to m being concentrated on the unit sphere F. (However, it is not even clear whether a nontrivial part of m may be concentrated on F.) We expect that many geometric properties of d may be expressed in terms of this correspondence. As an example we formulate the following conjecture (compare with [Ba2]):

Conjecture 6.2. *Let $p \in F$ be a rational point and assume that F is smooth and strictly convex in a neighborhood of p. Then minimizing geodesics of the direction p foliate all $\mathbf{R}^n$ as in section 1.5. In particular, if F is smooth and strictly convex then d is flat.*

6.3. Denote the volume of the ball of radius r in $(\mathbf{R}^n, \tilde d)$ centered at the origin by $V(r)$. Let ε_n be the (standard) volume of the unit ball in $(\mathbf{R}^n, \text{standard metric})$. The following theorem gives the multidimensional generalization of the main result of [B] (compare also with [C]):

Theorem 2. $\lim V(r)/\varepsilon_n r^n \geq 1$ *and the equality holds only if d is flat.*

The proof of this theorem will be published in the near future. The main idea is that we construct Lipschitz-1 functions B_i such that $\forall y \in \mathbf{R}^n$, $|B_i(y) - B_{p_i}(y)| \leq 2c$, where smooth points $p_i \in F$ are from the proof of Lemma 4.2, that is such points that the unit ball of a quadratic form $n\sum a_i B_{p_i}^2$, $a_i \geq 0$, $\sum a_i = 1$ is inscribed into F. Then we apply to these functions B_i a generalization of Derrick's proof of the Besikovitch-Almgren inequality (see [BurZ, pp. 294–296] and references there). The case of the equality $\lim V(r)/\varepsilon_n r^n = 1$ requires some simple additional considerations which resemble the final arguments in the proof of the Hopf conjecture.

6.4. Let (M, ρ) be a Finsler manifold. By $R(M)$ we denote the space of all Riemannian metrics on M. Let d_h be the Hausdorff distance between metric spaces. We define $V_R(M, \rho)$ setting

$$V_R(M, \rho) = \liminf\{\text{vol}(M, d) : d \in R(M), d_h((M, \rho), (M, d)) \to 0\}.$$

The same arguments as described in the previous section lead us to the following

Theorem 3. $V_R(M, \rho) \geq \text{vol}(M, \rho)$ *and the equality holds only if ρ is a Riemannian metric.*

By vol we mean here the Hausdorff measure (normalized to have Riemannian volume for Riemannian metrics).

It is known that every Finsler metric is a limit of a sequence of Riemannian metrics of a rather special type (quasi-periodic sequences, see [Bu2]). However, the above theorem shows that the volumes of these Riemannian metrics never converge to the volume of the Finsler one.

References

[A] A. Avez. Variétés riemanniennes sans points focaux. C.R.A.S. A-B 270 (1970), A 188–191.

[B] I. Babenko. Asymptotic volume of tori and geometry of convex bodies, Mat. Zametki 44:2 (1988), 177–188.

[Ba1] V. Bangert. Minimal Geodesics. Erg. Theory and Dyn. Syst. 10 (1990), 263–286.

[Ba2] V. Bangert. Geodesic Rays, Busemann functions and monotone twist maps, Calc. Var. 2 (1994), 49–63.

[Bu1] D. Burago. Periodic Metrics. Advances in Soviet Math. 9 (1992), 205-210.

[Bu2] D. Burago. Periodic metrics, In "Seminar on Dynamical Systems", Progress in Nonlinear Differential Equations (H. Brezis, ed.) 12, 90–96, Birkhauser, 1994.

[BurZ] Yu. Burago, V. Zalgaller. Geometric Inequalities, Springer-Verlag, 1988.

[Bus] H. Busemann. The Geometry of Geodesics. Acad. Press, New York, 1955.

[C] Volumes of balls in manifolds without conjugate points. Int. J. Math 3:4 (1992), 455–467.

[CF] C. Croke, A. Fathi. An inequality between energy and intersections. Bull. London Math. Soc. 22 (1990), 489-494.

[CK] C. Croke, B. Kleiner. On tori without conjugate points. Preprint.

[G] L. Green. A theorem of E. Hopf. Mich. Math. J. 5 (1958), 31–34.

[Gr] M. Gromov. Dimension, non-linear spectra and width, Springer Lecture Notes in Mathematics 1317 (1988), 132–184.

[H] J. Heber. Personal communication.

[HeM] G. Hedlund, M. Morse. Manifolds without conjugate points. Trans. Amer. Math. Soc. 51 (1942), 362–386.

[Ho] E. Hopf. Closed surfaces without conjugate points. Proc. Nat. Acad. of Sci. 34 (1948), 47–51.

[Kn] A. Knauf. Closed orbits and converse KAM Theory. Nonlinearity, 3 (1990), 961-973.

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