ON CONVERGING METRICS OF CURVATURE BOUNDED ABOVE ON 2-POLYHEDRA

SERGEI IVANOV

(Draft translation, 1998)

We define (two-dimensional) *polyhedron* as a topological space homeomorphic to a locally finite two-dimensional simlplicial complex. The *essential 1-skeleton* of a polyhedron X, denoted by $\operatorname{esk}_1 X$, is the set of points of X having no neighborhood homeomorphic to the disc. The 1-skeleton is a (locally finite) graph. Its edges adjacent to exactly one face of X are called *boundary* edges. By a *metric* on a polyhedron X we mean a length metric that induces the standard topology on X. A metric is a distance function on $X \times X$, hence the notion of uniform convergence in naturally defined on the set of metrics.

The purpose of this note is to weaken the assumptions of the Limit Metric Theorem of [BB]. This theorem ([BB, theorem 0.8]) claims the following. Let $\kappa \in \mathbf{R}$, and let $\{d_n\}_{n=1}^{\infty}$ be a sequence of metrics of curvature $\leq \kappa$ (in Alexandrov sense) on a polyhedron X converging uniformly to a metric d. Assume that the metrics d_n have uniformly bounded positive curvature parts and determine uniformly bounded lengths of $\operatorname{esk}_1 X$ and uniformly bounded turn variations of the boundary edges. Then the metric d has curvature $\leq \kappa$.

We will show that the assumption that the lengths of internal edges of $\operatorname{esk}_1 X$ are uniformly bounded is not essential since it follows from the other conditions and the fact that the metrics converge. More precisely, we will prove

Theorem 1. *Let $\{d_n\}_{n=1}^{\infty}$ be a sequence of metrics of curvature $\leq \kappa$ on a polyhedron X converging uniformly to some metric d on X. Assume that the metrics d_n have uniformly bounded positive curvature parts on $X \setminus \operatorname{esk}_1 X$ and determine uniformly bounded lengths and turn variations of boundary edges of X. Then for any compact set $K \subset X$, the lengths $\ell_n(K \cap \operatorname{esk}_1 X)$ are uniformly bounded, where ℓ_n denotes the length with respect to d_n.*

This immediately implies that Theorem 0.8 from [BB] can be stated as follows.

Corollary. *If d_n and d are the same as in Theorem 1, then d has curvature $\leq \kappa$.*

Remark. It is still unclear if the remaining assumptions on the metrics $\{d_n\}$ are essential. It seems plausible that one can omit at least the one about the lengths of boundary edges. It is easy to see that a uniform bound on their turn variations imply a uniform bound on their lengths within any compact set containing no (essential) vertices of the polyhedron.

The author was supported by RFFR grant 96-01-00674 and CRDF grant RM1-169.

The plan of the proof is the following. First, note that the edges with no adjacent faces can be removed from the polyhedron. It suffices to obtain a uniform bound on the length of the 1-skeleton locally, i.e., in a neighborhood of every point. Every point $x \in X$ has a neighborhood homeomorphic to a cone over a finite graph, namely the link of the point. The edges of X emanating from x correspond to the vertices of the link, and the boundary edges correspond to the vertices of degree one. Every vertex of the link can be included in a simple cycle or in a path whose endpoints are vertices of degree one or vertices that can be included in cycles. The cone over this cycle or path is an embedded disc D in X. Every metric d_n induces a length metric $d_{n,D}$ of curvature $\le \kappa$ on D. The set $D \cap \operatorname{esk}_1 X$ is a union of simple curves emanating from x. The assumption that positive parts of of curvature are uniformly bounded imply (by Gauss-Bonnet formula, cf. [BB]) that turn variations of these curves with respect to $d_{n,D}$ are uniformly bounded. We will prove that, first, the collection of metrics $d_{n,D}$ is pre-compact in a certain topology, and second, that the pre-compactness of the set of metrics and a uniform bound on turn variations of curves imply a uniform bound on their lengths.

We have to make use of the condition that the metrics d_n converge. This condition does not immediately imply anything about induced length metrics $d_{n,D}$ because attaching other faces of the polyhedron to $D \cap \operatorname{esk}_1 X$ can arbitrarily reduce distances between points of D. Nevertheless, the distances from a point $y \in D$ to the set $\partial D \cup (D \cap \operatorname{esk}_1 X)$ in the metrics d_n and $d_{n,D}$ are the same, since such a distance is determined by lengths of curves contained in D. Thus if D is small enough with respect to the limit metric d (and hence with respect to d_n for large n), the set $D \setminus \operatorname{esk}_1 X$ cannot contain a large ball of a metric $d_{n,D}$. The proof of Theorem 1 is based mainly on general properties of curves of bounded turn variation in a disk of curvature $\le \kappa$ whose complements have small maximal radius of an inscribed ball.

We will assume an upper bound for the curvature to be 1 (this can always be achieved by scaling). The metrics on discs will be assumed Riemannian and the curves will be assumed piecewise smooth. The results generalize to arbitrary metrics by means of approximation.

Now let us fix some notations and conventions. We call a *domain* a two-dimensional smooth manifold D with a piecewise smooth boundary, homeomorphic to the disc and equipped with a Riemannian metric of curvature ≤ 1. Such a domain is a length metric space of curvature ≤ 1 in the Alexandrov sense. The distance between points x and y with respect to the domain's length metric will be denoted by $|xy|$. A curve whose length equals the distance between its endpoints is called a *shortest path*, a locally shortest path is called a *geodesic*. (This covers more than only Riemannian geodesics. For example, concave boundary arcs are geodesics in the above sense.) If points x and y can be joined by a unique shortest path, this shortest path is denoted by $\overline{xy}$. We define the *injectivity radius* of D, $\rho_{\text{inj}}(D)$, as the maximal number r such that any geodesic of length $< r$ is a unique shortest path between its endpoints. If all geodesics are unique shortest paths, we define $\rho_{\text{inj}}(D) = \infty$. The injectivity radius is positive and the angle comparison property of curvature ≤ 1 holds for all geodesic triangles in D of perimeter less than $2\min\{\pi, \rho_{\text{inj}}(D)\}$. We will denote by $\tau(\gamma)$ the *turn* of a (piecewise smooth) curve γ, defined as the integral of the absolute value of the geodesic curvature plus the sum of turn angles at nonsmooth points. Finally, denote by $\rho(D)$ the maximal radius of a ball contained in the interior of D, $\rho(D) = \sup_{x\in D} \operatorname{dist}(x, \partial D)$.

Proposition 1. *If $\rho(D) < \pi/2$ then $\rho_{\mathrm{inj}}(D) \geq \pi$.*

Proof. If $\rho_{\mathrm{inj}}(D) < \pi$, then D contains a geodesic loop of length $L = 2\,\rho_{\mathrm{inj}}(D) < 2\pi$. Let D' be the domain bounded by this loop. Substituting $K = 1$ to theorems 2 and 3 from [I], we obtain that $L^2 - 4\pi F + F^2 \geq 0$ and

$$(*) \qquad \rho(D') \geq \min\left\{\frac{\pi}{2}, 2\operatorname{arctg}\left(\frac{F}{L + \sqrt{L^2 - 4\pi F + F^2)}}\right)\right\},$$

where F is the area of D'. Since D' is a domain with a concave boundary (in the Riemannian sense) and curvature ≤ 1, the Gauss-Bonnet formula imply that $F \geq 2\pi$. It is straightforward that for $L < 2\pi$ and $F \geq 2\pi$ the right-hand side of (*) is no less than $\pi/2$. Hence $\rho(D) \geq \rho(D') \geq \pi/2$. □

Proposition 1 implies that in a domain D with $\rho(D) < \pi/2$ every geodesic of length $< \pi$ is a unique shortest path between its endpoints, and angles of a geodesic triangle of perimeter $< 2\pi$ are not larger than the respective angles of a spherical triangle with sides of the same length. We will use these properties without explicit reference to Proposition 1.

From now on, we fix some sufficiently small positive r_0. For example, one can let $r_0 = \pi/100$.

Proposition 2. *For any $\varepsilon > 0$ there is a $\delta > 0$ such that the following holds. Let D be a domain with $\rho(D) \leq r_0$ whose boundary consists of two simple curves γ_0 and γ with common endpoints p and q, let γ_0 be a shortest path, $\ell(\gamma_0) \leq 4r_0$, and $\tau(\gamma) < \delta$. Then*

(i) *the angles at p and q between the curves γ and γ_0 are less than ε,*
(ii) $\ell(\gamma) < (1+\varepsilon)\ell(\gamma_0)$,
(iii) $\operatorname{dist}(x, \gamma_0) < \varepsilon\ell(\gamma_0)$ *for any point $x \in \gamma$.*

Proof. We may assume that γ is a smooth curve (otherwise use a smooth approximation). The proof is based on several lemmas.

Lemma 1. $\operatorname{dist}(x, \gamma) \leq 2r_0$ *for all $x \in D$.*

Proof. Suppose that $\operatorname{dist}(x, \gamma) > 2r_0$ for some $x \in D$. One may assume that $x \notin \gamma_0$. Let y be a point of γ_0 nearest to x. Clearly $|xy| \leq \rho(D) \leq r_0$. Observe that the angles at y between the shortest path $\overline{yx}$ and γ_0 are not less than $\pi/2$. Let γ_1 be the maximal Riemannian geodesic starting at y and extending $\overline{yx}$ beyond x. Then $\ell(\gamma_1) > 2r_0$. Indeed, if $\ell(\gamma_1) < \infty$ then γ_1 ends at the boundary of D. If it ends at γ, one has $\ell(\gamma_1) > \operatorname{dist}(x, \gamma) > 2r_0$. On the other hand, if it ends at γ_0 then γ_1 and an interval of γ_0 form a geodesic biangle and hence $\ell(\gamma_1) \geq \pi$. Let γ_1 be parameterized by arc length, $\gamma_1\colon [0, \ell(\gamma_1)) \to D$, $\gamma_1(0) = y$. Consider a point $z = \gamma_1(r_0+\sigma)$ where $0 < \sigma < |yx|$. The curve $\gamma_1|_{[0,r_0+\sigma]}$ realize the minimum distance from z to γ_0. Indeed, otherwise this curve, the shortest path $\overline{zy'}$ that realize the distance from z to γ_0, and the interval of γ_0 between y and y' form a geodesic triangle with sides $< \pi/2$ and two angles $\geq \pi/2$ which contradicts the angle comparison property. Thus $\operatorname{dist}(z, \gamma_0) = r_0 + \sigma > r_0$. On the other hand, $\operatorname{dist}(z, \gamma) \geq \operatorname{dist}(x, \gamma) - |xz| > 2r_0 - (r_0 + \sigma - |xy|) > r_0$. Hence $\rho(D) \geq \operatorname{dist}(x, \partial D) > r_0$ with a contradiction. □

Fix some small enough ε. Let $p_0p_1p_2$ be a triangle in the unit sphere such that $|p_0p_1| = |p_0p_2| = 2r_0$ and $\measuredangle p_0p_1p_2 = \measuredangle p_0p_2p_1 = \frac{\pi}{2} - \varepsilon$. Define $\theta = \measuredangle p_1p_0p_2$. Clearly $\theta \to 0$ as $\varepsilon \to 0$, so we may assume that $\theta < \pi/3$.

Suppose that one of the angles between γ and γ_0 is at least ε. We will then derive that $\tau(\gamma_1) \ge \varepsilon$ for an interval γ_1 of γ which is constructed below.

For two points $p_1, q_1 \in \gamma$ we denote by $[p_1, q_1]$ the interval of γ between them. Consider the intervals $[p_1, q_1] \subset \gamma$ such that $|p_1q_1| \le 4r_0$ and the shortest path $\overline{p_1q_1}$ form an angle $\ge \varepsilon$ with $[p_1, q_1]$ at p_1 or q_1. The entire γ is one of such intervals, and the lengths of such intervals have a positive lower bound (otherwise γ would not be smooth). Therefore this set of intervals contains one minimal by inclusion. Let $[p_1, q_1]$ be such a minimal interval. Denote it by γ_1 and the shortest path $\overline{p_1q_1}$ by γ_2. Let the angle between γ_1 and γ_2 at q_1 be at least ε. Clearly γ_2 has only points p_1 and q_1 in common with γ_1, otherwise $[p_1, q_1]$ would not be minimal. Thus γ_1 and γ_2 bound a domain D'. Note that the distances in the induced length metric of D' are the same as in D because D' is separated from D by a shortest path. The domain D' satisfy the assumptions of Proposition 2, so Lemma 1 remains true if one replaces D and γ by D' and γ_1.

Lemma 2. *If $x \in D'$, $y, z \in \gamma_1$ and $|xy| = |xz| = \operatorname{dist}(x, \gamma_1)$ then $\measuredangle_x(\overline{xy}, \overline{xz}) \le \theta$.*

Proof. By Lemma 1, $\operatorname{dist}(x, \gamma_1) \le 2r_0$ and hence $|yz| \le 4r_0$. If $\measuredangle(\overline{xy}, \overline{xz}) > \theta$ than $\measuredangle_y(\overline{yx}, \overline{yz}) < \frac{\pi}{2} - \varepsilon$ by triangle comparison. Hence the angle at y between the shortest path $\overline{yz}$ and the interval $[x, y]$ of γ_1 is at least ε. This contradicts the choice of $[p_1q_1]$. $\square$

Lemma 3. $\operatorname{dist}(x, \gamma_2) \le 4r_0$ *for every point $x \in D'$.*

Proof. We may assume that x is an interior point of D'. Consider the set $S = \{y \in D' : |xy| \le 2\operatorname{dist}(y, \gamma_1)\}$. Note that S is compact, $S \cap \gamma_1 = \varnothing$ and $S \ne \varnothing$ (because $x \in S$). Let y be a point of S furthest from γ_1. We will prove that $y \in \gamma_2$. Supposing the contrary, let y be an internal point of D'. Then define $C = \{z \in \gamma_1 : |yz| = \operatorname{dist}(y, \gamma_1)\}$ and let $W \subset T_yD$ be the set of directions of shortest paths joining y to points of C. By Lemma 2, W is contained within an angle less or equal to θ. Hence there is a unit vector $v \in T_yD$ such that $\measuredangle(v, w) \ge \pi - \theta/2$ for all $w \in W$. Pick a smooth curve $s\colon [0, \infty) \to D$ parameterized by arc length with $s(0) = y$ and $s'(0) = v$. The first variation formula implies that

$$\liminf_{t\to 0} \frac{\operatorname{dist}(s(t), \gamma_1) - \operatorname{dist}(y, \gamma_1)}{t} = \inf\{-\cos\measuredangle(v, w) : w \in W\}$$
$$\ge \cos(\theta/2) > 1/2.$$

Hence for sufficiently small t one has $\operatorname{dist}(s(t), \gamma_1) \ge \operatorname{dist}(y, \gamma_1) + t/2$ and

$$|xs(t)| \le |xy| + t \le 2\operatorname{dist}(y, \gamma_1) + t \le 2\operatorname{dist}(s(t), \gamma_1)$$

which contradicts the choice of y. Thus $y \in \gamma_2$, hence $\operatorname{dist}(x, \gamma_2) \le |xy| \le 2\operatorname{dist}(y, \gamma_1)$ (the second inequality holds because $y \in S$). The lemma follows, since $\operatorname{dist}(y, \gamma_1) \le 2r_0$ by Lemma 1. $\square$

Lemma 3 implies that $|p_1x| \le \operatorname{dist}(x, \gamma_2) + \ell(\gamma_2) \le 8r_0 < \pi/2$ for all $x \in D'$. In particular, for every point $x \in D'$ there is a unique shortest path joining it to p_1. Let γ_1 be parameterized by arc length, $\gamma_1\colon [0, \ell] \to D$, $\gamma_1(0) = p_1$, $\gamma_1(\ell) = q_1$. For every t, $0 < t \le \ell$, denote by $\alpha(t)$ the angle at $\gamma_1(t)$ between $\overline{\gamma_1(t)p_1}$ and $\gamma_1|_{[0,t]}$. Clearly the function $t \mapsto \alpha(t)$ is continuous and $\alpha(t) \to 0$ as $t \to 0$.

Lemma 4. *For every interval $J = [t_1, t_2] \subset (0, \ell]$,*

$$\alpha(t_2) - \alpha(t_1) \le \tau(\gamma_1|_J).$$

Proof. The statement is additive with respect to J. Thus it can be reduced, by means of approximation by broken geodesics, to the following: if $\gamma_1|_J$ is a shortest path then the function $t \mapsto \alpha(t)$ is non-decreasing. This is equivalent to that for all $t, t' \in J$ the sum of two angles of the triangle $p_1\gamma_1(t)\gamma_1(t')$ at $\gamma_1(t)$ and $\gamma_1(t')$ is less or equal to π. This in turn follows from triangle comparison, since the sides $|p_1\gamma_1(t)|$ and $|p_1\gamma_1(t')|$ are shorter than $\pi/2$, and for any spherical triangle with lateral sides $< \pi/2$ the sum of its angles at the base side is less or equal to π (because the area in not larger than the angle at the third vertex). $\square$

Applying Lemma 4 to intervals $J = [t, \ell]$ as $t \to 0$ we obtain that $\measuredangle_{q_1}(\gamma_2, \gamma_1) \le \tau(\gamma_1)$. Hence $\tau(\gamma) \ge \tau(\gamma_1) \ge \measuredangle_{q_1}(\gamma_2, \gamma_1) \ge \varepsilon$ which proves the statement (i) of Proposition 2 for $\delta = \varepsilon$.

Now let us prove the statement (ii). Let γ be parameterized by arc length, $\gamma\colon [0, \ell] \to D$, $\gamma(0) = p$, $\gamma(\ell) = q$. Denote by Σ the set of parameters $t \in [0, \ell]$ such that $|p\gamma(t)| \le 4r_0$. For every $t \in \Sigma$ one has

$$\frac{d}{dt^+}|\gamma(t)q| \le -\cos\alpha(t),$$

where d/dt^+ denotes the right derivative and $\alpha(t)$ is the angle at $\gamma(t)$ between $\overline{\gamma(t)q}$ and $\gamma|_{[t,\ell]}$. Applying (i) to the domain bounded by the curves $\gamma|_{[t,\ell]}$ and $\overline{\gamma(t)q}$ in place of D, we obtain that $\alpha(t) \le \varepsilon$. (If these curves have common internal points, they bound several domains, in which case the one containing $\gamma(t)$ should be chosen.) It follows that

$$\frac{d}{dt^+}|\gamma(t)q| \le -\cos\varepsilon < -(1+\varepsilon)^{-1}$$

for all $t \in \Sigma$. Then the left-hand side is negative and therefore Σ contains a right-side neighborhood of every its point. In addition, we have $0 \in \Sigma$, hence $\Sigma = [0, \ell]$. Now by integrating the above inequality we obtain that $|pq| \ge \ell(1+\varepsilon)^{-1}$ which proves the statement (ii) of Proposition 2.

The statement (ii) implies that for all $x \in \gamma$ we have $|px| + |qx| \le (1+\varepsilon)|pq|$, in particular, $|px|$ and $|qx|$ are less than $\pi/4$. By triangle comparison it easily follows that the altitude of the triangle pqx is small compared to the length of the base $\overline{pq} = \gamma_0$. The statement (iii) follows (for another but still arbitrarily small value of ε). This completes the proof of Proposition 2. $\square$

For the rest of the paper we fix a $\delta_0 > 0$ that is suitable for $\varepsilon = 1/2$ in Proposition 2.

Proposition 3. *Let γ be a curve in a domain D with endpoints p and q such that $|pq| < 4r_0$ and $\tau(\gamma) < \delta_0$. Let γ_0 be a shortest path joining p to q. Assume that any domain bounded by an interval of γ and an interval of γ_0 has the radius less or equal to r_0. Then $\ell(\gamma) \le 2|pq|$.*

Proof. By a slight variation of γ we may achieve that it intersects γ_0 only finitely many times. Every domain bounded by an interval of γ between two consecutive (on γ) intersection points and the corresponding interval of γ_0 satisfy the assumptions of Proposition 2. Therefore every such domain form acute angles at these intersection points. This implies that the intersection points have the same order on γ_0 and γ. By the last statement of Proposition 2, the length of an interval of γ between consecutive intersection points is at most twice the length of the corresponding interval of γ_0. Adding up these inequalities over all intervals, we obtain that $\ell(\gamma) \le 2|pq|$. $\square$

Proposition 4. *For every $T > 0$ here is an $\varepsilon > 0$ such that the following holds. Let D be a domain with $\rho(D) \ge r_0$, $\gamma_0 \subset \partial D$ be a shortest path, $\ell(\gamma_0) \le r_0$. Let $\Gamma \subset D$ be a set containing ∂D and representable as a union of γ_0 and an at most countable collection of disjoint simple curves with endpoints at γ_0. Assume that the sum of turns of the curves that form Γ is less or equal to T. Then there exists a point $x \in D$ such that $\operatorname{dist}(x, \Gamma) > \varepsilon$.*

Proof. Since it suffices to prove the statement for a set Γ composed of finitely many curves, we may assume that the curves are separated from one another. Consider all subdomains $D' \subset D$ with $\rho(D') \ge r_0$ bounded by an interval of one of the curves that form Γ and a shortest path of length $\le r_0$ joining the endpoints of this interval. Among these, there is a domain minimal by inclusion. A minimal domain D' and the set $\Gamma' = \Gamma \cap D'$ satisfy the assumptions of the proposition, so we may assume that D itself is minimal. This implies that propositions 2 and 3 apply to all curves contained in Γ and having the distance between endpoints no larger than r_0, except the curve $\partial D \smallsetminus \gamma_0$. Since $\rho(D) \ge r_0$, there is a point $p \in D$ such that the Riemannian exponential map $\exp_p$ is defined and injective in a ball of radius r_0 (cf. [B]). Let the ball of radius r_0 in T_pD be equipped with a metric of constant curvature 1. Then the exponential map does not increase lengths of curves and hence it is distance non-increasing in the ball of radius $r_0/2$. The curves composing $\Gamma \smallsetminus \partial D$ fall into two classes, those whose turns are less or equal to δ_0, and those whose turns are larger than δ_0. By Proposition 2, the curves of the former class are contained within the $(r_0/2)$-neighborhood of γ and therefore do not intersect the ball $B(p, r_0/2)$. The latter class contains at most T/δ_0 curves. Split each of them into T/δ_0 intervals whose turns are not greater than δ_0. Thus we obtain a collection of curves $s_1, \dots, s_N$, $N \le (T/\delta_0)^2$ such that $\tau(s_i) < \delta_0$ and

$$B(p, r_0/2) \cap \Gamma \subset \bigcup_{i=1}^{N} s_i.$$

Since $\operatorname{diam} B(p, r_0/2) \le r_0$, Proposition 3 implies that for every $i = 1, \dots, N$ the set $s_i \cap B(p, r_0/2)$ is contained within an interval s_i' of s_i with $\ell(s_i') \le r_0$. For a sufficiently small ε pick a maximal set of points $\{y_1, \dots, y_M\}$ in $B(p, r_0/3)$ separated by distances at least 3ε from one another. Comparing $B(p, r_0/3)$ with a ball in S^2 by means of the exponential map, we obtain an estimate $M \ge c(r_0/\varepsilon)^2$ for some (universal) constant $c > 0$. Every curve s_i' intersects at most r_0/ε balls $B(y_i, \varepsilon)$ because $\ell(s_i') \le r_0$. For $\varepsilon < cr_0/N$ we have $Nr_0/\varepsilon < c(r_0/\varepsilon)^2 < M$ and hence at least one of the M balls $B(y_i, \varepsilon)$ does not intersect Γ. $\square$

Proposition 5. *For any $T > 0$ and $N \in \mathbf{N}$ there is an $\varepsilon > 0$ such that the following holds. Let D be a domain, $\Gamma \subset D$ be an embedded tree with at most N*

edges whose turns are less or equal to T. Assume $dist(x, \Gamma \cup \partial D) < \varepsilon$ for all $x \in D$. Then

(i) *For any simple curve γ contained in one of the edges of Γ, joining points p and q with $|pq| \le r_0$, and such that $\tau(\gamma) < \delta_0$, one has $\ell(\gamma) \le 2|pq|$.*
(ii) *Any ball of radius $r < r_0$ lying in the interior of D contains a ball of radius εr not intersecting Γ.*

Proof. Approximate Γ by a simple closed curve Γ_0 which is a boundary of a small neighborhood of Γ. This curve can be chosen so that its turn is estimated from above in terms of N and the turns of the edges, and lengths of edges are close to lengths of some intervals of Γ_0. It suffices to prove the proposition for Γ_0 instead of Γ.

(i) Let ε be the same as in Proposition 4. If $\ell(\gamma) > |pq|$ then Proposition 3 fails for γ. This means that some interval of the shortest path joining p and q, and some interval of γ, bound a domain D' with $\rho(D') \ge r_0$. By Proposition 4 applied to D' and the set $|pq| \cup (\Gamma_0 \cap D')$, there is a point x such that $\operatorname{dist}(x, \Gamma_0 \cup \partial D) \ge \varepsilon$.

(ii) Let $B(p, r)$ be a ball contained in the interior of D. First assume that the Riemannian exponential map is injective within the radius r. Then (i) implies that the set $B(p, r/2) \cap \Gamma_0$ can be covered by a bounded number of intervals of Γ_0 each having length $\le r$ and turn $\le \delta_0$. Applying the selection procedure from the proof of Proposition 4 (for r in place of r_0), we obtain a desired ball of radius εr.

On the other hand, if $\exp_p$ is not injective within a radius r, then $\rho_{\rm inj}(D) \le r < \pi$, and hence $\rho(D) \ge \pi/2 > r_0$ by Proposition 1. It follows that there is a point $q \in D$ for which $\exp_q$ is defined and injective within the ball of radius r_0 (cf. [B]). Applying the selection procedure to this ball, we can obtain a ball of radius ε not intersecting Γ_0. This contradicts the assumption that $\sup_{x \in D} \operatorname{dist}(x, \Gamma_0 \cup \partial D) < \varepsilon$. $\square$

Proof of Theorem 1. Let $\{d_n\}$ be a sequence of metrics on a polyhedron X satisfying the assumptions of the theorem, and let $x \in X$ be an arbitrary point. Fix a homeomorphism identifying a neighborhood of x with a cone over a graph and construct there two small embedded discs, D and D' which are homothetic with respect to the cone structure. Let these discs be cones over a cycle or a path in the link of x, moreover in the latter case the endpoints of the path correspond to boundary edges or can be included in cycles. Then $D' \cap \operatorname{esk}_1 X$ is a union of curves emanating from x whose turns with respect to the metrics $d_{n,D}$ are uniformly bounded by some T. To prove the theorem it suffices to show the the lengths $\ell_n(D' \cap \operatorname{esk}_1 X)$ are uniformly bounded.

We may assume that the diameter of D with respect to the limit metric d, and hence with respect to d_n for large n, is less than ε, where ε is chosen by Proposition 5 for T and the number of edges of $D \cap \operatorname{esk}_1 X$. Then every point $y \in D$ lies within a distance less than ε from $(D \cap \operatorname{esk}_1 X) \cup \partial D$ with respect to d_n.

Recall that a family of metric spaces is (Gromov–Hausdorff) pre-compact if for any $\delta > 0$ the minimal cardinality of a δ-net is uniformly bounded over all spaces of the family. We will prove that the restrictions of the metrics $d_{n,D}$ to D' form a pre-compact family. Let $0 < \delta < \inf_n \operatorname{dist}_{d_n}(D', \partial D \setminus \partial X)$ (the infimum is positive because the metrics d_n converge). First consider the case when x is an internal point of the disc (i.e., the disc is spanned by a cycle in the link of x). In every space $(D', d_{n,D}|_{D'})$ pick a maximal collection of points separated from one another by distance at least $\delta/2$. These points obviously form a δ-net. The balls of radius $\delta/5$

(with respect to $d_{n,D}$) centered at these points are disjoint. By proposition 5, each of these balls contains a ball of radius $\varepsilon\delta/5$ not intersecting $\operatorname{esk}_1 X$. The number of such balls is uniformly bounded over n because they are also balls of the same radius with respect to metrics d_n, and metrics d_n form a pre-compact family.

The case when x is a boundary point is similar except that those $(\delta/5)$-balls that intersect the disc's boundary should be excluded. But the number of these balls is uniformly bounded because they intersect ∂D only in edges of X that correspond the the endpoints of the path that spans the disc, and the lengths of these edges are uniformly bounded. For boundary edges this is included in the assumptions of the theorem and for those included in link's cycles this follows (see below) from the cycle case that has already been considered.

Now we will derive that the lengths of $D' \cap \operatorname{esk}_1 X$ in metrics $d_{n,D}$ are uniformly bounded. Suppose the contrary. Then, since the turns of the edges are uniformly bounded, it follows that for any $L > 0$ there is a curve γ contained in a graph's edge such that, in one of the spaces $(D', d_{n,D}|_{D'})$, one has $\ell(\gamma) = L$ and $\tau(\gamma) < \delta_0$. Pick points $x_1, \dots, x_N$, $N \geq L/2r_0$, on this curve such that the length of the curve's interval between any two of them is at least $2r_0$. By the statement (i) of Proposition 5, applied to D and $D \cap \operatorname{esk}_1 X$, all distances between these points in the $d_{n,D}$ metric are not less than r_0. Since N is arbitrarily large, this contradicts the above pre-compactness of the family of metrics $d_{n,D}|_{D'}$. $\square$

References

[B] Yu. D. Burago, *On injectivity radius on surfaces of curvature bounded above*, Ukr. Mat. Sbornik **21** (1978), 10–14. (Russian)

[BB] Yu. D. Burago, S. V. Buyalo, *Metrics of curvature bounded above on 2-polyhedra. II.*

[I] V. K. Ionin, *On isoperimetric and some other inequalities for a manifold of bounded curvature*, Sib. Mat. J. **10** (1969), no. 2. (Russian)

Steklov Institute of Mathematics, 191011, Fontanka 27, St. Petersburg, Russia
E-mail address: svivanov@pdmi.ras.ru