ON CONVERGING METRICS OF CURVATURE
BOUNDDED ABOVE ON 2-POLYHEDRA

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We define (two-dimensional) polyhedron as a topological space homeomorphic to a locally finite two-dimensional simplicial complex. The essential 1-skeleton of a polyhedron $X$, denoted by $\text{esk}_1 X$, is the set of points of $X$ having no neighborhood homeomorphic to the disc. The 1-skeleton is a (locally finite) graph. Its edges adjacent to exactly one face of $X$ are called boundary edges. By a metric on a polyhedron $X$ we mean a length metric that induces the standard topology on $X$. A metric is a distance function on $X \times X$, hence the notion of uniform convergence in naturally defined on the set of metrics.

The purpose of this note is to weaken the assumptions of the Limit Metric Theorem of [BB]. This theorem ([BB, theorem 0.8]) claims the following. Let $\kappa \in \mathbb{R}$, and let $\{d_n\}_{n=1}^{\infty}$ be a sequence of metrics of curvature $\leq \kappa$ (in Alexandrov sense) on a polyhedron $X$ converging uniformly to a metric $d$. Assume that the metrics $d_n$ have uniformly bounded positive curvature parts and determine uniformly bounded lengths of $\text{esk}_1 X$ and uniformly bounded turn variations of the boundary edges. Then the metric $d$ has curvature $\leq \kappa$.

We will show that the assumption that the lengths of internal edges of $\text{esk}_1 X$ are uniformly bounded is not essential since it follows from the other conditions and the fact that the metrics converge. More precisely, we will prove

**Theorem 1.** Let $\{d_n\}_{n=1}^{\infty}$ be a sequence of metrics of curvature $\leq \kappa$ on a polyhedron $X$ converging uniformly to some metric $d$ on $X$. Assume that the metrics $d_n$ have uniformly bounded positive curvature parts on $X \setminus \text{esk}_1 X$ and determine uniformly bounded lengths and turn variations of boundary edges of $X$. Then for any compact set $K \subset X$, the lengths $\ell_n(K \cap \text{esk}_1 X)$ are uniformly bounded, where $\ell_n$ denotes the length with respect to $d_n$.

This immediately implies that Theorem 0.8 from [BB] can be stated as follows.

**Corollary.** If $d_n$ and $d$ are the same as in Theorem 1, then $d$ has curvature $\leq \kappa$.

**Remark.** It is still unclear if the remaining assumptions on the metrics $\{d_n\}$ are essential. It seems plausible that one can omit at least the one about the lengths of boundary edges. It is easy to see that a uniform bound on their turn variations imply a uniform bound on their lengths within any compact set containing no (essential) vertices of the polyhedron.

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The plan of the proof is the following. First, note that the edges with no adjacent faces can be removed from the polyhedron. It suffices to obtain a uniform bound on the length of the 1-skeleton locally, i.e., in a neighborhood of every point. Every point $x \in X$ has a neighborhood homeomorphic to a cone over a finite graph, namely the link of the point. The edges of $X$ emanating from $x$ correspond to the vertices of the link, and the boundary edges correspond to the vertices of degree one. Every vertex of the link can be included in a simple cycle or in a path whose endpoints are vertices of degree one or vertices that can be included in cycles. The cone over this cycle or path is an embedded disc $D$ in $X$. Every metric $d_n$ induces a length metric $d_{n,D}$ of curvature $\leq \kappa$ on $D$. The set $D \cap \text{esk}_1 X$ is a union of simple curves emanating from $x$. The assumption that positive parts of curvature are uniformly bounded imply (by Gauss-Bonnet formula, cf. [BB]) that turn variations of these curves with respect to $d_{n,D}$ are uniformly bounded. We will prove that, first, the collection of metrics $d_{n,D}$ is pre-compact in a certain topology, and second, that the pre-compactness of the set of metrics and a uniform bound on turn variations of curves imply a uniform bound on their lengths.

We have to make use of the condition that the metrics $d_n$ converge. This condition does not immediately imply anything about induced length metrics $d_{n,D}$ because attaching other faces of the polyhedron to $D \cap \text{esk}_1 X$ can arbitrarily reduce distances between points of $D$. Nevertheless, the distances from a point $y \in D$ to the set $\partial D \cup (D \cap \text{esk}_1 X)$ in the metrics $d_n$ and $d_{n,D}$ are the same, since such a distance is determined by lengths of curves contained in $D$. Thus if $D$ is small enough with respect to the limit metric $d$ (and hence with respect to $d_n$ for large $n$), the set $D \cap \text{esk}_1 X$ cannot contain a large ball of a metric $d_{n,D}$.

The proof of Theorem 1 is based mainly on general properties of curves of bounded turn variation in a disk of curvature $\leq \kappa$ whose complements have small maximal radius of an inscribed ball.

Now let us fix some notations and conventions. We call a domain a two-dimensional smooth manifold $D$ with a piecewise smooth boundary, homeomorphic to the disc and equipped with a Riemannian metric of curvature $\leq 1$. Such a domain is a length metric space of curvature $\leq 1$ in the Alexandrov sense. The distance between points $x$ and $y$ with respect to the domain's length metric will be denoted by $|xy|$. A curve whose length equals the distance between its endpoints is called a shortest path, a locally shortest path is called a geodesic. (This covers more than only Riemannian geodesics. For example, concave boundary arcs are geodesics in the above sense.) If points $x$ and $y$ can be joined by a unique shortest path, this shortest path is denoted by $xy$. We define the injectivity radius of $D$, $\rho_{\text{inj}}(D)$, as the maximal number $r$ such that any geodesic of length $< r$ is a unique shortest path between its endpoints. If all geodesics are unique shortest paths, we define $\rho_{\text{inj}}(D) = \infty$. The injectivity radius is positive and the angle comparison property of curvature $\leq 1$ holds for all geodesic triangles in $D$ of perimeter less than $2 \min\{\pi, \rho_{\text{inj}}(D)\}$. We will denote by $\tau(\gamma)$ the turn of a (piecewise smooth) curve $\gamma$, defined as the integral of the absolute value of the geodesic curvature plus the sum of turn angles at nonsmooth points. Finally, denote by $\rho(D)$ the maximal
radius of a ball contained in the interior of $D$, $\rho(D) = \sup_{x \in D} \text{dist}(x, \partial D)$.

**Proposition 1.** If $\rho(D) < \pi/2$ then $\rho_{\text{inj}}(D) \geq \pi$.

**Proof.** If $\rho_{\text{inj}}(D) < \pi$, then $D$ contains a geodesic loop of length $L = 2 \rho_{\text{inj}}(D) < 2\pi$. Let $D'$ be the domain bounded by this loop. Substituting $K = 1$ to theorems 2 and 3 from [I], we obtain that $L^2 - 4\pi F + F^2 \geq 0$ and

$$\rho(D') \geq \min \left\{ \frac{\pi}{2}, 2 \arctan \left( \frac{F}{L + \sqrt{L^2 - 4\pi F + F^2}} \right) \right\},$$

where $F$ is the area of $D'$. Since $D'$ is a domain with a concave boundary (in the Riemannian sense) and curvature $\leq 1$, the Gauss-Bonnet formula imply that $F \geq 2\pi$. It is straightforward that for $L < 2\pi$ and $F \geq 2\pi$ the right-hand side of (*) is no less than $\pi/2$. Hence $\rho(D) \geq \rho(D') \geq \pi/2$. \(\Box\)

Proposition 1 implies that in a domain $D$ with $\rho(D) < \pi/2$ every geodesic of length $< \pi$ is a unique shortest path between its endpoints, and angles of a geodesic triangle of perimeter $< 2\pi$ are not larger than the respective angles of a spherical triangle with sides of the same length. We will use these properties without explicit reference to Proposition 1.

From now on, we fix some sufficiently small positive $r_0$. For example, one can let $r_0 = \pi/100$.

**Proposition 2.** For any $\varepsilon > 0$ there is a $\delta > 0$ such that the following holds. Let $D$ be a domain with $\rho(D) \leq r_0$ whose boundary consists of two simple curves $\gamma_0$ and $\gamma$ with common endpoints $p$ and $q$, let $\gamma_0$ be a shortest path, $\ell(\gamma_0) \leq 4r_0$, and $\tau(\gamma) < \delta$. Then

(i) the angles at $p$ and $q$ between the curves $\gamma$ and $\gamma_0$ are less than $\varepsilon$,
(ii) $\ell(\gamma) < (1 + \varepsilon)\ell(\gamma_0)$,
(iii) $\text{dist}(x, \gamma_0) < \varepsilon \ell(\gamma_0)$ for any point $x \in \gamma$.

**Proof.** We may assume that $\gamma$ is a smooth curve (otherwise use a smooth approximation). The proof is based on several lemmas.

**Lemma 1.** $\text{dist}(x, \gamma) \leq 2r_0$ for all $x \in D$.

**Proof.** Suppose that $\text{dist}(x, \gamma) > 2r_0$ for some $x \in D$. One may assume that $x \notin \gamma_0$. Let $y$ be a point of $\gamma_0$ nearest to $x$. Clearly $|xy| \leq \rho(D) \leq r_0$. Observe that the angles at $y$ between the shortest path $\overline{yx}$ and $\gamma_0$ are not less than $\pi/2$. Let $\gamma_1$ be the maximal Riemannian geodesic starting at $y$ and extending $\overline{yx}$ beyond $x$. Then $\ell(\gamma_1) > 2r_0$. Indeed, if $\ell(\gamma_1) < \infty$ then $\gamma_1$ ends at the boundary of $D$. If it ends at $\gamma$, one has $\ell(\gamma_1) > \text{dist}(x, \gamma) > 2r_0$. On the other hand, if it ends at $\gamma_0$ then $\gamma_1$ and an interval of $\gamma_0$ form a geodesic biangle and hence $\ell(\gamma_1) \geq \pi$. Let $\gamma_1$ be parameterized by arc length, $\gamma_1 : [0, \ell(\gamma_1)] \to D$, $\gamma_1(0) = y$. Consider a point $z = \gamma_1(r_0 + \sigma)$ where $0 < \sigma < |yx|$. The curve $\gamma_1[0, r_0 + \sigma]$ realize the minimum distance from $z$ to $\gamma_0$. Indeed, otherwise this curve, the shortest path $\overline{yz}$ that realize the distance from $z$ to $\gamma_0$, and the interval of $\gamma_0$ between $y$ and $y'$ form a geodesic triangle with sides $< \pi/2$ and two angles $\geq \pi/2$ which contradicts the angle comparison property. Thus $\text{dist}(z, \gamma_0) = r_0 + \sigma > r_0$. On the other hand, $\text{dist}(z, \gamma) \geq \text{dist}(x, \gamma) - |xz| > 2r_0 - (r_0 + \sigma - |xy|) > r_0$. Hence $\rho(D) \geq \text{dist}(x, \partial D) > r_0$ with a contradiction. \(\Box\)
Fix some small enough $\epsilon$. Let $p_0p_1p_2$ be a triangle in the unit sphere such that $|p_0p_1| = |p_0p_2| = 2r_0$ and $\angle p_0p_1p_2 = \angle p_0p_2p_1 = \frac{\pi}{2} - \epsilon$. Define $\theta = \angle p_1p_0p_2$. Clearly $\theta \to 0$ as $\epsilon \to 0$, so we may assume that $\theta < \pi/3$.

Suppose that one of the angles between $\gamma$ and $\gamma_0$ is at least $\epsilon$. We will then derive that $r(\gamma_1) \geq \epsilon$ for an interval $\gamma_1$ of $\gamma$ which is constructed below.

For two points $p_1, q_1 \in \gamma$ we denote by $[p_1, q_1]$ the interval of $\gamma$ between them. Consider the intervals $[p_1, q_1] \subset \gamma$ such that $|p_1q_1| \leq 4r_0$ and the shortest path $\overline{p_1q_1}$ form an angle $\geq \epsilon$ with $[p_1, q_1]$ at $p_1$ or $q_1$. The entire $\gamma$ is one of such intervals, and the lengths of such intervals have a positive lower bound (otherwise $\gamma$ would not be smooth). Therefore this set of intervals contains one minimal by inclusion. Let $[p_1, q_1]$ be such a minimal interval. Denote it by $\gamma_1$ and the shortest path $p_1q_1$ by $\gamma_2$. Let the angle between $\gamma_1$ and $\gamma_2$ at $q_1$ be at least $\epsilon$. Clearly $\gamma_2$ has only points $p_1$ and $q_1$ in common with $\gamma_1$, otherwise $[p_1, q_1]$ would not be minimal. Thus $\gamma_1$ and $\gamma_2$ bound a domain $D'$. Note that the distances in the induced length metric of $D'$ are the same as in $D$ because $D'$ is separated from $D$ by a shortest path. The domain $D'$ satisfy the assumptions of Proposition 2, so Lemma 1 remains true if one replaces $D$ and $\gamma$ by $D'$ and $\gamma_1$.

**Lemma 2.** If $x \in D'$, $y, z \in \gamma_1$ and $|xy| = |xz| = \text{dist}(x, \gamma_1)$ then $\angle x(\overline{xy}, \overline{xz}) \leq \theta$.

**Proof.** By Lemma 1, $\text{dist}(x, \gamma_1) \leq 2r_0$ and hence $|yz| \leq 4r_0$. If $\angle \overline{xy}, \overline{xz}) > \theta$ than $\angle \overline{xy}, \overline{xz}) < \frac{\pi}{2} - \epsilon$ by triangle comparison. Hence the angle at $y$ between the shortest path $\overline{xy}$ and the interval $[x, y]$ of $\gamma_1$ is at least $\epsilon$. This contradicts the choice of $[p_1q_1]$.

**Lemma 3.** $\text{dist}(x, \gamma_2) \leq 4r_0$ for every point $x \in D'$.

**Proof.** We may assume that $x$ is an interior point of $D'$. Consider the set $S = \{y \in D' : |xy| \leq 2\text{dist}(y, \gamma_1)\}$. Note that $S$ is compact, $S \cap \gamma_1 = \emptyset$ and $S \neq \emptyset$ (because $x \in S$). Let $y$ be a point of $S$ furthest from $\gamma_1$. We will prove that $y \in \gamma_2$. Supposing the contrary, let $y$ be an internal point of $D'$. Then define $C = \{z \in \gamma_1 : |yz| = \text{dist}(y, \gamma_1)\}$ and let $W \subset T_yD$ be the set of directions of shortest paths joining $y$ to points of $C$. By Lemma 2, $W$ is contained within an angle less or equal to $\theta$. Hence there is a unit vector $v \in T_yD$ such that $\angle (v, w) \geq \pi - \theta/2$ for all $w \in W$. Pick a smooth curve $s : [0, \infty) \to D$ parameterized by arc length with $s(0) = y$ and $s'(0) = v$. The first variation formula implies that

$$\lim_{t \to 0} \inf \left\{ \frac{\text{dist}(s(t), \gamma_1) - \text{dist}(y, \gamma_1)}{t} \right\} = \inf \{-\cos \angle(v, w) : w \in W\} \geq \cos(\theta/2) > 1/2.$$  

Hence for sufficiently small $t$ one has $\text{dist}(s(t), \gamma_1) \geq \text{dist}(y, \gamma_1) + t/2$ and

$$|xs(t)| \leq |xy| + t \leq 2\text{dist}(y, \gamma_1) + t \leq 2\text{dist}(s(t), \gamma_1)$$

which contradicts the choice of $y$. Thus $y \in \gamma_2$, hence $\text{dist}(x, \gamma_2) \leq |xy| \leq 2\text{dist}(y, \gamma_1)$ (the second inequality holds because $y \in S$). The lemma follows, since $\text{dist}(y, \gamma_1) \leq 2r_0$ by Lemma 1.

Lemma 3 implies that $|p_1x| \leq \text{dist}(x, \gamma_2) + \ell(\gamma_2) \leq 8r_0 < \pi/2$ for all $x \in D'$. In particular, for every point $x \in D'$ there is a unique shortest path joining it to $p_1$. Let $\gamma_1$ be parameterized by arc length, $\gamma_1 : [0, \ell] \to D$, $\gamma_1(0) = p_1$, $\gamma_1(\ell) = q_1$. For every $t$, $0 < t \leq \ell$, denote by $\alpha(t)$ the angle at $\gamma_1(t)$ between $\gamma_1(t)p_1$ and $\gamma_1[0, t]$. Clearly the function $t \mapsto \alpha(t)$ is continuous and $\alpha(t) \to 0$ as $t \to 0$. 

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Lemma 4. For every interval $J = [t_1, t_2] \subset (0, \ell]$,
\[
\alpha(t_2) - \alpha(t_1) \leq \tau(\gamma_1 | J).
\]

Proof. The statement is additive with respect to $J$. Thus it can be reduced, by means of approximation by broken geodesics, to the following: if $\gamma_1 | J$ is a shortest path then the function $t \mapsto \alpha(t)$ is non-decreasing. This is equivalent to that for all $t, t' \in J$ the sum of two angles of the triangle $p_1 \gamma_1(t) \gamma_1(t')$ at $\gamma_1(t)$ and $\gamma_1(t')$ is less or equal to $\pi$. This in turn follows from triangle comparison, since the sides $|p_1 \gamma_1(t)|$ and $|p_1 \gamma_1(t')|$ are shorter than $\pi/2$, and for any spherical triangle with lateral sides $< \pi/2$ the sum of its angles at the base side is less or equal to $\pi$ (because the area in not larger than the angle at the third vertex). \hfill \Box

Applying Lemma 4 to intervals $J = [t, \ell]$ as $t \to 0$ we obtain that $\omega_{q_1}(\gamma_2, \gamma_1) \leq \tau(\gamma_1)$. Hence $\tau(\gamma) \geq \tau(\gamma_1) \geq \omega_{q_1}(\gamma_2, \gamma_1) \geq \varepsilon$ which proves the statement (i) of Proposition 2 for $\delta = \varepsilon$.

Now let us prove the statement (ii). Let $\gamma$ be parameterized by arc length, $\gamma : [0, \ell] \to D, \gamma(0) = p, \gamma(\ell) = q$. Denote by $\Sigma$ the set of parameters $t \in [0, \ell]$ such that $|p \gamma(t)| \leq 4\nu_0$. For every $t \in \Sigma$ one has
\[
\frac{d}{dt^+}|\gamma(t)q| \leq -\cos \alpha(t),
\]
where $d/dt^+$ denotes the right derivative and $\alpha(t)$ is the angle at $\gamma(t)$ between $\gamma(t)_q$ and $\gamma(t)_r$. Applying (i) to the domain bounded by the curves $\gamma|_{[t, \ell]}$ and $\gamma(t)_q$ in place of $D$, we obtain that $\alpha(t) \leq \varepsilon$. (If these curves have common internal points, they bound several domains, in which case the one containing $\gamma(t)$ should be chosen.) It follows that
\[
\frac{d}{dt^+} |\gamma(t)q| \leq -\cos \varepsilon < -(1 + \varepsilon)^{-1}
\]
for all $t \in \Sigma$. Then the left-hand side is negative and therefore $\Sigma$ contains a right-side neighborhood of every its point. In addition, we have $0 \in \Sigma$, hence $\Sigma = [0, \ell]$. Now by integrating the above inequality we obtain that $|pq| \geq \ell(1 + \varepsilon)^{-1}$ which proves the statement (ii) of Proposition 2.

The statement (ii) implies that for all $x \in \gamma$ we have $|px| + |qx| \leq (1 + \varepsilon)|pq|$, in particular, $|px|$ and $|qx|$ are less than $\pi/4$. By triangle comparison it easily follows that the altitude of the triangle $pqx$ is small compared to the length of the base $pq = \gamma_0$. The statement (iii) follows (for another but still arbitrarily small value of $\varepsilon$). This completes the proof of Proposition 2. \hfill \Box

For the rest of the paper we fix a $\delta_0 > 0$ that is suitable for $\varepsilon = 1/2$ in Proposition 2.

Proposition 3. Let $\gamma$ be a curve in a domain $D$ with endpoints $p$ and $q$ such that $|pq| < 4\nu_0$ and $\tau(\gamma) < \delta_0$. Let $\gamma_0$ be a shortest path joining $p$ to $q$. Assume that any domain bounded by an interval of $\gamma$ and an interval of $\gamma_0$ has the radius less or equal to $\nu_0$. Then $\ell(\gamma) \leq 2|pq|$.

Proof. By a slight variation of $\gamma$ we may achieve that it intersects $\gamma_0$ only finitely many times. Every domain bounded by an interval of $\gamma$ between two consecutive
(on $\gamma$) intersection points and the corresponding interval of $\gamma_0$ satisfy the assumptions of Proposition 2. Therefore every such domain form acute angles at these intersection points. This implies that the intersection points have the same order on $\gamma_0$ and $\gamma$. By the last statement of Proposition 2, the length of an interval of $\gamma$ between consecutive intersection points is at most twice the length of the corresponding interval of $\gamma_0$. Adding up these inequalities over all intervals, we obtain that $\ell(\gamma) \leq 2|pq|$.

**Proposition 4.** For every $T > 0$ here is an $\varepsilon > 0$ such that the following holds. Let $D$ be a domain with $\rho(D) \geq r_0$, $\gamma_0 \subset \partial D$ be a shortest path, $\ell(\gamma_0) \leq r_0$. Let $\Gamma \subset D$ be a set containing $\partial D$ and representable as a union of $\gamma_0$ and an at most countable collection of disjoint simple curves with endpoints at $\gamma_0$. Assume that the sum of turns of the curves that form $\Gamma$ is less or equal to $T$. Then there exists a point $x \in D$ such that $\text{dist}(x, \Gamma) > \varepsilon$.

**Proof.** Since it suffices to prove the statement for a set $\Gamma$ composed of finitely many curves, we may assume that the curves are separated from one another. Consider all subdomains $D' \subset D$ with $\rho(D') \geq r_0$ bounded by an interval of one of the curves that form $\Gamma$ and a shortest path of length $\leq r_0$ joining the endpoints of this interval. Among these, there is a domain minimal by inclusion. A minimal domain $D'$ and the set $\Gamma' = \Gamma \cap D'$ satisfy the assumptions of the proposition, so we may assume that $D$ itself is minimal. This implies that propositions 2 and 3 apply to all curves contained in $\Gamma$ and having the distance between endpoints no larger than $r_0$, except the curve $\partial D \setminus \gamma_0$. Since $\rho(D) \geq r_0$, there is a point $p \in D$ such that the Riemannian exponential map $\exp_p$ is defined and injective in a ball of radius $r_0$ (cf. [B]). Let the ball of radius $r_0$ in $T_pD$ be equipped with a metric of constant curvature 1. Then the exponential map does not increase lengths of curves and hence it is distance non-increasing in the ball of radius $r_0/2$. The curves composing $\Gamma \setminus \partial D$ fall into two classes, those whose turns are less or equal to $\delta_0$, and those whose turns are larger than $\delta_0$. By Proposition 2, the curves of the former class are contained within the $(r_0/2)$-neighborhood of $\gamma$ and therefore do not intersect the ball $B(p, r_0/2)$. The latter class contains at most $T/\delta_0$ curves. Split each of them into $T/\delta_0$ intervals whose turns are not greater than $\delta_0$. Thus we obtain a collection of curves $s_1, \ldots, s_N$, $N \leq (T/\delta_0)^2$ such that $\tau(s_i) < \delta_0$ and

$$B(p, r_0/2) \cap \Gamma \subset \bigcup_{i=1}^{N} s_i.$$ 

Since $\text{diam} B(p, r_0/2) \leq r_0$, Proposition 3 implies that for every $i = 1, \ldots, N$ the set $s_i \cap B(p, r_0/2)$ is contained within an interval $s'_i$ of $s_i$ with $\ell(s'_i) \leq r_0$. For a sufficiently small $\varepsilon$ pick a maximal set of points $\{y_1, \ldots, y_M\}$ in $B(p, r_0/3)$ separated by distances at least $3\varepsilon$ from one another. Comparing $B(p, r_0/3)$ with a ball in $S^2$ by means of the exponential map, we obtain an estimate $M \geq c(r_0/\varepsilon)^2$ for some (universal) constant $c > 0$. Every curve $s'_i$ intersects at most $r_0/\varepsilon$ balls $B(y_i, \varepsilon)$ because $\ell(s'_i) \leq r_0$. For $\varepsilon < cr_0/N$ we have $N r_0/\varepsilon < c(r_0/\varepsilon)^2 < M$ and hence at least one of the $M$ balls $B(y_i, \varepsilon)$ does not intersect $\Gamma$. □

**Proposition 5.** For any $T > 0$ and $N \in \mathbb{N}$ there is an $\varepsilon > 0$ such that the following holds. Let $D$ be a domain, $\Gamma \subset D$ be an embedded tree with at most $N$
edges whose turns are less or equal to $T$. Assume $\text{dist}(x, \Gamma \cup \partial D) < \varepsilon$ for all $x \in D$. Then

(i) For any simple curve $\gamma$ contained in one of the edges of $\Gamma$, joining points $p$ and $q$ with $|pq| \leq r_0$, and such that $\tau(\gamma) < \delta_0$, one has $\ell(\gamma) \leq 2|pq|$.

(ii) Any ball of radius $r < r_0$ lying in the interior of $D$ contains a ball of radius $\varepsilon r$ not intersecting $\Gamma$.

Proof. Approximate $\Gamma$ by a simple closed curve $\Gamma_0$ which is a boundary of a small neighborhood of $\Gamma$. This curve can be chosen so that its turn is estimated from above in terms of $N$ and the turns of the edges, and lengths of edges are close to lengths of some intervals of $\Gamma_0$. It suffices to prove the proposition for $\Gamma_0$ instead of $\Gamma$.

(i) Let $\varepsilon$ be the same as in Proposition 4. If $\ell(\gamma) > |pq|$ then Proposition 3 fails for $\gamma$. This means that some interval of the shortest path joining $p$ and $q$, and some interval of $\gamma$, bound a domain $D'$ with $\rho(D') \geq r_0$. By Proposition 4 applied to $D'$ and the set $|pq| \cup (\Gamma_0 \cap D')$, there is a point $x$ such that $\text{dist}(x, \Gamma_0 \cup \partial D) \geq \varepsilon$.

(ii) Let $B(p, r)$ be a ball contained in the interior of $D$. First assume that the Riemannian exponential map is injective within the radius $r$. Then (i) implies that the set $B(p, r/2) \cap \Gamma_0$ can be covered by a bounded number of intervals of $\Gamma_0$ each having length $\leq r$ and turn $\leq \delta_0$. Applying the selection procedure from the proof of Proposition 4 (for $r$ in place of $r_0$), we obtain a desired ball of radius $\varepsilon r$.

On the other hand, if $\exp_p$ is not injective within a radius $r$, then $\rho_{\partial D}(D) \leq r < \pi$, and hence $\rho(D) \geq \pi/2 > r_0$ by Proposition 1. It follows that there is a point $q \in D$ for which $\exp_q$ is defined and injective within the ball of radius $r_0$ (cf. [B]). Applying the selection procedure to this ball, we can obtain a ball of radius $\varepsilon$ not intersecting $\Gamma_0$. This contradicts the assumption that $\sup_{x \in D} \text{dist}(x, \Gamma_0 \cup \partial D) < \varepsilon$. $\square$

Proof of Theorem 1. Let $\{d_n\}$ be a sequence of metrics on a polyhedron $X$ satisfying the assumptions of the theorem, and let $x \in X$ be an arbitrary point. Fix a homeomorphism identifying a neighborhood of $x$ with a cone over a graph and construct there two small embedded discs, $D$ and $D'$ which are homothetic with respect to the cone structure. Let these discs be cones over a cycle or a path in the link of $x$, moreover in the latter case the endpoints of the path correspond to boundary edges or can be included in cycles. Then $D' \cap \text{esk}_1 X$ is a union of curves emanating from $x$ whose turns with respect to the metrics $d_{n,D}$ are uniformly bounded by some $T$. To prove the theorem it suffices to show the the lengths $\ell_n(D' \cap \text{esk}_1 X)$ are uniformly bounded.

We may assume that the diameter of $D$ with respect to the limit metric $d$, and hence with respect to $d_n$ for large $n$, is less than $\varepsilon$, where $\varepsilon$ is chosen by Proposition 5 for $T$ and the number of edges of $D \cap \text{esk}_1 X$. Then every point $y \in D$ lies within a distance less than $\varepsilon$ from $(D \cap \text{esk}_1 X) \cup \partial D$ with respect to $d_n$.

Recall that a family of metric spaces is (Gromov–Hausdorff) pre-compact if for any $\delta > 0$ the minimal cardinality of a $\delta$-net is uniformly bounded over all spaces of the family. We will prove that the restrictions of the metrics $d_{n,D}$ to $D'$ form a pre-compact family. Let $0 < \delta < \inf \text{dist}_{d_n}(D', \partial D \setminus \partial X)$ (the infimum is positive because the metrics $d_n$ converge). First consider the case when $x$ is an internal point of the disc (i.e., the disc is spanned by a cycle in the link of $x$). In every space $(D', d_{n,D}|_{D'})$ pick a maximal collection of points separated from one another by distance at least $\delta/2$. These points obviously form a $\delta$-net. The balls of radius $\delta/5$
(with respect to \( d_{n,D} \) centered at these points are disjoint. By proposition 5, each of these balls contains a ball of radius \( \varepsilon \delta / 5 \) not intersecting \( \text{esk}_1 X \). The number of such balls is uniformly bounded over \( n \) because they are also balls of the same radius with respect to metrics \( d_n \), and metrics \( d_n \) form a pre-compact family.

The case when \( x \) is a boundary point is similar except that those \( (\delta/5) \)-balls that intersect the disc’s boundary should be excluded. But the number of these balls is uniformly bounded because they intersect \( \partial D \) only in edges of \( X \) that correspond to the endpoints of the path that spans the disc, and the lengths of these edges are uniformly bounded. For boundary edges this is included in the assumptions of the theorem and for those included in link’s cycles this follows (see below) from the cycle case that has already been considered.

Now we will derive that the lengths of \( D' \cap \text{esk}_1 X \) in metrics \( d_{n,D} \) are uniformly bounded. Suppose the contrary. Then, since the turns of the edges are uniformly bounded, it follows that for any \( L > 0 \) there is a curve \( \gamma \) contained in a graph’s edge such that, in one of the spaces \( (D', d_{n,D}|_{D'}) \), one has \( \ell(\gamma) = L \) and \( \tau(\gamma) < \delta_0 \).

Pick points \( x_1, \ldots, x_N, N \geq L/2r_0 \), on this curve such that the length of the curve’s interval between any two of them is at least \( 2r_0 \). By the statement (i) of Proposition 5, applied to \( D \) and \( D \cap \text{esk}_1 X \), all distances between these points in the \( d_{n,D} \) metric are not less than \( r_0 \). Since \( N \) is arbitrarily large, this contradicts the above pre-compactness of the family of metrics \( d_{n,D}|_{D'} \). \( \square \)

References


[BB] Yu. D. Burago, S. V. Buyalo, Metrics of curvature bounded above on 2-polyhedra. II.