A CONTRACTIBLE GEODESICALLY COMPLETE SPACE OF CURVATURE ≤ 1 WITH ARBITRARILY SMALL DIAMETER

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Abstract. For every $\varepsilon > 0$, a contractible spherical 2-polyhedron is constructed which is a geodesically complete space of curvature ≤ 1 and has diameter $< \varepsilon$. As an application, a sequence of compact spaces of curvature ≤ 1 is constructed which homotopically converges to a compact length space with no upper curvature bound.

Introduction

0.1. It is well known that a complete length space X with nonnegative curvature in the Alexandrov sense can not contain a contractible geodesic loop. Moreover, the injectivity radius at any point $x \in X$ equals one half of the length of the shortest nontrivial element of the fundamental group $\pi_1(X, x)$. This easily implies the following convergence properties: if a sequence $\{X_i\}_{i=1}^{\infty}$ of compact spaces of curvature $\leq k$, where $k \leq 0$, homotopically converges to a length space X, then the injectivity radii of the spaces X_i are bounded from below, and therefore X is also a space of curvature $\leq k$. (The homotopy convergence of metric spaces is intermediate between the uniform convergence and the general Gromov–Hausdorff convergence, see e.g. [4]).

One may ask similar questions about spaces of curvature $\leq k$ with $k > 0$ (due to rescaling, it is sufficient to consider the case $k = 1$). Even on the two-dimensional sphere, it is easy to construct examples of Riemannian metrics of curvature ≤ 1 with arbitrarily small injectivity radius at some points. On the other hand, it is known [3] that for any metric of curvature ≤ 1 on the sphere S^2 or the disc D^2 there exist at least one point at which the injectivity radius is no less than $\pi/2$. As a consequence, a uniform limit of metrics of curvature ≤ 1 on any two-dimensional manifold also has curvature ≤ 1. This property of uniform limits also holds for two-dimensional polyhedra under certain additional assumptions about the one-dimensional skeleton [4, 5]. Whether these assumptions are necessary is not known.

In dimension 3, there are examples [2] of Riemannian metrics on the sphere S^3 with sectional curvature ≤ 1 and arbitrarily small diameter (and hence with arbitrarily small injectivity radius at every point). Note that such metrics do not yet allow one to construct a counter-example in which an upper curvature bound is not preserved under a uniform or homotopy convergence; for this one would need similar examples of metrics on the disc D^3. (Observe that a complement of a small

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ball of a Riemannian metric in S^3 is not a space of curvature ≤ 1 in the Alexandrov sense—due to the extreme concavity of the boundary.)

0.2. In this paper we construct examples of metrics with similar properties on *contractible* two-dimensional polyhedra without boundaries and, as an application, examples of homotopy convergence of two-dimensional polyhedra where an upper curvature bound is not preserved. By a *polyhedron* we mean a (two-dimensional) finite simlicial complex equipped with a length metric. In fact, all our examples are *spherical polyhedra*, that is, spaces that admit triangulations into spherical triangles (with metrics of constant curvature 1 and geodesic sides).

The precise formulations follow.

0.3. Theorem. *For every $\varepsilon > 0$ there exist a contractible two-dimensional spherical polyhedron which is a geodesically complete space with curvature ≤ 1 and diameter $< \varepsilon$.*

Note that the geodesic completeness above is equivalent to having no boundary, i.e., that every point of the one-dimensional skeleton belongs to at least two two-dimensional faces.

0.4. Corollary. *There is a sequence of two-dimensional polyhedra of curvature ≤ 1 homotopically converging* (*in the sense of* [4]) *to a two-dimensional polyhedron which is not a space of curvature bounded above.*

0.5. Remark. As ε goes to zero, the topological type of a polyhedron in Theorem 0.3 changes (and the combinatorial complexity of the construction grows rapidly). It remains unclear whether it is possible to construct such examples of metrics on a polyhedron of a fixed topological type. This is related to the question whether an upper curvature bound is preserved under a uniform convergence of metrics on polyhedra (not the homotopy convergence dealt with in Corollary 0.4).

0.6. Organization of the paper. In §1 we describe a construction of a space $P_{N,F}$ (depending on $\varepsilon > 0$ and combinatorial parameters N and F), which is a contractible spherical polyhedron of diameter $< 5\varepsilon$. In §3 we prove Lemma 2.3, which provides a sufficient condition for $P_{N,F}$ to have curvature ≤ 1. Finally, in §3 we describe how to choose parameters N and F so that $P_{N,F}$ satisfies all the conditions from Theorem 0.3, and also prove Corollary 0.4.

§1. Constructing the polyhedron $P_{N,F}$

1.1. The block K. The desired polyhedron will be obtained by gluing together many identical blocks. We begin with a description of a single block.

Fix an $\varepsilon > 0$ and construct an embedded graph Γ in the standard sphere S^2 with the following properties:

(1) it is a tree;

(2) every edge is a geodesic of length ε;

(3) its ε-neighborhood contains the whole sphere.

(For instance, one may let Γ be the union of several meridian segments of length $k\varepsilon$ starting at the north pole and divided into intervals of length ε, where k is the maximal integer such that $k\varepsilon < \pi$.)

Let n denote the number of edges of Γ; then the number of its vertices is $n + 1$.

Then find a geodesic segment $[pq]$ of length $\varepsilon/2$ in the sphere such that p is one of the leaf nodes of Γ and $[pq]$ have no common points with Γ except p. (In order to satisfy the last requirement one may, for instance, choose the segment $[pq]$ to be sufficiently close to an edge of Γ.)

Let X be the space obtained by completion of the intrinsic metric of the set $S^2 \setminus [pq]$. Clearly X is a spherical polyhedron homeomorphic to the disc, and its boundary is the doubling of the segment $[pq]$. Now identify all nodes of Γ in the space X. The resulting quotient space (with the length metric induced from X) will be referred to as the *block* and denoted by K.

The block K comes with the following cell decomposition. The null-dimensional skeleton consists of one point obtained by gluing together the nodes of Γ. We call this point the *origin* of the block. One-dimensional cells are the n loops obtained from the edges of Γ, and one loop obtained from the doubling of $[pq]$. We call them *internal edges* and the *boundary* of the block, respectively. For later use, fix some orientation on all edges of this one-dimensional skeleton and enumerate the internal edges by numbers from 1 to n. Finally, the cell decomposition contains one two-dimensional cell obtained from the set $S^2 \setminus (\Gamma \cup [pq])$.

As a metric space, K is a spherical polyhedron, and all its edges are closed geodesics of length ε. By the property (3) of Γ, the distance from any point of K to the one-dimensional skeleton is no greater than ε, therefore the distance from any point to the origin is no greater than 2ε.

1.2. Polyhedra $P_{N,F}$. The polyhedron in Theorem 0.3 will be obtained from a large number of copies of K by gluing every internal edge of every block to the boundary of some other block. Formally, a *gluing scheme* is a pair consisting of a natural number N and a map

$$F\colon \{1,\dots,N\} \times \{1,\dots,n\} \to \{1,\dots,N\}.$$

Given a gluing scheme (N,F), one constructs a polyhedron $P_{N,F}$ as follows: pick N copies of the block K, numbered from 1 to N, glue all their vertices into one point (for which we continue using the notation p), and then for all $i \leq N$ and $j \leq n$ glue the jth internal edges of the ith block to the boundary of the block number $F(i,j)$, by a unique orientation-preserving isometry which maps the point p to itself.

Later we will show that, for a suitable choice of N and F, the polyhedron $P_{N,F}$ satisfies the conditions given in Theorem 0.3 with ε substituted by 5ε. First observe that $P_{N,F}$ is a spherical polyhedron and its diameter is no greater than 4ε (since the distance from any point to p is no greater than 2ε).

1.3. Lemma. *The polyhedron $P_{N,F}$ is contractible for any N and F.*

Proof. Let X be as in section 1.1 (that is, the space from which the block K is obtained). The polyhedron $P_{N,F}$ can be thought of as the result of attaching N copies of X to a bouquet of N circles by certain maps from copies of $\Gamma \cup \partial X$ to the bouquet. Each copy of the boundary ∂X is mapped homeomorphically to one of the circles in the bouquet, and to each circle exactly one copy of the boundary is attached. Since Γ is contractible, the cell pair $(X, \Gamma \cup \partial X)$ is homotopy equivalent to $(X, \partial X)$. Therefore $P_{N,F}$ is homotopy equivalent to the result of attaching N copies of the disc X to the bouquet of circles by homeomorphisms from copies of ∂X to the corresponding circles in the bouquet, i.e., to a bouquet of two-dimensional discs. $\square$

§2. Gluing graph and boundedness of curvature

2.1. Now we turn our attention to the conditions under which the polyhedron $P_{N,F}$ has curvature ≤ 1. A well-known criterion of an upper curvature bound (see e.g. [6] or [1]) in the case of spherical polyhedra simplifies to the following: a spherical polyhedron has curvature ≤ 1 if and only if all simple cycles in the link at any point have length $\ge 2\pi$.

(Recall that the *link* of a two-dimensional polyhedron P at a point x is a space of directions of curves starting at x. The link is a metric graph whose nodes correspond to edges of P starting at x and whose edges correspond to faces of P containing x.)

It is clear that the above condition on the link is automatically satisfied for all points of P except the origin p. We denote by L and $L_{N,F}$ the links of p in the block K and the polyhedron $P_{N,F}$ respectively.

The graph L is a disjoint union of a segment of length 2π (which corresponds to the boundary point) and n circles of length 2π (corresponding to internal nodes of Γ). Every edge of the cell decomposition described in section 1.1 produces two nodes of L. If the edge is an internal one, then these two nodes belong to different connected components of the link, and the nodes corresponding to the boundary of the block are the endpoints of the segment of length 2π. (Note that this segment contains one more node, namely the point corresponding to the edge of Γ starting at $p \in S^2$.)

The link $L_{N,F}$ is obtained from the disjoint union of N copies of L by gluing together some nodes. Namely every single gluing of an edge of one block to an edge of another block makes the corresponding two nodes in links glued together.

Let us introduce the notion of *gluing graph* associated to a polyhedron $P_{N,F}$. Nodes of this graph correspond to blocks from which the polyhedron is glued, and each gluing of an internal edge of one block to the boundary of another block corresponds to an edge of the gluing graph. Formally we give the following

2.2. Definition. Let (N, F) be a gluing scheme (cf. 1.2). We refer to as the *gluing graph* of this scheme and denote by $\Gamma_{N,F}$ the multi-graph (that is a generalized graph in which loops and double edges are allowed) with a set of nodes $\{v_i : 1 \le i \le N\}$ and a set of edges $\{e_{ij} : 1 \le i \le N, 1 \le j \le n\}$ where every edge e_{ij} connects vertices v_i and $v_{F(i,j)}$.

2.3. Lemma. *Suppose that the combinatorial length of any simple cycle in the gluing graph $\Gamma_{N,F}$ is greater than $4\pi/\delta$, where δ is the minimal distance between distinct nodes in L. Then the polyhedron $P_{N,F}$ has curvature ≤ 1.*

Proof. First observe that $4\pi/\delta \ge 2$; hence the assumption on $\Gamma_{N,F}$ implies that this graph contains no loops and double edges (i.e., it is a graph is the usual sense). Thus $F(i,j) \ne i$ for all i, j, and $F(i,j) \ne F(i,j')$ for $j \ne j'$.

Consider N copies $L^1, \dots, L^N$ of the link L. Introduce the following notation for the nodes of L^i: let V^i_{j-} and V^i_{j+}, where $1 \le j \le n$, denote the directions of the outgoing and incoming ends of the internal edge number j, and V^i_{0-} and V^i_{0+} be the directions of the outgoing and incoming ends of the boundary edge. We refer to V^i_{0-} and V^i_{0+} as *boundary* nodes, and to other nodes as *internal* ones.

The link $L_{N,F}$ is obtained from the disjoint union $\bigsqcup L^i$ by means of the following identifications: for every i and j, $1 \le i \le N$, $1 \le j \le n$, the node V^i_{j-} is glued to $V^{F(i,j)}_{0-}$, and V^i_{j+} to $V^{F(i,j)}_{0+}$. Denote by $\sim$ the equivalence relation generated by these gluing rules. It is easy to see that $V^i_{0+} \not\sim V^j_{0+}$ and $V^i_{0-} \not\sim V^j_{0-}$ for $i \neq j$. The equivalence class of a node V^i_{0+} consists of this node itself and the nodes $V^{i'}_{j+}$ for which $F(i', j) = i$. The fact that $\Gamma_{N,F}$ has no loops and double edges implies that different points in one equivalence class belong to different copies of L.

To prove the lemma, it is sufficient to show that every simple cycle in $L_{N,F}$ has length $\geq 2\pi$. Suppose the contrary, and let γ be a simple cycle of length $< 2\pi$. Represent γ as a product of paths $\gamma_1 \dots \gamma_m$ where every path γ_k is contained in L^{i_k} for some i_k, $1 \leq i_k \leq N$, and the ending point of γ_k is $\sim$-equivalent to starting point of γ_{k+1} but not coincide with it in the disjoint union $\bigsqcup L^i$. (Here and later on all indices at γ are taken modulo m.) Clearly $m > 1$ because L does not contain simple cycles of length less than 2π. Also observe that $i_k \neq i_{k+1}$ for all k, because different points in L^{i_k} are never $\sim$-equivalent.

Then construct a closed path $s = s_1 \dots s_m$ in the graph $\Gamma_{N,F}$, in which every subpath s_k connects v_{i_k} to $v_{i_{k+1}}$, consists of no more than 2 edges and is constructed as follows. Consider two points, the ending point of γ_k and the starting point of γ_{k+1}. Since they are $\sim$-equivalent, there are two cases:

(1) One of these points is a boundary node. Then the nodes v_{i_k} and $v_{i_{k+1}}$ are connected by an edge in $\Gamma_{N,F}$. In this case, s_k is this edge.

(2) Both points are internal nodes. Then their equivalence means that they are attached to one boundary node of some component L^i. Hence in $\Gamma_{N,F}$ the corresponding node v_i is connected by edges to both v_{i_k} and $v_{i_{k+1}}$. In this case, let s_k be the path consisting of these two edges. We will refer to v_i as an *intermediate* node.

Let us show that the resulting cycle $s = s_1 \dots s_m$ is not contractible. To do this, it is sufficient to prove that any two adjacent edges in this cycle are different. Suppose the contrary: let some edge e is passed two times one after the other, and let v_i be the node visited between these two passes. This node v_i is not intermediate because (in the notation from case (2) above) $i_k \neq i_{k+1}$. Hence e is the first edge of the path s_k and the last edge of the path s_{k-1} for some k (in particular, $i = i_k$). Hence the beginning and the end of γ_k are involved in one gluing operation, namely the one corresponding to the edge e. Two different points in one component L^i are involved in one gluing operation only if these points have the form V^i_{j-} and V^i_{j+} for some j. Thus the original path γ contains an interval connecting points V^i_{j-} and V^i_{j+} in L^i for some i and j. But this is impossible: if $j \neq 0$, these points are in different connected components of L^i, and if $j = 0$, the distance between them equals 2π (recall that the length of γ is less than 2π by our assumption).

Thus the cycle s is not contractible, therefore it contains a simple cycle. Since the combinatorial length of s is no greater than $2m$, and the length of the simple cycle is no less than $4\pi/\delta$, it follows that $m \geq 2\pi/\delta$. It remains to recall that the original cycle γ consists of m intervals each of length $\geq \delta$, hence the length of γ is no less than 2π. This contradiction proves Lemma 2.3. □

§3. Proof of Theorem 0.3 and Corollary 0.4

The next lemma (independent of the previous arguments) guarantees the existence of graphs to which Lemma 2.3 applies.

3.1. Lemma. *For any natural numbers m and n there exists a finite graph G,* *possessing the following properties*:

(1) *the combinatorial length of any simple cycle in G is no less than m*;
(2) *the edges of G can be oriented so that the incoming and outgoing degree of every node equal n (i.e., every node is a beginning of exactly n edges and an ending of exactly n edges)*.

Proof. We construct the desired graph (actually, multi-graph) by induction in m. For $m = 1$ one may let G be a bouquet of n circles (note that any orientation of this bouquet satisfies the second requirement of the lemma).

Now suppose that a graph G satisfies the requirements of the lemma, and let us construct a graph G' satisfying the same requirements with m replaced by $m+1$. Let N be the number of edges of G and $M = 2^N$. Enumerate the edges of G from 0 to $N-1$ and fix an orientation on each of them. The number assigned to an edge e will be denoted by $\nu(e)$, its beginning and end with respect to that orientation by $v_-(e)$ and $v_+(e)$ respectively. Let V and E denote the set of nodes and the set of edges of G. The set of nodes of G' is defined as the product $V \times \mathbf{Z}_M$, where $\mathbf{Z}_M$ is the residue set modulo M. The edges of G' are indexed by the set $E \times \mathbf{Z}_M$, and the edge corresponding to a pair $(e, j) \in E \times \mathbf{Z}_M$ connects the nodes $(v_-(e), j)$ and $(v_+(e), j + 2^{\nu(e)})$.

There is a natural map $p\colon G' \to G$, projecting the products $V \times \mathbf{Z}_M$ and $E \times \mathbf{Z}_M$ to their first components. It is easy to see that p is a covering map. It follows that G' satisfies the requirement (2) of the lemma: in order to obtain a desired orientation on G', lift such an orientation from G.

Consider a simple cycle s in G' and its projection $p \circ s$ in G. Since p is a covering map, the cycle $p \circ s$ is not contractible. Then observe that $p \circ s$ is not a simple cycle. Indeed, if s is a lift of a simple cycle, then the second coordinates of its endpoints differ by a quantity of the form

$$\pm 2^{\nu_1} \pm 2^{\nu_2} \pm \cdots \pm 2^{\nu_r} \pmod M,$$

where $\nu_1, \dots, \nu_r$ are numbers assigned to those edges of G that are contained in the cycle. This quantity cannot be zero modulo M because

$$\left| \pm 2^{\nu_1} \pm \cdots \pm 2^{\nu_r} \right| \le \left| \sum_{j=0}^{N-1} 2^j \right| = 2^N - 1 < M,$$

and a sum of distinct powers of two with alternating signs can not be zero in $\mathbf{Z}$. Since the cycle $p \circ s$ is not simple, it contains a strictly shorter simple cycle. By inductive assumption, the length of any simple cycle in G is no less than m, therefore the cycle $p \circ s$ (and hence s) has length $\ge m+1$. Thus the graph G' possesses the properties stated in the lemma with m replaced by $m+1$. $\square$

3.2. Proof of Theorem 0.3. Fix an $\varepsilon > 0$ and construct a block K as in section 1.1. Let n, as before, denotes the number of internal edges of K, and let δ be the same as in Lemma 2.3. By Lemma 3.1, there exists a graph G in which all simple cycles have combinatorial length $> 4\pi/\delta$, and an orientation of this graph such that all incoming and outgoing degrees of nodes are equal to n. Let N be the number of nodes of G. Enumerate the vertices of G by numbers from 1 to N and choose a gluing scheme (N, F) so that for every i, $1 \le i \le N$, the values $F(i, j)$,

$1 \leq j \leq n$, range over the numbers assigned to end-nodes of oriented edges starting at the node number i in G. Then the gluing graph $\Gamma_{N,F}$ is obviously isomorphic to G, hence by Lemma 2.3 the polyhedron $P_{N,F}$ has curvature ≤ 1. As noted above (see 1.2 and 1.3), this polyhedron is contractible and has diameter $\leq 4\varepsilon$. Finally, observe that $P_{N,F}$ has no boundary because for every boundary edge in the disjoint union of N blocks there is at least one internal edge glued to it. This follows from the fact that the incoming degree of every node in G is at least 1. Thus the resulting polyhedron $P_{N,F}$ possesses all properties given in Theorem 0.3, with 5ε in place of ε. $\square$

3.3. Proof of Corollary 0.4. By Theorem 0.3 there exists a sequence $\{X_n\}$ of contractible geodesically complete spherical polyhedra of curvature ≤ 1 with $\operatorname{diam} X_n \to 0$. One may assume that $\operatorname{diam} X_n < \pi$ for all n. Then every polyhedron X_n contains a closed geodesic of length $\leq 2 \operatorname{diam} X_n$. Let γ_n be a shortest closed geodesic in X_n, and ε_n its length.

Consider a two-dimensional Euclidean cone with total angle at the vertex smaller than 2π. Denote the cone's vertex by p, and let Y be a sufficiently large closed ball of cone's intrinsic metric centered at p. Clearly the complement of any neighborhood of p in Y has curvature ≤ 1, but the whole Y is not a space of curvature bounded above. For every n, construct a space Y_n as follows: remove from Y a metric ball B_n centered at p, the length of whose boundary is equal to ε_n, and then attach to this boundary the polyhedron X_n by the curve γ_n. This space Y_n is a result of gluing two spaces of curvature ≤ 1 by an isometry between convex subsets, hence X_n has curvature ≤ 1.

Since every polyhedron X_n is contractible, the space Y_n is homotopy equivalent to Y, moreover, the maps and homotopies realizing this homotopy equivalence can be chosen to be identity outside an arbitrary neighborhood of B_n. Since $\operatorname{diam} B_n \to 0$ and $\operatorname{diam} X_n \to 0$, it follows that the spaces Y_n homotopically converge to Y. $\square$

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