INVARIANT MEASURES: NEW ASPECTS OF DYNAMICS, COMBINATORICS AND REPRESENTATION THEORY

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Abstract

Invariant Measures, Branching Graphs, Central Measures, Adic transformation and Rokhlin Towers, Young Diagrams, Random Walks, Random Groups, Random Representations, Exit- and Martin- Boundaries, Asymptotic Representation Theory, Asymptotic Dynamics and Combinatorics, Projective Limit of Simplices, Filtrations, Standardness, Lacunarity, Intrinsic (Kantorovich) Metrics, Limit shapes, Scaling Entropy.

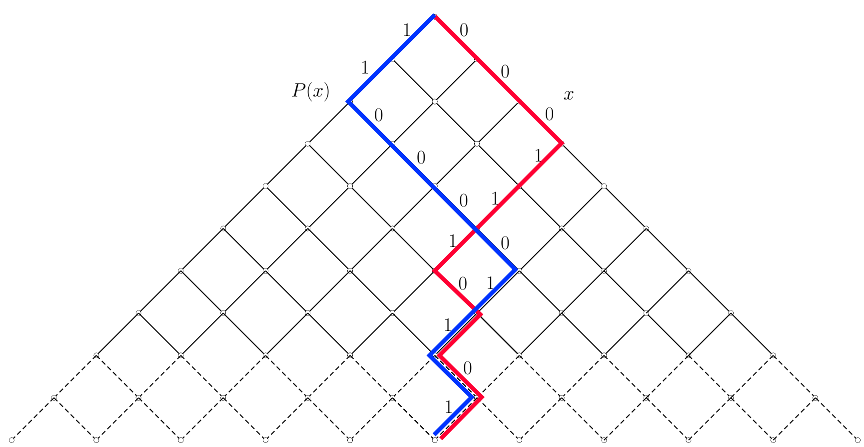
CONTENT WITH THE SHORT COMMENTS

0.**INTRODUCTION**

COMMENTS

I will describe several facts which belong to the various mathematical areas -dynamical systems, representations, combinatorics, Markov processes etc, and which are combined and based on distinguished, simple and very important object — graded or branching graph (or Bratteli diagram) with invariant measure on the space of paths of that graph.

Primarily this object was popular in the theory of AF-algebras, but later it became clear how many other mathematical areas are related with it, and in my talk I will try to enumerate and shortly describe the role of those links.



$$x \mapsto Px; \quad P(0^{m-s}1^s\mathbf{10}\dots) = 1^s0^{m-s}\mathbf{01}\dots \tag{1}$$

Figure 1: Pascal automorphism (Pascal 17th century, Takagi (1903), Kakutani (1975), Vershik (1981), Janvresse et al. (2005)).

First of all I want emphasize that there are two main problems which appeared with graded graph:

1/ To account so called central or invariant measures (probability or not) on the paths of graph;

This problem will be one of the fundamental for us. We will see that many questions from the representation theory, theory of Markov processes

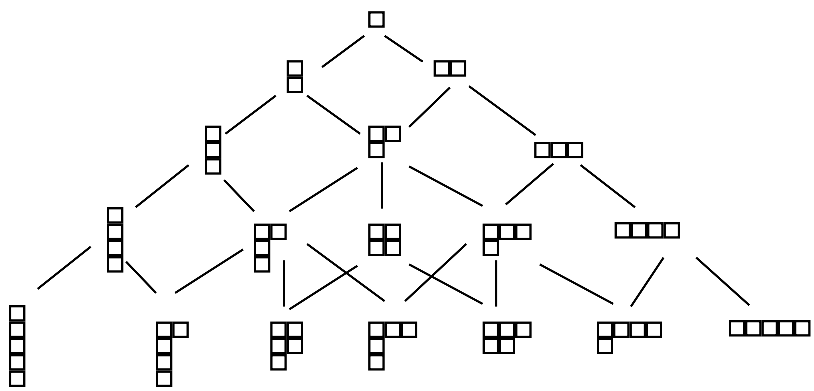


Figure 2: Young graph which is branching diagram of the irreducible representations of the symmetric group.

as well as from ergodic theory, group theory, asymptotic combinatorics were reduced to that problem.

2/ To calculate the typical objects (representations, Young diagrams, generic configurations etc.) its asymptotic, limit shape with respect to statistics and invariant measures on the space of paths.

There are many other problems related to above problem; like calculation of K-functor of the corresponding algebra (group) with given branching graph, analysis of the generating functions of the "generalized binomial coefficients" —which appears in combinatorics and statistical physics etc.

These questions are related to what I called in the 70-th "Asymptotic Representation Theory" but in this talk I can only shortly mention about this. Now this theory turn out very big area with many interesting results.

We will see that the notion of branching graph we can associate with

1) a new approach to ergodic theory (adic dynamics),

with

2)a new look on the theory of various boundaries as set of invariant measures, and consequently on the classification of traces and characters in the asymptotic theory of the representations;

and with — most deep part of that — with

3)the theory of filtration (=decreasing sequences of sigma-fields in measure theory). notion of standardness, generalized Kantorovich metric (inner metric) on the invariant measures etc.

PART ONE: DYNAMICS AND BOUNDARIES OF SPACE OF PATHS.

1.**BRANCHING (OR ℕ-GRADED) GRAPHS. THE SPACES OF PATHS, MARKOV INTERPRETATION, INVARIANT AND CENTRAL MEASURES, ADIC DYNAMICS AS A NEW FORMALISM.**

COMMENTS

In this section we define the main structure on the branching graph equipped structure and Markov interpretation of graph; lexicographic order and adic transformation, formulate the list concrete problems about invariant and central measures.

We formulate the main problem -description of ergodic markov measures with given set of cotransition probabilities and in particular, description of the set of central measures on the space of paths. The notion of adic transformation and "Bratteli-Vershik diagrams" gives the kind of new universal dynamics and open the new direction in ergodic theory. The Markov interpretation of graph gives new approach to the problem about different kind of boundaries in harmonic and probabilistic analysis, we also obtain a universal model in the metric theory of filtrations.

2.**EXAMPLES: PASCAL GRAPH AND TRANSFORMATION; YOUNG GRAPH (THOMA'S THEOREM); APPLICATION: CHARACTERS AND RANDOM SUBGROUPS, CHARACTERS AND REALIZATIONS OF THE REPRESENTATIONS**

COMMENTS

We illustrate our definitions and problems on two important examples: Pascal and Young graphs. Then we formulate the problem about random subgroups and give the solution for infinite symmetric group. One of the illustrations concerned the notion of "random subgroups" and its relation with totally non-free actions, in turn link with von Neumann factor-representation of semidirect products.

3.THEORY OF PROJECTIVE LIMIT OF SIMPLICES, PROBLEM DECIDABILITY

COMMENTS

We classify the problems of description of invariant measures on standard an non-standard parts. The link with the notion of affine type of simplecies (Bauer and Poulsen). Graph UP and Tower of measures.

[pictures:random partition of N; graph of unordered pairs]

PART TWO: FILTRATIONS, STANDARDNESS AND LIMIT SHAPE THEOREM, CLASSIFICATION AND RANDOM MATRICES

4.FILTRATIONS, STANDARDNESS, ITERATED KANTOROVICH METRICS, DICHOTOMY IN THE CLASSIFICATION PROBLEM CHARACTERS AND TRACES. CONJECTURES.

COMMENTS

Definition of dyadic filtrations and the main theorem: lacunary theorem and criteria if standardness. Graph UP and Tower of measures, Poulsen and Bauer simplices, intrinsic (Kantorovich iterated) metric, limit shape and dichotomy. General criteria of Standardness. Super-convergence of Martingales for standard filtrations.

5.THE APPLICATIONS OF INVARIANT MEASURES TO THE CLASSIFICATION OF METRIC-MEASURE SPACES, AND MEASURABLE FUNCTIONS.

COMMENTS

General problem of description of invariant measures for the action of infinite symmetric group. Classification of measurable functions and of classification of mm-spaces (=measure-metric spaces) (Gromov-Vershik theorem), random matrices, generalization of Aldous theorem.

PART ONE:NEW (ADIC) DYNAMICS AND THEORY OF BOUNDARIES

1 BRANCHING OR N-GRADED GRAPHS. THE SPACES OF PATHS, TAIL FILTRATION.

1.1 A graded graph, the path space, topology, main examples.

Main definitions, additional structures on the branching graphs, and the input data.

Consider a locally finite, infinite $\mathbb{N}$-graded graph Γ (= Bratteli diagram). The set of vertices of degree n, $n = 0, 1, \dots$, will be denoted by Γ_n and called the nth *level* of Γ:

$$\Gamma = \coprod_{n \in \mathbb{N}} \Gamma_n;$$

the level Γ_0 consists of the single initial vertex $\{\emptyset\}$. We assume that the edge join two vertices of adjoint levels and every vertex has at least one successor, and every vertex except the initial one has at least one predecessor. In what follows, we also assume that the edges of Γ are simple.[1] No other assumptions are imposed. A locally semi-simple algebra $A(\Gamma)$ over $\mathbb{C}$ is canonically associated to a graded graph Γ; however, here we do not consider this algebra and do not discuss the relation of the notions introduced below with this algebra and its representations; this problem is worth a separate study.

A path t in Γ is by definition (finite or infinite) sequence of edges of Γ in which end of every edge is beginning of the next edge (for graphs without multiple edges, this is the same as a sequence of vertices with the same condition). The space of all infinite paths in Γ is denoted by $T(\Gamma)$; it is, in

[1]For our purposes, allowing Bratteli diagrams to have multiple edges does not give anything new, since cotransition probabilities introduced below replace and generalize multiplicities of edges.

a natural sense, the inverse limit of the spaces of finite paths (leading from the initial vertex to vertices of some fixed level), and thus is a Cantor-like compact set. Cylinder sets in $T(\Gamma)$ are sets defined in terms of conditions on initial segments of paths up to level n; they are clopen and determine a base of the topology of $T(\Gamma)$. There is a natural notion of *tail equivalence relation* τ_Γ on $T(\Gamma)$: two infinite paths are tail-equivalent if they eventually coincide; one also says that such paths lie in the same block of the *tail partition*.

The *tail filtration* $\Xi(\Gamma) = \{\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \dots\}$ is the decreasing sequence of σ-algebras $\mathfrak{A}_n$, $n \in \mathbb{N}$, where $\mathfrak{A}_n$ consists of all Borel sets $A \subset T(\Gamma)$ such that along with every path A contains all paths coinciding with it up to the nth level. In an obvious sense, $\mathfrak{A}_n$ is complementary to the finite σ-algebra of cylinder sets of order n. The key idea of author is to apply the theory of decreasing filtrations (see, e.g.) to the analysis of the structure of path spaces and measures on them. Below we touch on this problem.

List of related notions and examples: stationary graphs, Graph Fibonacci, Pascal graph, Young graph, Many dimensional generalization, Graph of unordered pairs, Graph of ordered pairs, Hasse diagrams of the posets etc.

1.2 Markov interpretation of the branching graph, invariant and central measures, system of cotransition probabilities (matrices) —equipped structure

We can consider the space of paths $T(\Gamma)$ of the branching graph Γ as a Markov compact, e.g. space of sequences $\{v_i\}_{i\in\mathbb{N}} : v_i \in \Gamma_i$, where (v_i, v_{i+1}) is an edge of Γ for all $i \in \mathbb{N}$; (Markov compact is not stationary). We use the same denotation $T(\Gamma)$ for that Markov compact.

Definition 1. *A Markov measures ν on the compact $T(\Gamma)$ called a central measure if for each vertex v the conditional measure ν_v induce my measure ν on the finite set of paths which join vertex $\emptyset$ and v is uniform measure. For the case of stationary compact central measure called measure with maximal entropy.*

Any continuous transformations of the compact $T(\Gamma)$ which sent any path to the path which differs only on the finite many of vertexes preserves all central measures, we called the group of all such transformations adic group.

The notion of central measure on space $T(\Gamma)$ is defined intrinsically by structure of branching graph Γ. Set of all central measures on the space of paths $T(\Gamma)$ we denote as $\Sigma(\Gamma)$ and the set of ergodic central measures as $Erg(\Gamma)$.

Now we introduce an additional structure on the graph, namely, a *system of cotransition probabilities*

$$\Lambda = \{\lambda = \lambda_v^u;\ u \in \Gamma_n, v \in \Gamma_{n+1}, (u,v) \in \mathrm{edge}(\Gamma_n, \Gamma_{n+1}),\ n = 0, 1, \dots\},$$

by associating with each vertex $v \in \Gamma_n$ a probability vector whose component λ_v^u is the probability of an edge $u \prec v$ entering v from the previous level; here $\sum\limits_{u:u\prec v} \lambda_v^u = 1$ and $\lambda_v^u \geq 0$.

Definition 2. *An equipped graph is a pair* (Γ, Λ) *where* Γ *is a graded graph and* Λ *is a system of cotransition probabilities on its edges.*

The term "cotransition probabilities" is borrowed from the theory of Markov chains: if we regard the vertices of Γ as the states of a Markov chain starting from the state $\varnothing$ at time $t = 0$, and the numbers of levels as moments of time, then $\Lambda = \{\lambda_v^u\}$ is interpreted as the system of cotransition probabilities for this Markov chain:

$$\mathrm{Prob}\{x_t = u | x_{t+1} = v\} = \lambda_v^u.$$

The most important special case of a system of cotransition probabilities, which we already have defined, and also studied in combinatorics, representation theory, and algebraic settings, is as follows:

$$\lambda_v^u = \frac{\dim(u)}{\sum\limits_{u:u\prec v} \dim(u)},$$

where $\dim(u)$ is the number of paths leading from the initial vertex $\emptyset$ to u (i.e., the dimension of the representation of the algebra $A(\Gamma)$ corresponding to the vertex u). In other words, the probability to get from v to u is equal to the fraction of paths that lead from $\emptyset$ to u among all the paths that lead from $\emptyset$ to v. This system of cotransition probabilities is canonical, in that it is determined only by the graph. The corresponding Markov measures on the path space $T(\Gamma)$ are called *central measures*; up to now, they have been studied only in the literature on Bratteli diagrams. In terms of the theory of C*-algebras, central measures are traces on the algebra $A(\Gamma)$, and ergodic central measures are indecomposable traces. Note that already for central measures, the asymptotic behavior can be very different; the example of the graph of unordered pairs which shows how much the answer can differ from the case of familiar graphs, such as the Young graph.

It is convenient to regard the system of cotransition probabilities as a system of $d_n \times d_{n+1}$ Markov matrices:

$$\{\lambda_v^u\}, \quad u \in \Gamma_n, v \in \Gamma_{n+1}; \quad |\Gamma_n| = d_n,\ |\Gamma_{n+1}| = d_{n+1},\ n \in \mathbb{N};$$

these matrices generalize the $(0 \vee 1)$ incidence matrices of the graph Γ. Our main interest lies in the asymptotic properties of this sequence of matrices. In this sense, the whole theory developed here is a part of the asymptotic theory of infinite products of Markov matrices, which is important in itself.

Every Markov measure μ on the path space of a graph determines a system of cotransition probabilities as the system of conditional measures of natural measurable partitions. The equipped structure allow us to define the generalization of the notion of central measures.

A measure on the path space of a graph is called *ergodic* if the tail σ-algebra (i.e., the intersection of all σ-algebras of the tail filtration) is trivial mod0, i.e., consists of two elements. A Markov measure μ agreed upon given system Λ of cotransition probabilities if the collection of cotransition probabilities of μ (for all vertices) coincides with Λ

Definition 3. *Denote as $\Sigma_\Gamma(\Lambda)$ the set of all Markov measures on $T(\Gamma)$ with cotransition probability Λ. The set of ergodic markov measures from $\Sigma_\Gamma(\Lambda)$ called $Erg_\Gamma(\Lambda)$. The most important list of cotransition probabilities corresponds to the* ***central measures****; the notion central measure depends on graph Γ itself. The set of all central measures on space of paths of the graph Γ we denote as $\Sigma(\Gamma)$, and the set of ergodic central measures as Erg_Γ.*

The list of measures $Erg_\Gamma(\Lambda)$ will be called exit boundary of equipped graph (Γ, Λ). The set of ergodic central measures we call ***exit boundary of the graph*** Γ***.***

Recall that in general a system of cotransition probabilities does not define uniquely markov process or its system of transition probabilities: $\mathrm{Prob}\{x_{t+1} = v | x_t = u\}$. so in general $\Sigma_\Gamma(\Lambda)$ is not one-point set.

We will see that $\Sigma_\Gamma(\Lambda)$ is a projective limit of finite-dimensional simplices.

The exit boundary is a topological boundary, and, as we will see, it is the Choquet boundary of a certain simplex (a projective limit of finite-dimensional simplices).

Our goal is to enumerate set $\Sigma_\Gamma(\Lambda)$ *of all Markov measures with a given system of cotransition probabilities* Λ *and in particular set of ergodic measures* $Erg_\Gamma(\Lambda)$*. To describe a Markov measure on the space of paths means to describe its transition probabilities.*

In the probability literature (e.g., in the theory of random walks), cotransition probabilities are usually defined not explicitly, but as the cotransition probabilities of a given Markov process. We prefer to define them directly, i.e., include them into the input data of the problem.

Recall that the *Poisson–Furstenberg boundary of the given Markov measure*, is a measure space which by definition is the tail-space as a space, and with measure which is induced by measure μ on tail-space. This boundary, regarded as a measure space; it is only a part of the exit boundary.

It is useful to point out the following terminology. The system of cotransition probabilities determines a cocycle on the tail equivalence relation, i.e., a function $(\gamma_1, \gamma_2) \to c(\gamma_1, \gamma_2)$ of a pair of equivalent paths, which is equal to the ratio of the products of cotransition probabilities along these paths (such a ratio is finite, since the paths are equivalent). In statistical physics and the theory of configurations, one also considers more general cocycles called Radon–Nikodym cocycles. In our case, the cocycle has a special form (the product of probabilities over edges) and is called a Markov cocycle. *A measure with given cotransition probabilities is a measure with a given Radon–Nikodym cocycle for a transformation group whose orbit partition coincides with the tail partition.*

REMARK

An analog of a system of cotransition probabilities, and the notion of an equipped graph, can also be defined in more greater generality: instead of a graded graph, it suffices to have a directed graph or multi-graph whose vertexes (except possibly one) has a nonempty set of ingoing edges; one can define an arbitrary system of probabilities on the set of ingoing edges of every vertex; the problem is still to describe the Dynkin boundary, i.e., the collection of all measures on the set of directed paths with given conditional entrance probabilities. This generalization could give the interesting new examples of exit-boundaries for general graphs.

1.3 A lexicographic ordering and adic transformation

Now we define the linear order on the set of eventually coincided paths (or on the elements of the tail partition $\tau(\Gamma)$. For this we choose a linear order on the finite set of edges of Γ which came to vertex v. Then we consider any path $t \in T(\Gamma)$ and countable or finite set of all paths which are eventually coincided with t, or element of tail partition which contains t. We can define a lexicographic linear ordering from "below" on this set: if two paths which are coincided starting from the vertex v on the level exactly n , then the first is greater than the second if its edge goes to v is greater than the edge of the second path (they are different). We consider the subset $T_0(\Gamma)$ of the set of all paths $T(\Gamma)$, which with this linear ordering has type $\mathbb{Z}$ (not finite or $\mathbb{N}$ or $-\mathbb{N}$. For the large and interesting class of graphs $T_0(\Gamma)$ is generic (dense open subset of $T(\Gamma)$)

Definition 4. *We define the action of $\mathbb{Z}$ of the set $T_0(\Gamma)$ as a transformation P which put the path t to the next one in the sense of our ordering. The transformation P call adic transformation and it is an element of Adic group which was defined above (e.g. change only finite number of edges of path.*[2]

This type of dynamics was defined by author in 1981 and independently but not in the same generality by J.Ito.

The main fact is the following theorem (A.Vershik, DAN 1981).

Theorem 1. *For each measure preserving ergodic transformation S of the standard (Lebesgue) measure space with continuous measure (X, μ) there exists a branching graph γ with a Borel probability measure ν on the space of paths $T(\Gamma)$ which are invariant under the adic transformation P and*

$$(X, \mu, S) \sim (T(\Gamma), \nu, P),$$

here $\sim$ means isomorphism $\mod 0$ *in the sense of the theory of measure space.*

This means that adic realization gives another (with comparison of so called symbolic dynamics) universal model for dynamics of the group $\mathbb{Z}$. This kind of dynamics is nothing more than presentation of the sequence

[2] sometime adic transformation called as "Vershik transformation", and a branching graph with lexicographic ordering as "Bratteli-Vershik diagram"

of the subsequent periodic approximations which in a sense exhaust the automorphism. The classical Rokhlin Lemma gave the universal periodic approximation of aperiodic automorphism, but it does not give the knowledge about metric type of the automorphism. Adic realization in a sense put one Rokhlin tower to the comprehensive sequence of towers. It is possible to say that we globalized Rokhlin towers.

Remark, that there are two theories of approximations of automorphisms. The first — weak approximation - was very popular in 60-70-th (F.Berezin's ideas, A.Katok, A.Stepin etc.) gave many concrete results in ergodic theory; it used Rokhlin towers (approximation) periodic approximations of which converges in the sense of weak topology of automorphisms. The second theory of approximation which I suggested in the same time (70-th), based on the uniform convergence of automorphisms which means the approximation preserves the partitions on the orbits. This is just what was defined above as adic realization of automorphisms. During last years I and some authors try to make this ideas more popular and made it in sufficiently developed form.

The idea of adic transformation as well as Rokhlin Lemma can be applied to the action of the arbitrary amenable group. More exactly, the adic structure means that on the almost orbits of the action of the group like $\mathbb{Z}$) we have sequence of increasing hierarchies of the finite sets, and this sequence is a copy our branching graph. We will write about this elsewhere. For locally finite group this graph looks like graph of classes of cosets over consequent subgroups.

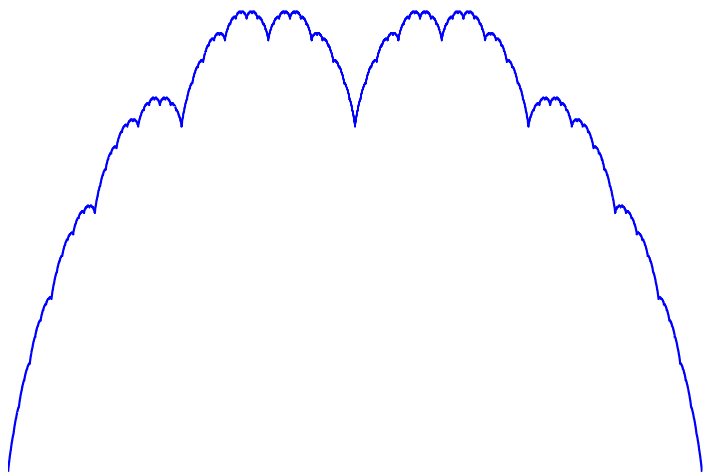


Figure 3: Takagi curve=Pascal bridge

1.4 New results about adic actions and The Tower of measures

The graph of unordered pairs UP could be equipped with adic structure: we can choose for all given v vertex the order on the pair of edges which came v. So we can define an adic transformation on UP with help of this order. <Picture UP>.

Using this structure we can strengthen the theorem above on adic transformations for group $\mathbb{Z}$ and for locally finite groups like $\sum_1^\infty \mathbb{Z}_2$.

Theorem 2. *(V.2015) For each measure preserving ergodic transformation S of the Lebesgue space (X,μ) which has two-fold generator (this equivalent to the property: entropy of S is less or equal to 1), and for each ergodic action of the group $\sum_1^\infty \mathbb{Z}_2$ which has two-fold generator there exist adic structure on the graph $Erg(UP)$ and a central measure $\nu \in Erg(UP)$ such that corresponding adic transformation P on the space of paths $T(UP)$ equipped with that measure ν, or triple $T(UP),\nu,P)$ is metrically isomorphic to the triple*

(X, μ, S). *The same is true for action of the group* $\sum_1^\infty \mathbb{Z}_2$ *on* $T(UP)$.

Thus we have universal branching graph UP *which allows using an adic structure to realize any ergodic action of* $\mathbb{Z}$ *with two-fold generator.* For the general automorphism we have to slightly change the graph.

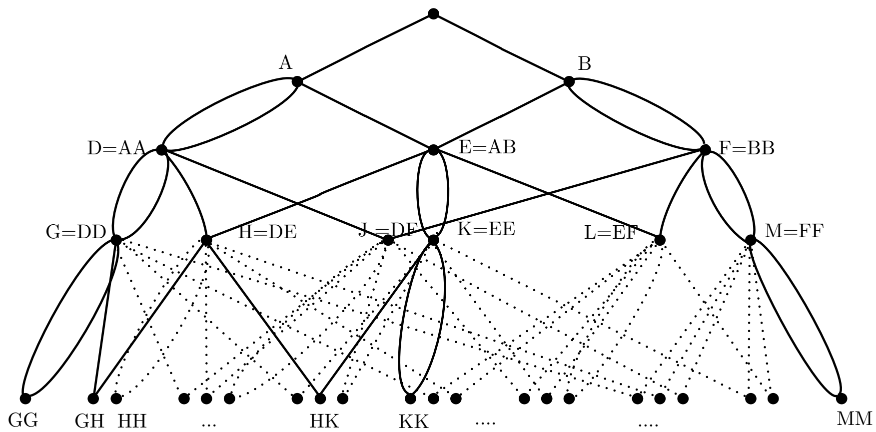


Figure 4: Unordered pairs graph

The proof use the important theorem about filtration which we will mention later, but formulate it here:

Theorem 3. *Any ergodic dyadic filtration in he Lebesgue space with continuous measure is isomorphic to the tail filtration of the graph* UP *with suitable ergodic central measure.*

This means that space of paths of the graph UP and its tail filtration is universal one. The simplex $\Sigma_\infty(UP)$ of all central measures is Poulsen simplex and also as projective limit is an Tower of measures in the sense of authors notion (Vershik 2013).

The adic realization of actions of this kind seems very different from the classical symbolic realization as a group of shifts (left or right) in the space of functions on the group (f.e.$\mathbb{Z}$). Many problems, related to approximation, to the rank transformation or so on must be considered from the point of view of adic dynamics.

2 EXAMPLES AND APPLICATION OF INVARIANT MEASURES: CHARACTERS, "RANDOM SUBGROUPS", TNF-ACTION AND FACTOR- REPRESENTATIONS.

2.1 List of central measures for well-known graphs

Theorem 4. *1)The exit boundary of Pascal graph is the unit interval:de Finetti theorem. Each ergodic central measure is Bernoulli measure $(p, 1-p), p \in [0.1]$.*

2)The exit boundary of d-dimensional Pascal(= d-dimension of orthant) —is $d-1$-dimensional simplex; (Bernoulli measures $(p_1, p_2 \dots p_d), \sum_i p_i = 1, h_i \geq 0$)

3)The exit boundary of Young graph =Thoma's theorem -simplex Thoma. $\{\alpha_n\}_{n \in \mathbb{Z}}$; where

$$\{0 \leq \dots \alpha_{-n} \leq \alpha_{-(n-1)} \leq \dots \leq \alpha_{-1}; \quad \alpha_0, \quad \alpha_1 \geq \dots \geq \alpha_{n-1} \geq \alpha_n \dots \geq 0; \sum_{i \in \mathbb{Z}} \alpha_i = 1\}$$

Formula

4)The exit boundary of dynamical Cayley graph of free group is the direct product of Poisson-Furstenberg boundary and unit interval.

We will discuss this example in the nest paragraph.

Conjecture 1. *The exit boundary of the branching graph Γ which is Hasse diagram of distributive lattice $L(Y)$ is the set of all monotonic positive functions $f \leq 1$ on the space of all minimal infinite ideals of the poset Y. Remember that each distributive lattice L is the a poset of all finite ideals a poset $Y, L = L(Y)$).*

For example, Young lattice is the lattice finite ideals of $[\mathbb{Z}_+]^2$ and infinite minimal ideals are union of rows and columns, so the monotonic function is $f : \mathbb{N} \cup \mathbb{N} \cup \{\infty\} \to [0, 1]$, its value $f(n, 0) = \alpha_n, f(0, n) = \alpha_{-n}, f(\infty) = \gamma$, where $\{\alpha_{\pm n}, \gamma\}$ are Thoma's parameters, and characters of infinite symmetric group in terms of frequency of rows and columns of the increasing sequence of Young diagrams. For many-dimensional Young Lattices this conjecture was suggested by Vershik-Kerov on 80-th, but only now it is close to be proved.

2.2 Exit boundary of random walks on the trees

Let T_{q+1} a tree with $q > 0$, of valency $q+1$. The case when $q = 2k$ means Cayley graph of the free groups with k generators. We consider the simple random walk starting from the origin and with equal probabilities on all edges. The branching graph $\Gamma(T, v_0)$ (or *dynamic* graph Cayley) is the $\mathbb{N}$-graded graph whose nth level is a copy of the set of vertices of T connected with the distinguished vertex v_0 by walks. We want to find the set $Erg(\Gamma(T_{q+1}))$ of all central measures of the graph $\Gamma(T, v_0)$? or in other words — exit boundary $Erg(\Gamma(T_{q+1}))$ of this branching graph $\Gamma(T, v_0) = \Gamma(T)$.

Theorem 5. *(Vershik-Malyutin 2015) For $q \geq 2$, the set $Erg(\Gamma(T_{q+1}))$ of all ergodic central measures on the space $Paths(\Gamma(T_{q+1}))$ of infinite paths in the dynamic graph $\Gamma(T_{q+1})$ over the $(q+1)$-homogeneous tree T_{q+1} (i. e., the exit boundary) coincides with the following family of Markov measures:*

$$\Lambda_q := \{\lambda_{\omega,r} \mid \omega \in \partial T_{q+1}, r \in [1/2, 1]\}.$$

Thus the exit boundary is homeomorphic (in the weak topology) to the product

$$\partial T_{q+1} \times [1/2, 1].$$

From other side this set can be identified with the set of all minimal positive eigenfunctions of Laplace operator on the tree with eigenvalue greater than some constant ($\sqrt{q}$). For eigenvalues which are less than this constant, we obtain non-ergodic measure, so we have some kind of phase-transition (a loss of ergodicity).

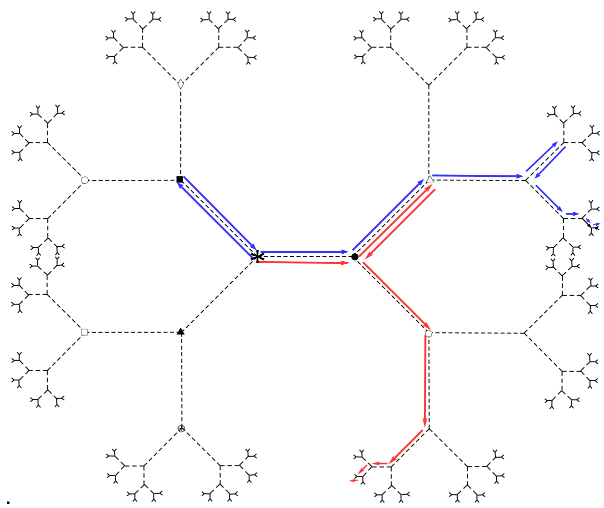


Figure 5: Random walk on the tree.

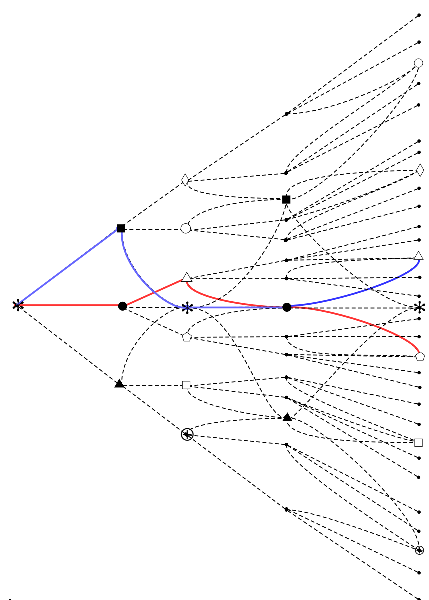


Figure 6: Boundary of T_3

2.3 Random subgroup, characters and representations of the infinite symmetric group. (V2010,V2013)

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Now we will discuss another application of the invariant measures.

Let G is group and $L(G)$ the lattice of all its subgroups. Group G acts on $L(G)$ as conjugacy:

$L(G) \ni H \mapsto gHg^{-1} \in L(G)$

What is the continuous measures (finite or sigma-finite) on $L(G)$ invariant under the conjugacy? Sometime such measures called as "random subgroup".

Important observation:

Proposition 1. *The function on the group*

$$\chi(g) = \mu\{x : gx = x\},$$

where μ is an invariant measure on the $G-space$ (X, μ) is a character on the group G. It means that $\chi(e) = 1, \chi(hgh^{-1}) = \chi(g)$, and χ is a positive definite function.

(Not all characters have this form, but for some group it is universal formula.

If the action of G is conjugacy action on the lattice $L(G)$ and a measure μ is invariant and concentrated on the *self-normalizers:* $gH = Hg \Rightarrow g \in H$ then the formula above looks as

$$\chi_\mu(g) = \mu\{H : g \in H\}$$

Measure of the set of subgroups which contain element g.

From this point of view two measures μ_1 and μ_2 are congruent if $\chi_{\mu_1}(g) = \chi_{\mu_2}(g)$.

Now our main theorem:

Theorem 6. *(Description of the random subgroups of $\mathfrak{S}_{\mathbb{N}}$) upto congruency)*

The following list of measures on the lattice of all subgroups $L(\mathfrak{S}_{\mathbb{N}})$ is the list of ergodic measures upto congruency on the Lattice which are invariant under conjugacy.

2.3.1 SIGNED PARTITIONS AND SIGNED YOUNG SUBGROUPS OF SYMMETRIC GROUPS

We consider the countable group $S_{\mathbb{N}}$, the infinite symmetric group of all finite permutations of the set of positive integers $\mathbb{N}$ (or an arbitrary countable set). In this section, we will give the list of all AD-measures on the lattice $L(S_{\mathbb{N}})$

of subgroups of this group and, in particular, the list of TNF measures. We will use some classical facts about permutation groups and the probabilistic approach.

The lattice $L(S_{\mathbb{N}})$ is very large and contains very different types of subgroups. Nevertheless, the support of an AD-measure consists of subgroups of a very special kind: so-called signed Young groups. The topology and the Borel structure on $L(S_{\mathbb{N}})$ are defined as usual; this is a compact (Cantor) space.

Definition 5 (Signed partitions)**.** *A signed partition η of the set $\mathbb{N}$ is a finite or countable partition $\mathbb{N} = \cup_{B\in\mathcal{B}} B$ of $\mathbb{N}$ together with a decomposition $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \cup \mathcal{B}^0$ of the set of its blocks, where $\mathcal{B}^0$ is the set of all single-point blocks; elements of $\mathcal{B}^+$ are called positive blocks, and elements of $\mathcal{B}^-$ are called negative blocks (thus each positive or negative block contains at least two points), and we denote by B_0 the union of all single-point blocks: $B_0 = \cup_{\{x\}\in\mathcal{B}^0}\{x\}$.*

Denote the set of all signed partitions of $\mathbb{N}$ by $\mathrm{SPart}(\mathbb{N})$.

Recall that in the theory of finite symmetric groups, the Young subgroup Y_η corresponding to an ordinary partition $\eta = \{B_1, B_2, \ldots, B_k\}$ is $\prod_{i=1}^k S_{B_i}$, where S_B is the symmetric group acting on B. We will define the more general notion of a *signed Young subgroup*, which makes sense both for finite and infinite symmetric groups. We will use the following notation: $S^+(B)$ is the symmetric group of all finite permutations of elements of a set $B \subset \mathbb{N}$, and $S^-(B)$ is the alternating group on B.[3]

Definition 6 (Signed Young subgroups)**.** *The signed Young subgroup Y_η corresponding to a signed partition η of $\mathbb{N}$ is*

$$Y_\eta = \prod_{B\in\mathcal{B}^+} S^+(B) \times \prod_{B\in\mathcal{B}^-} S^-(B).$$

Note that on the set $B_0 \subset \mathbb{N}$, the subgroup Y_η act identically, so that the partition into the orbits of Y_η coincides with η.

It is not difficult to describe the conjugacy class of Young subgroups with respect to the group of inner automorphisms: $Y_\eta \sim Y_{\eta'}$ if and only if η

[3]Traditionally, the alternating group is denoted by A_n; V. I. Arnold was very enthusiastic about the idea to denote it by S_n^- in order not to confuse it with the Lie algebra A_n; I agree with this idea.

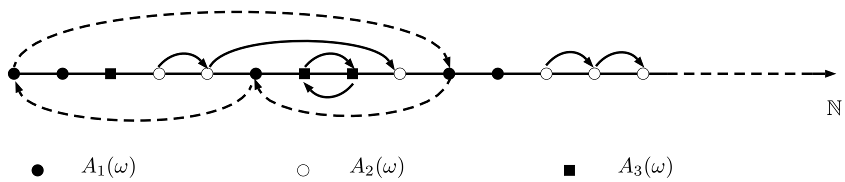


Figure 7: Random subgroup

and η' are equivalent up to the action of $S_{\mathbb{N}}$. But it is more important to consider the conjugacy with respect to the group of outer automorphisms. This is the group $S^{\mathbb{N}}$ of all permutations of $\mathbb{N}$. Denote by $r_0^{\pm}$ the number of infinite positive (respectively, negative) blocks, and by $r_s^{\pm}$ the number of finite positive (respectively, negative) blocks of length $s > 1$. Obviously, the list of numbers $\{r_0^{\pm}, r_1^{\pm}, \dots\}$ is a complete set of invariants of the group of outer automorphisms.

2.3.2 STATEMENT OF THE MAIN RESULT

Consider a sequence of positive numbers $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$ such that

$$\alpha_i \geq \alpha_{i+1} \geq 0 \text{ for } i > 0; \quad \alpha_{i+1} \geq \alpha_i \geq 0 \text{ for } i < 0; \quad \alpha_0 \geq 0; \quad \sum_{i\in\mathbb{Z}} \alpha_i = 1.$$

Consider a sequence of $\mathbb{Z}$-valued independent random variables ξ_n, $n \in \mathbb{N}$, with the distribution

$$\mathrm{Prob}\{\xi_n = v\} = \alpha_v \quad \text{for all} \quad n \in \mathbb{N}, v \in \mathbb{Z}.$$

Thus we have defined a Bernoulli measure μ_α on the space of integer sequences

$$\mathbb{Z}^{\mathbb{N}} = \{\xi = \{\xi_n\}_{n\in\mathbb{N}} : \xi_n \in \mathbb{Z}\}.$$

Definition 7 (A random signed Young subgroup and the measures ν_α)**.** *Fix a sequence $\alpha = \{\alpha_i, i \in \mathbb{Z}\}$, and corresponding Bernoulli measure μ_α; for each realization of the random sequence $\{\xi_n\}$, $n \in \mathbb{N}$, with the distribution μ_α, define a random signed partition $\eta(\xi)$ of $\mathbb{N}$ as follows:*

$$\eta(\xi) = \{B_i \subset \mathbb{N}, i \in \mathbb{Z}\}, \qquad B_i := \{n \in \mathbb{N} : \xi_n = i\},$$

here $\mathcal{B}^+ = \{B_i, i > 0\}; \mathcal{B}^- = \{B_i, i < 0\}$, and B_0 is understood as the union of one-point blocks. The correspondence $\xi \mapsto \eta(\xi)$ defines a probability *measure on the set* SPart($\mathbb{N}$) *of signed partitions, or random signed partition; the image of the Bernoulli measure μ_α. The correspondence $\xi \mapsto Y_{\eta(\xi)}$ defines a measure, which we denote by ν_α, on the set of signed Young subgroups, i.e., a measure on the lattice $L(S_\mathbb{N})$ of subgroups of $S_\mathbb{N}$.*

Note that all nonempty blocks of the random signed partition $\eta(\xi)$ that consist of more than one point are infinite with ν_α-probability one.

Now we describe the list of all AD and TNF-measures for the group $S_\mathbb{N}$.

Theorem 7. 1.Every measure ν_α is a Borel ergodic adjoint invariant (=invariant under conjugacy; AD-) measure on the lattice $L(S_\mathbb{N})$; every ergodic probability Borel conjugacy invariant measure on this lattice upto congruence coincides with the measure ν_α for some α.

2. Adjoint action of the group $S_\mathbb{N}$ on the lattice $L(S_\mathbb{N})$ with any AD-measure is TNF-(totally non-free) action.

2.4 The link to Representation-theory:von Neumann representations and the groups with tame trace problem

Why totally non-free actions are important for representation theory?

Theorem 8. *Suppose that there is an ergodic, measure preserving ergodic action of the countable group G on the standard measure space (X, μ). Consider von Neumann construction of the W^*-factor of type II_1 which is generated by this action of the group G on (X, μ). This factor as W^*-algebra is in general a weak closer of the set of all operators of semidirect product of the group of unitary operators corresponds to the elements of the group G and commutative algebra $L^\infty(X, \mu)$ of the measurable bounded functions on (X, μ). If the action is TNF-action, then this factor generated by operators of group G only; in another words: the multiplicators from W^*-algebra $L^\infty(X, \mu)$ belongs to weak closer of the algebra generated by operators from the group G.*

This is a new source of the factor-representations of the group. For the group $S_\mathbb{N}$ this involves the following result by Vershik-Kerov('81):

Theorem 9. *Each factor-representation of type II_1 of the group $S_\mathbb{N}$ can be realized in framework of groupoid construction based on the action of $S_\mathbb{N}$ on $[0, 1]^\mathbb{N}, \nu_\alpha$, where ν_α is a Bernoulli measure.*

Now we can ask about class of groups (which in general are not of type I) for which the set of the representations of type II_1 is parameterizable precompact space. In other words: the space of indecomposable finite traces (or characters if we consider representation of the group) is totally bounded? Of course this question is natural if the group has enough number of traces e.g. each pair of elements of algebra which are not conjugate can be distinguished by some indecomposable trace. Infinite symmetric group is one of such groups.

The question could be included in our general problem about central measures on the branching graphs.

3 INVARIANT MEASURES: GEOMETRY OF PROJECTIVE LIMIT OF SIMPLICES, KANTOROVICH METRIC AND INTRINSIC TOPOLOGY ON THE PROJECTIVE LIMIT, SMOOTH AND NON-SMOOTH CASES AND THE MAIN CONJECTURE.

3.1 Setting of the problem on invariant measures

We turn to describing the main problem.

Assume that we are given a Markov compactum $\mathcal{X}$ (or the path space $T(\Gamma)$ of a Bratteli diagram); the set $\mathrm{Meas}(\mathcal{X})$ of all Borel probability measures on $\mathcal{X}$ is an affine compact (in the weak topology) simplex, whose extreme points are delta measures. Since $\mathcal{X}$ is an inverse (projective) limit of finite spaces (namely, the spaces of finite paths), it obviously follows that $\mathrm{Meas}(\mathcal{X})$ is also an inverse limit of finite-dimensional simplices $\hat{\Sigma}_n$, where $\hat{\Sigma}_n$ is the set of formal convex combinations of finite paths (or just the set of probability measures on these paths) leading from the initial vertex to vertices of level n, $n = 1, 2, \dots$, and the projections $\hat{\pi}_n : \hat{\Sigma}_n \to \hat{\Sigma}_{n-1}$ correspond to "forgetting" the last vertex of a path. Every measure is determined by its finite-dimensional projections to cylinder sets (i. e., is a so-called cylinder measure). We will be interested only in invariant (central) measures, which form a subset of $\mathrm{Meas}(\mathcal{X})$.

Definition 8. *A Borel probability measure μ on a Markov compactum (= on the path space of a graph) is called central if for any vertex of an arbitrary level, the projection of this measure to the subfield of cylinder sets of finite paths ending at this vertex is the uniform measure on this (finite) set of paths.*

Other, equivalent, definitions of a central measure $\mu \in \mathrm{Meas}(\mathcal{X})$ are as follows.

1. The conditional measure of μ obtained by fixing the "tail" of infinite paths passing through a given vertex, i. e., the conditional measure of μ on the elements of the partition ξ_n, is the uniform measure on the initial segments of paths for any vertex.

2. The measure is invariant under any adic shift (for any choice of orderings on the edges).

3. The measure is invariant with respect to the tail equivalence relation.

The term "central measure" stems from the fact that in the application to representation theory of algebras and groups, measures with these properties determine traces on algebras (respectively, characters on groups). In the theory of stationary (homogeneous) topological Markov chains, central measures are called measures of maximal entropy.

The set of central measures on a Markov compactum $\mathcal{X}$ (on the path space $T(\Gamma)$ of a graph Γ) will be denoted by $Inv(\mathcal{X})$ or $Inv(\Gamma) = \Sigma(\Gamma)$. Clearly, the central measures form a convex weakly closed subset of the simplex of all measures: $Inv(\mathcal{X}) \subset Meas(\mathcal{X})$. The set $Inv(\mathcal{X})$ of central measures is also a simplex, which can be naturally presented as a projective limit of the sequence of finite-dimensional simplices of convex combinations of uniform measures on the n-cofinality classes. In more detail:

Proposition 2. *The simplex of central measures can be written in the form*

$$Inv(\mathcal{X}) = \lim_{\leftarrow}(\Sigma_n; p_{n,m}),$$

or

$$\Sigma_1 \leftarrow \Sigma_2 \leftarrow \cdots \leftarrow \Sigma_n \leftarrow \Sigma_{n+1} \leftarrow \cdots \leftarrow \Sigma_\infty \equiv Inv(\mathcal{X}),$$

where Σ_n is the simplex of formal convex combinations of vertices of the nth level Γ_n (i. e., points of X_n), and the projection $p_{n,n-1} : \Sigma_n \to \Sigma_{n-1}$ sends a vertex $\gamma_n \in \Gamma_n$ to the convex combination $\sum \lambda^{\gamma_n}_{\gamma_{n-1}} \delta_{\gamma_{n-1}} \in \Sigma_{n-1}$ where the numbers $\lambda^{\gamma_n}_{\gamma_{n-1}}$ are uniquely determined by the condition that $\lambda^{\gamma_n}_{\gamma_{n-1}}$ is proportional to the number of paths leading from $\emptyset$ to γ_{n-1} (which is denoted, as already mentioned, by $\dim \gamma_{n-1}$).[4] *The general form of the projection is $p_{n,m} = \prod_{i=m}^{n+1} p_{i,i-1}$, $m > n$.*

Proof. The set of all Borel probability measures on the path space is a simplex which is a projective limit of the simplices generated by the spaces of finite paths of length n in the graph, which follows from the fact that the path space itself is a projective limit with the obvious projections of "forgetting"

[4] In the general (noncentral) case, the coefficients λ are the cotransition probabilities (see above).

the last edge of a path. The space of invariant measures is thus a weakly closed subset of this simplex, and we will show that it is also a projective limit of simplices (the fact that it is a simplex is well known. The projection μ_n of any invariant measure μ to a finite cylinder of level n is a measure invariant under changes of initial segments of paths and hence lies in the simplex defined above; since the projections preserve this invariance, $\{\mu_n\}$ is a point of the projective limit. It remains to observe that a measure is uniquely determined by its projections, which establishes a bijection between the points of the projective limit and the set $\Sigma(\Gamma)$ of invariant measures. $\square$

Recall that points of the simplex Σ_n are probability measures on the points of X_n (i.e., on the vertices of the nth level Γ_n), and the extreme points of Σ_n are exactly these vertices. Remark 1 means that distinct vertices of the graph correspond to distinct vertices of the simplex.

Extreme points of the simplex $\Sigma(\Gamma)$ of invariant measures on the whole path space $T(\Gamma)$ are indecomposable invariant measures, i. e., measures that cannot be written as nontrivial convex combinations of other invariant measures. Then it follows from the theorem on the decomposition of measures invariant with respect to a hyperfinite equivalence relation into ergodic components that an indecomposable measure is ergodic (= there are no invariant subsets of intermediate measure). It is these measures that are of most interest to us, since the other measures are their convex combinations, possibly continual. The set of ergodic central measures of a Markov compactum $\mathcal{X}$ (of a graph Γ) will be denoted by $\mathrm{Erg}(\mathcal{X})$ or $\mathrm{Erg}(\Gamma)$.

Problem 1. *Describe all central ergodic measures for a given Markov compactum (respectively, all indecomposable central measures for a given graph). A meaningful question is for what Markov compacta or graphs the set of ergodic central measures has an analytic description in terms of combinatorial characteristics of this compactum or graph, and what are these characteristics; and in which cases such a description does not exist. The role of such characteristics may be played by some properties of the sequence of matrices* $\{M_n\}$ *determining the compactum (graph), frequencies, spectra, etc.*

This problem includes those of describing unitary factor representations of finite type of discrete locally finite groups, finite traces of some C^*-algebras, Dynkin's entrance and exit boundaries; it is very closely related to the problems of finding Martin boundaries, Poisson–Furstenberg boundaries, etc. The answer to the question stated in Problem 1 may be either "tame" (there exists a Borel parametrization of the ergodic measures or the factor representations of finite type) or "wild" (such a parametrization does not exist). As is well known since the 1950s, in the representation theory such is the state of affairs in the theory of irreducible representations of groups and algebras. However, this also happens, though more rarely, in the theory of factor representations. But in many classical situations the answer in "tame," which is a priori far from obvious.

For example, the characters of the infinite symmetric group, i.e., the invariant measures on the path state of the Young graph (see Fig. 1), have a nice parametrization, and this is a deep result; however, for the graph of unordered pairs (see Fig. 2) there is no nice parametrization. We emphasize that the presentation of $\Sigma(\Gamma)$ as a projective limit of simplices relies essentially on the approximation, i. e., on the structure of the Markov compactum (graph). Obviously, the answer to the stated question also depends on the approximation. The fact is that we can change the approximation without changing the stock of invariant measures, which is determined only by the tail equivalence relation. The dependence of our answers on the approximation will be discussed later (see the remark on the lacunary isomorphism theorem in the last section). But since in actual problems the approximation is explicit already in the setting of the problem, the answer should also be stated in its terms. See examples below.

3.2 Geometric formulations

We will recall some well known geometric formulations, since the language of convex geometry is convenient and illustrative in this context.

1. The set of all Borel probability measures on a separable compact set invariant under the action of a countable group (or equivalence relation) is a simplex (= Choquet simplex), i. e., a separable affine compact set in the weak topology whose any point has a unique decomposition into an integral with respect to a measure on the set of extreme points.[5] The set of ergodic measures is the Choquet boundary, i. e., the set of extreme points, of this simplex; it is always a G_δ set.

[5] Choquet's theorem on the decomposition of points of a convex compact set into an integral with respect to a probability measure on the set of extreme points is a strengthening, not very difficult, of the previous fundamental Krein–Milman theorem saying that a convex affine compact set is the weak closure of the set of convex combinations of extreme points.

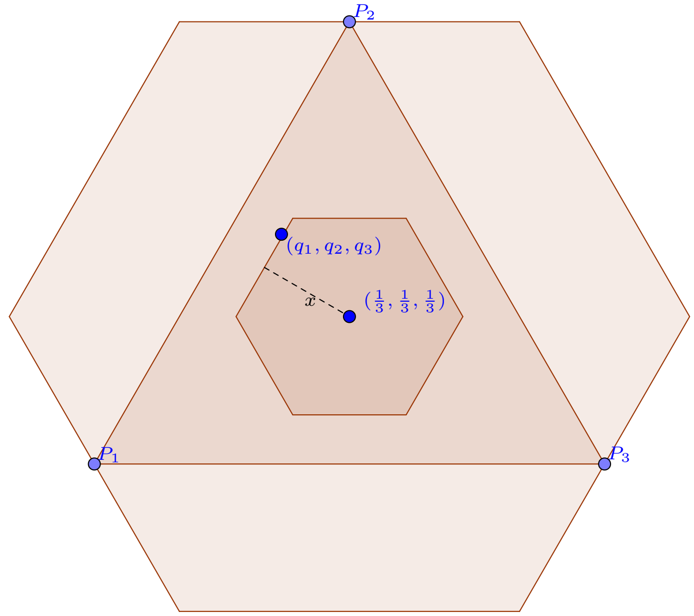


Figure 8: Unit ball in the intrinsic metric.

2. Terminology (somewhat less than perfect): a Choquet simplex is called a Poulsen simplex [?] if its Choquet boundary is weakly dense in it, and it is called a Bauer simplex if the boundary is closed. Cases intermediate between these two ones are possible.

3. A projective limit of simplices (see below) is a Poulsen simplex if and only if for any n the union of the projections of the vertex sets of the simplices with greater numbers to the nth simplex is dense. The universality of a Poulsen simplex was observed and proved much later by several authors:

Proposition 3. *All separable Poulsen simplices are topologically isomorphic as affine compacta; this unique, up to isomorphism, simplex is universal in the sense of model theory.*[6]

One can easily check that every projective limit of simplices arises when studying *quasi-invariant measures* on the path space of a graph, or Markov measures with given cotransition probabilities (see above). But in what follows we consider only central measures, i. e., take a quite special system of projections in the definition of a projective limit. However, there is no significant difference in the method of investigating the general case compared with the case of central measures. We will return to this question elsewhere.

We add another two simple facts, which follow from definitions.

4. Every ergodic central measure on a Markov compactum (on the path space of a graph) is a Markov measure with respect to the structure of the Markov compactum (the ergodicity condition is indispensable here).

5. The tail filtration is semi-homogeneous with respect to every ergodic central measure, which means exactly that almost all conditional measures for every partition ξ_n, $n = 1, 2, \dots$, are uniform.

The metric theory of semi-homogeneous filtrations will be treated in a separate paper.

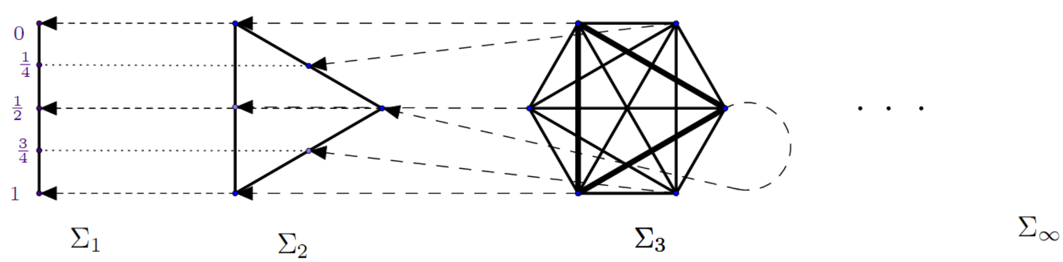


Figure 9: Projective limit of simplecies

[6]That is, every separable simplex can be mapped infectively into the Poulsen simplex, and an isomorphism of any two isomorphic faces of the Poulsen simplex can be extended to an automorphism of the whole simplex.

3.3 Extremality of points of a projective limit, and ergodicity of Markov measures

We give a criterion for the ergodicity of a measure in terms of general projective limits of simplices, in other words, a criterion for the extremality of a point of a projective limit of simplices.

Assume that we are given an arbitrary projective limit of simplices $\Sigma_1 \leftarrow \Sigma_2 \leftarrow \cdots \leftarrow \Sigma_n \leftarrow \Sigma_{n+1} \leftarrow \cdots \leftarrow \Sigma_\infty$ with affine projections $p_{n,n-1} : \Sigma_n \to \Sigma_{n-1}$, $n = 1, 2, \dots$ (the general projection $p_{m,n} : \Sigma_m \to \Sigma_n$ is given above).

Consider an element $x_\infty \in \Sigma_\infty$ of the projective limit; it determines, and is determined by, the sequence of its projections $\{x_n\}_{n=1,2,\dots}$, $x_n \in \Sigma_n$, to the finite-dimensional simplices. Fix positive integers $n < m$ and take the (unique) decomposition of the element x_m, regarded as a point of the simplex Σ_m, into a convex combination of its extreme vertices e_i^m:

$$x_m = \sum_i c_m^i \cdot e_i^m, \quad \sum_i c_m^i = 1, \quad c_m^i \geq 0;$$

denote by $\mu_m = \{c_m^i\}_i$ the measure on the vertices of Σ_m corresponding to this decomposition. Project this measure μ_m to the simplex Σ_n, $n < m$, and denote the obtained projection by μ_m^n; this is a *measure on* Σ_n*, and thus a random point of* Σ_n; note that this measure is not in general concentrated on the vertices of the simplex Σ_n.

Proposition 4 (Extremality of a point of a projective limit of simplices)**.** *A point* $x_\infty = \{x_n\}_n$ *of the limit simplex* Σ_∞ *is extreme if and only if the sequence of measures* μ_n^m *weakly converges, as* $m \to \infty$*, to the delta measure* δ_{x_n} *for all values of* n*:*

$$\text{for every } \epsilon > 0, \text{ for every } n \text{ there exists } K = K_{\epsilon,n} \text{ such that}$$
$$\mu_n^m(V_\epsilon(\mu_n)) > 1 - \epsilon \quad \text{for every } m > K,$$

where $V_\epsilon(\cdot)$ *is the* ϵ*-neighborhood of a point in the usual (for instance, Euclidean) topology.*

It suffices to use the continuity of the decomposition of an arbitrary point x_∞ into extreme points in the projective limit topology, and project this decomposition to the finite-dimensional simplices; then for extreme points, and only for them, the sequence of projections must converge to a delta measure.

One can easily rephrase this criterion for our case $\Sigma_\infty = \Sigma(\Gamma) = \Sigma(\mathcal{X})$. Now it is convenient to regard the coordinates (projections) of a central measure μ_∞ not as points of finite-dimensional simplices, but as measures $\{\mu_n\}_n$ on their vertices (which is, of course, the same thing). Then the measures μ_m^n should be regarded as measures on probability vectors indexed by the vertices of the simplex, and the measure μ on the Markov compactum $\mathcal{X}$ (or on $T(\Gamma)$), as a point of the limit simplex Σ. The criterion then says that μ is an ergodic measure (i. e., an extreme point of Σ) if and only if the sequence of measures μ_m^n (on the set of probability measures on the vertices of the simplex Σ_n) weakly converges as $m \to \infty$ to the measure μ_n (regarded as a measure on the vertices of Σ_n) for all n.

In probabilistic terms, our assertion is a topological version of the theorem on convergence of martingales in measure and has a very simple form: for every n, the conditional distribution of the coordinate x_n given that the coordinate x_m, $m > n$, is fixed converges in probability to the unconditional distribution of x_n as $m \to \infty$.

According to this proposition, in order to find the finite-dimensional projections of ergodic measures, one should enumerate all delta measures that are weak limits of measures μ_n^m as $m \to \infty$. But, of course, this method is inefficient and tautological. The more efficient *ergodic method* requires, in order to be justified, a strengthening of this proposition, namely, replacing convergence in measure with convergence almost everywhere, i. e., the individual ergodic theorem, or pointwise convergence of martingales (Vershik 74,02). The point of this paper is the enhancement of the ergodic method via a new type of convergence of martingales.

3.4 All boundaries in geometric terms

The following definition is a paraphrase of the definition of Martin boundary in terms of projective limits.

Definition 9. *A point $\{x_n\} \in \Sigma_n$ of a projective limit of simplices belongs to the Martin boundary if there is a sequence of vertices $\alpha_n \in ex(\Sigma_n)$, $n = 1, 2, \dots,$ such that for every m and an arbitrary neighborhood $V_\epsilon(x_m) \subset \Sigma_m$ there exists N such that*

$$\pi_{n,m}(\alpha_n) \in V_\epsilon(x_m)$$

for all $n > N$. Less formally, a point of the limiting simplex belongs to the Martin boundary if there exists a sequence of vertices that weakly converges *to this point ("from the outside").*

This sequence itself does not in general correspond to a point of the projective limit Σ_∞, but it is a point of the space $\mathcal{M}$ (the direct product of the simplices Σ_n), and it makes sense to say that its components approach the components of a point of the projective limit, which belongs to the Martin boundary by definition. The condition of belonging to the Martin boundary is a weakening of the almost extremality criterion, hence the following assertion is obvious.

Proposition 5. *The Martin boundary contains the closure of the Choquet boundary.*

However, there are examples where the Martin boundary contains the closure of the Choquet boundary as a proper subset. Such an example, related to random walks, will be described in a joint paper by the author and A. V. Malyutin, which is now in preparation. A question arises: can one describe the Martin boundary in terms of the limiting simplex itself? In other words, can one say what other points (except those lying in the closure of the Choquet boundary) belong to the Martin boundary? The author tends to believe that this cannot be done, since the answer to the latter question depends not only on the geometry of the limiting simplex itself, but also on how it is represented as a projective limit.

3.5 The probabilistic interpretation of properties of projective limits

Parallelism between considering pairs {a graded graph, a system of cotransition probabilities} on the one hand and considering projective limits of simplices on the other hand means that the latter subject has a probabilistic interpretation. It is useful to describe it without appealing to the language of pairs. Recall that in the context of projective limits a path is a sequence $\{t_n\}_n$ of vertices $t_n \in \mathrm{ex}\Sigma_n$ that agrees with the projections $\pi_{n,n-1}$ for all $n \in \mathbb{N}$ in the following sense: $\pi_{n,n-1}t_n$ has a nonzero barycenter coordinate with respect to t_{n-1}. First of all, every point $x_\infty \in \Sigma_\infty$ of the limiting simplex is a sequence $\{x_n\}$ of points of the simplices Σ_n that agrees with the projections: $\pi_{n,n-1}x_n = x_{n-1}$, $n \in \mathbb{N}$. As an element of the simplex, x_n determines a measure on its vertices, and, since all these measures agree with the projections, x_∞ determines a measure μ_x on the path space with fixed cotransition probabilities. Conversely, every such measure comes from a point x_∞. Thus the limiting simplex is the simplex of all measures on the path space with given cotransition probabilities. The extremality of a point $\mu \in \mathrm{ex}(\Sigma_\infty)$ means the ergodicity of the measure μ, i.e., the triviality with respect to μ of the tail σ-algebra on the path space. The above extremality criterion has a simple geometric interpretation, on which we do not dwell.

So, we have considered the following boundaries of a projective limit of simplices (or an equipped graph):

the Poisson–Furstenberg boundary $\subset$ *the Dynkin boundary* $=$ *the Choquet boundary* $\subset$ *the closure of the Choquet boundary* $\subset$ *the Martin boundary* $\subset$ *the limiting simplex*.

The first boundary is understood as a measure space; all inclusions are in general strict; the answer to the question of whether the Martin boundary is a geometric object (i.e., whether it can be defined in purely geometric terms, rather than via approximation) is most probably negative.

We summarize this section with the following conclusion: *the theory of equipped graded graphs* (*i.e., pairs* $\{$*a graded graph* $+$ *a system of cotransition probabilities*$\}$) *is identical to the theory of Choquet simplices regarded as projective limits of finite-dimensional simplices.*

4 THEORY OF FILTRATIONS, CRITERIA OF STANDARTNESS, COMBINATORICS OF DYADIC FILTRATION, INTRINSIC METRIC AND LIMIT SHAPE THEOREMS.

4.1 Filtrations in measure theory and in topology

We get together several definitions and preliminary non-trivial facts about filtrations.

Theorem-Definition 1. *Let (X, μ, A) is a standard measure space (Lebesgue space) with continuous probability measure μ and sigma-field $\mathfrak{A}$ of classes* $\mod 0$ *of all measurable sets.*

1. A filtration is the decreasing sequence of the sigma-fields of measurable sets:

$$\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \dots .$$

The sigma field $\mathfrak{A}_0 = \mathfrak{A}$.

Each sigma field canonically corresponds to measurable partition in sense of Rokhlin: the elements of sigma-field are the sets which are measurable with respect to that partition. So filtration uniquely $\mod 0$ *generates the decreasing sequence of the measurable partitions $\{\xi_n\}_n$.*

2.The filtration called "discrete" if the conditional filtration $\{\mathfrak{A}_i/\mathfrak{A}_n; i = 0, 1, \dots n-1\}$ over sigma-field $\mathfrak{A}_n$ are filtration of the finite space with measure. In other words, the discrete filtration is the filtration for which all the partitions ξ_n, have finite number of points in almost all its elements.

3.Filtration called Markov filtration if it is the @pasr@ filtration for a Markov chain.

4.Filtration called homogeneous if for each n almost all elements of the partition ξ_n are finite measure- space with the uniform conditional measure, and number of point are the same for given n. Filtration called semi-homogeneous if conditional measure of almost all elements of partition ξ_n is uniform.

Homogeneous filtration whose number of points in almost all elements of partition xi_n equal to r^n called r-adic filtration (diadic for $r = 2$). Filtration called ergodic if intersection $\bigcap_n \mathfrak{A}_n = \mathfrak{N}$ is trivial sigma-fields.

5. Each discrete filtration correctly define ***an ergodic equivalence relation****. Two points x, y belongs to the same class if there exists such n that they belongs to the same element of partition ξ_n.*

6.Two filtrations $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ and $\{\mathfrak{A}'_n\}_{n=0}^{\infty}$ called finitely isomorphic if for each N the finite fragments for $n = 0, 1 \dots N$ of its are metrically isomorphic.

The fundamental problem is to classify the filtrations in the given class of the finitely isomorphic filtrations.

We also consider the filtration in the standard separable Borel space.

Examples:

1)Tail filtration in the space of paths of branching graph above.

Theorem 10. *(universality of tail filtrations) 1.Each discrete filtration in the Lebesgue space is isomorphic to the tail filtration of the equipped branching graph with some system of cotransition probabilities.*

2.Each semi-homogeneous filtration isomorphic to the tail filtration of a branching graph Γ equipped with a central measure.

2)Filtration of the "past" of the random process with discrete time $\{\xi_n\}, -n \in \mathbb{N}$.

The main problem is to give a rough metric invariants of the filtration upto measure-theoretic (or Borel) equivalence. For example: to distinguish filtrations from a class of finitely isomorphic ergodic homogeneous filtration. The theory started from the example of the author who gave in 1970 the first example of non-isomorphic ergodic dyadic filtration.

Definition 10. *1.Filtration in the Lebesgue space called* ***filtration of the product type*** *if it is isomorphic metrically to the filtration of the past of Bernoulli sequence of random variables: $\mathfrak{A}_n$ is sigma-field generated by variables $\omega_r, n > r$ where $\{\omega_n\}_{n \in \mathbb{N}}$ are independent random variables each of which took finite values.*

2.Homogeneous standard filtration is a filtration which is isomorphic to the homogeneous filtration of product type. Dyadic filtration of product type has form: $\mathfrak{A}_n, n = 0, 1 \dots$ where $\mathfrak{A}_n$ is sigma-field of the measurable sets on $[0, 1]$ with Lebesgue measure, depending on digits $\epsilon_r(x)$ with number $r > n$ in the dyadic decompositions of x:

$$x = \sum_{r=1}^{\infty} \frac{\epsilon_r(x)}{2^r}.$$

So the standard dyadic filtration is filtration which is isomorphic to this example.

It is possible to give analogous definition of standard filtration of semi-homogenous type or general, but such definition will more cumbersome. So we give the general definition of standardness using generalization of criteria of standardness for homogeneous filtration. This leads to many new questions bout filtrations, which we will discuss. The crucial question is how to characterize the branching graphs for which tail filtration is standard for all central ergodic measures (or more general— for all measures with any cotransition probabilities.) We will give answer on this question.

4.2 The definition of the intrinsic topology on an inductive limit

We proceed to our main goal, which is to construct an approximation of a projective limit of simplices, i.e., a simplex of measures with a given cocycle, and to define the "intrinsic metric (topology)" on this limit. This metric was defined in the recent papers of author about path-spaces of graphs, only for central measures and under some additional conditions on the graph (the absence of vertices with the same predecessors). Here we give this definition in its natural generality, for an arbitrary graded graph and an arbitrary system of cotransition probabilities (see Sec. 2), and, most importantly, we consider the whole limiting simplex and not only its Choquet boundary. This allows us to study the boundary for graphs with nonstandard (noncompact) intrinsic metrics. We formulate definitions and results both in terms of equipped graded graphs and in terms of projective limits of simplices spanned by the vertices of different levels.

We start with the definition of an important topological operation which will be repeatedly used, that of "transferring a metric."

Let (X, ρ_X) be a metric space and $\phi : X \to Y$ be a (Borel-)measurable map from X to a Borel space Y; assume that the preimages of points $\phi^{-1}(y)$, $y \in \phi(X) \subset Y$, are endowed with Borel probability measures ν_y that depend on y in a Borel-measurable way; ϕ will be called an *equipped map*.

Definition 11. *The result of transferring the metric ρ_X on the space X to the Borel space Y along the equipped map*

$$\phi : X \to Y$$

is the metric ρ_Y on Y defined by the formula

$$\rho_Y(y_1, y_2) = k_{\rho_X}(\nu_{y_1}, \nu_{y_2}),$$

where k_ρ is the classical Kantorovich metric on Borel probability measures on (X, ρ_X).

1. Consider an equipped graph (Γ, Λ) and the corresponding projective limit of simplices $\Sigma_\infty(\Gamma)$. Define an arbitrary metric $\rho = \rho_1$ on the path space $T(\Gamma)$ that agrees with the Cantor topology on $T(\Gamma)$; denote by k_{ρ_1} the Kantorovich metric on the space $\Delta(\Gamma)$ of all Borel probability measures on $T(\Gamma)$ constructed from the metric ρ_1 (see the definition below).

2. Given an arbitrary path $v \equiv \{v_n\}$, consider the finite set of paths $v(u) = \{u, v_2, \dots\}$ whose coordinates coincide with the corresponding coordinates of v starting from the second one, and assign each of these paths the measure $\lambda^u_{v_2}$. We have defined an equipped map $\phi_1 : T(\Gamma) \to \Delta(\Gamma) = \Delta_1$, which sends a path to the measure $\sum\limits_{u:u\prec v_2} \lambda^u_{v_2}\delta_v(u)$. It is more convenient to regard it as a map from the simplex $\Delta(\Gamma)$ to itself, by identifying a path with the δ-measure at it.

Transferring the metric ρ_1 along the equipped map ϕ_1, we obtain a metric ρ_2 on a subset $\Delta_2 = \phi(\Delta_1)$ of the simplex $\Delta(\equiv \Delta_1(\Gamma))$.

3. In a similar way we define the map ϕ_2 that sends every measure from Δ_2 concentrated on paths of the form $\{u_1, v_2, \dots\}$, $u_1 \prec v_2$, to the measure on the finite collection of paths of the form $\{u_1, u_2, v_3, \dots\}$ whose coordinates coincide with v_i starting from the third one and the second coordinate u_2 runs over all vertices $u_2 \prec v_3$ with probabilities $\lambda^{u_2}_{v_3}$. Again transferring the metric ρ_2 from the space Δ_2 along the equipped map ϕ_2, we obtain a metric ρ_3 on the image $\Delta_3 \equiv \phi_2(\Delta_2) = \phi_2\phi_1(\Delta)$.

Note that the images of the maps ϕ_n, i.e., the sets Δ_n, are simplices, but their vertices are no longer δ-measures on the path space, but measures with finite supports of the form $\sum\limits_{u_1,u_2,\dots,u_k} \lambda^{u_1}_{u_2} \cdots \lambda^{u_k}_{v_{k+1}} \cdot \delta_{u_1,\dots,u_k,v_{k+1},\dots}$. The definition of the simplices Δ_n does not depend on the metrics ρ_n.

4. Continuing this process indefinitely, we obtain an infinite sequence of metrics on the decreasing sequence of simplices

$$\Delta_n = \phi_{n-1}(\Delta_{n-1}) = \phi_n\phi_{n-1}\dots\phi_1(\Delta_1),$$
$$\Delta = \Delta_1 \supset \Delta_2 \supset \Delta_3 \dots, \quad \bigcap_n \Delta_n = \Delta_\infty.$$

Thus we have a sequence of equipped maps of the decreasing sequence of simplices

$$\Delta_1 \to \Delta_2 \to \cdots \to \Delta_n \to \cdots \to \Delta_\infty.$$

First we mention an assertion that does not involve the metric.

Proposition 6. *The intersection Δ_∞ of all simplices Δ_n consists exactly of those measures on the path space $T(\Gamma)$ (i.e., those points of the simplex $\Delta(\Gamma)$ of all measures) that have given cotransition probabilities (given cocycle), and, therefore, this intersection coincides with the projective limit of the simplices:*

$$\Delta_\infty = \Sigma_\infty(\Gamma).$$

Of more importance is the following fact.

Theorem 11. *There exists a limit $\lim_{n\to\infty} \rho_n = \rho_\infty$ of metrics on the space $\Delta_\infty(= \Sigma_\infty(\Gamma))$. The limiting simplex $\Sigma_\infty(\Gamma)$ equipped with this metric is not in general compact, so that ρ_∞ does not generate the projective limit topology.*

Proof. We will give an explicit description of the limiting "intrinsic" metric, using more detailed information on the metrics ρ_n. To this end, we should remind the definition of the Kantorovich metric on measures and the notion of coupling, which is actually used in the definition of transferring metrics.

Definition 12. *A coupling of two Borel probability measures μ_1, μ_2 defined on two (in general, different) Borel spaces X_1, X_2 is an arbitrary Borel measure ψ on the product $X_1 \times X_2$ whose projections to the factors X_1, X_2 coincide with μ_1, μ_2. The set of all couplings for μ_1, μ_2 will be denoted by $\Psi(\mu_1, \mu_2)$. (Other names for this notion are "bi-stochastic measure," "polymorphism," "Young measure," "correspondence," etc.)*

The Kantorovich metric on the simplex of measures on a metric space (X, μ) is defined as follows:

$$k_\rho(\mu_1, \mu_2) = \inf \left\{ \int\limits_{X\times X} \rho(x_1, x_2)\, d\psi(x_1, x_2) : \psi \in \Psi(\mu_1, \mu_2) \right\}.$$

Above we defined metrics (i.e., distances between measures) by recursion on n, each time applying coupling. But one can do this consistently, combining all conditions on successive couplings together. In the infinite case, this gives at once a formula for the limiting metric.

Assume that the metric space X in the previous definition is endowed with a sequence of equipped maps

$$X = X_1 \to X_2(\subset X_1) \to X_3(\subset X_2) \to \dots \to X_n(\subset X_{n-1})$$

(here n is finite or infinite; in the second case, the last space should be replaced by the intersection $\bigcap_n X_n \equiv X_\infty$) and we want to define the distance between measures on the last space (X_n or X_∞). This is exactly our situation, where the spaces $X_n = \Delta_n$ are the simplices determined by the maps that replace an initial segment of a path by a measure distributed on initial segments. The formula remains the same as in the classical case, the difference being in what one means by a coupling:

$$K_n(\mu_1, \mu_2) = \inf \left\{ \int\limits_{X \times X} \rho(x_1, x_2)\, d\psi(x_1, x_2) : \psi_n \in \Psi_n (\text{or } \in \Psi_\infty) \right\}.$$

Here the coupling ψ_n runs over the set Ψ_n consisting of measures on the space $X \times X$ that not only have given projections but are such that the projection of ψ_n to each component agrees with the structure of the sequence of projections of the space $X = X_1 \to X_2 \to \dots$ itself. In other words, for every n the coupling ψ_n is a mixture of the couplings ψ_{n-1}: this strict constraint is the difference with the usual procedure. Thus the above formula correctly defines all metrics, including the limiting metric on the simplex $\Delta_\infty = \Sigma_\infty(\Gamma)$.

Although the limiting intrinsic metric depends on the initial metric, nevertheless the formula shows also that the topology determined by the limiting metric is the same for all initial metrics that agree with the topology of the simplex. □

4.3 Main results and conjectures

Now we can formulate the alternative on the problem about invariant measures. For simplicity we formulate the problem only about central measures but the case of equipped graph and measures with given cotransition probabilities can be considered in the same way.

Theorem 12. *Consider the branching graph Γ and let Σ_∞ is simplex of all central measures. the following two properties of simplex are equivalent.*

1.Intrinsic metric on the space $T(\Gamma)$ of paths of graph Γ is precompact.

2.The simplex Σ_∞ has the following property: the Choquet boundary (=set of extremal points) is open in its closure;

We say that the problem of description of central measures for given graph Γ is smooth (or tame) if the properties above took place for Γ. In this case graph will be called a standard.

The problem is how to calculate intrinsic topology for graphs and to distinguish standard and nonstandard graphs.

4.4 Criteria of standardness. Super-strong convergence of decreasing martingales

Definition 13. *An equipped graph* (Γ, Λ)*, as well as a projective limit of simplices* $\lim_n(\Sigma, \pi_{n,m})$*, are called standard if the limiting simplex of measures* Σ_∞ *endowed with the intrinsic metric is compact. In this (and only this) case the projective limit topology coincides with the intrinsic topology.*

A (non-equipped) graph will be called standard if the limiting simplex of central measures is compact in the intrinsic metric. The standardness or non-standardness of an equipped graph depends in general on the system Λ*.*

This definition generalizes the definition of a standard graph given earlier by author. More exactly, if we restrict Definition 13 to the spaces of paths of length n regarded as sequences of vertices, then we obtain exactly the definition from the authors's paper in 70-th. One may say that the new definition is a linearization (extending to linear combinations) of the previous one. This can also be stated as follows: we consider (instead of vertices of a given level) measures on the set of paths leading to these vertices with given cotransition probabilities. Therefore, all metrics and their limit are defined on sets of measures (rather than sets of vertices), which provides a natural generality for the definition removing the restrictions on the graph previously imposed. From a practical point of view, of course, it is more convenient to check the standardness by considering vertices (diagrams) if this is possible.

The examples of a graph with a noncompact intrinsic metric is given above: we only mention that this is, for instance, the graph of unordered pairs related to the notion of tower of measures.

We state without proofs the main facts, which were partially reported in the previous article by author but under additional assumptions.

1. For a standard graph (projective limit of simplices), every ergodic measure on paths enjoys a concentration property: for every $\epsilon > 0$, for all sufficiently large n, the nth level vertices lying on a set of paths of measure $> 1 - \epsilon$ are contained in a ball of radius at most $\epsilon > 0$ in the intrinsic metric

(this is also called the "limit shape" property). This allows one, in the case of an arbitrary standard equipped graph, to search for all ergodic measures among the limits along paths in the intrinsic metric (rather than among the weak limits, according to the ergodic method). In the nonstandard situation, the ergodic method cannot be strengthened in this way: the set of weak limits in this case is in fact greater than the set of limits in the intrinsic metric.

2. The tail filtration on the path space of a standard graph with respect to every ergodic measure is standard in the metric sense. The definition of a standard filtration and criteria of standardness in the metric category were considered in author's papers. 3. The most important fact, which reproduces the theorem on lacunary isomorphism in the paper Vershik '68, but in the topological situation is as follows.

Theorem 13 (Lacunarization theorem)**.** *For every equipped graph* $(\Gamma = \bigcup_n \Gamma_n, \Lambda)$ *(respectively, for every projective limit of simplices* $\lim_n\{\Sigma_n, \{\pi_{n,m}\}_{n,m}\}$*), one can choose a subsequence of positive integers* n_k*,* $k = 1, 2, \dots$*, such that the equipped multi-graph* $\Gamma' = \bigcup_k \Gamma_{n_k}$ *obtained by removing all levels between* n_k *and* n_{k+1}*,* $k = 1, 2, \dots$*, and preserving all paths connecting them (respectively, the projective limit* $\lim_k\{\Sigma_{n_k}, \{\pi'_{k,s}\}_{k,s}\}$ *with the lumped system of projections* $\pi'_{k,s}$*, where*

$$\pi'_{k,k+1} = \prod_{i=n_k}^{i=n_{k+1}-1} \pi_{i,i+1})$$

is standard.

This means that standardness is a property of the projective limit, and not of the limiting simplex: by changing (lumping) the approximation one can change the intrinsic topology and make it equivalent to the projective limit topology, even if they were distinct before lumping.

The interrelations between standardness and property to be Bauer simplex of the limiting simplex need be further studied.

4.5 SHORT SURVEY OF FILTRATION THEORY:STANDARD FILTRATIONS AND SUPER-STRONG CONVERGENCE OF DECREASING MARTINGALES

We will give very schematic picture of the metric theory of filtration.

4.5.1 Filtrations,parametrization

Definition 14. $\{\xi_n; -n \in \mathbb{N}\}$*; sequence of random variables with values from* $[0,1]$*, A filtration is decreasing sequence of sigma-fields:*

$$\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \ldots.$$

The sigma field $\mathfrak{A}_0 = \mathfrak{A}$*. where sigma-field* $\mathfrak{A}_n$ *is generated with* $\xi_r, r < n$ *("past").*

Suppose that the filtration is ergodic (=Kolvogorovian = regular = satisfied to $0-1$ Law): if

$$\bigcap_n \mathfrak{A}_n = \mathfrak{N}$$

(trivial sigma-field).

We will consider as space $X = \prod_{n=1}^{\infty}[0,1]$ with any probability Borel measure μ, and as filtration the filtration $\{\mathfrak{A}_n\}_n$ where $\mathfrak{A}_n$ generated by condition on the coordinates with number $k > n$ where $n = 1, \ldots.$ We do not assume the stationarity of measures on X.

Now we define the main tool - dynamic of metric.

Choose any initial metric (or even generating semi-metric) — ρ_1 which is compatible with weak topology on $X = \prod_1^{\infty}[0,1]$.

Define sequence of consecutive (Kantorovich) metric by induction as follow:

$$\rho_{n+1}(\{x_m\}_m, \{x'_m\}_m) = k_{\rho_n}(\mu(. \mid x_{n+1}, x_{n+2}, \ldots), \mu(. \mid x'_{n+1}, x'_{n+2}, \ldots)),$$

where $\mu(.|...)$ is conditional measure on the $[0,1]^n$ under conditions ..., and $k_r(.,.)$ is Kantorovich distance between measures on the metric space with metric r, and $n = 1 \ldots.$

4.5.2 Standardness, Coupling

Definition 15. *(Theorem) The filtration, defined by sequence of random variables* $\{x_n\}_{-n\in\mathbb{N}}$ *called standard if it is ergodic, and the each of the following equivalent condition are true:*

1.For any metric ρ_1*,*

$$\lim_{n\to\infty} \int_X \int_X \rho_n(x,x')d\mu(x)d\mu(x') = 0$$

2.For some metric ρ_1 *the same is true.*

Definition 16. *Consider a filtration $\{\mathfrak{A}_k\}_{k=1}^n$ the cube $\prod_1^n[01]$ and define two measures ν_1^n, ν_2^n on the cube. The coupling is a measure Ψ_n on the space $\prod_1^n[01] \times \prod_1^n[01]$ which satisfies two conditions:*

1. has the measures ν_1^n and ν_2^n as marginal measures, and

2. "agree with filtrations or tower of coupling" which means that coordinate projection $\prod_1^n[01]\times\prod_1^n[01]$ to $\prod_1^{n-1}[01]\times\prod_1^{n-1}[01]$ sent measure Ψ_n to a measure Ψ_{n-1} which satisfies to the condition 1 with marginal measures ν_1^{n-1} and $\nu^n - 1_2$, which are coordinate projection of ν_1^n, ν_2^n on cube $\prod_1^{n-1}[01]$.

4.5.3 Criteria of standardness and super-strong convergence of the martingales, examples

Theorem 14. *The ergodic filtration $\{\mathfrak{A}_n\}_{n=1}^\infty$ with resect to a measure μ on cube $X = [0,1]^\infty$ is standard iff for any measurable function f on X (it is enough to get only cylindric functions with finitely many values) the following quantity:*

$$\inf_{\Psi_n} \int_{X_n} \int_{X_n} |f(x_1, \dots x_n, x_n{+}1 \dots) - f(x'_1, \dots x'_n, x'_{n+1}, \dots)| d\Psi_n(x_i, x'_i); (i \leq n)$$

as a function of the tails $\{x_k\}_{k>n}, \{x'_k\}_{k>n}$ tends (in measure $\mu \times \mu$) to zero. The inf *of Ψ in the formula above run over all coupling of two conditional measures: $\mu(.|x_{n+1}, \dots)$ and $\mu(.|x'_{n+1}, \dots)$.*

and for each $\epsilon > 0$ the number of the points in ϵ-net with respect to metric $\rho_n, n > 1$ is bounded for all n;

Definition 17. *A filtration called homogeneous if all conditional measures under condition $\{...|x_{n+1}, \dot\}$ are uniform measures on X_n (discrete or continuous) independently on the tail $\{x_{n+1}, \dot\}$*

and called semi-homogeneous is those measure are uniform but could be different for different tails.

Theorem 15. *Suppose that measure μ on space $X = [0,1]^\infty$ the filtration is homogeneous (f.e. dyadic), then μ is Bernoulli measure e.g. product-measure of one dimensional components, which could be various for various so called $\{r_n\}$-adic standard filtration; for dyadic case — $(1/2, 1/2)$). So this means that standardness is analogous of independence.*

This is a hard theorem which called criteria of standardness.

Corollary 1. *In the class finite isomorphism of homogeneous filtrations there is only one standard filtration (product type)*

The same is true for semi-homogeneous filtrations.

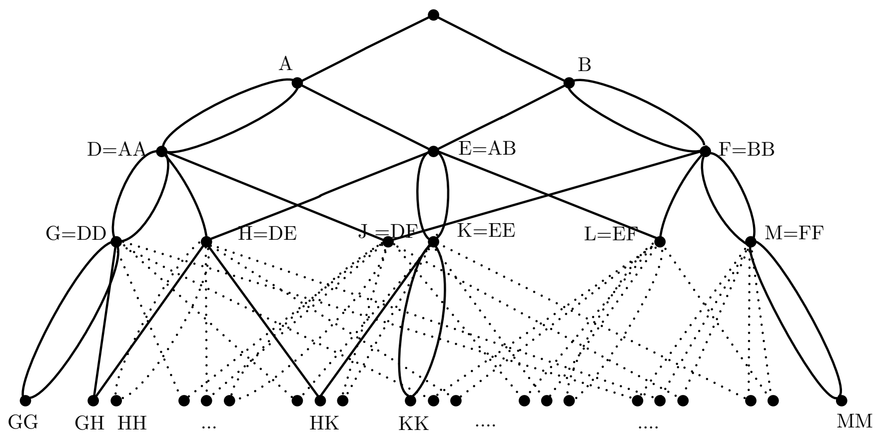


Figure 10: Unordered pairs graph

There are many homogeneous filtrations (f.e. dyadic filtrations) which are not standard. (Vershik 1970 and many recent papers). In the class of general (non semi-homogeneous) filtrations even filtration of the past of Markov process could be nonstandard filtration.

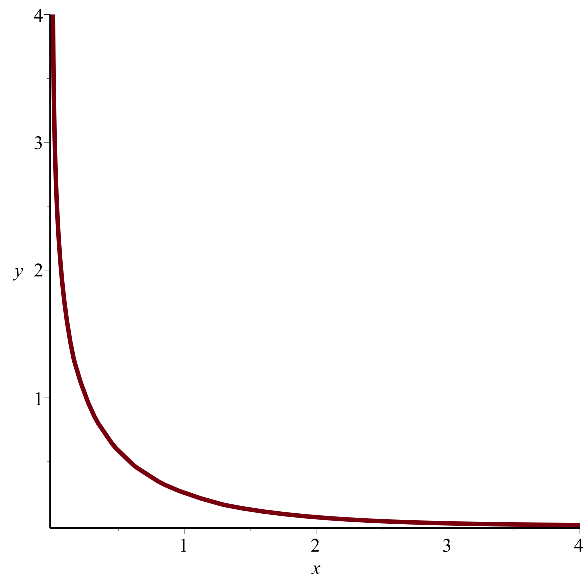
The notion of standardness has similarity with the Ornstein's notion of weak Bernoulli stationary measures. That notion is invariant formulation of Bernoulli automorphism. But this is not the same (even for stationary case as standardness. There are two difference: weak Bernoulli used the Hamming metric which is discrete at infinity, secondly the coupling in Ornstein case is arbitrary, but we use (condition 2 above) the tower of coupling in opposite to the definition of Kantorovich metric.

4.6 Limit Shape and Standardness

Now we return back to the tail filtrations of branching graphs.

Remark that tail filtration with respect to central measure is semi=homogneous filtration and we will consider only this case.

Theorem 16. *Consider the branching graph* Γ *and central measure* μ *on the space of paths* $T(\Gamma)$.



$$\exp(-(\pi/\sqrt(6)x)) + exp(-(\pi/\sqrt(6)y)) = 1$$

Figure 11: Limit shape for uniform statistics of diagram.

The following assertions are equivalent: 1.Tail filtration is standard with respect to the measure μ*;*

2.For each $\varepsilon > 0$ *there exist* N *such that for* $n > N$ *and for some vertex* $v_n \in \Gamma_n$*:*

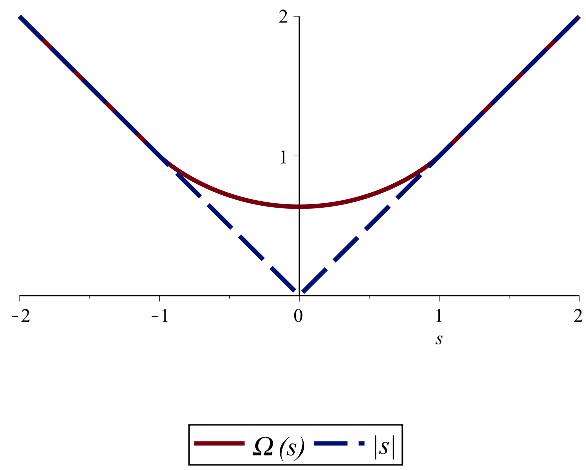
$$\mu\{\gamma \in \Gamma : r(\gamma_n, v_n) < \varepsilon\} > 1 - \varepsilon,$$

here r *is intrinsic metric of level* n *and* γ_n *is the vertex of the path* γ *on the level* Γ_n*. The second assertion usually called "limit shape theorem".*

This theorem express the behavior of the random path with respect to the central measure and could be considered as analogous Law of Large Number. Depending of metric this theorem shows how close the random path to the someone typical. It is possible to formulate individual limit shape theorem in which the convergence took place for almost all paths.

A central measure called standard if tail filtration is standard with respect to this measure.

Very important notion which concerns the central measures (especially standard measures) is notion of entropy. Partial case of this question —



$$\Omega(s) = \begin{cases} (2/\pi)(s \arcsin s + \sqrt{1-s^2}), \text{ for } \ |s| \le 1 \\ |s|, \text{ for } |s| \ge 1 \end{cases}$$

Figure 12: Limit shape for Plancherel measure.

about the entropy of Plancherel measure on the space of Young tableaux was suggested by author in 80-th and was published in the paper in '85 with S.V.Kerov ("entropy conjecture"). In that article the two-sided estimation of entropy was obtained and the question was about existence of limit of entropy a.e. Recently A.Bufetov had proved existence of limit in L^2.

The general problem in the spirit of theory by C.Shannon is the following:

Problem 2. *Consider a branching graph Γ and a central measure μ on the space of paths $T(\Gamma)$. Let μ_n is a projection of the measure μ on the level Γ_n and let $h_n = H(\mu_n$ is entropy of measure μ_n. Then does the limit*

$$\lim \frac{\ln(\mu(\gamma_n))}{h_n} = const$$

exists for almost all paths γ with respect to central measure μ? Here γ_n is the vertex of path γ on the n-th level. For the case of the Plancherel measure on Young graph we have is just $\sqrt{n}$:

$$\sum_{\lambda \vDash n} \frac{dim(\lambda)^2}{n!} \ln \frac{dim(\lambda)^2}{n!} \equiv E_{\mu_n}[\mu_n(\lambda)] \approx \sqrt{n}$$

So we obtain the conjecture:

$$\lim_n \frac{1}{\sqrt{n}} \ln \mu(t_n) = const, t_n \in Y_n, t \in T(Y),$$

for almost all infinite Young tableaux t in Plancherel measure μ.

It can happened that the positive answer takes place for standard central measures. Remark also the connection our problem with theory of entropy of random walks on the groups and graphs.

5 THE APPLICATIONS OF INVARIANT MEASURES TO THE CLASSIFICATION OF METRIC-MEASURE SPACES, AND MEASURABLE FUNCTIONS.

5.1 A GENERAL PROBLEM, ANSWER FOR INFINITE SYMMETRIC GROUP

Consider the action of the countable group G on the separable Borel space X. Our main problem is to describe all G-invariant ergodic measures on X or to distinguish the cases of the classification of the pairs (X, G) up to isomorphism of X which preserve the orbit partition of G. If we consider this problem up to *Borel isomorphism* of the space, then by Kechris's theorem all the cases with given number of invariant measures are isomorphic. From other side topological classification is too detailed in order to be useful. It is more reasonable question is to concentrate the efforts on the class of amenable groups G, or even more on the locally finite groups like infinite symmetric group. In this case we can distinguish the cases on "smooth" (standard) and "wild" (nonstandard) using a finite approximation of orbit partition, or using structure of associated branching graph (see above). More exactly we can consider the standard and non-standard approximations of orbit partition, and divide the cases depending on existence of standard approximation.

Consider for example diagonal (coordinate) action of the group $S_{\mathbb{N}}$ of all finite permutations on the space of infinite tensors $\{t_{i_1,i_2,\dots i_n}\}; i_k \in \mathbb{N}, k = 1, \dots n$ of rank n with values in any Borel space $E : t_{i_1,i_2,\dots i_n} \in E$ (it is enough to consider interval $E = [0,1]$ and most of interesting case $E = \{0;1\}$).

We can consider special types of tensors (symmetric, antisymmetric etc.) or consider multi-coordinate actions of group $S_{\mathbb{N}}^n$, for example separate action on each coordinate, etc. All this type of actions we called *tensor actions* of infinite symmetric group. Because of inductive (locally finite) structure of the group $S_{\mathbb{N}}$ the above problem about invariant measures could be included to the above context about the central measures in the space of paths of the branching graphs.

Theorem 17. *All tensor actions of infinite symmetric group with natural approximation are standard e.g. the list of all ergodic central measures is precompact.*

Remark 1. *1."Natural approximation" means that we consider approximation with inductive family of action of the finite group S_n on the space of finite dimensional tensors. For the group $S_{\mathbb{N}}$ the tensor actions plays the role which is similar to the role of actions with discrete spectra for commutative groups ($\mathbb{Z}$).*

2.There are many examples of non-standard actions of $S_{\mathbb{N}}$, f.e. Bernoulli action on the space $2^{S_{\mathbb{N}}}$. It is interesting problem to describe all standard actions of this group.

The reason why standardness took place related to that fact that action can be extended on the group $S^{\mathbb{N}}$ of all permutations of $\mathbb{N}$ (which is completion of the group $S_{\mathbb{N}}$ in weak topology) and the representations of this bigger group exhausted by tensor representations (G.Libermann). But here we have in a sense equivalence of these two facts: possibility to enumerate ergodic invariant measures and irreducible representations of the group $S^{\mathbb{N}}$.

The similar problem and corresponding result can be formulated for infinite unitary group and other inductive limits of finite or compact groups. There is also the link with the theory of representation of those groups.

5.2 Classification of functions and metrics, matrix distributions as random matrices.

We start with the following problem:

Problem

To classify measurable (or continuous, smooth etc.) functions of two or more arguments on some space with standard measure structure (Lebesgue space) upto metric isomorphism e.g. upto the group which is direct product of groups of measure preserving transformations acting on each variable separately.

More exactly: two real-valued measurable functions f, g on the Lebesgue space X with continuous measure μ are *separately metrically isomorphic* iff

$$g(x, y) = f(T_1 x, T_2 y)$$

and *jointly metrically isomorphic* iff

$$g(x, y,) = f(T_1 x, T_1 y),$$

where T_1, T_2 are the arbitrary invertible measure preserving transformations of (X, μ). In the last case it is natural to suppose that f, g are symmetric functions. It is clear how to put this problem for many arguments, in the case of more than two arguments it is possible to consider various conditions of symmetry, We return to this later. very important partial case of this problem is the following:

Very important special subcase:

Problem. *To classify Polish (=separable complete metric) spaces* (X, ρ, μ) *with metric* ρ *with Borel probability measure* μ *— upto measure preserving isometry:*

$$(X, \mu, \rho) \sim (X_1, \mu_1, \rho_1) \Leftrightarrow \rho_1(Tx, Ty) = \rho(x, y),$$

where $T : X \to X_1; T\mu = \mu_1$.

A metric space with measure M.Gromov called "mm-space", I used term "Gromov triple" or "metric triples". It is evident that that classification of the metric spaces is the same as classification of metrics as measurable functions of two variables on the square of the space, and this is partial case of the previous problem.

We will speak about the symmetric case only ($T_1 = T_2$).

The function f called *pure function* if equality $\mu\{x : f(gx, .) = f(x, .) \mod 0\} > 0$ valid only when $g = Id \mod 0$, and *completely pure* if equality $\mu\{(x, y) : f(gx, hy) = f(x, y) \mod 0\} > 0$ valid only when $g = h = Id$.

Let f is a real symmetric measurable function of two variables on the space $(X \times X, \mu \times \mu)$ with values in some standard borel space R and $(X^{\mathbf{N}}, \mu^{\mathbf{N}})$ is infinite direct product of domain spaces. Let $M_{\mathbf{N}}(R)$ is the space of all symmetric matrices with entries from R. We will omit $[0, 1]$ in the denotation: $M_{\mathbf{N}}([0, 1]) = M_{\mathbf{N}}$. Define a map:

$$F_f : X^{\mathbf{N}} \to M_{\mathbf{N}}(R)$$

by formula

$$F_f(x_1, \dots) = \{f(x_i, x_j)\}_{i,j=1}^{\infty}.$$

Definition 18. *A pushforward measure* $F_f(\mu^{\mathbf{N}})$ *on the space of matrices* $M_{\mathbf{N}}(R)$ *(image of the measure* $\mu^{\mathbf{N}}$*) under the map* F_f *we will call a matrix distribution of measurable function* f *and denote* D_f*.*

The matrix distribution is a generalization of the notion of distribution of the function of one variable. It is easy to check that the matrix distribution is ergodic $S_{\mathbb{N}}$-invariant measure with respect to diagonal action of the group on the space of matrices.

Theorem 18. *(Classification of pure measurable functions) Suppose f is pure symmetric measurable function on the space $(X \times X, \mu \times \mu)$ with values in a standard borel space R, then matrix distribution measure D_f is complete invariant of the function f in the sense of the equivalence above. In another words*

1)If two (not necessary pure) real functions f_1 and f_2 are isomorphic then $D_{f_1} = D_{f_2}$;

and

2)If for two pure measurable functions f_1, f_2 which are defined in the spaces $(X_1 \times X_1, \mu_1 \times \mu_1)$ and $(X_2 \times X_2, \mu_2 \times \mu_2)$ correspondingly and have the same measures $D_{f_1} = D_{f_2}$ on the space $M_{\mathbf{N}}(\mathbf{R})$ then they are isomorphic e.g there exist measure preserving automorphisms $T : (X_1, \mu_1) \to (X_2, \mu_2)$ such that $f_2(x, y) = f_1(Tx, Ty)$ for almost all $(x, y) \in (X_2 \times X_2)$

The special case of the problem which was posed by M.Gromov.

Theorem 19. *(Gromov [**?**], Vershik [**?**]) The Gromov's proof used some analytic tools, the proof by Vershik is very simple and based on individual ergodic theorem.*

Let (X, ρ, μ) a metric separable space with non-degenerated Borel probability measure (=there is no nonempty open set with positive measure). Then matrix distribution D_ρ is complete invariant with respect to measure preserving isometries. Here matrix distribution D_ρ is a measure on the space of infinite distance matrices $\{r_{i,j}\}$ which is the image of the map $F_\rho : X^\infty \to M_N(\mathbb{R})$: $F_\rho(\{x_n\}) = \{\rho(x_i, x_j)\}$.

The matrix distributions of the of symmetric function of two variables (f.e. metrics) is special class of of all ergodic invariant measures on the space $M_{\mathbb{N}}^{Symm}$ it could be directly characterized. We consider here only the case of metrics as measurable functions of two variables.

The previous theorem gives the uniqueness upto measure preserving isometry of the mm-space with given matrix distribution. More difficult problem is about existence of mm-space with given matrix distribution.

Denote as $\mathcal{R}$ the closed cone of all distance matrices in $M_\infty(\mathbb{R}_+)$:

$$\mathcal{R} = \{\{r_{i,j}\} : r_{i,j} \geq 0, r_{i,j} = r_{j,i}, \quad r_{i,j} + r_{j,k} \geq r_{i,k}, i, j, k \in \mathbb{N}\}$$

Theorem 20. *(Characterization of the matrix distributions (existence of of mm-spaces). see [?, ?].*

A probability measure τ on the space of distance matrices $\mathcal{R}$ is the matrix distribution $D(\rho)$ of some metric of the mm-space (X, ρ, μ) iff the following conditions are valid:

1)the measure τ is invariant under the action of infinite symmetric group $S_{\mathbb{N}}$ by simultaneous permutation of the rows and columns;

2) the measure τ is "simple" which means: the following map T is isomorphism between measure space $(\mathcal{R}, \tau)$ and measure space $(Meas[R^{\infty}], \theta \equiv T_*\tau)$, where $Meas[R^{\infty}]$ is the space of all Borel probability measures on the space R^{∞} and the map T corresponds to the matrix $r \equiv \{r_{i,j}\}$ the empirical distribution of the columns (or rows) of the matrix r and $T_*\tau)$ is the T-image of the measure τ.

Remark that the map T is well-defined on the st of full τ-measure.

Proof. Necessity of the first condition is trivial because of definition of the matrix distribution $D(\rho)$. The condition 2 means that the almost all matrices with respect to measure $D(\rho$ could be restored if we know empirical joint distribution of the distances from one given point. In another words it means that with probability 1 the coincidence of the joint distributions of the sequence of functions of the second argument $\{\rho(x_i, .)\}_i$ and $\{\rho(x'_i, .)\}_i$ implies the equality $x_i = x'_i, i = 1 \dots$. But this follows from pureness of the function f (see details in [?, ?]).

Suppose we have simple $S_{\mathbb{N}}$-invariant measure τ on the cone $\mathcal{R}$. Then for τ-almost all matrices r which is distance matrix on $\mathbb{N}$ we can define a metric space X_r which is completion of the metric space $(\mathbb{N}, r)$ with respect to metric r, and to define a unique Borel measure μ_r on X_r. The group $g \in S_{\mathbb{N}}$ naturally acts on $\mathbb{N}$ and consequently this action can be extended by continuity on the space X_r. Denote the extension of the metric r onto X_r as ρ_r. We obtain mm-space (X_r, ρ_r, μ_r). Remark that upto measure preserving isometry triple (X_r, ρ_r, μ_r) does not depend on matrix r. Indeed, if we have another matrix of type $r' = grg^{-1}$, $g \in S_{\mathbb{N}}$ then action of g can be extended to X_r as measure preserving isometry. But because of ergodicity of action of $S_{\mathbb{N}}$ in X_r for τ-almost all distance matrix r and for τ-almost all distance matrix r' we can choose in X_r an every where dense sequence of of points $\{x_i\}$ whose distance matrix is r'. In order to finish the proof it is enough to prove that the matrix distribution of triple (X_r, ρ_r, μ_r) is measure τ. $\square$

5.3 Aldous theorem and its connection with the problem of the classification of functions of several arguments

The remarkable result by Aldous ('81) as well as some of subsequent papers by R.Hoover, O.Kallenberg, T.Tsankov, T.Ostin etc. gave the list of such invariant measures for the group $S_{\mathbb{N}}$. The answer express with help of some measurable function in undefined space. But the missing link is absence of canonical form of the answer and mainly the absence of the connection with intimately related classification problem for the functions and other objects which plays role of dual problem in a sense. In my papers (2002,2010) was done an example of such approach which looks like generalization of "ergodic" method.

More general problem of the classification of measurable functions of several arguments is the following: (we formulate it for simplicity for the functions of three argument)

Problem Consider the space of all measurable functions of three variables $(x_1, x_2, x_3) \mapsto f(x_1, x_2, x_3), x_i \in X, i = 1, 2, 3$, ($(X, \mu)$ is a standard measure space), which is symmetric in the sense $f(x_1, x_2, z) = f(x_2, x_1, z)$, and the following equivalence relation:

$$g \sim f \Leftrightarrow g(x_1, x_2, x_3) = f(T_1 x_1, T_2 x_2, S_{x_1,x_2} x_3),$$

where T_1, T_2, S_{x_1,x_2} are measure preserving transformation of (X, μ), and $\{S_{x_1,x_2}\}$ is a measurable function on $(X, \mu)\}^2$ with values in $Aut(X, \mu)$.

Consequently, the group of automorphisms which stays behind the classification is the group of skew-product

$$\{Aut(X, \mu)\}^2 \leftthreetimes (X^2 \to Aut(X, \mu)) \subset Aut(X, \mu)^3.$$

We called this equivalence as $(2-1)$-equivalence.

It is more convenient to reformulate this problem as a problem of the classification of the functions of two variables with values in the classes of metric equivalence of the functions of one variable (roughly speaking — with values in Borel probability measures on some Borel space).

Let f is a measurable function on the spaces $(X, \mu)^3$ with values in the standard Borel space E. Define a map

$$\{x_i, z_{i,j}\}_{i,j} \mapsto \{f(x_i, x_j, z_{i,j})\}_{i,j},$$

where $\{x_i\}, \{z_{i,j}\}$ are mutually independent random variables ($z_{i,j} = z_{j,i}$) with values in X and with common distribution — μ. The images of the sequences under this map are the matrix from space of matrices $M_{\mathbb{N}}(E)$, and the image Df of the measure $\mu^N \times \mu^{N^2}$ we called it "matrix distribution" of the function f (in framework of given problem).

The generalization of Aldous theorem can be formulated as follow:

Theorem 21. *1. The complete invariant with respect to* $2-1$ *classification of the measurable pure function* $f : X^3 \to E$ *of three variables with values in the space* E *is the matrix distribution* Df *of* f*;*

2. Each ergodic $S_{\mathbb{N}}$ *-invariant (with respect to diagonal action of he group) measure on the space of matrices* $M_{\mathbb{N}}(E)$ *with entries in a Borel space* E*; is the matrix distribution of some function* f *which is unique up to* $(2-1)$*-equivalence.*