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INTEGRATION OF VIRTUALLY CONTINUOUS FUNCTIONS OVER BISTOCHASTIC MEASURES AND THE TRACE FORMULA FOR NUCLEAR OPERATORS

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To Nina Nikolaevna Uraltseva on the occasion of her anniversary

ABSTRACT. Birman's definition of the integral trace of a nuclear operator as an integral over the diagonal is linked to the recent concept of virtually continuous measurable functions of several variables [2, 3]. Namely, it is shown that the construction of Birman is a special case of the general *integration of virtually continuous functions over polymorphisms (or bistochastic measures)*, which in particular makes it possible to integrate such functions over some submanifolds of zero measure. Virtually continuous functions have similar application to embedding theorems (see [2]).

§1. INTRODUCTION

In the paper [1] unpublished in his lifetime, M. S. Birman provided a meaning for the integral of the kernel $\mathcal{K}$ of a nuclear operator $K \in S_1$ over a diagonal measure on $X \times X$.[1]

Since any nuclear operator is represented as the product of two Hilbert–Schmidt operators, we can represent its kernel (by changing it on a null set) in the form suitable for integration over the diagonal. The value of this integral is independent of a factorization, but this needs a proof. We show that the reason for this independence is the fact that the kernel is a *virtually continuous function*, hence admitting integration not only over the diagonal, but also over a wide class of submanifolds of zero measure — more rigorously, over the polymorphisms, see [6, 7]. In §2 we recall the basic definitions from the theory of virtual continuity, see [2, 3], and introduce the necessary spaces and norms. In §3 we recall (without proof) the main duality theorem of [3].

The main property of virtually continuous functions, which is most completely expressed by the duality theorem, is that they admit integration over some subsets of zero measure. In the last section we show that this makes it possible to get the result of [1].

§2. VIRTUAL CONTINUITY OF FUNCTIONS OF TWO VARIABLES

We recall the necessary definitions from [2, 3]:

Definition 1. An *admissible metric* (or semimetric) ρ on a standard measure space $(X, \mathfrak{A}, \mu)$ is a measurable function on $X \times X$ with the following property: there exists

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[1]D. R. Yafaev turned our attention a similar construction that was given in the Appendix (without proofs) to M. A. Shubin's book [9].

a measurable subset $X_0 \subset X$ of full measure, $\mu(X \setminus X_0) = 0$, such that the metric (respectively, semimetric) space (X_0, ρ) is separable.

A standard measure space (X, μ) with admissible (semi)metric ρ is called an *admissible metric triple* or simply an *admissible triple* (X, μ, ρ).

Definition 2. A measurable function $f(\,\cdot\,,\,\cdot\,)$ on the product of measure spaces $(X, \mu) \times (Y, \nu)$ is said to be *properly virtually continuous* if for any $\varepsilon > 0$ there exist subsets $X' \subset X$, $Y' \subset Y$ of almost full measure (i.e. $\mu(X \backslash X') < \varepsilon$, $\mu(Y \backslash Y') < \varepsilon$), and admissible semimetrics ρ_X, ρ_Y on X', Y' (respectively) such that the function f is continuous on $(X' \times Y', \rho_X \times \rho_Y)$. *A virtually continuous* function is a function coinciding with some properly virtually continuous function on a full measure subset of $X \times Y$. Virtually continuous functions of several variables are defined similarly.

The class of virtually continuous functions is a linear subspace of the linear space of all measurable functions. It is invariant under all quasi-measure preserving maps of the form $T_1 \times T_2$, where T_1, T_2 are quasiautomorphisms of the measure spaces $(X, \mu), (Y, \nu)$, respectively. However, it is not invariant under arbitrary measure preserving maps of the space $(X \times Y, \mu \times \nu)$.

Now we specify an important subclass of virtually continuous functions. The functions of the form $f(x, y) = a(x) + b(y)$ are *separate*. The next construction defines a norm (the so-called *norm with regulator*) of a function of two variables, where the regulator is a separate function and the norm is taken in L^1.

Definition 3. For a measurable function f on the space $(X \times Y, \mu \times \nu)$, we define the following (finite or infinite) norm:

$$\|f\|_{\mathrm{SR}^1} := \inf \Big\{ \int_X a(x)\, d\mu(x) + \int_Y b(y)\, d\nu(y) \;: \\ a(x) \geq 0, b(y) \geq 0, |f(x, y)| \leq a(x) + b(y) \text{ a.e.} \Big\}.$$

Let VC^1 denote the space of virtually continuous functions with finite SR^1-norm. This space turns out to be complete (see [2, 3]), i.e., virtual continuity is SR^1-limit-preserved.

The space VC^1 can be described as the closure in SR^1 of "rectangular-step" functions.

Theorem 1 ([2], Theorem 10)**.** *The subspace of finite linear combinations of characteristic functions of all sets of the form $X_1 \times Y_1$, where $X_1 \subset X$ and $Y_1 \subset Y$ are arbitrary measurable subsets, is dense in the space VC^1 (in the SR^1-norm).*

What is the dual space of VC^1? Roughly speaking, what are the objects with which we may couple (integrate) such functions? A related question arose in the generalization of the Kantorovich optimal transportation theory. The Kantorovich theorem about duality between an appropriate space of measures and the space of Lipschitz functions has a side interpretation: the optimal price may be viewed as the norm of a metric, and this norm coincides with the norm defined above. Thus, the duality theorem presented below can be viewed as a generalization of the Kantorovich duality to cost functions different from metrics. This question was considered in [8, 5]. However, in those works, only part of the corresponding duality was described; in our terms this is the part corresponding to absolutely continuous quasibistochastic measures (in other words, polymorphisms, couplings, transport plans, etc., see [7, 6]). The answer is that the space dual to VC^1 is the space of *quasibistochastic polymorphisms*, as defined in the next section.

§3. Quasibistochastic signed measures and duality theorem

Definition 4. Let η be a finite measure on the product $X \times Y$ of measure spaces $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$, defined on the direct product of sigma-algebras $\mathfrak{A} \times \mathfrak{B}$. We say that it is *bistochastic* if $\eta(X_1 \times Y) = \mu(X_1)$, $\eta(X \times Y_1) = \nu(Y_1)$ for all measurable subsets $X_1 \subset X$, $Y_1 \subset Y$. Equivalently, it is bistochastic if and only if the pushforwards $P^x_*\eta, P^y_*\eta$ under the projections P_x, P_y of the direct product $X \times Y$ onto X, Y coincide with the measures μ and ν, respectively.

If the projections mentioned above are absolutely continuous with respect to μ, ν and have bounded Radon–Nikodym density, then we say that the measure η is *quasibistochastic*. A signed measure η is *said to be quasibistochastic* if its variation $|\eta|$ is quasibistochastic.

We refer the reader to [7] for more information on bistochastic measures and the relationship with the theory of polymorphisms.

The space QB^∞ of quasibistochastic signed measures admits a norm defined as an essential supremum of the densities of projections of the total variation:

$$\|\eta\|_{\mathrm{qb}} = \max\left(\sup_{X_1\subset X} |\eta|(X_1 \times Y)/\mu(X_1), \sup_{Y_1\subset Y} |\eta|(X \times Y_1)/\nu(Y_1)\right) = \max\left\{\Big\|\frac{\partial P^x_*|\eta|}{\partial\mu}\Big\|_{L^\infty(X,\mu)}, \Big\|\frac{\partial P^y_*|\eta|}{\partial\nu}\Big\|_{L^\infty(Y,\nu)}\right\}.$$

Definition 5. A signed measure η on $X\times Y$ is *subbistochastic* if $\|\eta\|_{\mathrm{qb}} \le 1$. A measurable function $h(\,\cdot\,,\,\cdot\,)$ on the space $(X \times Y, \mu \times \nu)$ is *subbistochastic* if the signed measure $h(\,\cdot\,,\,\cdot\,)\mu \times \nu$ is subbistochastic. We denote the set of all subbistochastic functions by $\mathcal{S}$. In other words, $\mathcal{S}$ is the "absolutely continuous" part of the unit ball of the space QB^∞.

Note that a measure η is subbistochastic if and only if a signed measure $\eta_0 - \eta$ is nonnegative for some bistochastic measure η_0.

The bistochastic (and quasibistochastic) (signed) measures may be viewed as a "multivalued" generalization of the measurable maps from (X, μ) to (Y, ν) that transfer the measure μ to ν (respectively, to a measure absolutely continuous with respect to ν). In this interpretation, which generalizes the theory of dynamical systems, a (quasi)bistochastic measure is a polymorphism from (X, μ) into (Y, ν), see [7].

But here we need a direct interpretation of (quasi)bistochastic measures as functionals on appropriate spaces of functions on $X \times Y$. So, the SR^1-norm defined above can be defined in a dual way as the supremum of couplings with absolutely continuous (with respect to $\mu \times \nu$) subbistochastic measures.

Theorem 2. *For any measurable function f on the space $(X \times Y, \mu \times \nu)$ we have*

$$\|f\|_{\mathrm{SR}^1} = \sup\left\{\iint_{X\times Y} |f(x, y)|h(x, y)\, d\mu(x)\, d\nu(y) \;:\; h \in \mathcal{S}\right\}. \tag{1}$$

This theorem was proved in [2] (see Theorem 8 therein) by using the Komlos lemma. Earlier, Kellerer [4] established a more general duality theorem in the language of the descriptive set theory, see [5, Theorem 2.4.3].

The following theorem describes the Banach space dual to VC^1.

Theorem 3 ([2], [3, Theorem 11])**.** *The space dual to VC^1 is QB^∞. The action of a signed measure η on a virtually continuous function f is defined as $\int \widetilde{f}\, d\eta$, where $\widetilde{f}$ is a proper virtually continuous function equivalent to f.*

In a sense, the classical Kantorovich duality is a "factorization" of duality between VC^1 and QB^∞, see [3].

If two proper virtually continuous functions $\widetilde{f}_1$ and $\widetilde{f}_2$ on $X \times Y$ coincide $(\mu \times \nu)$-a.e., then there exist subsets $X_0 \subset X$, $Y_0 \subset Y$ of full measure such that $\widetilde{f}_1 = \widetilde{f}_2$ on $X_0 \times Y_0$ (see [2, Proposition 6]). Thus, the action of a signed measure η with finite norm $\|\eta\|_{\mathrm{qb}}$

on a virtually continuous function f does not depend on the choice of $(\mu \times \nu)$-equivalent proper virtually continuous function $\widetilde{f}$; hence, it is well defined.

Consider a particular case: $(X, \mu) = (Y, \nu)$ and η is a diagonal measure (the push-forward of μ under the map $x \mapsto (x, x)$). Since it is bistochastic and $\|\eta\|_{\mathrm{qb}} = 1$, the integral $\int_X f(x,x)\, d\mu(x) = \int_{X\times Y} f\, d\eta$ is well defined for any virtually continuous function $f \in VC^1$.

Here is a criterion of $*$-weak convergence for quasibistochastic polymorphisms.

Proposition 1. *A sequence of quasibistochastic polymorphisms η_n on $X \times Y$ $*$-weakly converges to a polymorphism η in the space $QB^\infty = (VC^1)^*$ if and only if*

(i) *the norms $\|\eta_n\|_{\mathrm{qb}}$ are uniformly bounded, and*

(ii) *for any measurable rectangle $R = X_1 \times Y_1$ we have the convergence of measures $\eta_n(R) \to \eta(R)$.*

Proof. The boundedness of the norms follows from weak convergence. Condition (ii) implies the convergence of couplings with the characteristic functions of measurable rectangles. The linear combinations of such functions are dense in VC^1; thus, whenever the norms are bounded, (ii) is necessary and sufficient for weak convergence. □

§4. Approximation of the diagonal and Birman's theorem

Now we show how the approach of [1] to regularization of the integral over the diagonal is related to the above theory.

Let $K \in S_1$ be a nuclear operator in the Hilbert space $L^2(X, \mu)$, and let $\mathcal{K}$ be its kernel. As in [1], we define a regular kernel corresponding to the operator K.

Definition 6. A measurable function $\mathcal{K}_0$ defined on a square $X' \times X'$ of some subset $X' \subset X$ of full measure is called a *regular kernel* corresponding to the operator $K \in S_1$ if there exist two Hilbert–Schmidt operators $L, M \in S_2$ with kernels $\mathcal{L}, \mathcal{M} \in L^2(X \times X, \mu \times \mu)$, respectively, such that $K = LM$ and

$$\mathcal{K}_0(x, y) = \int_X \mathcal{L}(x, z)\mathcal{M}(z, y)\, d\mu(z) \tag{2}$$

for all $x, y \in X'$.

Clearly, the functions $\mathcal{K}$ and $\mathcal{K}_0$ coincide a.e. on $X \times X$. Note that, in general, a regular kernel for a nuclear operator is not unique, because the operator may admit different factorizations into two Hilbert–Schmidt operators. But, as we shall see below, any two regular kernels coincide on the square of a full measure set.

Proposition 2. *A regular kernel $\mathcal{K}_0$ of a nuclear operator K is a properly virtually continuous function, and the kernel $\mathcal{K}$ is a virtually continuous function.*

Proof. Let $L, M \in S_2$ be Hilbert–Schmidt operators with kernels $\mathcal{L}, \mathcal{M} \in L^2(X \times X, \mu \times \mu)$, respectively, such that $K = LM$ and (2) is true. Consider two semimetrics on X:

$$\rho_{\mathcal{L}}(x_1, x_2) = \|\mathcal{L}(x_1, \cdot\,) - \mathcal{L}(x_2, \cdot\,)\|_{L^2(X,\mu)}; \tag{3}$$

$$\rho_{\mathcal{M}}(x_1, x_2) = \|\mathcal{M}(\,\cdot, x_1) - \mathcal{M}(\,\cdot, x_2)\|_{L^2(X,\mu)}. \tag{4}$$

By the Fubini theorem, they are well defined on some subset $X' \subset X$ of full measure. The semimetric spaces $(X', \rho_{\mathcal{L}})$ and $(X', \rho_{\mathcal{M}})$ are separable, because the Hilbert space $L^2(X, \mu)$ is separable, so that the triples $(X', \mu, \rho_{\mathcal{L}})$ and $(X', \mu, \rho_{\mathcal{M}})$ are admissible.

The continuity of the function $\mathcal{K}_0(x,y)$ with respect to the metric $\rho_{\mathcal{L}} \times \rho_{\mathcal{M}}$ on $X' \times X'$ follows from the following estimate, which is true for all $x, x', y, y' \in X'$:

$$
\begin{aligned}
&|\mathcal{K}_0(x,y) - \mathcal{K}_0(x',y')| \le |\mathcal{K}_0(x,y) - \mathcal{K}_0(x',y)| + |\mathcal{K}_0(x',y) - \mathcal{K}_0(x',y')| \\
&\le \int_X |\mathcal{L}(x,z) - \mathcal{L}(x',z)|\, |\mathcal{M}(z,y)|\, d\mu(z) + \int_X |\mathcal{L}(x',z)|\, |\mathcal{M}(z,y) - \mathcal{M}(z,y')|\, d\mu(z) \\
&\qquad \le \rho_{\mathcal{L}}(x,x') \|\mathcal{M}(\,\cdot\,,y)\|_{L^2(X)} + \rho_{\mathcal{M}}(y,y') \|\mathcal{L}(x',\,\cdot\,)\|_{L^2(X)} \\
&\qquad \le \rho_{\mathcal{L}}(x,x') \|\mathcal{M}(\,\cdot\,,y)\|_{L^2(X)} + \rho_{\mathcal{M}}(y,y') \left(\|\mathcal{L}(x,\,\cdot\,)\|_{L^2(X)} + \rho_{\mathcal{L}}(x,x') \right).
\end{aligned}
$$

So, the function $\mathcal{K}_0$ is properly virtually continuous, and the $(\mu\times\mu)$-equivalent function $\mathcal{K}$ is virtually continuous. □

Birman interpreted the integral of the kernel $\mathcal{K}$ of the nuclear operator K as the integral of the regular kernel $\mathcal{K}_0$. In [2], the authors defined the integral of a virtually continuous function over a measure as the integral of an equivalent proper virtually continuous function. In the case of integration of kernels of nuclear operators and diagonal measures, this coincides with Birman's approach.

Birman suggested to calculate the trace of a nuclear operator as the limit of averages over the neighborhood of the diagonal.

Theorem 4 (Birman [1])**.** *If (X,μ) is $\mathbb{R}^m$ with Lebesgue measure, and $\mathcal{K}$ is a regular kernel of a nuclear operator in $L^2(X,\mu)$, then for the integral over the diagonal we have*

$$
\int_X \mathcal{K}(x,x)\, d\mu(x) = \lim_{h\to 0+} \frac{1}{\kappa_m h^m} \iint_{|x-y|\le h} \mathcal{K}(x,y)\, d\mu(x)\, d\mu(y),
$$

where κ_m is the Lebesgue measure of the unit ball in $\mathbb{R}^m$.

In the context of virtual continuity, this is a claim about the $*$-weak convergence of absolutely continuous measures in the neighborhood of the diagonal to the measure on the diagonal.

First, we recall a result on embedding of the space of nuclear operators to VC^1.

One of applications of virtual continuity given in [2] is the following embedding theorem.

Theorem 5 ([2, Theorem 14])**.** *The map $K \mapsto \mathcal{K}$ that takes an integral operator K to its kernel $\mathcal{K}$ is an embedding with norm 1 of the space $S_1(L^2(X))$ to the space VC^1.*

The main theorem of [1] cited above (Theorem 4) is a direct consequence of this embedding and the following general fact on $*$-weak convergence of polymorphisms to the diagonal measure.

Proposition 3. *Let (X,ρ,μ) be a separable metric space with sigma-compact Borel measure μ. Let η be a measure on the diagonal* $\mathrm{diag} = \{(x,x) : x \in X\} \subset X \times X$ *defined as a pushforward of the measure μ under the map $x \mapsto (x,x)$. Assume that a sequence of bistochastic measures η_n on $X \times X$ is such that $\eta_n(K) \to 0$ for any compact subset $K \subset X^2 \setminus \mathrm{diag}$. Then the sequence of measures η_n converges to the measure η in $QB^\infty = (VC^1)^*$ in the $*$-weak topology. In other words, for any function $f \in VC^1$ we have $\int_{X^2} f\, d\eta_n \to \int_{X^2} f\, d\eta$.*

Proof. We check the conditions of Proposition 1. Since the bistochastic measures have norm 1, it remains to check condition (ii). Fix a rectangle $R = X_1 \times Y_1 \subset X^2$ and $\varepsilon > 0$. The rectangle R is partitioned onto 4 rectangles. One of them is the square $(X_1 \cap Y_1)^2$, and the other three do not meet the diagonal. For a bistochastic measure λ we have

$$
\lambda(X_1 \times X_1) = \mu(X_1) - \lambda\left(X_1 \times (X \setminus X_1)\right),
$$

so that the convergence of values of bistochastic measures on squares reduces to convergence on rectangles that do not meet the diagonal. So, we may suppose that our rectangle $R = X_1 \times Y_1$ does not meet the diagonal. By inner regularity, there exist compact subsets $X_2 \subset X_1$, $Y_2 \subset Y_1$ such that $\mu(X_1 \setminus X_2) + \mu(Y_1 \setminus Y_2) < \varepsilon/2$. Define a compact rectangle $R' = X_2 \times Y_2$. For any bistochastic measure λ we have

$$\lambda(R) \leq \lambda(R') + \lambda\left((X_1 \setminus X_2) \times X\right) + \lambda\left(X \times (Y_1 \setminus Y_2)\right) \leq \lambda(R') + \varepsilon.$$

By our assumption $\eta_n(R') \to 0$; hence, $\limsup \eta_n(R) \leq \varepsilon$. Since ε is arbitrary, we get $\lim \eta_n(R) = 0 = \eta(R)$, as required. □

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